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COMPLETE PERIODICITY OF PRYM EIGENFORMS

BY Erwan LANNEAU AND Duc-Manh NGUYEN

Abstract. – This paper deals with Prym eigenforms which are introduced previously by McMullen.We prove several results on the directional flow on those surfaces, related to complete periodicity (introduced by Calta). More precisely we show that any homological direction is algebraically periodic, and any direction of a regular closed geodesic is a completely periodic direction. As a consequence we draw that the limit set of the Veech group of every Prym eigenform in some Prym loci of genus $3, 4$, and 5 is either empty, one point, or the full circle at infinity. We also construct new examples of translation surfaces satisfying the topological dichotomy (without being lattice surfaces). As a corollary we obtain new translation surfaces whose Veech group is infinitely generated and of the first kind.

Résumé. – Dans cet article nous démontrons plusieurs résultats topologiques sur les formes propres des lieux Prym, formes différentielles abéliennes découvertes par McMullen dans des travaux antérieurs. Nous obtenons une propriété dite de complète périodicité (introduite par Calta), ainsi que de nouvelles familles de surfaces de translation vérifiant la dichotomie topologique de Veech (sans être une surface de Veech) . Comme conséquences nous montrons que l'ensemble limite des groupes de Veech de formes propres de certaines strates en genre $3, 4$, et 5 est soit vide, soit un point, soit tout le cercle à l'infini. Ceci nous permet de plus de construire de nouveaux exemples de surfaces de translation ayant un groupe de Veech infiniment engendré et de première espèce.

Notre preuve repose sur une nouvelle approche de la notion de feuilletage périodique par les involutions linéaires.

1. Introduction

1.1. Periodicity and Algebraic Periodicity

In his 1989 seminal work [35], Veech introduced an important class of translation surfaces (now called *Veech surfaces*) providing first instances of translation surfaces whose directional flows satisfy a remarkable property: for a given direction, the flow is either uniquely ergodic (all the flow lines are dense and uniformly distributed) or completely periodic (all the flow lines are closed or a saddle connection). This property is subsequently called the *Veech* *dichotomy*. Since then numerous efforts have been made in the study of the linear flows on translation surfaces, to name a very few: [22, 30, 27, 31, 11, 7]. Veech's theorem raised the issue of what can be said about the dynamics of the directional flows on non Veech surfaces.

This paper deals with the question of completely periodic linear flows. This aspect has been initiated in [5], and then developed later in [6]. A useful invariant to detect completely periodic flows (i.e., all the flow lines are closed or connect singularities), introduced in Arnoux's thesis [1], is the Sah-Arnoux-Fathi (SAF) invariant. It is well known that the linear flow $\mathcal{F}_\theta$ in a direction $\theta \in \mathbb{RP}^1$ on a translation surface (X,ω) (equipped with a transversal interval I) provides an interval exchange transformation T_θ, which is the first return map to I. The invariant of the flow in direction θ can be informally defined by

$$SAF(T_\theta) = \int_I 1 \otimes (T_\theta(x) - x)dx \in \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$$

(the integral is actually a finite sum). If $\mathcal{F}_\theta$ is periodic, that is when every leaf of $\mathcal{F}_\theta$ is either a closed curve or an interval joining two zeros of ω, then $SAF(T_\theta) = 0$. However the converse is not true in general. Following this remark, the direction θ will be called *algebraically periodic* if the SAF-invariant of the flow $\mathcal{F}_\theta$ vanishes.

A translation surface (X,ω) is *completely periodic (in the sense of Calta)* if for every $\theta \in \mathbb{RP}^1$ for which $\mathcal{F}_\theta$ has a closed regular orbit, the flow $\mathcal{F}_\theta$ is completely periodic. We have the corresponding "algebraic" definition: the surface (X,ω) is *completely algebraically periodic* if the SAF-invariant of $\mathcal{F}_\theta$ vanishes in any homological direction ($\theta \in \mathbb{RP}^1$ is *homological* if it is the direction of a vector $\int_c \omega \in \mathbb{C} \simeq \mathbb{R}^2$ for some $c \in H_1(X,\mathbb{Z})$). These notions are introduced in [5] and [6].

Flat tori and their ramified coverings are both completely periodic and completely algebraically periodic; in this case, up to a renormalization by $\mathrm{GL}^+(2,\mathbb{R})$, the set of homological directions is $\mathbb{Q} \cup \{\infty\}$. In [5], Calta proved that these two properties also coincide for genus 2 translation surfaces, in which case the set of homological directions is $K\mathbb{P}^1$, where K is either $\mathbb{Q}$ or a real quadratic field over $\mathbb{Q}$, and moreover a surface in $\mathcal{H}(2)$ is completely periodic if and only if it is a Veech surface (see also [25]). However there are completely periodic surfaces in $\mathcal{H}(1,1)$ that are not Veech surfaces (actually, most of them are not Veech surfaces).

We will say that a quadratic differential is algebraically completely periodic (respectively, completely periodic in the sense of Calta) if its orientation double cover is. Translation surfaces in genus two are closely related to quadratic differentials over $\mathbb{CP}^1$, since we have the following identifications (which are $\mathrm{GL}^+(2,\mathbb{R})$ invariant) $\mathcal{H}(2) \simeq \mathcal{Q}(-1^5,1)$, $\mathcal{H}(1,1) \simeq \mathcal{Q}(-1^6,2)$. Note that $\dim_{\mathbb{C}} \mathcal{Q}(-1^5,1) = 4$, and $\dim_{\mathbb{C}} \mathcal{Q}(-1^6,2) = 5$. We record all strata of quadratic differentials of dimension 5 in Table 1. In this paper, our first aim is to extend Calta's result to all of these strata.

THEOREM A. – *Let (Y,q) be quadratic differential in one of the strata in Table 1. If (Y,q) is completely algebraically periodic then it is completely periodic in the sense of Calta.*

THEOREM B. – *Let K be either $\mathbb{Q}$ or a real quadratic field. For any stratum $\mathcal{Q}(\kappa)$ in Table 1, the set of algebraically completely periodic quadratic differentials in $\mathcal{Q}(\kappa)$, with homological directions in $K\mathbb{P}^1$ up to renormalization by $\mathrm{GL}^+(2,\mathbb{R})$ is a union of $\mathrm{GL}^+(2,\mathbb{R})$-invariant* *submanifolds of complex dimension* 3*. Such invariant submanifolds are called* Prym eigenform loci *(see Section 2 for precise definitions).*

The techniques developed in this paper for the proof of Theorems A and B actually provide us with some precise information on the flow in directions for which the SAF-invariant vanishes: we get some topological properties of the directional flows on surfaces in some particular strata. Here we introduce the terminology of [7].

We say that a translation surface satisfies the *topological dichotomy* if for every direction, either the flow is minimal, or every flow line is closed or a saddle connection. Observe that this is equivalent to saying that if there is a saddle connection in some direction, then there is a cylinder decomposition of the surface in that direction. Obviously a Veech surface satisfies the topological dichotomy. First examples of surfaces satisfying topological dichotomy without being Veech surfaces have been constructed in [12] (see also [23]). All examples are ramified coverings above "true" Veech surfaces. Our next theorem provides us with new examples which do not arise from a covering construction above Veech surfaces (see Theorem 1.13).

THEOREM C. – *Let (Y,q) be a quadratic differential in $\mathcal{Q}(8)$ or $\mathcal{Q}(-1,2,3)$. Assume that all the periods (relative and absolute) of the orientation double cover of (Y,q) belong to $K(\imath)$, where K is either $\mathbb{Q}$ or a real quadratic field. If (Y,q) is algebraically completely periodic then it satisfies the topological dichotomy. In particular, if (Y,q) is stabilized by a pseudo-Anosov homeomorphism, then it satisfies the topological dichotomy.*

Observe that Theorem C is false for other strata. Moreover, "most" of surfaces of Theorem C are not Veech surfaces, namely:

THEOREM D. – *The following two hold:*

(1) *There are quadratic differentials in the strata $\mathcal{Q}(8)$ and $\mathcal{Q}(-1,2,3)$ satisfying the topological dichotomy without being Veech surfaces.*
(2) *There are quadratic differentials in each of the strata $\mathcal{Q}(-1^3,1,2)$, $\mathcal{Q}(-1,2,3)$, and $\mathcal{Q}(8)$ whose Veech group is infinitely generated and of the first kind.*

Finally our techniques also provide us with the following result for quadratic differentials in a slightly larger family of strata (compared to Theorem C).

THEOREM E. – *For any quadratic differential in the collection of strata $\mathcal{Q}(-1^3,1,2)$, $\mathcal{Q}(-1,2,3)$, and $\mathcal{Q}(8)$ the limit set of its Veech group is either the empty set, a single point, or the full circle at infinity.*

1.2. Prym loci and Prym eigenforms

From the work of McMullen [25], it turns out that all completely periodic surfaces in genus two belong to the loci of *eigenforms for real multiplication*. Later McMullen [24] proved the existence of similar loci in genus $3, 4$ and 5. These loci are of interest since they are closed $\mathrm{GL}^+(2,\mathbb{R})$-invariant sub-manifolds in the moduli spaces of Abelian differentials. We briefly recall the definitions of those objects here below.

1.2.1. *Prym forms.* – If X is a compact Riemann surface, and $\tau : X \to X$ is a holomorphic involution of X, we will denote by $\Omega(X)$ the set of holomorphic 1-forms (Abelian differentials) on X and by $\Omega^-(X,\tau) := \ker(\tau + \mathrm{Id}) \subset \Omega(X)$.

For any integer vector $\kappa = (k_1, \dots, k_n)$ with $k_i \geq 0$ and $\sum k_i = 2g-2$, we will denote by $\mathcal{H}(\kappa)$ the moduli space of translation surfaces having n singularities with multiplicities κ. The set of Prym forms $\mathrm{Prym}(\kappa) \subset \mathcal{H}(\kappa)$ is the subset of pairs $(X,\omega) \in \mathcal{H}(\kappa)$ such that there exists an involution $\tau : X \to X$ satisfying $\tau^*\omega = -\omega$ i.e., $\omega \in \Omega^-(X,\tau)$, and $\dim_{\mathbb{C}} \Omega^-(X,\tau) = 2$. We sometimes add a superscript to the vector κ, which could be "even", "odd", or "hyp", to specify the corresponding component of $\mathcal{H}(\kappa)$ in which the Prym locus lies (see [15] for the classification of connected components of $\mathcal{H}(\kappa)$).

Any translation surface of genus two is a Prym form: $\mathrm{Prym}(2) = \mathcal{H}(2)$ and $\mathrm{Prym}(1,1) = \mathcal{H}(1,1)$ (the hyperelliptic involution is by definition a Prym involution, which is actually unique). See Figure 7 for an example.

Let Y be the quotient of X by the Prym involution and π the corresponding (possibly ramified) double covering from X to Y. By pushforward, there exists a meromorphic quadratic differential q on Y (with at most simple poles) so that $\pi^*q = \omega^2$. Let $\kappa' = (d_1, \dots, d_r)$ be the integer vector that records the orders of the zeros and poles of q. Then there is a $\mathrm{GL}^+(2,\mathbb{R})$-equivariant bijection between $\mathcal{Q}(\kappa')$ and $\mathrm{Prym}(\kappa)$ [16, p. 6].

All the strata of quadratic differentials of dimension 5 are recorded in Table 1. It turns out that if (Y,q) is a quadratic differential in one of those strata, and (X,ω) is its orientation double cover, then (letting τ be the deck transformation) $\dim_{\mathbb{C}} \Omega^-(X,\tau) = \mathrm{genus}(X) - \mathrm{genus}(Y) = 2$. Hence (X,ω) is by definition a Prym form.

EXAMPLE 1.1. – *Let q be a quadratic differential on a Riemann surface Y having at most simple poles. We assume that q is not the global square of any Abelian differential. Let $\pi : X \to Y$ be the orientation double cover. If* $\mathrm{genus}(X) - \mathrm{genus}(Y) = 2$ *then the deck transformation τ on X provides a natural Prym form (X,τ,ω) where $\omega = \sqrt{\pi^*q} \in \Omega(X)^-$.*

For instance, if $(Y,q) \in \mathcal{Q}(-1^2,6)$ then the orientation cover belongs to $\mathrm{Prym}(3,3)$. *The same is true for $(Y,q) \in \mathcal{Q}(-1^6,2)$: the orientation cover $(X,\omega) \in$* $\mathrm{Prym}(1,1) = \mathcal{H}(1,1)$. *On the other hand, we have a one to one map from $\mathcal{Q}(-1^6,2)$ to $\mathcal{Q}(-1^2,6)$ (given by taking double cover ramified over the double zero and five poles* [16]*). This explains the notation* $\mathrm{Prym}(3,3) \simeq \mathcal{H}(1,1)$.

$\mathcal{Q}(\kappa')$	$\mathrm{Prym}(\kappa)$	$g(X)$	$\mathcal{Q}(\kappa')$	$\mathrm{Prym}(\kappa)$	$g(X)$
$\mathcal{Q}(-1^6,2)$	$\mathrm{Prym}(1,1) = \mathcal{H}(1,1)$	2	$\mathcal{Q}(-1^3,1,2)$	$\mathrm{Prym}(1,1,2)$	3
$\mathcal{Q}(-1^2,6)$	$\mathrm{Prym}(3,3) \simeq \mathcal{H}(1,1)$	4	$\mathcal{Q}(-1,2,3)$	$\mathrm{Prym}(1,1,4)$	4
$\mathcal{Q}(1,1,2)$	$\mathrm{Prym}(1^2,2^2) \simeq \mathcal{H}(0^2,2)$	4	$\mathcal{Q}(8)$	$\mathrm{Prym}(4,4)^{\mathrm{even}}$	5
$\mathcal{Q}(-1^4,4)$	$\mathrm{Prym}(2,2)^{\mathrm{odd}}$	3	$\mathcal{Q}(-1,1,4)$	$\mathrm{Prym}(2,2,2)^{\mathrm{even}}$	4

TABLE 1. Prym loci for which the associated stratum of quadratic differentials $\mathcal{Q}(\kappa')$ has (complex) dimension 5.

1.2.2. *Prym eigenforms.* – We now give the definition of Prym eigenforms. We define

$$\mathrm{Prym}(X,\tau) := (\Omega^-(X,\tau))^*/H_1(X,\mathbb{Z})^-,$$

where $H_1(X,\mathbb{Z})^- := \{c \in H_1(X,\mathbb{Z}) \ : \ \tau(c) = -c\}$. $\mathrm{Prym}(X,\tau)$ will be called the *Prym variety* of X, it is a sub-Abelian variety of the Jacobian variety $\mathbf{Jac}(X) := \Omega(X)^*/H_1(X,\mathbb{Z})$.

Recall that a quadratic order is a ring isomorphic to $\mathcal{O}_D = \mathbb{Z}[X]/(X^2+bX+c)$, where $D = b^2 - 4c > 0$ (quadratic orders being classified by their discriminant D).

Definition 1.2 (Real multiplication). – *Let A be an Abelian variety of dimension* 2*. We say that A admits a real multiplication by $\mathcal{O}_D$ if there exists an injective homomorphism* $\mathfrak{i} : \mathcal{O}_D \to \mathrm{End}(A)$*, such that* $\mathfrak{i}(\mathcal{O}_D)$ *is a self-adjoint, proper subring of* $\mathrm{End}(A)$ *(i.e., for any* $f \in \mathrm{End}(A)$*, if there exists* $n \in \mathbb{Z}\backslash\{0\}$ *such that* $nf \in \mathfrak{i}(\mathcal{O}_D)$ *then* $f \in \mathfrak{i}(\mathcal{O}_D)$*).*

Definition 1.3 (Prym eigenform). – *For any quadratic discriminant $D > 0$, we denote by $\Omega E_D(\kappa)$ the set of* $(X,\omega) \in \mathrm{Prym}(\kappa)$ *such that* $\dim_{\mathbb{C}} \mathrm{Prym}(X,\tau) = 2$, $\mathrm{Prym}(X,\tau)$ *admits a multiplication by $\mathcal{O}_D$, and ω is an eigenvector of $\mathcal{O}_D$. Surfaces in $\Omega E_D(\kappa)$ are called* Prym eigenforms.

Prym eigenforms exist in each Prym locus described in Table 1, as real multiplications arise naturally with pseudo-Anosov homeomorphisms commuting with τ (see Theorem 7.1). It follows from the work of McMullen [24], that each $\Omega E_D(\kappa)$ is a $\mathrm{GL}(2,\mathbb{R})$-invariant submanifold of $\mathrm{Prym}(\kappa)$. It turns out that for $\mathrm{Prym}(\kappa)$ in Table 1, the loci $\Omega E_D(\kappa)$ have complex dimension 3 (see Proposition 3.1).

1.3. Other formulations of main results

Because of the correspondence between quadratic differentials in Table 1 and Prym forms, we can now reformulate our main results in terms of Prym forms.

Theorem 1.4. – *Any Prym eigenform in the Prym loci of Table 1 is completely algebraically periodic.*

Note that the cases (1), (2), (3) follow from the work of Calta and McMullen. Conversely, we have

Theorem 1.5. – *Let* $(X,\omega) \in \mathrm{Prym}(\kappa)$ *where* $\mathrm{Prym}(\kappa)$ *is given by Table 1. Assume that (X,ω) is completely algebraically periodic, and the set of homological directions of (X,ω) is $K\mathbb{P}^1$, where K is either $\mathbb{Q}$, or a real quadratic field. Then the surface (X,ω) is a Prym eigenform i.e., $(X,\omega) \in \Omega E_D(\kappa)$ for some discriminant D.*

To prove Theorem 1.5, we need the following theorem which relates complete algebraic periodicity and complete periodicity.

Theorem 1.6. – *Let (X,ω) be a translation surface in one of the Prym loci given by the cases of Table 1. If (X,ω) is completely algebraically periodic, then it is completely periodic in the sense of Calta.*

As a consequence of Theorems 1.4 and 1.6, we draw

COROLLARY 1.7. – *Every Prym eigenform in the loci shown in Table 1 is completely periodic in the sense of Calta.*

REMARK 1.8. – *In a recent preprint* [36]*, A. Wright obtains an independent proof of Corollary 1.7, the result of Wright is actually more general than this as it applies to Prym eigenform loci of any dimension.*

The key ingredient of Wright's approach is the fact that the tangent space of $\Omega E_D(\kappa)$ at a point (X,ω) projects to a subspace of complex dimension two in $H^1(X,\mathbb{C})$. Our approach to prove Theorem 1.6 (which implies Corollary 1.7) is different from Wright's, it is based on a careful investigation of the geodesic foliation in directions for which the SAF-invariant vanishes. In particular, it does not require any assumption on the $\mathrm{GL}^+(2,\mathbb{R})$*-orbit closure of the surface, and hence can be used to prove the complete periodicity of surfaces which are not Prym eigenforms.*

In the appendix, using similar ideas, we will show that for surfaces in $\mathcal{H}^{\mathrm{hyp}}(4)$, algebraic complete periodicity also implies complete periodicity in the sense of Calta, this implies the existence of completely periodic surfaces whose $\mathrm{GL}^+(2,\mathbb{R})$*-orbit is dense in* $\mathcal{H}^{\mathrm{hyp}}(4)$.

To prove Theorem 1.6, we will consider linear involutions defined over 6 letters (see Section 4 for more details) for which the SAF-invariant vanishes. It turns out that in some particular cases, one can improve the complete periodicity in the sense of Calta. Namely, as by-products of our strategy, we will show the following theorems, which only involve some strata in Table 1.

THEOREM 1.9. – *Let (X,ω) be a Prym form in* $\mathrm{Prym}(4,4)^{\mathrm{even}}$ *or* $\mathrm{Prym}(1,1,4)$ *having all the periods (relative and absolutes) in $K(\imath)$, where K is either $\mathbb{Q}$ or a real quadratic field. If (X,ω) is completely algebraically periodic then (X,ω) satisfies the topological dichotomy. Moreover, a direction θ is periodic on (X,ω) if and only if $\theta \in K\mathbb{P}^1$. In particular, if the Veech group* $\mathrm{SL}(X,\omega)$ *of (X,ω) contains a hyperbolic element, then (X,ω) satisfies the topological dichotomy.*

REMARK 1.10. – *The statement of Theorem 1.9 is not true for all Prym loci. For instance, every completely algebraically periodic surface $(X,\omega) \in \Omega E_D(1,1)$ with relative and absolute periods in $K(\imath)$, which is not a Veech surface, admits an irrational splitting into two isogenous tori* [8]. *In particular, the topological dichotomy fails for such surfaces.*

We will show (see Section 9) that if $(X,\omega) \in \mathrm{Prym}(1,1,2)\sqcup\mathrm{Prym}(1,1,4)\sqcup\mathrm{Prym}(4,4)^{\mathrm{even}}$ is stabilized by an affine pseudo-Anosov homeomorphism, then the set of directions $\theta \in K\mathbb{P}^1$ that are fixed by parabolic elements in the Veech group is dense in $\mathbb{R}\mathbb{P}^1$. As corollaries, we get

THEOREM 1.11. – *Let (X,ω) be a Prym form in*

$$\mathrm{Prym}(4,4)^{\mathrm{even}} \sqcup \mathrm{Prym}(1,1,4) \sqcup \mathrm{Prym}(1,1,2).$$

Then the limit set of the Veech group $\mathrm{SL}(X,\omega)$ *is either the empty set, a single point, or the full circle at infinity.*

And

THEOREM 1.12. – (1) *There exist surfaces in* $\mathrm{Prym}(4,4)^{\mathrm{even}} \sqcup \mathrm{Prym}(1,1,4)$ *which satisfy the topological dichotomy but without being Veech surfaces.*

(2) *There exist Teichmüller discs generated by Prym forms in the loci* $\mathrm{Prym}(1,1,2)$*,* $\mathrm{Prym}(1,1,4)$*, and* $\mathrm{Prym}(4,4)^{\mathrm{even}}$ *whose Veech group is infinitely generated and of the first kind.*

Finally, we will show that our results give rise to surfaces which have connection points (in the sense of Hubert and Schmidt [12]) but are not lattice surfaces. Recall that a (non singular) point p is a connection point of a translation surface if every separatrix passing through p is a *saddle connection*.

THEOREM 1.13. – *Let* (X,ω) *be a Prym form in* $\mathrm{Prym}(4,4)^{\mathrm{even}}$ *or* $\mathrm{Prym}(1,1,4)$ *having all the periods (relative and absolutes) in* $K(\imath)$*, where* K *is either* $\mathbb{Q}$ *or a real quadratic field. Then any (non singular) point* $p \in X$ *having coordinates in* $K[\imath]$ *is a connection point.*

Proof of Theorem 1.13. – If p has homology in $K[\imath]$ then any segment from a singularity to p has slope in K. By Theorem 1.9, it is a periodic direction, hence the segment is part of a saddle connection. □

1.4. Outline of the paper

We conclude by sketching the proof of our results. It involves the dynamics of interval exchange transformations and linear involutions, the SAF-invariant and the kernel foliation in Prym loci.

1. To prove Theorem 1.4 we use an invariant introduced by McMullen similar to the SAF-invariant: the Galois flux. Let T be an interval exchange transformation (IET), and let λ_α, t_α, $\alpha \in \mathcal{A}$, be respectively the lengths of the exchanged intervals and their translation lengths. The Galois flux of T is defined *only* if the translation lengths t_α lie in a real quadratic field $K \subset \mathbb{R}$, namely

$$\mathrm{flux}(T) = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha t'_\alpha, \text{ where } t'_\alpha \text{ is the Galois conjugate of } t_\alpha.$$

It turns out (see Theorem 2.6) that if $(X,\omega) \in \Omega E_D(\kappa)$ and having all absolute periods in $K(\imath)$ then for any $\theta \in K\mathbb{P}^1$, $\mathrm{flux}(T_\theta) = 0$, where T_θ is the IET defined by the first return map of the flow in direction θ to a transversal interval in X. The two invariants are related by Proposition 2.8. Namely, under the additional assumption: if the relative periods of ω are also in $K(\imath)$ then $\mathrm{flux}(T_\theta) = 0$ implies $SAF(T_\theta) = 0$.

Now if the relative periods of ω are not in $K(\imath)$ then we can "perturb" ω in $\Omega E_D(\kappa)$ to get a new form ω' (by using the kernel foliation, see Section 3) so that the relative periods of ω' belong to $K(\imath)$. Thus by the preceding discussion $SAF(T'_\theta) = 0$ (T'_θ is the IET defined by flow in direction θ on X').We then conclude with Proposition 3.3: which states that the "perturbation" leaves the SAF-invariant unchanged. This proves Theorem 1.4.

2. It is well known that linear flows on translation surfaces are encoded by interval exchange transformations. Since we will work with non-orientable measured foliations defined by quadratic differentials, it will be more convenient to use the coding provided by *linear involutions*. By [3] one can still define a "first return" of the non-orientable foliation (that is no longer a flow) to a transverse interval, which gives a linear involution defined

over d intervals (see Section 4). It turns out that the number d of exchanged intervals is related to the dimension of the Prym locus, namely $d = \dim_{\mathbb{C}} \mathrm{Prym}(\kappa) + 1$.

Obviously complete periodicity for a foliation or for its associated linear involution is the same. In view of this, we will deduce Theorem 1.6 from results on linear involutions (see Section 5). We briefly sketch a proof here. Let $(X, \omega) \in \mathrm{Prym}(\kappa)$ be a Prym form which has a vertical cylinder (we normalize so that any homological direction belongs to a quadratic field). We will consider the cross section T of the vertical *foliation* to some full transversal interval on the quotient $X/\langle\tau\rangle$. Let us resume the situation: T is a linear involution defined over $d = 6$ letters (for all the Prym loci in Table 1, $\dim_{\mathbb{C}} \mathrm{Prym}(\kappa) = 5$) having a periodic orbit, and $SAF(T) = 0$. We want to show that T is completely periodic.

If T is defined over 2 or 3 intervals then the proof is immediate. We prove the assertion for $d = 6$ by induction on the number of intervals, we pass from d intervals to $d-1$ intervals by applying the Rauzy induction (which preserves the SAF-invariant, see Section 5).

3. Theorem 1.9 is a refinement of Theorem 1.6 by inspecting the possible degenerations of linear involutions (see Section 8). We actually show that for surfaces in $\mathrm{Prym}(4,4)^{\mathrm{even}}$ and $\mathrm{Prym}(1,14)$, complete algebraic periodicity implies topological dichotomy, and the set of periodic directions coincides with the set of homological directions.

If the Veech group of a Prym form (X, ω) contains a hyperbolic element A, then (X, ω) belongs to some Prym eigenform locus ΩE_D (see [24], Theorem 3.5), and all the periods of ω belong to $K(\imath)$, where $K = \mathbb{Q}(\mathrm{Tr}(A))$ (see [23] Theorem 9.4). Thus (X, ω) is completely algebraically periodic by Theorem 1.4, and the arguments above show that (X, ω) satisfies the topological dichotomy.

4. For the proof of Theorem 1.11, we first remark that if the limit set of the Veech group $\mathrm{SL}(X, \omega)$ of (X, ω) contains at least two points, then (X, ω) is stabilized by an affine pseudo-Anosov homeomorphism ϕ. It follows that all the relative and absolute periods of ω belong to $K(\imath)$, where $K = \mathbb{Q}(\mathrm{Tr}(D\phi))$. To show that the limit set of $\mathrm{SL}(X, \omega)$ is the full circle at infinity, it is sufficient to show that the set of directions which are fixed by parabolic elements of $\mathrm{SL}(X, \omega)$ is dense in $\mathbb{RP}^1$. For the cases of $\mathrm{Prym}(1,1,4)$ and $\mathrm{Prym}(4,4)^{\mathrm{even}}$, this follows from Theorem 1.9 together with a criterion for a periodic direction to be parabolic (that is to be fixed by a parabolic element in $\mathrm{SL}(X, \omega)$, see Proposition 9.2). For the case $\mathrm{Prym}(1,1,2)$, this follows from a similar result to Theorem 1.9 (see Corollary 8.6), and a careful inspection of topological models for cylinder decompositions of surfaces in $\mathrm{Prym}(1,1,2)$ (see Section 9).

5. To prove Theorem 1.12 we will construct explicitly Prym eigenforms in

$$\mathrm{Prym}(1,1,2) \sqcup \mathrm{Prym}(1,1,4) \sqcup \mathrm{Prym}(4,4)^{\mathrm{even}}$$

with periods in a real quadratic field, for which there are periodic directions such that the associated cylinder decomposition is not parabolic (that is the ratios of the cylinder moduli are not all rational numbers). By Theorem 1.9 and Theorem 1.11 such surfaces satisfy the topological dichotomy and the limit set of the Veech group is dense in $\mathbb{RP}^1$. However, such surfaces are not Veech surfaces, from which we get the desired conclusions.

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2. Interval exchange, Sah-Arnoux-Fathi invariant and McMullen's flux

In this section we recall necessary background on interval exchange transformations and we will make clear the relations between the SAF-invariant introduced by Arnoux in his thesis [1], the J-invariant introduced by Kenyon-Smillie [14] and Calta [5], and the flux introduced by McMullen [22].

2.1. Interval exchange transformation and SAF-invariant

An interval exchange transformation (IET) is a map T from an interval I into itself defined as follows: we divide I into finitely many subintervals of the form $[a,b)$. On each of such interval, the restriction of T is a translation: $x \mapsto x+t$. By convention, the map T is continuous from the right at the endpoints of the subintervals. Any IET can be encoded by a combinatorial data $(\mathcal{A},\pi)$, where $\mathcal{A}$ is a finite alphabet, $\pi=(\pi_0,\pi_1)$ is a pair of one-to-one maps $\pi_\varepsilon : \mathcal{A} \to \{1,\dots,d\},\ d=|\mathcal{A}|$, together with a vector in the positive cone $\lambda=(\lambda_\alpha)_{\alpha\in\mathcal{A}} \in \mathbb{R}_{>0}^{|\mathcal{A}|}$. The permutations (π_0,π_1) encodes how the intervals are exchanged, and the vector λ encodes the lengths of the intervals. Following Marmi, Moussa, Yoccoz [18], we denote these intervals by $\{I_\alpha,\ \alpha\in\mathcal{A}\}$, the length of the interval I_α is λ_α. Hence the restriction of T to I_α is $T(x)=x+t_\alpha$ for some translation length t_α. Observe that t_α is uniquely determined by π and λ.

A useful tool to detect periodic IET is given by the Sah-Arnoux-Fathi invariant (SAF-invariant). It is defined by (see [1]):

$$SAF(T)=\sum_{\alpha\in\mathcal{A}} \lambda_\alpha \wedge_{\mathbb{Q}} t_\alpha.$$

It turns out that if T is periodic then $SAF(T)=0$. However the converse is not true in general.

In the case where T is defined by the first return map of the vertical flow of a translation surface (X,ω) to a transversal interval I which crosses all the vertical leaves, Arnoux proved the following in his thesis

THEOREM 2.1 ([1, Theorem 3.5]). – *Set $\rho:=\mathrm{Re}(\omega)$. Let $\{a_1,b_1,\dots,a_g,b_g\}$ be a symplectic basis of $H_1(X,\mathbb{Z})$. Then the SAF-invariant of T satisfies*

$$SAF(T)=\sum_{i=1}^{g}\rho(a_i)\wedge_{\mathbb{Q}}\rho(b_i).$$

In particular, $SAF(T)$ only depends on the cohomology class of ρ in $H^1(X,\mathbb{R})$.

2.2. J-invariant, SAF, and algebraic periodic direction

Let (X,ω) be a translation surface. If $\mathbf{P}$ is a polygon in $\mathbb{R}^2$ with vertices $v_1,\dots,v_n$ which are numbered in counterclockwise order about the boundary of P, then the J-invariant of $\mathbf{P}$ is $J(\mathbf{P}) = \sum_{i=1}^n v_i \wedge v_{i+1}$ (with the dummy condition $v_{n+1} = v_1$). Here $\wedge$ is taken to mean $\wedge_{\mathbb{Q}}$ and $\mathbb{R}^2$ is viewed as a $\mathbb{Q}$-vector space. $J(\mathbf{P})$ is a translation invariant (e.g., $J(\mathbf{P}+\overrightarrow{v}) = J(\mathbf{P})$), thus this permits to define $J(X,\omega)$ by $\sum_{i=1}^k J(\mathbf{P}_i)$ where $\mathbf{P}_1\cup\dots\cup\mathbf{P}_k$ is a cellular decomposition of (X,ω) into planar polygons (see [14]).

The SAF-invariant of an interval exchange is related to the J-invariant as follows. We define a linear projection $J_{xx}:\mathbb{R}^2\wedge_{\mathbb{Q}}\mathbb{R}^2\to\mathbb{R}\wedge_{\mathbb{Q}}\mathbb{R}$ by

$$J_{xx}\left(\left(\begin{smallmatrix} a\\ b\end{smallmatrix}\right)\wedge\left(\begin{smallmatrix} c\\ d\end{smallmatrix}\right)\right) = a\wedge c.$$

If T is an interval exchange transformation induced by the first return map of the vertical foliation on (X,ω) (on a transverse interval I) then $SAF(T) = J_{xx}(X,\omega)$. Note that the definition does not depend of the choice of I if the interval meets every vertical leaf (see [1]). Hence this allows us to define

$$SAF(X,\omega) = SAF(T),$$

and we will say that $SAF(X,\omega)$ is the SAF-invariant of (X,ω) in the *vertical direction*.

Following Calta [5], one also defines the SAF-invariant of (X,ω) in any direction $k\in\mathbb{RP}^1$ ($k\neq\infty=\left(\begin{smallmatrix} 0\\ 1\end{smallmatrix}\right)$) as follows. Let $g\in GL^+(2,\mathbb{R})$ be a matrix that sends the vector $\left(\begin{smallmatrix} 1\\ k\end{smallmatrix}\right)$ to the vector $\left(\begin{smallmatrix} 0\\ 1\end{smallmatrix}\right)$. Then we define the SAF-invariant of (X,ω) *in direction* k to be $J_{xx}(g\cdot(X,\omega))$.

2.3. Galois flux

For the remaining of this section K will be a real quadratic field. There is a unique positive square-free integer f such that $K=\mathbb{Q}(\sqrt{f})$. The *Galois conjugation* of K is given by $u+v\sqrt{f}\mapsto u-v\sqrt{f},\ u,v\in\mathbb{Q}$. For any $x\in K$, we denote by x' its Galois conjugate. An interval exchange transformation T is *defined over* K if its translation lengths t_α are all in K. In [23], McMullen defines the *Galois flux* of an IET T defined over K to be

$$\mathrm{flux}(T) = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha t'_\alpha.$$

Observe that for all $n\in\mathbb{N}$, $\mathrm{flux}(T^n) = n\mathrm{flux}(T)$. In particular $\mathrm{flux}(T)=0$ if T is periodic. The flux is closely related to the SAF-invariant as we will see.

2.4. Flux of a measured foliation

Let (X,ω) be a translation surface. The real form $\rho=\mathrm{Re}(\omega)$ defines a measured foliation $\mathcal{F}_\rho$ on X: the leaves of $\mathcal{F}_\rho$ are vertical geodesics of the flat metric defined by ω. For any interval I, transverse to $\mathcal{F}_\rho$, the cross section of the flow is an IET. We say that I is *full transversal* if it intersects all the leaves of $\mathcal{F}_\rho$. If all the absolute periods of ρ belong to the field K, that is $[\rho]\in H^1(X,K)\subset H^1(X,\mathbb{R})$, then the first return map to I is defined over K, and we have

THEOREM 2.2 (McMullen [23]). – *Let T be the first return map of $\mathcal{F}_\rho$ to a full transversal interval. If $[\rho] \in H^1(X,K)$, then we have*

$$\mathrm{flux}(T) = -\int_X \rho \wedge \rho',$$

where $\rho' \in H^1(X,K)$ is defined by $\rho'(c) = (\rho(c))'$, for all $c \in H_1(X,\mathbb{Z})$. In particular, the flux is the same for any full transversal interval. In this case, we will call the quantity $-\int_X \rho\wedge\rho'$ the flux *of the measured foliation $\mathcal{F}_\rho$, or simply the flux of ρ, and denote it by* $\mathrm{flux}(\rho)$.

2.5. Complex flux

Let $K(\imath)$ be the extension of K by $\imath = \sqrt{-1}$. Elements of $K(\imath)$ have the form $k = k_1 + \imath k_2$, $k_1, k_2 \in K$. We define $(k_1+\imath k_2)' = k_1' + \imath k_2'$, and $\overline{k_1 + \imath k_2} = k_1 - \imath k_2$. Suppose that $\omega \in \Omega(X)$ satisfies $[\omega] \in H^1(X, K(\imath))$ and

$$\int_X \omega \wedge \omega' = 0,$$

(ω' is an element of $H^1(X,K(\imath))$). The *complex flux* of ω is defined by

$$\mathrm{Flux}(\omega) = -\int_X \omega \wedge \overline{\omega}'.$$

Note that we always assume that $\int_X \omega\wedge\omega' = 0$ when we consider $\mathrm{Flux}(\omega)$. This condition holds, for example, if $[\omega']$ is represented by a holomorphic 1-form.

In the following proposition, we collect the important properties of the complex flux:

PROPOSITION 2.3 (McMullen [23]). – a) *For any $k \in K(\imath)$,* $\mathrm{Flux}(k\omega) = k\overline{k}'\mathrm{Flux}(\omega)$.

b) *If $\rho = \mathrm{Re}(\omega)$, then*

$$\mathrm{flux}(\rho) = -\frac{1}{4}\int_X (\omega+\overline{\omega}) \wedge (\omega' + \overline{\omega}') = \frac{1}{2}\mathrm{Re}(\mathrm{Flux}(\omega)).$$

(here we used the condition $\int_X \omega\wedge\omega' = 0$).

c) *Assume that* $\mathrm{Flux}(\omega) = 0$. *Let $k = k_2/k_1 \in K\mathbb{P}^1$, $k_1, k_2 \in K$, and $\rho = \mathrm{Re}((k_1+\imath k_2)\omega)$. Then $\mathcal{F}_\rho$ is the foliation by geodesic of slope k in (X,ω), and we have*

$$\mathrm{flux}(\mathcal{F}_\rho) = \mathrm{flux}(\rho) = \frac{1}{2}\mathrm{Re}((k_1+\imath k_2)\omega) = \frac{1}{2}\mathrm{Re}((k_1+\imath k_2)(k_1' - \imath k_2')\mathrm{Flux}(\omega)) = 0.$$

2.6. Periodic foliation

Given a cylinder C in (X,ω), we denote its width and height by $w(C)$ and $h(C)$ respectively. If the vertical foliation is completely periodic, then X is decomposed into cylinders in this direction. It turns out that the imaginary part of $\mathrm{Flux}(\omega)$ provides us with important information on the cylinders. Namely the following is true:

THEOREM 2.4 (McMullen [23]). – *Assume that $[\omega] \in H^1(X,K(\imath))$, $\int \omega\wedge\omega' = 0$, and the foliation $\mathcal{F}_\rho$ is periodic, where $\rho = \mathrm{Re}(\omega)$. Let $\{C_j\}_{1\le j\le m}$ be the vertical cylinders of X. Then we have*

$$\sum_{1\le j\le m} h(C_j)w(C_j)' = \frac{1}{2}\mathrm{Im}(\mathrm{Flux}(\omega)).$$

Recall that we have $N(k) = kk' \in \mathbb{Q}$, for any $k \in K$. For any cylinder C, the modulus of C is defined by $\mu(C) = h(C)/w(C)$. A direct consequence of Theorem 2.4 is the following useful corollary.

COROLLARY 2.5. – *If $\mathcal{F}_\rho$ is periodic, and the complex flux of ω vanishes, then the moduli of the vertical cylinders satisfy the following rational linear relation*

$$\sum_{1\le j\le m} \mu(C_j)N(w(C_j)) = 0.$$

2.7. Prym eigenform and complex flux

A remarkable property of Prym loci is that the complex flux of a Prym eigenform (of a real quadratic order) vanishes.

THEOREM 2.6 ([23, Theorem 9.7]). – *Let (X,ω) be a Prym eigenform belonging to some locus $\Omega E_D(\kappa)$. After replacing (X,ω) by $A\cdot(X,\omega)$ for a suitable $A \in \mathrm{GL}^+(2,\mathbb{R})$, we can assume that all the absolute periods of ω are in $K(\imath)$, where $K = \mathbb{Q}(\sqrt{D})$. We have*

$$\int_X \omega\wedge\omega' = 0 \qquad \textit{and} \qquad \mathrm{Flux}(\omega) = 0.$$

Proof. – Let T be a generator of the order $\mathcal{O}_D$. We have a pair of 2-dimensional eigenspaces $S \oplus S' = H^1(X,\mathbb{R})^-$ on which T acts with eigenvalues t, t' respectively. Since T is self-adjoint, S and S' are orthogonal with respect to the cup product.

The eigenspace S is spanned by $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$. These forms lie in $H^1(X,K)$. The Galois conjugate of any form $\alpha \in H^1(X,K)\cap S$ satisfies $T\alpha' = t'\alpha'$, and hence belongs to S'. In particular $\mathrm{Re}(\omega)'$ and $\mathrm{Im}(\omega)'$ are orthogonal to $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$. This shows

$$\int_X \omega\wedge\omega' = 0 \qquad \text{and} \qquad \mathrm{Flux}(\omega) = -\int_X \omega\wedge\overline{\omega}' = 0. \qquad \square$$

COROLLARY 2.7. – *If (X,ω) is a Prym eigenform for a quadratic order $\mathcal{O}_D$ such that $[\omega] \in H^1(X,K(\imath))$, where $K = \mathbb{Q}(\sqrt{D})$, then for any $k \in K\mathbb{P}^1$, the flux of the foliation by geodesics in direction k vanishes.*

2.8. Relation between SAF-invariant and complex flux

PROPOSITION 2.8. – *Let (X,ω) be as in Theorem 2.6. Assume that all the relative periods of ω are also in $K(\imath)$. Then* $\mathrm{flux}(\omega) = 0$ *implies $SAF(\omega) = 0$. Here,* $\mathrm{flux}(\omega)$ *and $SAF(\omega)$ denote the corresponding invariants of the vertical flow on (X,ω).*

Proof. – Let I be a full transversal interval for the vertical flow, and T be the IET induced by the first return map on I. We denote the lengths of the permuted intervals I_α by λ_α and the translation lengths by t_α so that $T(x) = x + t_\alpha$ for any $x \in I_\alpha$. The assumption on relative periods implies that $\forall\alpha\in\mathcal{A}, \lambda_\alpha \in K$. Since $K = \mathbb{Q}(\sqrt{f})$, then we can write $t_\alpha = x_\alpha + y_\alpha\sqrt{f}$ with $x_\alpha, y_\alpha \in \mathbb{Q}$. Then $t'_\alpha = x_\alpha - y_\alpha\sqrt{f}$. The condition $\mathrm{flux}(T) = 0$ is then equivalent to

$$\sum_{\alpha\in\mathcal{A}} \lambda_\alpha x_\alpha = \sqrt{f}\sum_{\alpha\in\mathcal{A}} \lambda_\alpha y_\alpha.$$

Since $\lambda_\alpha \in K$, it follows

$$\sum_{\alpha\in\mathcal{A}} \lambda_\alpha y_\alpha = A + B\sqrt{f} \quad \text{and} \quad \sum_{\alpha\in\mathcal{A}} \lambda_\alpha x_\alpha = fB + A\sqrt{f} \text{ with } A, B \in \mathbb{Q}.$$

By definition of the SAF-invariant, we have

$$SAF(T) = (\sum_{\alpha\in\mathcal{A}} \lambda_\alpha x_\alpha) \wedge_{\mathbb{Q}} 1 + (\sum_{\alpha\in\mathcal{A}} \lambda_\alpha y_\alpha) \wedge_{\mathbb{Q}} \sqrt{f} = A\sqrt{f} \wedge_{\mathbb{Q}} 1 + A \wedge_{\mathbb{Q}} \sqrt{f} = 0. \qquad \square$$

COROLLARY 2.9. – *Let (X,ω) be a Prym eigenform in $\Omega E_D(\kappa)$. Assume that all the periods of ω belong to $K(\imath)$, where $K = \mathbb{Q}(\sqrt{D})$. Then (X,ω) is completely algebraically periodic.*

Proof. – We first remark that the set of homological directions of (X,ω) is $K\mathbb{P}^1$. For any direction $\theta \in K\mathbb{P}^1$, there exists a matrix $g_\theta \in \mathrm{GL}^+(2,K)$ that maps θ to the vertical direction. Note that all the periods of $g_\theta \cdot \omega$ are in $K(\imath)$. From the properties of flux, we know that $\mathrm{flux}(g_\theta \cdot \omega) = 0$, thus $SAF(g_\theta \cdot \omega) = 0$, which implies that the SAF-invariant vanishes in direction θ. $\square$

3. Invariance of SAF along kernel foliation leaves

3.1. Kernel foliation

Here we briefly recall the kernel foliation for Prym loci (see [10, 21, 5, 17, 26] and [37, §9.6] for related constructions). The kernel foliation was introduced by Eskin-Masur-Zorich, and was certainly known to Kontsevich.

Let (X,ω) be a translation surface having several distinct zeros. The intersection of the leaf of the kernel foliation through (X,ω) with a neighborhood of (X,ω) consists of surfaces (X',ω') that share the same absolute periods as (X,ω), i.e., for any $c \in H_1(S,\mathbb{Z})$, where S is the base topological surface homeomorphic to both X and X', we have $\omega(c) = \omega'(c)$.

One can get such a surface by the following construction: choose a zero P of ω and $\varepsilon > 0$ small enough so that the set $D(P,\varepsilon) = \{x \in X, \mathbf{d}(P,x) < \varepsilon\}$ is an embedded disc in X and disjoint from all the other zeros of ω. Assume that P is a zero of order k, then $D(P,\varepsilon)$ can be constructed from $2(k+1)$ half-discs as described in the left part of Figure 1. Pick a vector $w \in \mathbb{C}$, $0 < |w| < \varepsilon$, and cut $D(P,\varepsilon)$ along the rays in direction $\pm w$, we get $2(k+1)$ half-discs which are glued together such that all the centers are identified with P. We modify the metric structure of $D(P,\varepsilon)$ as follows: in the diameter of each half-disc, there is a unique point P' such that $\overrightarrow{PP'} = w$, we can glue the half-discs in such a way that all the points P' are identified. Let us denote by D' the domain obtained from this gluing. We can glue D' to $X \setminus D(P,\varepsilon)$ along $\partial D'$, which is the same as $\partial D(P,\varepsilon)$. We then get a translation surface (X',ω') which has the same absolute periods as (X,ω), and satisfies the following condition: if c is a path in X joining another zero of ω to P, and c' is the corresponding path in X', then we have $\omega(c') = \omega(c) + w$. In this situation, we will say that P is moved by w. By definition (X',ω') lies in the kernel foliation leaf through (X,ω).

Let us now describe the kernel foliation in Prym loci in Table 1. Let (X,ω) be a translation surface in a Prym locus given in Table 1. We first observe that either ω has two zeros, in which case the zeros are permuted by the Prym involution τ, or ω has 3 zeros, two of which are permuted by τ, the third one is fixed. In both cases, let us denote the pair of permuted zeros by P_1, P_2. Given ε and w as above, to get a surface (X', ω') in the same leaf, it suffices to move P_1 by $w/2$ and move P_2 by $-w/2$. Indeed, by assumptions, the Prym involution exchanges $D(P_1, \varepsilon)$ and $D(P_2, \varepsilon)$. Let D'_1 and D'_2 denote the new domains we obtain from $D(P_1, \varepsilon)$ and $D(P_2, \varepsilon)$ after modifying the metric. It is easy to check that D'_1 and D'_2 are symmetric, thus the involution in $X \setminus (D(P_1, \varepsilon) \sqcup D(P_2, \varepsilon))$ can be extended to $D'_1 \sqcup D'_2$. Therefore we have an involution τ' on X' such that ${\tau'}^*\omega' = -\omega'$, which implies that (X', ω') also belongs to the same Prym locus as (X, ω).

In what follows we will denote the surface (X', ω') obtained from this construction by $(X, \omega) + w$ (from w small). Let c be a path on X joining two zeros of ω, and c' be the corresponding path in X'. Then we have

- if two endpoints of c are exchanged by τ then $\omega'(c') - \omega(c) = \pm w$,
- if one endpoint of c is fixed by τ, but the other is not, then $\omega'(c') - \omega(c) = \pm w/2$.

The sign of the difference is determined by the orientation of c.

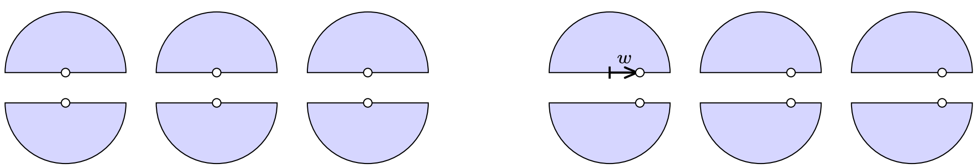


FIGURE 1. Moving a zero by a vector $w \in \mathbb{R}^2$

3.2. Neighborhood of a Prym eigenform

We first show

LEMMA 3.1. – *For any Prym locus* $\mathrm{Prym}(\kappa)$ *in Table 1, and any discriminant* $D \in \mathbb{N}$, $D \equiv 0, 1 \mod 4$, *if* $\Omega E_D(\kappa) \neq \varnothing$, *then* $\dim_{\mathbb{C}} \Omega E_D(\kappa) = 3$.

Proof. – Denote by Σ the set of zeros of ω. Let $H^1(X, \mathbb{C})^-$ and $H^1(X, \Sigma; \mathbb{C})^-$ denote the eigenspaces of τ with the eigenvalue -1 in $H^1(X, \mathbb{C})$ and $H^1(X, \Sigma; \mathbb{C})$ respectively. In a local chart which is given by a period mapping, a Prym form in $\mathrm{Prym}(\kappa)$ close to (X, ω) corresponds to a vector in $H^1(X, \Sigma; \mathbb{C})^-$. Note that $\dim_{\mathbb{C}} H^1(X, \mathbb{C})^- = 4$ and $\dim_{\mathbb{C}} H^1(X, \Sigma; \mathbb{C})^- = 5$, and we have a natural surjective linear map $\rho : H^1(X, \Sigma; \mathbb{C})^- \to H^1(X, \mathbb{C})^-$.

Let $\hat{S} = \mathbb{C} \cdot [\mathrm{Re}(\omega)] \oplus \mathbb{C} \cdot [\mathrm{Im}(\omega)] \subset H^1(X, \mathbb{C})^-$, where $[\mathrm{Re}(\omega)]$ and $[\mathrm{Im}(\omega)]$ are the cohomology classes in $H^1(X, \mathbb{R})$ represented by $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$. Since ω is an eigenform of a quadratic order $\mathcal{O}_D$, there exists an endomorphism T of $H^1(X; \mathbb{C})^-$ which generates $\mathcal{O}_D$ such that $\hat{S}$ is an eigenspace of T for some real eigenvalue. A Prym eigenform in $\Omega E_D(\kappa)$ close to (X, ω) corresponds to a vector in $\rho^{-1}(\hat{S})$. Since $\dim_{\mathbb{C}} \hat{S} = 2$ and $\dim_{\mathbb{C}} \ker \rho = 1$, the lemma follows. $\square$

COROLLARY 3.2. – *For any* $(X,\omega)\in\Omega E_D(\kappa)$*, if* (X',ω') *is a Prym eigenform in* $\Omega E_D(\kappa)$ *close enough to* (X,ω)*, then there exists a unique pair* (g,w)*, where* $g\in \mathrm{GL}^+(2,\mathbb{R})$ *close to* Id*, and* $w\in\mathbb{R}^2$ *with* $|w|$ *small, such that* $(X',\omega')=g\cdot((X,\omega)+w)$.

Proof. – Let $(Y,\eta)=(X,\omega)+w$, with $|w|$ small, be a surface in the leaf of the kernel foliation through (X,ω). We denote by $[\omega]$ and $[\eta]$ the classes of ω and η in $H^1(X,\Sigma;\mathbb{C})^-$. Then we have

$$[\eta]-[\omega]\in\ker\rho,$$

where $\rho: H^1(X,\Sigma;\mathbb{C})^-\to H^1(X,\mathbb{C})^-$ is the natural surjective linear map. On the other hand, the action of $g\in\mathrm{GL}^+(2,\mathbb{R})$ on $H^1(X,\Sigma;\mathbb{C})^-$ satisfies

$$\rho(g\cdot[\omega])=g\cdot\rho([\omega]).$$

Therefore the leaves of the kernel foliation and the orbits of $\mathrm{GL}^+(2,\mathbb{R})$ are transversal. Since their dimensions are complementary, the corollary follows. □

3.3. Kernel foliation and SAF-invariant

In the remaining of this section, (X,ω) is a translation surface in $\mathrm{Prym}(\kappa)$ where

$$\kappa\in\left\{(1,1),(3,3),(2,2)^{\mathrm{odd}},(1,1,2),(4,4)^{\mathrm{even}},(2,2,2)^{\mathrm{even}},(1,1,4)\right\}.$$

As we have seen, moving in the kernel foliation leaves does not change the cohomology class $[\omega]\in\mathcal{H}^1(X,\mathbb{C})$. Therefore, the following proposition is an immediate consequence of Theorem 2.1 (see also [5]).

PROPOSITION 3.3. – *For any* $(X,\omega)\in\mathrm{Prym}(\kappa)$ *there exists* $\varepsilon>0$ *such that for any* $w\in\mathbb{C}$*, with* $|w|<\varepsilon$

$$SAF(X,\omega)=SAF((X,\omega)+w).$$

As a consequence, we draw our first theorem.

Proof of Theorem 1.4. – We want to show that every Prym eigenform (X,ω) is completely algebraically periodic. Since ω is an eigenform for a real quadratic order $\mathcal{O}_D$, up to action of $\mathrm{GL}^+(2,\mathbb{R})$ we can assume that all the absolute periods of ω are in $K(\imath)$, where $K=\mathbb{Q}(\sqrt{D})$. As a consequence, the set of homological directions of (X,ω) is $K\mathbb{P}^1$.

If D is a square, then $K=\mathbb{Q}$, in which case, we can assume that all the absolute periods of ω belong to $\mathbb{Z}\oplus\imath\mathbb{Z}$. Thus (X,ω) is a ramified covering of the standard torus $\mathbb{C}/(\mathbb{Z}\oplus\imath\mathbb{Z})$. It follows that for every direction $\theta\in\mathbb{Q}\cup\{\infty\}$, the linear flow in direction θ is periodic, which means that the SAF-invariant vanishes. Therefore, (X,ω) is completely algebraically periodic.

For the case where K is a real quadratic field, given a direction $k\in K\mathbb{P}^1$, as usual we normalize so that k is the vertical direction $(0:1)$. Let T be the first return map of the vertical flow to a full transversal interval. All we need to show is that $SAF(T)=0$.

Theorem 2.6 ensures that $\mathrm{flux}(\omega)=0$. However, in view of applying Proposition 2.8 we need ω to have relative periods in $K(\imath)$, which is not necessarily true. To bypass this difficulty we first apply Proposition 3.3. One remarks that all the relative periods of ω are determined by a chosen relative period and the absolute ones. Hence we can choose a small suitable vector $w\in\mathbb{R}^2$ such that all the relative coordinates of $(X,\omega)+w$ are in $K(\imath)$ and

Proposition 3.3 applies i.e., $SAF(X,\omega) = SAF((X,\omega)+w)$. Since $(Y,\eta) = (X,\omega)+w$ is still an eigenform, again Theorem 2.6 gives $\mathrm{flux}(\eta) = 0$. But now by Proposition 2.8, we draw $SAF((Y,\eta)) = 0$. Hence the SAF-invariant of the vertical flow on (X,ω) also vanishes and Theorem 1.4 is proven. □

4. Interval exchange transformations and linear involutions

4.1. Linear involutions

The first return map of the vertical flow on a translation surface (X,ω) to an interval I defines an interval exchange transformation (see Section 2.1). Such a map is encoded by a partition of I into d subintervals that we label by letters in some finite alphabet $\mathcal{A}$, and a permutation π of $\mathcal{A}$. The length of these intervals is recorded by vector λ with positive entries. The vector λ is called the continuous datum of T and π is called the combinatorial datum (we will write $T = (\pi,\lambda)$). We usually represent π by a table of two lines (here $\mathcal{A} = \{1,\dots,d\}$):

$$\pi = \begin{pmatrix} 1 & 2 & \dots & d \\ \pi^{-1}(1) & \pi^{-1}(2) & \dots & \pi^{-1}(d) \end{pmatrix}.$$

When the measured foliation is not oriented, the above construction does not make sense. Nevertheless a generalization of interval exchange maps for any measured foliation on a surface (oriented or not) was introduced by Danthony and Nogueira [9]. The generalizations (*linear involutions*) corresponding to oriented flat surfaces with $\mathbb{Z}/2\mathbb{Z}$ linear holonomy were studied in detail by Boissy and Lanneau [3] (see also Avila-Resende [2] for a similar construction). We briefly recall the objects here.

Roughly speaking, a linear involution encodes the successive intersections of the foliation with some transversal interval I. We choose I and a positive vertical direction (equivalently, a choice of left and right ends of I) that intersect every vertical geodesics. The first return map $T_0 : I \to I$ of vertical geodesics in the positive direction is well defined, outside a finite number of points (called singular points) that correspond to vertical geodesics that stop at a singularity before intersecting again the interval I. This equips I with is a finite open partition (I_α) so that $T_0(x) = \pm x + t_\alpha$.

However the map T_0 alone does not properly correspond to the dynamics of vertical geodesics since when $T_0(x) = -x + t_\alpha$ on the interval I_α, then $T_0^2(x) = x$, and $(x, T_0(x), T_0^2(x))$ does not correspond to the successive intersections of a vertical geodesic with I starting from x. To fix this problem, we have to consider T_1 the first return map of the vertical geodesics starting from I in the negative direction. Now if $T_0(x) = -x + c_i$ then the successive intersections with I of the vertical geodesic starting from x will be $x, T_0(x), T_1(T_0(x))\dots$

Definition 4.1. – *Let f be the involution of $I \times \{0,1\}$ given by $f(x,\varepsilon) = (x, 1-\varepsilon)$. A* linear involution *is a map T, from $I \times \{0,1\}$ into itself, of the form $f \circ \tilde{T}$, where $\tilde{T}$ is an involution of $I \times \{0,1\}$ without fixed point, continuous except on a finite set of point Σ_T, and which preserves the Lebesgue measure. In this paper we will only consider linear involutions with the following additional condition: the derivative of $\tilde{T}$ is 1 at (x,ε) if (x,ε) and $\tilde{T}(x,\varepsilon)$ belong to the same connected component, and -1 otherwise.*

REMARK 4.2. – *A linear involution T that preserves $I \times \{0\}$ corresponds precisely to an interval exchange transformation map T_0 (by restricting T to $I \times \{0\}$). Therefore, we can identify the set of interval exchange maps with a subset of the linear involutions.*

As for interval exchange maps, a linear involution T is encoded by a combinatorial datum called *generalized permutation* and by continuous data. This is done in the following way: $I \times \{0\} \backslash \Sigma_T$ is a union of l open intervals $I_1 \sqcup \ldots \sqcup I_l$, where we assume by convention that I_i is the interval at the place i, when counted from the left to the right. Similarly, $I \times \{1\} \backslash \Sigma_T$ is a union of m open intervals $I_{l+1} \sqcup \ldots \sqcup I_{l+m}$. For all i, the image of I_i by the map $\tilde{T}$ is a interval I_j, with $i \neq j$, hence $\tilde{T}$ induces an involution without fixed points on the set $\{1, \ldots, l+m\}$. To encode this involution, we attribute to each interval I_i a symbol such that I_i and $\tilde{T}(I_i)$ share the same symbol. Choosing the set of symbol to be $\mathcal{A}$, we get a two-to-one map $\pi : \{1, \ldots, l+m\} \to \mathcal{A}$, with $d := |\mathcal{A}| = \frac{l+m}{2}$. Note that π is not uniquely defined by T since we can compose it on the left by any permutation of the alphabet $\mathcal{A}$. The continuous data of T is a real vector $\lambda = (\lambda_\alpha)_{\alpha \in \mathcal{A}}$ with positive entries, which records the lengths of the permuted intervals.

DEFINITION 4.3. – *A* generalized permutation *of type (l, m), with $l + m = 2d$, is a two-to-one map $\pi : \{1, \ldots, 2d\} \to \mathcal{A}$ to some finite alphabet $\mathcal{A}$. We will usually represent such generalized permutations by a table of two lines of symbols, with each symbol appearing exactly two times.*

$$\pi = \begin{pmatrix} \pi(1) & \ldots & \pi(l) \\ \pi(l+1) & \ldots & \pi(l+m) \end{pmatrix}.$$

We will call top *(respectively* bottom*) the restriction of a generalized permutation π to $\{1, \ldots, l\}$ (respectively $\{l+1, \ldots, l+m\}$) where (l, m) is the type of π.*

In the table representation of a generalized permutation, a symbol might appear two times in the same line (see Example 4.6 below). Therefore, we do not necessarily have $l = m$.

Convention. We will use the terminology generalized permutations for permutations that are *not* "true" permutations.

4.2. Irreducibility and suspension over a linear involution

Starting from a linear involution T, we want to construct a flat surface and a horizontal segment whose corresponding "first return" maps of the vertical foliation give T. Such surface will be called a *suspension* over T, and the parameters encoding this construction will be called *suspension data* (see [3, §2.3] for details).

We say that a linear involution $T = (\pi, \lambda)$ admits a *suspension* data if there exists a collection of complex numbers $\zeta = \{\zeta_\alpha\}_{\alpha \in \mathcal{A}}$ such that

1. $\forall \alpha \in \mathcal{A} \quad Re(\zeta_\alpha) = \lambda_\alpha$.
2. $\forall 1 \leq i \leq l-1 \quad Im(\sum_{j \leq i} \zeta_{\pi(j)}) > 0$
3. $\forall 1 \leq i \leq m-1 \quad Im(\sum_{1 \leq j \leq i} \zeta_{\pi(l+j)}) < 0$
4. $\sum_{1 \leq i \leq l} \zeta_{\pi(i)} = \sum_{1 \leq j \leq m} \zeta_{\pi(l+j)}$.

Given such a collection of complex numbers, one can form two broken lines L_0 and L_1 (with a finite number of edges) on the plane: the edge number i of L_0 is represented by the complex number $\zeta_{\pi(i)}$, for $1 \leq i \leq l$, and L_1 starts on the same point as L_0, and the edge number j is represented by the complex number $\zeta_{\pi(l+j)}$ for $1 \leq j \leq m$.

If L_0 and L_1 only intersect on their endpoints, then L_0 and L_1 define a polygon whose sides come in pairs and for each pair the corresponding sides are parallel and have the same length. Then identifying these sides together, one gets a flat surface such that the first return map of the vertical foliation on the segment corresponding to X in S defines the same linear involution as T.

If π is a "true" permutation defined over d letters, it is well known(see [34, Formula 3.7, p. 207] and [19, p. 174]) that T admits a suspension if and only if π is irreducible, i.e., $\pi(\{1,\dots,k\}) \neq \{1,\dots,k\}, \quad 1 \leq k \leq d-1$.

It turns out that a similar characterization exists for generalized permutations. For the convenience of the reader, we state this criterion here.

DEFINITION 4.4. – *We will say that π is* reducible *if π admits a decomposition*

$$\left(\begin{array}{c|c|c} A\cup B & *** & D\cup B \\ \hline A\cup C & *** & D\cup C \end{array}\right), \ A,B,C,D \text{ disjoint subsets of } \mathcal{A},$$

where the subsets A,B,C,D are not all empty and one of the following statements holds

(i) *No corner is empty.*
(ii) *Exactly one corner is empty and it is on the left.*
(iii) *Exactly two corners are empty and they are both on the same side.*

A permutation that is not reducible is irreducible.

For example of irreducible and reducible permutations, see Claim 8.2.

THEOREM 4.5 ([3] Theorem 3.2). – *Let $T = (\pi, \lambda)$ be a linear involution. Then T admits a suspension if and only if the underlying generalized permutation π is irreducible.*

4.3. Rauzy induction

The *Rauzy induction* $\mathcal{R}(T)$ of a linear involution T is the first return map of T to a smaller interval $I' \times \{0,1\}$, where $I' \subsetneq I$. More precisely, if $T = (\pi, \lambda)$ and (l,m) is the type of π, we identify I with the interval $[0,1)$. If $\lambda_{\pi(l)} \neq \lambda_{\pi(l+m)}$, then the Rauzy induction $\mathcal{R}(T)$ of T is the linear involution obtained by the first return map of T to

$$\big(0, \max\{1-\lambda_{\pi(l)}, 1-\lambda_{\pi(l+m)}\}\big) \times \{0,1\}.$$

It is easy to see that this is again a linear involution, defined on d letters.

The combinatorial data of the new linear involution depends only on the combinatorial data of T and whether $\lambda_{\pi(l)} > \lambda_{\pi(l+m)}$ or $\lambda_{\pi(l)} < \lambda_{\pi(l+m)}$. We say that T has type 0 or type 1 respectively. The combinatorial data of $\mathcal{R}(T)$ only depends on π and on the type of T. This defines two operations $\mathcal{R}_0$ and $\mathcal{R}_1$ by $\mathcal{R}(T) = (\mathcal{R}_\varepsilon(\pi), \lambda')$, with ε the type of T (see [3] for details). We will not use these operations in this paper.

We stress that the Rauzy-Veech induction is well defined if and only if the two rightmost intervals do not have the same length i.e., $\lambda_{\pi(l)} \neq \lambda_{\pi(l+m)}$. However, when these intervals do have the same length, we can still consider the first return map of T to

$$\big(0, 1-\lambda_{\pi(l)}\big) \times \{0,1\}.$$

This is again a linear involution, denoted by $\mathcal{R}_{\mathrm{sing}}(T)$, defined over $d-1$ letters. The combinatorics of $\mathcal{R}_{\mathrm{sing}}(T)$ can be defined as follows: we apply the top operation of the Rauzy induction and then we erase the last letter of the top. Equivalently, we apply the bottom operation of the Rauzy induction and then we erase the last letter of the bottom.

Example 4.6. – *Let $T=(\pi,\lambda)$ with $\pi = \left(\begin{smallmatrix} A & A & B & C \\ D & C & B & D \end{smallmatrix}\right)$. Then the combinatorial datum of the Rauzy induction $\mathcal{R}(T)$ of T is:*

$$\begin{array}{ll} \left(\begin{smallmatrix} A & A & B & C \\ D & C & D & B \end{smallmatrix}\right) & \text{if } \lambda_C > \lambda_D \\ \left(\begin{smallmatrix} A & A & B & & \\ C & D & C & B & D \end{smallmatrix}\right) & \text{if } \lambda_C < \lambda_D \\ \left(\begin{smallmatrix} A & A & B \\ D & D & B \end{smallmatrix}\right) & \text{if } \lambda_C = \lambda_D \end{array}$$

We can formally define the converse of the (singular) Rauzy Veech operations. We proceed as follows: given some permutation π' defined over an alphabet $\mathcal{A}$, and a letter $\alpha \notin \mathcal{A}$, we put α at the end of the top or bottom line of π'. Then we choose some letter $\beta \in \mathcal{A}$. We replace β by α and we put the letter β at the end of the bottom of top line of π'. The new permutation π we have constructed is defined over the alphabet $\mathcal{A} \sqcup \{\alpha\}$ and satisfies $\mathcal{R}_{\mathrm{sing}}(\pi) = \pi'$. It turns out that all the permutations of $\mathcal{R}^{-1}_{\mathrm{sing}}(\pi')$ are constructed as above with one exception: the one given by putting at the end of the top and bottom line the letter α.

Example 4.7. – *Let $\pi' = \left(\begin{smallmatrix} A & B & C & D \\ B & A & D & C \end{smallmatrix}\right)$. For instance if we choose the letter $\beta = B$ and we put α at the end of the top line, we get $\left(\begin{smallmatrix} A & \alpha & C & D & \alpha \\ B & A & D & C & B \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} A & B & C & D & \alpha \\ \alpha & A & D & C & B \end{smallmatrix}\right)$. More precisely, if β ranges over all letters of $\mathcal{A}$ we get (up to a permutation of the letters of the alphabet $\mathcal{A} \sqcup \{\alpha\}$):*

$$\begin{aligned} &\mathcal{R}^{-1}_{\mathrm{sing}} \left(\begin{smallmatrix} A & B & C & D \\ B & A & D & C \end{smallmatrix}\right) \\ &= \Big\{ \left(\begin{smallmatrix} A & B & C & D & \alpha \\ \alpha & A & D & C & B \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B & C & D & \alpha \\ B & \alpha & D & C & A \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B & C & D & \alpha \\ B & A & \alpha & D & C \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B & C & D & \alpha \\ B & A & D & \alpha & C \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B & C & D & \alpha \\ B & A & D & C & \alpha \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B & C & \alpha & D \\ B & A & D & C & \alpha \end{smallmatrix}\right), \\ &\qquad \left(\begin{smallmatrix} \alpha & B & C & D & \alpha \\ B & A & D & C & A \end{smallmatrix}\right), \left(\begin{smallmatrix} A & \alpha & C & D & \alpha \\ B & A & D & C & B \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B & \alpha & D & \alpha \\ B & A & D & C & C \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B & C & \alpha & \alpha \\ B & A & D & C & D \end{smallmatrix}\right) \Big\}. \end{aligned}$$

We end this section with the following easy lemma that will be useful for the sequel.

Lemma 4.8. – *If π is an irreducible generalized permutation then $\mathcal{R}_{\mathrm{sing}}(\pi)$ is also a generalized permutation.*

4.4. Rauzy induction and SAF-invariant

We can naturally extend $SAF(.)$ to linear involutions by $SAF(T) := SAF(\hat{T})$ where the transformation $\hat{T}$ is the double of T (see [2] for details). As for interval exchange maps, if T is periodic then $SAF(T) = 0$. The converse is true if $|\mathcal{A}| = d \leq 3$ (see Lemma 5.1 for a proof in case $d = 3$).

Proposition 4.9. – *A linear involution T defined over $d \leq 2$ intervals is completely periodic if and only if $SAF(T) = 0$.*

Proof. – We fix some alphabet $\mathcal{A} = \{A, B\}$. There are three possibilities depending on the combinatorics of the associated permutation:

$$\left(\begin{smallmatrix} A & A \\ B & B \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B \\ A & B \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right).$$

In the first two cases, clearly T is completely periodic (no matter what $SAF(T)$ is). In the last case T is a rotation of $[0,1)$ of some angle θ:

$$T(x) = \begin{cases} x+\theta & \text{if } 0 < x < 1-\theta, \\ x+\theta-1 & \text{if } 1-\theta < x < 1. \end{cases}$$

A direct computation shows $SAF(T) = 2 \wedge_{\mathbb{Q}} \theta$; hence $SAF(T) = 0$ implies $\theta \in \mathbb{Q}$ and T is completely periodic. □

Since the SAF-invariant is a scissors congruence invariant, it is preserved by the Rauzy operations.

PROPOSITION 4.10. – *Let T be a linear involution. If the Rauzy induction is well defined then T then*

$$SAF(\mathcal{R}(T)) = SAF(T).$$

Otherwise (if the Rauzy induction is not well defined), then

$$SAF(\mathcal{R}_{\text{sing}}(T)) = SAF(T).$$

Moreover, T is completely periodic if and only if $\mathcal{R}(T)$ or $\mathcal{R}_{\text{sing}}(T)$ is completely periodic.

4.5. Rauzy induction and Keane property

We will say that $T = (\pi, \lambda)$ is *decomposed* if there exists $\mathcal{A}' \subsetneq A$ such that both following conditions hold:

$$\begin{cases} \pi = \left(\begin{smallmatrix} \alpha_1 & \dots & \alpha_{i_0} & | & *** \\ \beta_1 & \dots & \beta_{j_0} & | & *** \end{smallmatrix}\right), \text{ where } \{\alpha_1, \dots, \beta_{j_0}\} = \mathcal{A}' \sqcup \mathcal{A}', \\ \sum_{i=1}^{i_0} \lambda_{\pi(i)} = \sum_{j=1}^{j_0} \lambda_{\pi(j)}, \text{for some } 1 \le i_0 < l \text{ and } 1 \le j_0 < m. \end{cases}$$

This means exactly that T splits into two linear involutions. In this case, we will use the notation $T = T_1 \# T_2$. Since the SAF-invariant is additive we have

$$SAF(T) = SAF(T_1) + SAF(T_2). \tag{1}$$

DEFINITION 4.11. – *A linear involution has a connection (of length r) if there exist $(x, \varepsilon) \in I \times \{0,1\}$ and $r \ge 0$ such that*

- *(x, ε) is a singularity for T^{-1}.*
- *$T^r(x, \varepsilon)$ is a singularity for T.*

A linear involution with no connection is said to have the Keane property *(also called the infinite distinct orbit condition or i.d.o.c. property).*

An instance of a linear involution with a connection of length 1 is when T is decomposed.

If $T = (\pi, \lambda)$ is a linear involution, we will use the notation $(\pi^{(n)}, \lambda^{(n)}) := \mathcal{R}^{(n)}(T)$ if the n-th iteration of T by $\mathcal{R}$ is well defined, and $\lambda_\alpha^{(n)}$ for the length of the interval associated to the symbol $\alpha \in \mathcal{A}$. The next proposition is a slightly more precise statement of Proposition 4.2. of [3]:

PROPOSITION 4.12. – *The following statements are equivalent.*

1. *T satisfies the Keane property.*
2. *$\mathcal{R}^{(n)}(T)$ is well defined for any $n \geq 0$ and the lengths of the intervals $\lambda^{(n)}$ tends to 0 as n tends to infinity.*

Moreover in the above situation the transformation T is minimal.

In addition, if T has a connection and if the Rauzy induction $\mathcal{R}^{(n)}(T)$ is well defined for every $n \geq 0$, then there exists $n_0 > 0$ such that $\mathcal{R}^{(n_0)}(T)$ is decomposed.

The following proposition relates connections with vanishing SAF-invariant.

PROPOSITION 4.13. – *Let T be a linear involution such that the lengths of the exchanged intervals belong to a 2-dimensional space over $\mathbb{Q}$. If $SAF(T) = 0$ then T has a connection.*

Proof. – Let $\hat{T}$ be the double of T. Since the interval exchange map $\hat{T}$ has vanishing SAF, $\hat{T}$ is not ergodic (see Arnoux's thesis [1]). But by a result of Boshernitzan ([4], Theorem 1.1), $\hat{T}$ is neither minimal (otherwise $\hat{T}$ would be uniquely ergodic).
If T has no connection then it satisfies Keane property and Proposition 4.12 implies T would be minimal. So that $\hat{T}$ would also be minimal that is a contradiction. □

5. Complete periodicity of linear involution up to 5 intervals

In this section we specialize the analysis of complete periodicity to linear involutions. In the sequel, T will be a linear involution defined over d intervals. We prove several lemmas depending on the values of $d \in \{3, \dots, 6\}$. Section 5 is devoted to the case $d \leq 5$; as a corollary we will draw Theorem 1.6. In Section 8 we will consider the case $d = 6$ and deduce Theorem 1.9.

Since we proceed by induction on d, let us start with the case $d = 3$.

LEMMA 5.1 ($d = 3$). – *If T is a linear involution defined over 3 intervals with $SAF(T) = 0$ then T is completely periodic.*

Proof. – The condition $SAF(T) = 0$ implies that the lengths of the intervals exchanged by T span a 2-dimensional space over $\mathbb{Q}$. It follows from Proposition 4.13 T has a connection. Hence from Proposition 4.12, two possibilities can occur:

(a) either the Rauzy induction $\mathcal{R}^{(n_0)}(T)$ is not well defined for some $n_0 > 0$, or
(b) there exists $n_0 > 0$ such that the permutation $\mathcal{R}^{(n_0)}(T)$ is decomposed.

Let us first consider case (a). Since the Rauzy induction is not well defined on $T' := \mathcal{R}^{(n_0)}(T)$, there is a relation: $\lambda_{\pi^{(n_0)}(l)} = \lambda_{\pi^{(n_0)}(l+m)}$. We can consider the first return map of T' to

$$\left(0, 1 - \lambda_{\pi^{(n_0)}(l)}\right) \times \{0, 1\}.$$

We get a new $T'' = \mathcal{R}_{\text{sing}}(T')$ defined over 2 intervals. Since $0 = SAF(T) = SAF(T') = SAF(T'')$ we get that T'' has vanishing SAF and hence T'' is completely periodic. We conclude by Proposition 4.10 that T' and T are also completely periodic.

Let us now consider case (b). By assumption, T splits as $T = T_1\#T_2$. We will denote the alphabet by $\mathcal{A} = \{A, B, C\}$. The decomposition of the permutation $\pi^{(n_0)}$ involves the following decomposition (up to permutation of the letters and T_i):

$$\left(\begin{array}{c|cc} A & \alpha_1 & \alpha_2 \\ A & \beta_1 & \beta_2 \end{array}\right), \text{ where } \{\alpha_1, \dots, \beta_2\} = \{B, C\} \sqcup \{B, C\}.$$

Here $T_1 = \left(\begin{pmatrix} A \\ A \end{pmatrix}, \lambda_A\right)$. Thus obviously $SAF(T_1) = 0$. Reporting into Equation (1):

$$0 = SAF(T) = SAF(T_1) + SAF(T_2),$$

we draw $SAF(T_2) = 0$. Since T_2 is a linear involution defined over 2 letters, we again conclude that T_2 is completely periodic. This proves the lemma. □

In the next lemma, we continue this induction process. The idea is to consider the inverse of the singular Rauzy induction. Thus the number of intervals increases, and we need to avoid "bad" cases.

Lemma 5.2 ($d = 4$). – *If $T = (\pi, \lambda)$ is a linear involution defined over 4 intervals with $SAF(T) = 0$, and if $\pi \neq \left(\begin{smallmatrix} A & B & C & D \\ B & A & D & C \end{smallmatrix}\right)$ up to a permutation of the letters, then T is completely periodic.*

Proof. – We first show that T has a connection. We can assume that π is irreducible (otherwise there is a connection) and the Rauzy induction $\mathcal{R}^{(n)}(T)$ is well defined for any $n > 0$ (otherwise T would have a connection and we are done).

If π is a true permutation then by Proposition 4.12 the range of the Rauzy induction is the Rauzy class of π. Since the SAF is invariant along the Rauzy induction, we can assume that $\pi = \left(\begin{smallmatrix} A & B & C & D \\ D & C & B & A \end{smallmatrix}\right)$. For any $\alpha \in \mathcal{A}$, the translation lengths t_α are (in terms of the lengths of the subintervals):

$$(t_A, t_B, t_C, t_D) = (\lambda_B + \lambda_C + \lambda_D, \lambda_C + \lambda_D - \lambda_A, \lambda_D - \lambda_A - \lambda_B, -\lambda_A - \lambda_B - \lambda_C).$$

It follows that

$$0 = SAF(T) = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha \wedge_{\mathbb{Q}} t_\alpha = \lambda_A \wedge (\lambda_B + \lambda_C + \lambda_D) + \lambda_B \wedge (\lambda_C + \lambda_D) + \lambda_C \wedge \lambda_D.$$

We rewrite the above relation as follows:

$$(2) \qquad -\lambda_A \wedge \lambda_B = (\lambda_A + \lambda_B + \lambda_C) \wedge (\lambda_C + \lambda_D).$$

If $\lambda_A \wedge \lambda_B = 0$, which means that $\lambda_B \in \mathbb{Q}\lambda_A$, then (2) implies $\lambda_D \in \mathbb{Q}\lambda_A + \mathbb{Q}\lambda_C$. Therefore the span over $\mathbb{Q}$ of the lengths $\{\lambda_A, \lambda_B, \lambda_C, \lambda_D\}$ is equal to $\mathbb{Q}\lambda_A + \mathbb{Q}\lambda_C$, and it follows that the space $\mathrm{Span}_{\mathbb{Q}}(\lambda_A, \lambda_B, \lambda_C, \lambda_D)$ has rank at most 2.

Now if $\lambda_A \wedge \lambda_B \neq 0$, Equation (2) gives

$$(3) \qquad \lambda_A \wedge \lambda_B \wedge (\lambda_C + \lambda_D) = 0.$$

Since λ_A and λ_B are linearly independent over $\mathbb{Q}$ there exists $a, b \in \mathbb{Q}$ such that $\lambda_C + \lambda_D = a\lambda_A + b\lambda_B$. Reporting into Equation (2) we draw

$$(b + 1 - a) \cdot \lambda_A \wedge \lambda_B - a \cdot \lambda_A \wedge \lambda_C - b \cdot \lambda_B \wedge \lambda_C = 0.$$

If λ_C does not belong to the vector space (over $\mathbb{Q}$) generated by λ_A and λ_B then the vectors $\{\lambda_A \wedge \lambda_B, \lambda_A \wedge \lambda_C, \lambda_B \wedge \lambda_C\}$ are linearly independent in $\bigwedge^2_{\mathbb{Q}}(\mathbb{R})$. Thus $a = 0$, $b = 0$ and $1 + b - a = 0$ that is a contradiction. We can then conclude that λ_C belongs to $\mathbb{Q}\lambda_A + \mathbb{Q}\lambda_B$ and so does λ_D. Therefore the lengths of the exchanged intervals span a vector space of rank 2 over $\mathbb{Q}$. Now, by Proposition 4.13, the linear involution T has a connection.

If π is a generalized permutation, since it is irreducible, there is a suspension (Y, q) belonging to some stratum of quadratic differentials and inducing π. Since the number of intervals is 4, the dimension of the stratum is 3. Hence the only possibility is $\mathcal{Q}(-1,-1,2)$ and, up to the Rauzy induction, the permutation is $\pi = \left(\begin{smallmatrix} A & A & B & C \\ C & B & D & D \end{smallmatrix}\right)$. The same computation shows that the lengths of the exchanged intervals have linear rank 2 over $\mathbb{Q}$ so that Proposition 4.13 applies and T has a connection.

We now repeat the same strategy as in the proof of the previous lemma. From Proposition 4.12 only two possibilities (a) and (b) can occur (see the proof of Lemma 5.1). In case (a), the Rauzy induction is not well defined and we can reduce the problem to some $T' = \mathcal{R}^{(n_0)}(T)$ defined over 3 letters. We then conclude using Lemma 5.1.
Thus let us assume that there exists $n_0 > 0$ such that $\mathcal{R}^{(n_0)}(T)$ breaks into two linear involutions T_1 and T_2 with $SAF(T_1) + SAF(T_2) = 0$. Again if T_1 or T_2 is defined over only one interval then we are done (by the same argument as above). So assume that T_1 and T_2 are defined over 2 intervals. If $SAF(T_1) = 0$ then we are done by Proposition 4.9. Hence we will assume that $SAF(T_1) = -SAF(T_2) \neq 0$ and we will get a contradiction. This can be achieved only if the permutation associated to T_1 has the form $\left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$. The same is true for T_2: the permutation is $\left(\begin{smallmatrix} C & D \\ D & C \end{smallmatrix}\right)$. Hence $\pi^{(n_0)} = \left(\begin{smallmatrix} A & B & C & D \\ B & A & D & C \end{smallmatrix}\right)$. From this observation, it is not hard to see that $\pi = \pi^{(n_0)}$, that is the desired contradiction. The lemma is proven. $\square$

For the case $d = 5$, again new pathological cases appear as shown in the next lemma:

Lemma 5.3 ($d = 5$). – *Let $T = (\pi, \lambda)$ be a linear involution defined over 5 intervals with $SAF(T) = 0$. We assume that either the Rauzy induction is not well defined, or T decomposes. If π does not belong to one of the following sets (up to permutation of the letters of $\mathcal{A}$):*

$$\mathcal{E}_1 = \left\{\left(\begin{smallmatrix} A & B & C & D & \alpha \\ \alpha & A & D & C & B \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B & C & D & \alpha \\ B & \alpha & D & C & A \end{smallmatrix}\right)\right\}, \qquad \mathcal{E}_2 = \left\{\left(\begin{smallmatrix} \alpha & B & C & D & \alpha \\ B & A & D & C & A \end{smallmatrix}\right), \left(\begin{smallmatrix} A & \alpha & C & D & \alpha \\ B & A & D & C & B \end{smallmatrix}\right)\right\},$$
$$\mathcal{E}_3 = \left\{\left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix} \; \pi'\right), \left(\pi' \; \begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right), \ \pi' \text{ permutation defined over } 3 \text{ letters}\right\},$$

then T is completely periodic.

Proof of Lemma 5.3. – If the Rauzy induction is not well defined for T then we get a new linear involution $T' = (\pi', \lambda') = \mathcal{R}_{\mathrm{sing}}(T)$ defined over 4 intervals with vanishing SAF-invariant. If $\pi' \neq \left(\begin{smallmatrix} A & B & C & D \\ B & A & D & C \end{smallmatrix}\right)$ then we are done by Lemma 5.2. Hence we can assume that $\mathcal{R}_{\mathrm{sing}}(\pi) = \left(\begin{smallmatrix} A & B & C & D \\ B & A & D & C \end{smallmatrix}\right)$. But by Example 4.7 permutations in $\mathcal{R}^{-1}_{\mathrm{sing}}\left(\begin{smallmatrix} A & B & C & D \\ B & A & D & C \end{smallmatrix}\right)$ are exactly those lying in $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$. Thus the lemma is proved for this case.
If $T = T_1 \# T_2$ is decomposed into two linear involutions, where T_i is defined over d_i intervals with $d_1 + d_2 = 5$, then T_i satisfy $SAF(T_1) + SAF(T_2) = 0$. There are two possible partitions for the number $\{d_1, d_2\}$ of intervals of T_i: namely, up to permuting T_i: (d_1, d_2) equals $(1, 4)$ or $(2, 3)$. In the first situation T_1 is completely periodic as any linear involution defined over only 1 interval is periodic. Hence $SAF(T_2) = 0$. One wants to use Lemma 5.2. For that we need to avoid the case $\pi_2 = \left(\begin{smallmatrix} A & B & C & D \\ B & A & D & C \end{smallmatrix}\right)$. But in the latter situation one would have

$\pi = \left(\begin{smallmatrix} E & A & B & C & D \\ E & B & A & D & C \end{smallmatrix}\right)$. This permutation belongs to the set $\mathcal{E}_3$. Hence the lemma is proved in this situation.

The last remaining case is $(d_1, d_2) = (2,3)$. If $SAF(T_1) = 0$ then we are done. On the other hand $SAF(T_1) \neq 0$ implies $\pi_1 = \left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$, that is $\pi \in \mathcal{E}_3$. The lemma is proved. □

6. Complete Algebraic Periodicity implies Complete Periodicity

We begin with the following simple lemma.

LEMMA 6.1. – *Let $T = (\pi, \lambda)$ be a linear involution defined over 3 letters. We assume that $\pi \notin \{\left(\begin{smallmatrix} A & B & C \\ B & A & C \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B & C \\ A & C & B \end{smallmatrix}\right)\}$. If T has a periodic orbit then T is completely periodic.*

We postpone the proof of the lemma to the end of this section and show Theorem 1.6.

Proof of Theorem 1.6. – Let θ be a direction of a cylinder in X. The core curve of this cylinder represents an element of $H_1(X, \mathbb{Z})$, hence by assumption, the SAF-invariant of the foliation $\mathcal{F}_\theta$ vanishes. As usual one assumes that θ is the vertical direction. We want to show that the flow in the vertical direction is periodic. Let $T = (\pi, \lambda)$ be the linear involution given by the cross section of the vertical foliation to some full transversal interval. By assumption T is defined over 6 intervals and has a periodic orbit. Moreover π is an *irreducible* generalized permutation.

Obviously proving complete periodicity for the vertical foliation or for T is the same. Since T has a periodic orbit Proposition 4.12 implies that only two cases can occur (up to replacing T by $\mathcal{R}^{(n)}(T)$ for some suitable n), either

(a) $\mathcal{R}(T)$ is decomposed, or
(b) the Rauzy induction $\mathcal{R}(T)$ is not well defined.

CASE (a). In this situation, $T = (\pi, \lambda) = T_1 \# T_2$ is decomposed into two linear involutions, each defined over d_i intervals with $d_1 + d_2 = 6$, with opposite SAF. There are three possible (unordered) partitions for $\{d_1, d_2\}$, namely $\{1,5\}$, $\{2,4\}$ or $\{3,3\}$. In the first situation π is reducible that is a contradiction. In the second situation since π is irreducible, we necessarily have $\pi_1 = \left(\begin{smallmatrix} A & A \\ B & B \end{smallmatrix}\right)$. Hence T_1 is completely periodic, and $SAF(T_2) = 0$. We conclude with Lemma 5.2 (π_2 is not $\left(\begin{smallmatrix} C & D & E & F \\ D & C & F & E \end{smallmatrix}\right)$ otherwise π would be reducible). In the last case, i.e., $d_1 = d_2 = 3$, T_1 or T_2 has a closed orbit, say T_1. Again, by the irreducibility of π, the two permutations π_1 and π_2 are generalized permutations. Then Lemma 6.1 implies that T_1 is completely periodic. Hence $SAF(T_2) = 0$ and we conclude by Lemma 5.1.

CASE (b). Since the Rauzy induction is not well defined $T' = \mathcal{R}_{\mathrm{sing}}(T) = (\pi', \lambda')$ is a linear involution defined over 5 letters, with a periodic orbit and vanishing SAF-invariant. Note that π' is not necessarily irreducible. Applying Proposition 4.12 again, we know that there exists n such that either $\mathcal{R}^{(n)}(T')$ decomposes, or $\mathcal{R}^{(n+1)}(T')$ is not well defined.

Set $T' = (\pi', \lambda')$ and $T'' = (\pi'', \lambda'') := \mathcal{R}^{(n)}(T')$. In the first case, we have $T' = T_1'' \# T_2''$, where T_i'' is a linear involution defined over d_i'' intervals, and $d_1'' + d_2'' = 5$. If $(d_1'', d_2'') = (1,4)$ then $SAF(T_1'') = 0$, hence $SAF(T_2'') = 0$. Since T_2'' is defined over 4 letters, it follows that T_2'' is periodic unless $\pi_2'' = \left(\begin{smallmatrix} A & B & C & D \\ B & A & D & C \end{smallmatrix}\right)$ (Lemma 5.2), consequently $\pi'' = \left(\begin{smallmatrix} \alpha & A & B & C & D \\ \alpha & B & A & D & C \end{smallmatrix}\right)$. But since T'' is obtained from T' by a sequence of Rauzy inductions, it follows that

$\pi' = \left(\begin{smallmatrix} \alpha & A & B & C & D \\ \alpha & B & A & D & C \end{smallmatrix}\right)$. In particular, π' is not a generalized permutation, which is a contradiction to Lemma 4.8. Therefore, we can conclude that T'' and hence T is periodic. The case $(d_1'', d_2'') = (4, 1)$ follows from the same argument.

If $(d_1'', d_2'') = (2, 3)$, then by assumption, we know that either T_1'' or T_2'' has a periodic orbit. If T_1'' has a periodic then T_1'' is periodic (since it is defined over 2 letters), therefore $SAF(T_1'') = SAF(T_2'') = 0$. Hence T_2'' is also periodic by Lemma 5.1. Assume that T_1'' is not periodic, then we must have $\pi_1'' = \left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$, and T_2'' has a periodic orbit. If T_2'' is periodic then $SAF(T_2'') = 0$, which implies that $SAF(T_1'') = 0$, and T_1'' is periodic. Therefore T_2'' is not periodic. By Lemma 6.1, we have $\pi_2'' \in \{\left(\begin{smallmatrix} C & D & E \\ D & C & E \end{smallmatrix}\right), \left(\begin{smallmatrix} C & D & E \\ C & E & D \end{smallmatrix}\right)\}$. Thus we have

$$\pi'' \in \{\left(\begin{smallmatrix} A & B & C & D & E \\ B & A & D & C & E \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B & C & D & E \\ B & A & C & E & D \end{smallmatrix}\right)\}.$$

But since T'' is obtained from T' by Rauzy induction, we have $\pi' = \pi''$, and in particular π' is not a generalized permutation, which contradicts Lemma 4.8. Obviously, the case $(d_1'', d_2'') = (3, 2)$ follows from similar arguments.

We are left with the case where $\mathcal{R}(T'')$ is not well defined. Set $\tilde{T} = (\tilde{\pi}, \tilde{\lambda}) := \mathcal{R}_{\mathrm{sing}}(T'')$, then $\tilde{T}$ is defined over 4 letters. We have $SAF(\tilde{T}) = SAF(T) = 0$, and $\tilde{T}$ has an periodic orbit. By Lemma 5.2 if $\tilde{\pi} \neq \left(\begin{smallmatrix} A & B & C & D \\ B & A & D & C \end{smallmatrix}\right)$ then $\tilde{T}$ is periodic. If $\tilde{\pi} = \left(\begin{smallmatrix} A & B & C & D \\ B & A & D & C \end{smallmatrix}\right)$ then $\tilde{T}$ decomposes into two IETs defined over 2 letters. Since one of them has a periodic orbit, both SAF-invariants vanish. Therefore $\tilde{T}$ is periodic by Proposition 4.9. □

Proof of Lemma 6.1. – If π is a "true" permutation, then T is an IET defined over 3 letters. The assumption implies that T is irreducible, therefore T can be realized as the first return map of the vertical flow to a full transversal interval on a flat torus with two marked points. If T has a periodic orbit, then the torus has a closed geodesic in the vertical direction, from which we deduce that the vertical flow is periodic, and T is also periodic.

In the case π is a generalized permutation, since T is not minimal, up to replacing T by some of its iterates under the Rauzy induction, either $\mathcal{R}(T)$ is not well defined, or T is decomposed. In the first case the problem reduces to some T' (defined over 2 intervals) with a periodic orbit, hence we are done. In the latter case T decomposes as two linear involutions T_i. Since π is a generalized permutation by assumption, one of the permutations π_i is of the form $\left(\begin{smallmatrix} A & A \\ B & B \end{smallmatrix}\right)$. Hence the corresponding linear involution is completely periodic and we are done. □

7. Complete algebraic periodicity implies real multiplication

The aim of this section is to prove the converse of Theorem 1.4, namely Theorem 1.5. Our proof is based on the following theorems.

THEOREM 7.1 (McMullen, [24] Theorem 3.5). – *Let (X, ω) be a Prym form with $\dim_{\mathbb{C}} \mathrm{Prym}(X) = 2$, and let us assume that there is a hyperbolic element A in $\mathrm{SL}(X, \omega)$, where $\mathrm{SL}(X, \omega)$ denotes the Veech group of (X, ω). Then (X, ω) is a Prym eigenform in ΩE_D, for some discriminant D satisfying $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\mathrm{Tr}(A))$.*

Sketch of proof. – Let $\phi : X \to X$ be a pseudo-Anosov affine with respect to the flat metric given by $\omega \in \Omega(X,\tau)^-$ (e.g., given by Thurston's construction [33]). By replacing ϕ by one of its powers if necessary, we can assume that ϕ commutes with τ. It follows that ϕ induces an isomorphism of $H_1(X,\mathbb{Z})^-$ preserving the intersection form. Therefore

$$T = \phi_* + \phi_*^{-1} : H_1(X,\mathbb{Z})^- \longrightarrow H_1(X,\mathbb{Z})^-,$$

is a self-adjoint endomorphism of $\mathrm{Prym}(X,\tau)$. Observe that T preserves the complex line S in $(\Omega(X,\tau)^-)^*$ spanned by the dual of $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$, and the restriction of T to this vector space is $\mathrm{Tr}(\mathrm{D}\phi)\cdot\mathrm{id}_S$. Since $\dim_{\mathbb{C}} \Omega(X,\tau)^- = 2$, one has $\dim_{\mathbb{C}} S^\perp = 1$. But T preserves the splitting $(\Omega(X,\tau)^-)^* = S \oplus S^\perp$, and acts by real scalar multiplication on each line, hence T is $\mathbb{C}$-linear, i.e., $T \in \mathrm{End}(\mathrm{Prym}(X,\tau))$. This equips $\mathrm{Prym}(X,\tau)$ with the real multiplication by $\mathbb{Z}[T] \simeq \mathcal{O}_D$ for a convenient discriminant D. Since $T^*\omega = \mathrm{Tr}(\mathrm{D}\phi)\omega$, the form ω becomes an eigenform for this real multiplication. Observe that $\mathbb{Q}(\sqrt{D}) = \mathbb{Q}(\lambda+\lambda^{-1})$ where λ being the expanding factor of the map ϕ. Note that the fact $T \notin \mathbb{Z}\mathrm{Id}$ follows from basic results in the theory of pseudo-Anosov homeomorphisms. □

THEOREM 7.2 (Calta [5]). – *Fix a real quadratic field $K \subset \mathbb{R}$. Let (X,ω) be a completely algebraically periodic translation surface such that all the periods (both relative and absolute) of ω belong to $K(\imath)$. Suppose that (X,ω) cannot be normalized by* $\mathrm{GL}^+(2,K)$ *such that all the absolute periods of ω belong to $\mathbb{Q}(\imath)$. Then if (X,ω) admits a decomposition into k cylinders in the horizontal direction, then the following equality holds*

$$\sum_{i=1}^{k} w_i' h_i = 0 \tag{4}$$

where w_i, h_i are respectively the width and the height of the i-th cylinder, and w_i' is the Galois conjugate of w_i in K.

REMARK 7.3. – *This statement is slightly more general than the statements* [5, Proposition 4.1, and Lemma 4.2] *but its proof is essentially the same. One can also remark that Equation* (4) *is the same as the one in Corollary 2.5.*

Sketch of proof. – We have $K = \mathbb{Q}(\sqrt{f})$, where f is a square-free positive integer. Recall that if $a, b \in \mathbb{Q}$ then $(a+b\sqrt{f})' = a - b\sqrt{f}$. Let $w, h \in K$.

$$\begin{aligned} 4w \wedge h &= (w + w' + w - w') \wedge (h + h' + h - h') \\ &= \frac{1}{\sqrt{f}}((w+w')(h-h') - (w-w')(h+h'))1 \wedge \sqrt{f} \\ &= \frac{2}{\sqrt{f}}(w'h - wh')1 \wedge \sqrt{f}. \end{aligned}$$

Let $C_i,\ i = 1,\dots,k$, denote the cylinders in the horizontal direction. We identify each C_i with a parallelogram P_i in $\mathbb{R}^2$ which is constructed from the pair of vectors $\{(w_i,0),(t_i,h_i)\}$ in K^2. We have (see Section 2.2):

$$J(X,\omega) = 2\sum_{i=1}^{k} \begin{pmatrix} w_i \\ 0 \end{pmatrix} \wedge \begin{pmatrix} t_i \\ h_i \end{pmatrix}.$$

By assumption, the vertical direction $(0:1)$ is algebraically periodic. Hence

$$J_{xx}(X,\omega) = \sum_{i=1}^{k} w_i \wedge t_i = 0.$$

Let $\overrightarrow{v}_q = (1,q)$ with $q \in K$, and $A_q = \begin{pmatrix} 1 & -1/q \\ 0 & 1/q \end{pmatrix}$ so that $A_q \cdot \overrightarrow{v}_q = (0,1)$. Thus

$$J_{xx}(A_q \cdot (X,\omega)) = \sum_{i=1}^{k} w_i \wedge (t_i - \frac{1}{q} h_i) = 0.$$

It follows that

$$\sum_{i=1}^{k} w_i \wedge s h_i = 0, \ \forall s \in K,$$

which implies

$$\sum_{i=1}^{k} w_i' s h_i - w_i s' h_i' = 0, \ \forall s \in K.$$

By evaluating the last equality for $s = 1$ and $s = \sqrt{f}$, we get

$$\sum_{i=1}^{k} w_i h_i' = \sum_{i=1}^{k} w_i' h_i = -\sum_{i=1}^{k} w_i' h_i.$$

Theorem 7.2 is then proved. □

Proof of Theorem 1.5. – We first observe that both properties of being completely algebraically periodic and being a Prym eigenform is invariant along the leaves of the kernel foliation in the Prym loci given in Table 1. In view of Theorem 7.1, we will show that there exists in the leaf of the kernel foliation through (X,ω) a surface whose Veech group contains a hyperbolic element.

By normalizing using $\mathrm{GL}^+(2,\mathbb{R})$ and moving in the kernel foliation leaf, we can suppose that all the periods of (X,ω) belong to $K(\imath)$. If $K = \mathbb{Q}$, then the $\mathrm{GL}^+(2,\mathbb{R})$-orbit of (X,ω) has a square-tiled surface. Thus the Veech group of (X,ω) contains a hyperbolic element, and we are done.

Now assume that K is a real quadratic field. By Theorem 1.6, we know that (X,ω) is completely periodic. We can assume that the horizontal and vertical directions are periodic. We want to find a suitable vector $v = (s,t) \in K^2$ such that the Veech group of $(X,\omega) + v$ has two parabolic elements, one preserves the horizontal direction, the other preserves the vertical direction, a product of some powers of such elements provides us with a hyperbolic element in $\mathrm{SL}(X,\omega)$.

Let $C_i, i = 1,\dots,k$, denote the horizontal cylinders, the width, height and modulus of C_i are denoted by w_i, h_i, and μ_i respectively. Let n be the number of cylinders up to involution, we choose the numbering of cylinders such that for every $i = 1,\dots,n$, if C_i and C_j are permuted by the Prym involution then either $j = i$, or $j > n$. Theorem 7.2 implies

$$\sum_{i=1}^{k} w_i' h_i = \sum_{i=1}^{n} \alpha_i \mu_i N(w_i) = 0, \tag{5}$$

where $\alpha_i = 1$ if C_i is preserved by the Prym involution, $\alpha_i = 2$ otherwise, and $N(w_i) = w_i w_i' \in \mathbb{Q}$. Remark that for all the Prym loci in Table 1, we have $n \leq 3$. By Lemma 9.1, if this number is maximal (i.e., equal to three) then the cylinder decomposition is stable (i.e., every saddle connection in this direction connects a zero to itself). In the case $n \leq 2$, Equation (5) implies that all the cylinders are commensurable, therefore there exists a parabolic element in $\mathrm{SL}(X,\omega)$ that fixes the vector $(1,0)$.

Assume that $n = 3$, since the cylinder decomposition is stable, in each cylinder the upper (resp. lower) boundary contains only one zero of ω. For $t \in \mathbb{R}$ such that $|t|$ small enough, the surface $(X,\omega) + (0,t)$ also admits a cylinder decomposition in the horizontal direction with the same topological properties as the decomposition of (X,ω). Let C_i^t denote the cylinder in $(X_t,\omega_t) = (X,\omega) + (0,t)$ corresponding to $C_i = C_i^0$. Note that $w(C_i^t) = w(C_i) = w_i$ for any t, but in general $h_i(t) = h(C_i^t)$ is a non-constant function of t. Namely, if the zeros in the upper and lower boundaries of C_i are the same then $h_i(t) = h_i,\ \forall t$, otherwise, either $h_i(t) = h_i \pm t$, or $h_i(t) = \pm t/2$.
In particular, we see that $h_i(t) = h_i + \alpha_i t$, where $\alpha_i \in \{-1, -1/2, 0, 1/2, 1\}$. There always exist two cylinders C_i, C_j such that $\alpha_i \neq \alpha_j$. Set $R_{ij}(t) := \mu(C_i^t)/\mu(C_j^t)$. We have

$$R_{ij}(t) \in \left\{ \frac{w_j(h_i+t)}{w_i h_j}, \frac{w_j(h_i+t)}{w_i(h_j-t)}, \frac{w_j(h_i+t)}{w_i(h_j+t/2)}, \frac{w_j(h_i+t)}{w_i(h_j-t/2)}, \frac{w_j(h_i+t/2)}{w_i h_j}, \frac{w_j(h_i+t/2)}{w_i(h_j-t/2)} \right\}.$$

One can easily see that there always exists $t \in K$ such that $R_{ij}(t) \in \mathbb{Q}$. For t small enough, the surface $(X,\omega)+(0,t)$ is also decomposed into k cylinders in the horizontal direction, and Equation (5) holds, thus the condition $R_{ij}(t) \in \mathbb{Q}$ implies that all the horizontal cylinders of $(X,\omega) + (0,t)$ are commensurable, which means that $\mathrm{SL}((X,\omega) + (0,t))$ contains a parabolic element preserving the vector $(1,0)$.

Observe that the vertical direction on $(X,\omega)+(s,t)$ (for small s) is still a periodic direction. Thus by the same arguments, we can conclude that there exists a vector $v = (s,t) \in K^2$ such that $\mathrm{SL}((X,\omega)+v)$ contains a parabolic element fixing the vertical direction. It follows that $\mathrm{SL}((X,\omega)+v)$ contains a hyperbolic element. By Theorem 7.1 $(X,\omega)+v$ is a Prym eigenform, and so is (X,ω). Theorem 1.5 is then proven. $\square$

8. Complete periodicity of quadratic differentials with periods in a quadratic field

In this section we prove Theorem 1.9. We will deduce the theorem from a stronger statement. We will concentrate on cases (4), (5), (6), (7), (8) of Table 1.

THEOREM 8.1. – *Let $T = (\pi,\lambda)$ be a linear involution defined over 6 intervals which is defined by the first return map of the vertical foliation on a quadratic differential $(Y,q) \in \mathcal{Q}(\kappa)$, where*

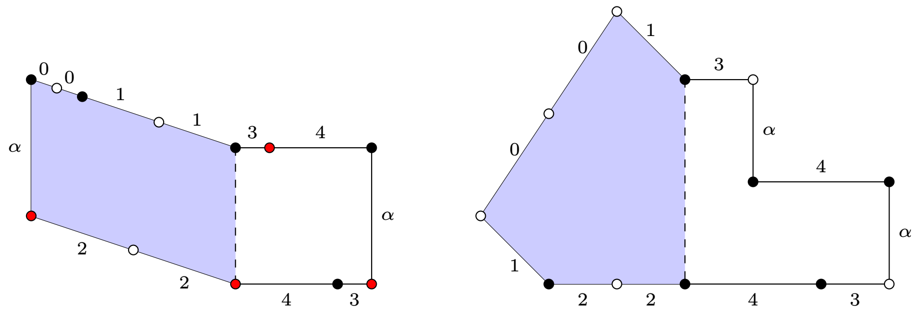
$$\kappa \in \{(-1^4,4),\ (-1^3,1,2),\ (-1,2,3),\ (-1,1,4),\ (8)\}.$$

We assume that the lengths of the intervals exchanged by T belong to a vector space of rank two over $\mathbb{Q}$, and $SAF(T) = 0$. Then we have the followings

1. *If $(Y,q) \in \mathcal{Q}(-1,2,3) \sqcup \mathcal{Q}(8)$ then T is completely periodic,*
2. *Otherwise, if T is not completely periodic then:*

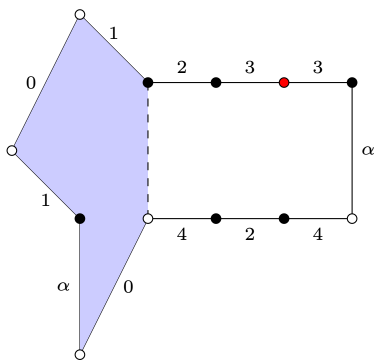
(a) *if* $(Y,q) \in \mathcal{Q}(-1^3,1,2) \sqcup \mathcal{Q}(-1^4,4)$ *then* (Y,q) *is the connected sum of a flat torus and a flat sphere, irrationally foliated with opposite SAF-invariants.*

(b) *if* $(Y,q) \in \mathcal{Q}(-1,1,4)$ *then* (Y,q) *is the connected sum of two flat tori, irrationally foliated with opposite SAF-invariants.*



$(Y_1,q_1) \in \mathcal{Q}(-1^3,1,2)$ $(Y_2,q_2) \in \mathcal{Q}(-1^4,4)$

FIGURE 2. Decompositions of (Y_i,q_i) in a connected sum of a flat torus and a flat sphere (colored in blue).



$(Y_3,q_3) \in \mathcal{Q}(-1,1,4)$

FIGURE 3. Decompositions of (Y_3,q_3) in a connected sum of two tori.

Examples of decompositions of quadratic differentials into connected sum of irrationally foliated components are shown in Figures 2, 3, and 5. We first show how Theorem 1.9 is obtained from Theorem 8.1.

Proof of Theorem 1.9 assuming Theorem 8.1. – If $K = \mathbb{Q}$ then (X,ω) is a square-tiled surface so are done. In the case K is a real quadratic field, let $\theta \in \mathbb{RP}^1$ be a direction. If the linear flow in direction θ is not minimal then θ is the direction of a saddle connection, hence $\theta \in K\mathbb{P}^1$ and up to renormalization by $\mathrm{GL}^+(2,K)$, we can assume that θ is the vertical direction. Recall that the Prym form (X,ω) covers a quadratic differential $(Y,q) \in \mathcal{Q}(-1,2,3) \sqcup \mathcal{Q}(8)$. Let T be a the linear involution associated to the vertical foliation on (Y,q). Then T satisfies the hypothesis of Theorem 8.1, therefore T is completely periodic, which implies that θ is a periodic direction on (X,ω). □

Proof of Theorem 8.1. – We begin by observing that replacing T by $\mathcal{R}^{(n)}(T)$ does not change the statement. We will assume that T is not completely periodic. By Proposition 4.13 we know that T has a connection, and by Proposition 4.12 only two possibilities can occur. Namely, up to replacing T by $\mathcal{R}^{(n)}(T)$ for some suitable n, we will assume in the sequel that

(a) T is decomposed, or
(b) the Rauzy induction is not well defined for T.

Case (a): T decomposes. – We have $T = T_1\#T_2$, where $T_i = (\pi_i, \lambda_i)$ is a linear involution defined over d_i intervals with $d_1 + d_2 = 6$. Since π is irreducible the only possible partitions for $\{d_1, d_2\}$ are $\{2, 4\}$ or $\{3, 3\}$. Recall that by assumption, we have $SAF(T) = SAF(T_1) + SAF(T_2) = 0$.

If $(d_1, d_2) = (2, 4)$, since T is irreducible, we must have $\pi_1 = \left(\begin{smallmatrix} A & A \\ B & B \end{smallmatrix}\right)$. Thus, $SAF(T_1) = SAF(T_2) = 0$. From Lemma 5.2, we know that T_2 is periodic unless $\pi_2 = \left(\begin{smallmatrix} C & D & E & F \\ D & C & F & E \end{smallmatrix}\right)$. But in this case T is reducible, so we must have $(d_1, d_2) \neq (2, 4)$. The case $(d_1, d_2) = (4, 2)$ is also ruled out by the same arguments.

In the case $(d_1, d_2) = (3, 3)$, π_1 and π_2 are generalized permutations, each defined over 3 letters, and we have naturally a decomposition of Y into a connected sum of two subsurfaces Y_1 and Y_2 corresponding to T_1 and T_2 respectively. One can check that Y_i is either a sphere or a torus. Actually, Y_i either belongs to $\mathcal{Q}(-1^4, 0)$ or $\mathcal{Q}(-1^2, 2)$, and the Y_1 and Y_2 are glued together along a geodesic loop which is obtained by cutting Y_i along a geodesic joining a pole of q_i to another singular point, where q_i is the quadratic differential defining the flat metric of Y_i. The assertions of the theorem can be easily verified by a case-by-case check.

Case (b): the Rauzy induction is not well defined. – We get a new linear involution $T' := \mathcal{R}_{\mathrm{sing}}(T) = (\pi', \lambda')$ defined over 5 intervals with zero SAF-invariant. Since the lengths of the intervals exchanged by T' still belong to a vector space of rank two over $\mathbb{Q}$, Propositions 4.13 and 4.12 imply the existence of $n_0 > 0$ such that either the Rauzy induction is not well defined for $\mathcal{R}^{(n_0)}(T')$, or $\mathcal{R}^{(n_0)}(T')$ is decomposed. Recall that by assumption T is not completely periodic, so that $\mathcal{R}^{(n_0)}(T')$ is not completely periodic either. Hence Lemma 5.3, applied to $\mathcal{R}^{(n_0)}(T')$, gives $\mathcal{R}^{(n_0)}(\pi') \in \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$.

We first claim that $\mathcal{R}^{(n_0)}(\pi') \notin \mathcal{E}_1$. Indeed since π is geometrically irreducible, by Lemma 4.8 π' is a generalized permutation (not a "true" permutation). Therefore, its image by any sequence of Rauzy inductions cannot belong do $\mathcal{E}_1$.

Secondly we claim that if $\mathcal{R}^{(n_0)}(\pi') \in \mathcal{E}_2$ then, up to exchanging of the lines of π', we have

$$\pi' \in \mathcal{F} = \left\{\left(\begin{smallmatrix} \alpha & B & C & D & \alpha \\ B & A & D & C & A \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha & B & C & D & \alpha & B \\ A & D & C & A & & \end{smallmatrix}\right), \left(\begin{smallmatrix} B & C & D & \alpha & B \\ A & D & C & A & \alpha \end{smallmatrix}\right)\right\}$$

(this list is obtained by iterating Rauzy inverse inductions to the permutations in $\mathcal{E}_2$). The next claim analyze these three permutations.

Claim 8.2. – *Assume that $\mathcal{R}_{\mathrm{sing}}(\pi) \in \mathcal{F}$. Then $(Y, q) \in \mathcal{Q}(-1, 1, 4)$, and (Y, q) is the connected sum of two flat tori, irrationally foliated with opposite SAF-invariants.*

Proof. – The first statement follows directly from above discussion. Next we will consider separately the three cases $\pi \in \mathcal{R}_{\mathrm{sing}}^{-1}\pi'$ where π' ranges over $\mathcal{F}$.

For the first one, up to permutation of the letters of the alphabet $\{A, B, C, D, \alpha, \beta\}$ we have:

$$
\begin{aligned}
&\mathcal{R}_{\mathrm{sing}}^{-1}\left(\begin{smallmatrix} \alpha & B & C & D & \alpha \\ B & A & D & C & A \end{smallmatrix}\right) \\
&= \left\{ \boxed{\begin{pmatrix} \alpha & B & C & D & \alpha & \beta \\ \beta & A & D & C & A & B \end{pmatrix}}, \boxed{\begin{pmatrix} \alpha & B & C & D & \alpha & \beta \\ B & \beta & D & C & A & A \end{pmatrix}}, \left(\begin{array}{cc|cccc} \alpha & B & C & D & \alpha & \beta \\ B & A & \beta & C & A & D \end{array}\right), \left(\begin{array}{cc|cccc} \alpha & B & C & D & \alpha & \beta \\ B & A & D & \beta & A & C \end{array}\right), \right. \\
&\quad \left(\begin{array}{cc|cccc} \alpha & B & C & D & \alpha & \beta \\ B & A & D & C & \beta & A \end{array}\right), \left(\begin{array}{cc|cccc} \alpha & B & C & D & \alpha & \beta \\ B & A & D & C & A & \beta \end{array}\right), \left(\begin{array}{cc|cccc} \beta & B & C & D & \alpha & \beta \\ B & A & D & C & A & \alpha \end{array}\right), \left(\begin{array}{cc|cc|cc} \alpha & \beta & C & D & \alpha & \beta \\ B & A & D & C & A & B \end{array}\right), \\
&\quad \left. \left(\begin{array}{cccc|cc} \alpha & B & \beta & D & \alpha & \beta \\ B & A & D & C & A & C \end{array}\right), \left(\begin{array}{cccc|cc} \alpha & B & C & \beta & \alpha & \beta \\ B & A & D & C & A & D \end{array}\right), \boxed{\left(\begin{smallmatrix} \alpha & B & C & D & \beta & \beta \\ B & A & D & C & A & \alpha \end{smallmatrix}\right)} \right\}.
\end{aligned}
$$

The boxed permutations correspond exactly to irreducible permutations, we also indicate the decomposition when the permutation is reducible (see Definition 4.4). For the above three irreducible permutations, the corresponding suspension (Y, q) belongs to the stratum $\mathcal{Q}(-1, 1, 4)$.

Observe that in these cases, we have either $|\alpha| = |A|$ and $|\beta| = |B|$, or $|\alpha| = |\beta| = |A|$, where $|.|$ denote the length of the intervals. It follows that we have a decomposition of Y into a connected sum of two tori which correspond to the subsets of letters $\{\alpha, \beta, A, B\}$ and $\{C, D\}$.

For the next one, the *irreducible* permutations in $\mathcal{R}_{\mathrm{sing}}^{-1}\left(\begin{smallmatrix} \alpha & B & C & D & \alpha & B \\ A & D & C & A & & \end{smallmatrix}\right)$ are

$$
\left\{ \boxed{\begin{pmatrix} \alpha & B & C & D & \alpha & \beta \\ \beta & D & C & A & A & \end{pmatrix}}, \boxed{\left(\begin{smallmatrix} \alpha & B & C & D & \beta & B & \beta \\ A & D & C & A & \alpha & & \end{smallmatrix}\right)}, \boxed{\left(\begin{smallmatrix} \alpha & B & C & D & \alpha & \beta & \beta \\ A & D & C & A & B & & \end{smallmatrix}\right)} \right\}.
$$

Again the corresponding suspension (Y, q) belongs to the stratum $\mathcal{Q}(-1, 1, 4)$ and we have a decomposition of Y into a connected sum of two tori which correspond to the subsets of letters $\{\alpha, \beta, A, B\}$ and $\{C, D\}$.

For the remaining case, the *irreducible* permutations in $\mathcal{R}_{\mathrm{sing}}^{-1}\left(\begin{smallmatrix} B & C & D & \alpha & B \\ A & D & C & A & \alpha \end{smallmatrix}\right)$ are

$$
\left\{ \boxed{\begin{pmatrix} B & C & D & \alpha & B & \beta \\ \beta & D & C & A & \alpha & A \end{pmatrix}}, \boxed{\left(\begin{smallmatrix} B & C & D & \beta & B & \beta \\ A & D & C & A & \alpha & \alpha \end{smallmatrix}\right)}, \boxed{\left(\begin{smallmatrix} B & C & D & \alpha & \beta & \beta \\ A & D & C & A & \alpha & B \end{smallmatrix}\right)} \right\}.
$$

We check that (Y, q) belongs to the stratum $\mathcal{Q}(-1, 1, 4)$ and has a decomposition into a connected sum of two tori. This proves the claim in this case. $\square$

We now turn into the case $\mathcal{R}^{(n_0)}(\pi') \in \mathcal{E}_3$.

CLAIM 8.3. – *Assume that $\mathcal{R}^{(n_0)}(\pi') \in \mathcal{E}_3$ then the followings hold:*

1. $(Y, q) \notin \mathcal{Q}(8) \sqcup \mathcal{Q}(-1, 2, 3)$.
2. *If $(Y, q) \in \mathcal{Q}(-1^3, 1, 2) \sqcup \mathcal{Q}(-1^4, 4)$ then (Y, q) is the connected sum of a flat torus and a flat sphere, irrationally foliated with opposite SAF-invariants.*
3. *If $(Y, q) \in \mathcal{Q}(-1, 1, 4)$ then (Y, q) is the connected sum of two flat tori, irrationally foliated with opposite SAF-invariants.*

Proof. – Set $\mathcal{E}_3' := \{\left(\begin{smallmatrix} \pi'' & A & B \\ & B & A \end{smallmatrix}\right)\}$, and $\mathcal{E}_3'' := \{\left(\begin{smallmatrix} A & B & \pi'' \\ B & A & \end{smallmatrix}\right)\}$, where π'' is a generalized permutation defined over 3 letters. We first remark that $\mathcal{E}_3'$ is invariant by Rauzy inverse induction, while $\mathcal{E}_3''$ is not, for instance $\mathcal{R}_1\left(\begin{smallmatrix} A & B & C & D & D & A \\ B & E & C & E & & \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & B & C & D & D \\ B & A & E & C & E \end{smallmatrix}\right)$.

If $\mathcal{R}^{(n_0)}(\pi') \in \mathcal{E}_3'$ then $\pi' \in \mathcal{E}_3'$, hence $\pi \in \mathcal{R}_{\mathrm{sing}}^{-1}\mathcal{E}_3'$. An irreducible generalized permutation π in $\mathcal{R}_{\mathrm{sing}}^{-1}\mathcal{E}_3'$ belongs to one of the following two families: either $\pi = (\pi_1 \mid \pi_2)$, where π_1, π_2 are generalized permutations defined over 3 letters, or $\pi = \left(\begin{smallmatrix} \dots & | & A & B & | & * \\ \dots & | & B & A & | & * \end{smallmatrix}\right)$. In the first case, the claim follows from the arguments of Case (a). In the second case, Y is a connected sum of a slit torus (corresponding to the permutation $\left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$), and another flat surface which is the suspension of the (irreducible) generalized permutation defined over 4 letters, which is obtained by deleting the letters A and B from π. The assertions of the claim can be verified by a case-by-case check.

Finally, assume that $\overline{\pi} = \mathcal{R}^{(n_0)}(\pi') \in \mathcal{E}_3''$. Set $\overline{T} := \mathcal{R}^{(n_0)}(T')$. Applying Rauzy inductions and using the assumption that T (hence $\overline{T}$) is not periodic, we can reduce to the case

$$\bar{\pi} \in \{\left(\begin{smallmatrix} A & B & C & C & D \\ B & A & D & E & E \end{smallmatrix}\right), \left(\begin{smallmatrix} A & B & C & D & D \\ B & A & E & C & E \end{smallmatrix}\right)\}.$$

We now use a lemma (see Lemma 8.4 below) saying that there exists an irreducible generalized permutation $\hat{\pi}$, which is obtained by adding a pair of letters $\{\alpha, \alpha\}$ to $\bar{\pi}$, such that (Y, q) is obtained by a zippered rectangle construction from a suspension of $\hat{\pi}$, where we allow the width of the rectangle labelled by α to be zero. Here again, the assertions of the claim can be verified by a case-by-case check. $\square$

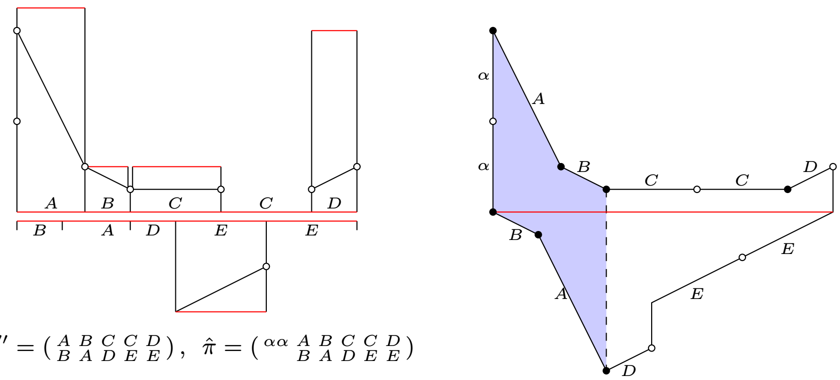
Hence the proof of Theorem 8.1 will be complete once we prove Lemma 8.4. $\square$

LEMMA 8.4. – *Let $T := (\pi, \lambda)$ be the linear involution defined by the first return map of the vertical foliation on a quadratic differential (Y, q) to a full transversal interval I, the left endpoint of which is a singular point. Assume that $\mathcal{R}(T)$ is not well defined, and let $T' = (\pi', \lambda') := \mathcal{R}_{\mathrm{sing}}(T)$. Then for any $T'' = (\pi'', \lambda'')$ which is obtained from T' by a sequence of Rauzy inductions, there exists an irreducible generalized permutation $\hat{\pi}$ that satisfies*

(a) *there is a pair of letters α such that if we delete this pair from $\hat{\pi}$, then we get π'',*
(b) *the surface (Y, q) can be represented by a zippered rectangle construction from $\hat{\pi}$ where the width of the rectangle labeled by α is zero.*

Proof of Lemma 8.4. – Assume that π is a generalized permutation defined over an alphabet $\mathcal{A}$ of d letters. By definition, the quadratic differential (Y, q) is constructed from (π, λ) by a zippered rectangle construction. The singular Rauzy induction consists of cutting the rightmost rectangle and gluing it to another one. Thus we get a zippered rectangle construction of (Y, q) with $d - 1$ rectangles (see Figure 4).

By construction, T' and T'' are the first return map of the vertical foliation on (Y, q) to some segments I' and I'' respectively, where $I'' \subset I' \subset I$. Since T'' is obtained from T' by Rauzy inductions, the surface (Y, q) is also constructed by a zippered rectangle construction with $d-1$ rectangles associated to T''. Note that there is a vertical saddle connection α which does not intersect the interior of I' and I'', this saddle connections must be contained in the border of some rectangle.



$$\pi'' = \left(\begin{smallmatrix} A & B & C & C & D \\ B & A & D & E & E \end{smallmatrix}\right), \quad \hat{\pi} = \left(\begin{smallmatrix} \alpha\alpha & A & B & C & C & D \\ & B & A & D & E & E \end{smallmatrix}\right)$$

FIGURE 4. Degeneration of a linear involution defined over 6 letters to a linear involution defined over 5 letters. The associated quadratic differential belongs to $\mathcal{Q}(-1^4, 4)$, and decomposes as connected sum of a torus (the colored part) and a sphere.

There exists a family $\mathcal{S}$ of saddle connections and segments joining the right endpoint of I'' to some singular points such that

- $\alpha \in \mathcal{S}$,
- for every segment s in this family, either s is contained in a vertical side of a rectangle, or s is a segment joining a point in the left side to a point in the right side of the same rectangle,
- cutting Y along the segments in $\mathcal{S}$, what we get is a polygon P in $\mathbb{R}^2$, each side of P is paired up with another one which is parallel and has the same length.

Note that the vertices of P are the singular points of Y and the right endpoint of I''. One can deform P slightly so that the paired sides are still parallel and have the same length, and the sides corresponding to α are no longer vertical. We then get a polygon P$'$ that gives rise to a quadratic differential (Y', q') close to (Y, q). By construction, there exists a full transversal segment J in Y' corresponding to I''. The first return map of the vertical foliation on Y' to J is a linear involution $\hat{T} := (\hat{\pi}, \hat{\lambda})$ defined over d letters, the additional letter (with respect to T'') arises from the sides of P$'$ corresponding to α. In particular, we see that $\hat{\pi}$ is an irreducible generalized permutation. Moreover, as one deforms P$'$ to get back P, $\hat{T}$ becomes T''. Thus, if we delete the pair of new letter from $\hat{\pi}$, we obtain π''. The lemma is then proved. $\square$

REMARK 8.5. – *It follows from a result by McMullen* [23], *Theorem 6.1, that if (X, ω) is an Abelian differential having relative periods in a real quadratic field and vanishing SAF-invariant in the vertical direction, then there exists a loop in X which is a union of vertical saddle connections. Thus, Theorem 8.1 gives a more precise description of this situation where (X, ω) is the orientation double cover of some quadratic differential in Table 1.*

– *Using the same analysis, one can also prove that if $(Y, q) \in \mathcal{Q}(-1^6, 2)$ satisfies the hypothesis of Theorem 8.1, then either the vertical flow is periodic, or Y decomposes as* *a connected sum of two flat spheres, which are glued together along a vertical loop. This statement was proved in* [23], *Theorem 8.2.*

We end this section by some consequences of Theorem 8.1 in terms of Prym eigenforms.

Corollary 8.6. – *Let $(X,\omega) \in \Omega E_D(\kappa)$ be a Prym eigenform and assume that ω has all its periods (absolute and relative) in $K(\imath)$, where $K = \mathbb{Q}(\sqrt{D})$. Let θ be a direction in $K\mathbb{P}^1$.*

1. *If $(X,\omega) \in \Omega E_D(1,1,4) \sqcup \Omega E_D(4,4)^{\text{even}}$ then $\mathcal{F}_\theta$ is completely periodic.*
2. *If $(X,\omega) \in \Omega E_D(1,1,2)$ and the spine of the foliation in direction θ contains a regular fixed point of the Prym involution then $\mathcal{F}_\theta$ is completely periodic. The* spine *of $\mathcal{F}_\theta$ is the union of geodesic rays emanating from the zeros of ω in direction $\pm\theta$.*

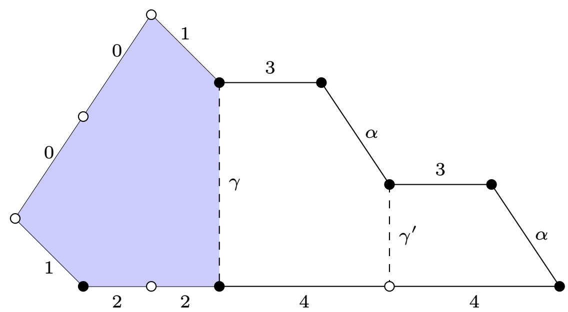
Again we emphasize that assertion (8.6) of above corollary is false for other Prym loci, see e.g., Example 8.8 and Figure 5 when $(X,\omega) \in \Omega E_D(2,2)$.

Proof of Corollary 8.6. – As usual we will assume that θ is vertical. We begin by observing that Theorem 1.4 implies $SAF(X,\omega) = 0$. Let T be the cross section of the vertical foliation to some full transversal interval on the quotient $(Y,q) = (X,\omega)/\langle\tau\rangle$. We have $SAF(T) = 0$. Complete periodicity of $\mathcal{F}_\theta$ is equivalent to complete periodicity of T.
Since $(X,\omega) \in \mathrm{Prym}(1,1,4)$ (respectively, $(X,\omega) \in \mathrm{Prym}(4,4)^{\text{even}}$) is equivalent to $(Y,q) \in \mathcal{Q}(-1,2,3)$ (respectively, $(Y,q) \in \mathcal{Q}(8)$), assertion (1) is a reformulation of Theorem 8.1.
Let us prove (2). Again $(X,\omega) \in \mathrm{Prym}(1,1,2)$ is equivalent to $(Y,q) \in \mathcal{Q}(-1^3,1,2)$. If T is not completely periodic then by Theorem 8.1 (Y,q) is the connected sum of a flat torus and a flat sphere, irrationally foliated with opposite SAF-invariants. Hence there exists a geodesic loop γ based at the zero of multiplicity 1 which cuts Y into a flat sphere Y_0, and a flat torus Y_1 (with geodesic boundary).

Observe that the three poles of q are contained in the interior of Y_0. Since $Y \in \mathcal{Q}(-1^3,1,2)$ the component Y_0 lifts to a fixed torus X_0 and Y_1 to two permuted tori $X_{1,j}$, $j = 1,2$, in X. One has $SAF(X_0) = -2SAF(X_{1,1})$. By assumption the spine of the foliation on X_0 contains a fixed point of the Prym involution; hence $\mathcal{F}_{\theta|X_0}$ is periodic. Thus $SAF(X_0) = 0$ and we conclude that $\mathcal{F}_\theta$ is periodic. $\square$

Remark 8.7. – *The above proof fails if $(Y,q) \in \mathcal{Q}(-1^4,4)$, even though we also have a connected sum of a flat sphere and a flat torus. This is because the existence of a pole in the torus component. Indeed the decomposition into three tori (of (X,ω)) still holds but it could happen that the pole on the spine of the foliation on Y is the one contained in Y_1 (see Figure 5). In this case, the foliation on Y_0 may not be periodic, for instance, in Example 8.8, if we choose the lengths $\lambda_\alpha, \lambda_3$ of the intervals labelled by α and 3 so that $\frac{\lambda_\alpha}{\lambda_3} \notin \mathbb{Q}$.*

Example 8.8. – *In Figure 5 below the surface (Y,q) decomposes along the saddle connection γ into a connected sum of a flat torus and a flat sphere, as we can notice by the underlying permutation $\left(\begin{smallmatrix} 0 & 0 & 1 & 3 & \alpha & 3 & \alpha \\ 1 & 2 & 2 & 4 & 4 & & \end{smallmatrix}\right)$. One can arrange the parameters so that $SAF(T) = 0$ and T is not completely periodic. Moreover, there exists a regular fixed point of the Prym involution in the spine of the double cover (X,ω). Namely (X,ω) decomposes into two permuted tori and one invariant torus along the lifts of γ and γ'.*



$(Y, q) \in \mathcal{Q}(-1^4, 4)$

FIGURE 5. Decompositions of (Y, q) into a connected sum of a flat torus and a flat sphere (colored in blue).

9. Limit set of Veech groups

In this section, we prove the result on the limit sets of Veech groups of Prym eigenforms i.e., Theorem 1.11. In the sequel we fix a form

$$(X, \omega) \in \Omega E_D(4,4)^{\text{even}} \sqcup \Omega E_D(1,1,4) \sqcup \Omega E_D(1,1,2).$$

A periodic direction is said to be *stable* if there is no saddle connection in this direction that connects two different zeros, it is said to be *unstable* otherwise.

LEMMA 9.1. – *Any direction θ that decomposes $(X, \omega) \in \mathcal{H}(\kappa)$ into $g + |\kappa| - 1$ cylinders, where g is the genus of X, is stable.*

Proof. – We begin by observing that any periodic direction decomposes the surface X into *at most* $g + |\kappa| - 1$ cylinders. Now if the direction θ is not stable then there exists necessarily a saddle connection between two different zeros that we can collapse to a point (in direction θ) without destroying any cylinder. But in this way we get a surface $(X', \omega') \in \mathcal{H}(\kappa')$ of genus g where $|\kappa'| < |\kappa|$, and having $g + |\kappa| - 1$ cylinders . This is a contradiction. □

We now prove the following proposition about unstable periodic directions on Prym eigenforms.

PROPOSITION 9.2. – *Let $(X, \omega) \in \mathrm{Prym}(4,4)^{\text{even}} \sqcup \mathrm{Prym}(1,1,4) \sqcup \mathrm{Prym}(1,1,2)$. Assume that (X, ω) is completely algebraically periodic, and all the relative periods of ω belong to $K(\imath)$, where K is a real quadratic field. Then any unstable periodic direction θ decomposes the surface into cylinders with commensurable moduli. As a consequence, $\mathrm{SL}(X, \omega)$ contains a parabolic element fixing θ.*

Proof. – Assume that $(X, \omega) \in \mathrm{Prym}(4,4)^{\text{even}} \sqcup \mathrm{Prym}(1,1,4)$ then the decomposition in direction θ has at most 6 cylinders by Lemma 9.1. Since the direction θ is not stable and none of the cylinders is fixed by the Prym involution (otherwise the quotient (Y, q) by the Prym involution would have at least 2 poles) one has $n \in \{2, 4\}$. We denote the cylinders by $C_i, i = 1, \dots, 2r = n$, so that $C_{i+r} = \tau(C_i)$. By Theorem 7.2 the moduli of the cylinders satisfies

$$2\sum_{i=1}^{r} h(C_i)w(C_i)' = 2\sum_{i=1}^{r} k_i \cdot \mu(C_i) = 0.$$

where $k_i = w(C_i)w(C_i)' \in \mathbb{Q} \setminus \{0\}$. Hence

$$\sum_{i=1}^{r} k_i \cdot \mu(C_i) = 0.$$

But $r \leq 2$ thus above equality implies that $\mu(C_i)$ are commensurable. The direction is parabolic and a suitable product of Dehn twist in each cylinder gives rise to an affine automorphism with parabolic derivative fixing θ.

The case $(X, \omega) \in \mathrm{Prym}(1, 1, 2)$ follows from similar arguments since the decomposition in direction θ has at most 5 cylinders. □

COROLLARY 9.3. – *Let* $(X, \omega) \in \mathrm{Prym}(1, 1, 2)$ *be a Prym form which is completely algebraically periodic with relative periods in* $K(\imath)$. *If* $\theta \in K\mathbb{P}^1$ *is the direction of a saddle connection between the two simple zeros that is invariant under the Prym involution, then* $\mathrm{SL}(X, \omega)$ *contains a parabolic element fixing* θ.

Proof of Corollary 9.3. – In view of the previous proposition it suffices to show that θ is an unstable periodic direction. Since θ is the direction of a saddle connection, we have $\theta \in K\mathbb{P}^1$. Necessarily the saddle connection contains a regular fixed point of the Prym involution. By Corollary 8.6, assertion (2), the flow $\mathcal{F}_\theta$ is completely periodic (the spine contains a regular fixed point of the Prym involution). Since there is a saddle connection connecting two different zeros, this periodic direction is unstable, and the corollary follows from Proposition 9.2. □

9.1. Proof of Theorem 1.11, Case $\mathrm{Prym}(4, 4)^{\mathrm{even}} \sqcup \mathrm{Prym}(1, 1, 4)$

Proof. – If the limit set has at least two points then there is a hyperbolic element in $\mathrm{SL}(X, \omega)$ represented by an affine pseudo-Anosov homeomorphism ϕ. By a result of McMullen ([23], Theorem 9.4) we can assume that all the periods of ω belong to $K(\imath)$.

By Theorem 1.9 and Proposition 9.2, any linear foliation on (X, ω) in the direction θ of a saddle connection between two different zeros is fixed by a parabolic element of $\mathrm{SL}(X, \omega)$. It remains to show that those directions fill out a dense subset of $\mathbb{RP}^1$, which implies that the limit set is the full circle at infinity.

Let $\theta_0 \in \mathbb{RP}^1$ and fix $\varepsilon > 0$. By Theorem 1.9, one can find $\theta \in K\mathbb{P}^1$ so that the foliation $\mathcal{F}_\theta$ is completely periodic and $|\theta - \theta_0| < \varepsilon/2$. If the direction θ is not stable then by Proposition 9.2 we are done. Otherwise X is decomposed into 6 cylinders in direction θ. Since X is a connected surface, we claim that there exists a cylinder C such that the top boundary of C is made of saddle connections between one zero P and the bottom boundary is made of saddle connections between one other zero $Q \neq P$. By a suitable Dehn twist, it is easy to find a new direction θ' satisfying $|\theta - \theta'| < \varepsilon/2$ such that there is a saddle connection contained in C between P and Q in direction θ'. This is the desired direction. □

9.2. Proof of Theorem 1.11, Case $\mathrm{Prym}(1,1,2)$

Proof. – We now show the result for $(X,\omega) \in \Omega E_D(1,1,2)$. By a result of Masur [20], we know that the set Θ of directions $\theta \in \mathbb{RP}^1$ such that θ is the direction of a regular closed geodesic is dense in $\mathbb{RP}^1$. Thus, by using Proposition 9.2, it suffices to show that any direction $\theta \in \Theta$ is contained in the closure of the set of unstable periodic directions.

Let θ_0 be a direction in Θ. By Theorem 1.6, we know that X is decomposed into cylinders in direction θ_0. We can assume that θ_0 is the horizontal direction. Obviously, we only need to consider the case where θ_0 is a stable periodic direction. Note that in this case X is decomposed into 5 cylinders in direction θ_0.

If γ is a geodesic segment connecting a regular fixed point of X to one simple zero of ω, then $\gamma \cup \tau(\gamma)$ is a saddle connection joining two simple zeros and invariant under τ. Following Corollary 9.3, the direction of γ is an unstable periodic direction. We claim that there exist such geodesic segments whose direction is arbitrarily close to θ_0.

We begin by observing that in a cylinder decomposition of X, only one cylinder (denoted by C_0) is invariant under τ. Recall that τ has three regular fixed points, two of which are contained in C_0, the third one is the midpoint of a saddle connection contained in the boundaries of two exchanged cylinders. We can divide those decompositions into three types:

(a) The boundary of the C_0 only contains the simple zeros, or
(b) The boundary of C_0 only contains the double zero, and C_0 is a simple cylinder, or
(c) The boundary of C_0 only contains the double zero, and C_0 is not a simple cylinder.

(A cylinder is simple if each of its boundary component consists of exactly one saddle connection.)

In Case (a) each simple zero is contained in a boundary component of C_0. Thus there exist saddle connections contained in C_0 and invariant under τ which connect the two simple zeros whose direction is arbitrarily close to θ_0.

In Case (b), let γ be the saddle connection (in direction θ_0) that contains the third fixed point of τ. There exists a pair of cylinder C_1, C_2 exchanged by τ such that γ is included in the lower boundary of C_1 (resp. in the upper boundary of C_2). Note that since the cylinder decomposition is stable γ must join the double zero to itself. Remark that the upper boundary of C_1 must contain a simple zero (otherwise the angle at the double zero exceeds 6π), and consequently the lower boundary of C_2 also contains a simple zero. Therefore, there exist saddle connections contained in $C_1 \cup C_2$ joining the simple zeros and invariant under τ (passing through the third fixed point) whose direction is arbitrarily close to θ_0.

In Case (c), the only topological model is presented in Figure 6 below. One can easily see that there always exists a geodesic segment from a fixed point of τ in the interior of C_0 to a simple zero in the boundary of a cylinder adjacent to C_0. Using Dehn twists, we see that there exist infinitely many such segments whose direction can be made arbitrarily close to θ_0. The theorem is then proved for this case. $\square$

REMARK 9.4. – *Actually we also proved a slightly different result: the limit set of the Veech group of any* $(X,\omega) \in \mathrm{Prym}(4,4)^{\mathrm{even}} \sqcup \mathrm{Prym}(1,1,4) \sqcup \mathrm{Prym}(1,1,2)$*, completely algebraically periodic, having all periods in* $K(\imath)$*, is the full circle at infinity.*

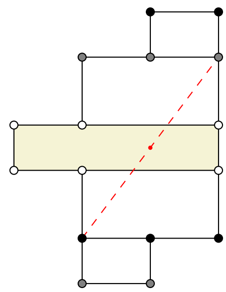


FIGURE 6. Stable cylinder decomposition in $\mathrm{Prym}(1,1,2)$, the double zero is colored in white (the "gray" cylinder is the unique cylinder invariant under τ).

10. Infinitely generated Veech groups

We end with the proof of Theorem 1.12

Proof of Theorem 1.12 . – Recall that a Fuchsian group is said to be of the first kind if its limit set is the full circle at infinity. Such a group is either a lattice, or infinitely generated (see e.g., [13]). Hence, in view of Remark 9.4, it suffices to exhibit Prym eigenforms (with relative periods in $K(\imath)$) whose Veech group is not a lattice. But a theorem of Veech [35] asserts that in the lattice case the directional flow $\mathcal{F}_\theta$ is either uniquely ergodic or parabolic (i.e., the surface is decomposed into cylinders of commensurable moduli in direction θ). Hence it suffices to give examples where $\mathcal{F}_\theta$ is periodic, but with incommensurable cylinders. In what follows we only focus on $\mathrm{Prym}(1,1,2)$ since similar constructions work for the two other loci.

We begin by choosing a discriminant D which is not a square, and a tuple (w,h,e) of integers such that:

$$\begin{cases} w>0, \ h>0, \\ e+2h<w, \\ \gcd(w,h,e)=1, \text{ and } D=e^2+8wh. \end{cases}$$

Let $\lambda := \frac{e+\sqrt{D}}{2} > 0$ (remark that $\lambda < w$). We also choose $t \in \mathbb{Q}(\sqrt{D})$ so that $0<t<\lambda$. Let (X,ω) be the surface represented in Figure 7 having the following coordinates

$$\begin{cases} \omega(\alpha_1)=(\lambda,0), & \omega(\beta_1)=(0,\lambda) \\ \omega(\alpha_{2,1})=\omega(\alpha_{2,2})=(w/2,0), & \omega(\beta_{2,1})=\omega(\beta_{2,1})=(0,h/2) \\ \omega(\eta)=(t,0). \end{cases}$$

By construction, there exists an involution τ on X which fixes the colored cylinder and exchanges the other two. It is not hard to check that $(X,\omega) \in \mathrm{Prym}(1,1,2)$. Letting $\alpha_2 := \alpha_{2,1}+\alpha_{2,2}$ and $\beta_2 := \beta_{2,1}+\beta_{2,2}$, the set $\{\alpha_i,\beta_i\}_{i=1,2}$ is a symplectic basis of $H_1(X,\mathbb{Z})^-$. Moreover, in these coordinates the restriction of the intersection form is given by the matrix $\left(\begin{smallmatrix} J & 0 \\ 0 & 2J \end{smallmatrix}\right)$. In particular it is straightforward to check that the endomorphism $T=\left(\begin{smallmatrix} e & 0 & w & 0 \\ 0 & e & 0 & h \\ 2h & 0 & 0 & 0 \\ 0 & 2w & 0 & 0 \end{smallmatrix}\right)$ (in the basis $(\alpha_i,\beta_i)_{i=1,2}$) is self-adjoint and satisfies $T^2=eT+2wh\mathrm{Id}_{\mathbb{R}^4}$

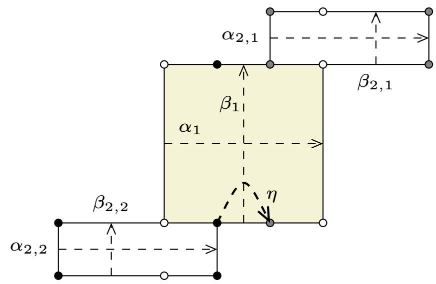


FIGURE 7. A translation surface $(X,\omega) \in \mathrm{Prym}(1,1,2)$. The double zero is represented in white color (the fixed cylinder is colored in grey). The identifications of the sides are the “obvious” identifications.

and $T^*(\omega) = \lambda\omega$. Hence $(X,\omega) \in \Omega E_D(1,1,2)$, and (X,ω) is completely algebraically periodic by Theorem 1.4.

Note that (X,ω) also admits a cylinder decomposition in the vertical direction. A straightforward computation shows that the moduli of the vertical cylinders are given by

$$\frac{t}{\lambda}, \qquad \frac{\lambda - t}{2\lambda + h}, \qquad \text{and} \qquad \frac{w - (\lambda - t)}{h}.$$

One can easily see that if $t/\lambda \in \mathbb{Q}$, then the first two moduli are incommensurable. Hence the Veech group of the corresponding surface (X,ω) is infinitely generated.

For the cases $\mathrm{Prym}(1,1,4)$ and $\mathrm{Prym}(4,4)^{\mathrm{even}}$, examples of surfaces having infinitely generated Veech groups can be obtained by similar constructions as shown in Figure 8. □

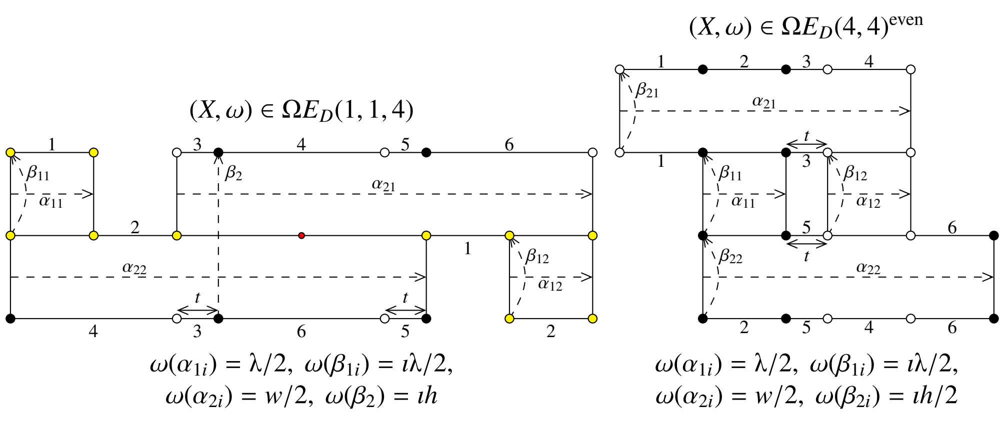


FIGURE 8. Constructions of Prym eigenforms in $\Omega E_D(1,1,4)$ and $\Omega E_D(4,4)^{\mathrm{even}}$. For almost all values of $t \in \mathbb{Q}(\sqrt{D})$, the vertical direction is periodic, but not parabolic.

Appendix

Complete algebraic periodicity in $\mathcal{H}^{\mathrm{hyp}}(4)$

In this section, we will sketch a proof of the following

THEOREM A.1. – *Let (X,ω) be a translation surface in the hyperelliptic component $\mathcal{H}^{\mathrm{hyp}}(4)$ of the stratum $\mathcal{H}(4)$. If (X,ω) is completely algebraically periodic then it is completely periodic in the sense of Calta.*

REMARK A.2. – *By a theorem by Calta-Smillie* [6]*, we know that if* $\mathrm{SL}(X,\omega)$ *contains a hyperbolic element then (X,ω) is completely algebraically periodic. Examples of surfaces in $\mathcal{H}^{\mathrm{hyp}}(4)$ whose Veech group contains hyperbolic elements can be found in* [28]*. In those examples the trace field of* $\mathrm{SL}(X,\omega)$ *is cubic (one can also construct examples with quadratic trace field). It is shown in* [28] *that such surfaces can be* generic *(see also* [29]*), that is their* $\mathrm{GL}^+(2,\mathbb{R})$*-orbit is dense in $\mathcal{H}^{\mathrm{hyp}}(4)$. Thus there exist completely periodic surfaces which are generic in $\mathcal{H}^{\mathrm{hyp}}(4)$.*

Proof of Theorem A.1. – By definition, there exists a double covering $\rho : X \to \mathbb{CP}^1$, and a quadratic differential q on $\mathbb{CP}^1$ (which has a unique zero of order 3, and 7 simple poles) such that $\rho^* q = \omega^2$. In our notations $(\mathbb{CP}^1, q) \in \mathcal{Q}(-1^7, 3)$. Let C be a cylinder on X, and c be its core curve. As usual we assume that the direction of C is the vertical direction. We want to show that the vertical flow is completely periodic. Since (X,ω) is completely algebraically periodic, we have $SAF(X,\omega) = 0$.

Let $\mathring{C}$ denote the open cylinder which is filled out by simple closed geodesics in the free homotopy class of c, and C denote the closure of $\mathring{C}$ in X. The set $\partial C := C \setminus \mathring{C}$ is a union of several, say k, vertical saddle connections. The case $k = 1$ only occurs when X is a torus, therefore we have $k \geq 2$. Since a surface in $\mathcal{H}^{\mathrm{hyp}}(4)$ has at most 5 saddle connections in a given direction, we have $2 \leq k \leq 5$. If $k = 5$ then the vertical flow is completely periodic (all vertical separatrices are saddle connections). Thus we only need to consider the cases $k = 2, 3, 4$.

Note that since (X,ω) belongs to the hyperelliptic component, *all* the cylinders are fixed by the hyperelliptic involution τ. It follows that $\tau(\partial C) = \partial C$. Therefore τ maps a vertical saddle connection s in ∂C either to itself or to another one in ∂C. If $\tau(s) \neq s$, then $\rho(s)$ is a geodesic loop in $\mathbb{CP}^1$ based at the unique zero of q and if $\tau(s) = s$ then $\rho(s)$ is a segment joining the unique zero to a pole. Since we only need to consider the cases where $C \neq X$ (otherwise X is filled by saddle connections in the free homotopy class of c), we can assume that there exists a saddle connection in ∂C that is not invariant by τ, which implies that $\rho(\partial C)$ contains (at least) a geodesic loop. The configurations of $\rho(\partial C)$ *containing a geodesic loop* are shown in Figure 9.

Let Y' be the metric completion of $\rho(X \setminus C)$. Remark that Y' is a union of flat discs with geodesic boundary, each boundary component corresponds to a geodesic loop in $\rho(\partial C)$. One can "fold up" the boundaries of Y' to get closed flat surfaces defined by quadratic differentials on the sphere. Let us denote this union by $\hat{Y}'$. Note that in each component of $\hat{Y}'$ we have a vertical saddle connection corresponding to a geodesic loop in $\rho(\partial C)$. By assumption, we have $SAF(\hat{Y}') = 0$. We need to show that the vertical direction is completely periodic on $\hat{Y}'$.

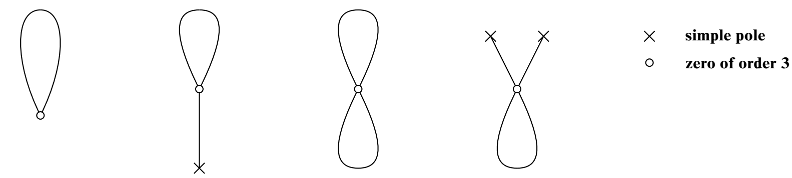


FIGURE 9. Configurations of $\rho(\partial C)$ having a geodesic loop. The projection of $\mathring{C}$ is the open disc represented by the unbounded component of $\mathbb{CP}^1 \setminus \rho(\partial C)$, note that this open disc contains two poles of q. The exterior angle between two consecutive rays at the zero of q is π. The projection of $X \setminus C$ is a union of open discs bounded by the loops based at the unique zero of q.

In the case $k = 4(a)$, $\hat{Y}'$ has two connected components: one belongs to $\mathcal{Q}(-1^4)$ and the other belongs to $\mathcal{Q}(-1^4, 0)$. The orientation double cover of both connected components are flat tori, one of which has a vertical closed geodesic, the other one has vanishing SAF for the vertical foliation and a vertical saddle connection. We easily draw that the vertical flow is completely periodic.

In the cases $k = 2, 3, 4(b)$, $\hat{Y}'$ has only one connected component, and $(\hat{Y}', q')$ belongs to one of the following components respectively $\mathcal{Q}(-1^6, 2)$, $\mathcal{Q}(-1^5, 1)$, $\mathcal{Q}(-1^4, 0)$. The orientation double cover belongs to $\mathcal{H}(1,1)$, $\mathcal{H}(2)$ and $\mathcal{H}(0,0)$ respectively. By assumption, we know that there exists a vertical saddle connection and the SAF-invariant of the vertical direction vanishes. From this we can easily conclude that the vertical direction is periodic if $(\hat{Y}', q') \in \mathcal{Q}(-1^5, 1)$ or $(\hat{Y}', q') \in \mathcal{Q}(-1^4, 0)$ (using Lemma 5.2 and Lemma 5.1). Thus the cases $k = 3$ and $k = 4(b)$ are done.

We are left with the case $k = 2$. We denote by $(X', \omega') \in \mathcal{H}(1,1)$ the orientation double cover of $(\hat{Y}', q')$. By above discussion $SAF(X', \omega') = 0$ and there exists a vertical saddle connection σ connecting two zeros of ω'. It is not difficult to see that σ is invariant by the hyperelliptic involution τ' of X'.

Assume that by moving vertically in the leaf of the kernel foliation, one can collapse the two zeros of ω' along σ to get a surface $(X'', \omega'') \in \mathcal{H}(2)$. We then have $SAF(X'', \omega'') = SAF(X', \omega') = 0$ by Proposition 3.3. But the first return map (to a full transversal interval) of the vertical flow on X'' gives an irreducible IET defined over 4 letters. It follows from Lemma 5.2 that the vertical foliation on X'' is periodic, and we are done.

The only obstruction to the collapsing of the zeros of ω' along σ (so that the resulting surface belongs to $\mathcal{H}(2)$) is the existence of another vertical saddle connection σ' joining the two zeros of ω' such that $|\sigma'| \leq |\sigma|$ (as σ is shortened, σ' is also shortened by the same amount). For a detailed account on collision of singularities along kernel foliation leaves we refer to [26]. Since there exist exactly two geodesic rays in the same direction (in $\mathbb{S}^1$) from each zero of ω', if σ' exists, then it is unique, and in particular it is also invariant by τ'. It follows that $\sigma \cup \sigma'$ is a non-separating curve on X'. If $|\sigma'| < |\sigma|$, one can collapse the zeros along σ' (shortening both σ' and σ until the two zeros collide). The resulting surface belongs to $\mathcal{H}(2)$, and the argument above shows that the vertical direction is periodic. In the case $|\sigma'| = |\sigma|$, we can cut X' along $\sigma \cup \sigma'$ and glue the pair of segments in each boundary component of the new surface, what we obtain is a flat torus (X'', ω'') (since the closed curve $\sigma \cup \sigma'$ is non-separating). By construction, $SAF(X'', \omega'') = SAF(X', \omega') = 0$. Hence the vertical foliation on X'' is periodic which implies that the vertical foliation on X' is also periodic. $\square$

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