<span id="page-0-0"></span>*quatrième série - tome 49 fascicule 2 mars-avril 2016*

a*NNALES SCIEN*n*IFIQUES SUPÉRIEU*k*<sup>E</sup> de L ÉCOLE*  $NORMALE$ 

Daniel JUTEAU & Carl MAUTNER & Geordie WILLIAMSON *Parity sheaves and tilting modules*

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

# **Annales Scientifiques de l'École Normale Supérieure**

Publiées avec le concours du Centre National de la Recherche Scientifique

#### **Responsable du comité de rédaction /** *Editor-in-chief*

Antoine CHAMBERT-LOIR





### **Rédaction /** *Editor*

Annales Scientifiques de l'École Normale Supérieure, 45, rue d'Ulm, 75230 Paris Cedex 05, France. Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80. annales@ens.fr

### **Édition /** *Publication* **Abonnements /** *Subscriptions*

Société Mathématique de France Maison de la SMF Institut Henri Poincaré Case 916 - Luminy 11, rue Pierre et Marie Curie 13288 Marseille Cedex 09 Fax : (33) 01 40 46 90 96

75231 Paris Cedex 05 Fax : (33) 04 91 41 17 51 Tél. : (33) 01 44 27 67 99 email : smf@smf.univ-mrs.fr

#### **Tarifs**

Europe : 515  $\in$ . Hors Europe : 545  $\in$ . Vente au numéro : 77  $\in$ .

© 2016 Société Mathématique de France, Paris

En application de la loi du 1er juillet 1992, il est interdit de reproduire, même partiellement, la présente publication sans l'autorisation de l'éditeur ou du Centre français d'exploitation du droit de copie (20, rue des Grands-Augustins, 75006 Paris). *All rights reserved. No part of this publication may be translated, reproduced, stored in a retrieval system or transmitted in any form or by any other means, electronic, mechanical, photocopying, recording or otherwise, without prior permission of the publisher.*

ISSN 0012-9593 Directeur de la publication : Marc Peigné Périodicité : 6 nos / an

## PARITY SHEAVES AND TILTING MODULES

# BY DANIEL JUTEAU, CARL MAUTNER AND GEORDIE WILLIAMSON

ABSTRACT. – We show that tilting modules and parity sheaves on the affine Grassmannian are related through the geometric Satake correspondence, when the characteristic is bigger than an explicit bound.

R. – Nous montrons que la correspondance de Satake géométrique fait correspondre les modules basculants aux faisceaux à parité sur la grassmannienne affine, lorsque la caractéristique est plus grande qu'une borne explicite.

#### **1. Introduction**

#### **1.1. Tilting modules for reductive groups**

Let G be a split reductive group over a field  $k$  of characteristic  $p$  with a chosen maximal torus and Borel subgroup  $T \subset B \subset G$ . Let  $\Lambda^+$  denote the set of dominant weights, where dominance is defined with respect to the system of positive roots which are opposite to the **T**-weights in the Lie algebra of **B**. To each dominant weight  $\lambda \in \Lambda^+$ , is associated an induced representation  $\nabla_\lambda := \text{ind}_{\mathbf{B}}^{\mathbf{G}} k_\lambda = H^0(\mathbf{G}/\mathbf{B}, \mathcal{O}(\lambda))$  and its dual  $\Delta_\lambda$ , the Weyl module.

The rational representations of G form a highest weight category in which the Weyl modules are the standard objects and the induced modules are the costandard objects. A rational representation i[s sa](#page-20-0)id to be tilting, if it admits two filtrations—one with successive quotients isomorphic to Weyl modules a[nd t](#page-19-0)he other with successive quotients isomorphic to induced modules.

A theorem of Ringel [30, Proposition 2] about general highest weight categories specializes in this setting to the following result [10, Theorem 1.1],

ANNALES SCIENTIFIQUES DE L'ÉCOLE NORMALE SUPÉRIEURE 0012-9593/02/© 2016 Société Mathématique de France. Tous droits réservés

D.J. was supported by ANR grant ANR-09-JCJC-0102-01 and C.M. by an NSF postdoctoral fellowship.

THEOREM 1.1. – *For each*  $\lambda \in \Lambda^+$  *(up to non-canonical isomorphism), there exists a unique indecomposable tilting module* T(λ) *which has a unique highest weight* λ*. Moreover,* λ *has multiplicity one as a weight of* T(λ)*. Every indecomposable tilting module is isomorphic to*  $T(\lambda)$  *for some*  $\lambda \in \Lambda^+$ *.* 

An interesting feature of the class of tilting modules is that it is closed under both tensor product and restriction to a Levi subgroup:

THEOREM 1.2. – If  $T$  and  $T'$  are tilting modules for  $G$ , then so is the tensor product  $T \otimes T'$ .

T 1.3. – *Let* L *be a Levi subgroup of* G*. If* T *is a tilting mo[dule](#page-0-0) for* G*, then the* restriction  $\operatorname{Res}^{{\bf G}}_{{\bf L}}T$  $\operatorname{Res}^{{\bf G}}_{{\bf L}}T$  $\operatorname{Res}^{{\bf G}}_{{\bf L}}T$  to  ${\bf L}$  is a tilting module for  ${\bf L}$ .

Results in this direction were originally motivated by a question of Humphreys, who asked if the tensor product of two Weyl modules has a Weyl filtration. Theorem 1.2 was first proven by Wang [32] in [type](#page-20-2) A and in large characteristic for other groups. Donkin [9] later proved both theorems in [alm](#page-20-3)[ost](#page-20-4) [full](#page-20-5) [gen](#page-19-1)[era](#page-3-0)lity (he excluded the case when  $p = 2$  and G has a component of type  $E_7$  or  $E_8$ ). The first complete and uniform proof of both theorems is due to Mathieu [25] and uses Frobenius splitting techniques. Other approaches to the first theorem appear in [21, 29, 28, 18].<sup>(1)</sup>

#### <span id="page-3-1"></span>**1.2. Parity sheaves for the affine Grassmannian**

Let  $\check{G} \supset \check{T}$  be the connected complex algebraic group and maximal torus with root datum dual to that of G.

Let  $\mathcal{K} = \mathbb{C}((t))$  and  $\mathcal{O} = \mathbb{C}[[t]]$ . The affine Grassmannian  $\mathcal{G}_r$  for  $\check{G}$  is an ind-scheme whose complex points form the [set](#page-20-6)  $\tilde{G}(\mathcal{K})/\tilde{G}(\mathcal{O})$ . We consider its complex points as an ind- $\check{G}(\mathcal{O})$ -variety. The  $\check{G}(\mathcal{O})$ -orbits are labeled by the set  $\Lambda^+$  of dominant weights of  ${\bf G}$  and we denote the orbit corresponding to a weight  $\lambda$  by  $\mathcal{G}r^{\lambda}$ .

The geometric Satake theorem [27] shows that the representation theory of G is encoded in a category of sheaves of  $k$ -vector spaces on  $\mathcal{G}_r$ . More precisely, the category of rational representations of G is equivalent to the category  $P(\mathcal{G}_r)$ : the category of  $\check{G}(\mathcal{O})$ -equivariant perverse sheaves on  $\mathcal{G}_r$  with coefficients in  $k$ .

The category  $P(\mathcal{G}_r)$  is the heart of a t-structure on  $D(\mathcal{G}_r)$ , the bounded  $\check{G}(\mathcal{O})$ -equivariant constructible derived category of sheaves of  $k$ -vector spaces on  $\mathcal{G}_r$ . There is a natural convolution product  $\star : D(\mathcal{G}_r) \to D(\mathcal{G}_r)$  which is t-exact and produces a tensor structure on  $P(\mathcal{G}_r)$ , corresponding under the equivalence to the tensor product of rational representations of G.

<span id="page-3-0"></span>Similarly, for [any L](#page-8-0)evi subgroup  $L \subset G$ , the restriction functor from G to L corresponds to a geometrically-defined t-exact functor  $R_{\tilde{L}}^{\tilde{G}}: D(\mathcal{G}r) \to D(\mathcal{G}r_{\tilde{L}})$ , where  $\mathcal{G}r_{\tilde{L}}$  is the affine Grassmannian for  $\check{L} \subset \check{G}$ , the Levi subgroup containing  $\check{T}$  whose roots are dual to those of L (see Section 2.3 for more details).

<sup>(1)</sup> In the literature, these theorems appear with the words 'tilting modules' replaced by 'modules admitting a good filtration' or 'modules admitting a Weyl filtration'. In the appendix to this paper, we explain the fact, well-known to experts, that these are equivalent formulations.

Recall the notion of a parity complex  $[17,$  Section 2.2].<sup>(2)</sup> The affine Grassmannian is a Kac-Moody flag variety and hence the results from [17, Section 4.1] (see in particular Example 4.2 of loc. cit.) c[an b](#page-19-2)e used to study  $D(\mathcal{G}_r)$ . Parity complexes on [the](#page-0-0) affine Grassmannian behave very much like the tilting modules for G. In particular, we [ha](#page-4-0)ve the following theorems which mirror the ones for tilting modules.

The starting point is a result [17, Theorem 4.6] that is very similar to Theorem 1.1:

THEOREM 1.4. – Assume that the characteristic of k is not a torsion prime for  $\check{G}$ <sup>(3)</sup>

*For each*  $\lambda \in \Lambda^+$  (*up to non-canonical isomorphism*), there exists a unique indecomposable *parity complex*  $\mathcal{E}(\lambda)$  *such that*  $\text{supp}(\mathcal{E}(\lambda)) = \mathcal{G}r^{\lambda}$  and  $\mathcal{E}(\lambda)|_{\mathcal{G}r^{\lambda}} = \underline{k}_{\mathcal{G}r^{\lambda}}[\dim \mathcal{G}r^{\lambda}]$ . Every *inde[com](#page-19-2)posable parity co[m](#page-0-0)plex is isomorphic to*  $\mathcal{E}(\lambda)[m]$  *for some*  $\lambda \in \Lambda^{\frac{1}{+}}$  *and*  $m \in \mathbb{Z}$ *.* 

The indecomposable parity complexes  $\mathcal{E}(\lambda)$  are known as parity sheaves. As a special case of [17, Theorem 4.8], we obtain an analogue of Theorem 1.2:

THEOREM 1.5. – If  $\mathcal{F} \in D(\mathcal{G}_r)$  and  $\mathcal{G} \in D(\mathcal{G}_r)$  are parit[y com](#page-0-0)plexes, th[en](#page-6-0) so is the *convolution product*  $\mathcal{F} \star \mathcal{G} \in D(\mathcal{G}r)$ *.* 

The first part of this paper establishes an analogue of Theorem 1.3. In Section 2, we prove the following result,

THEOREM 1.6. – Let L<sup> $i$ </sup> be the Langlands dual of **L** a Levi sub[gro](#page-19-3)up of **G**. If  $\mathcal{F} \in D(\mathcal{G}_r)$  is *a parity complex, then*  $R_L^\check{G}(\mathcal{F})$  *is a parity complex on the affine Grassmannian for L*̃.

The idea of the proof is to replace the purity argument of [7, Theorem 2] by a parity argument.

#### **1.3. Tilting equals parity**

In Sect[ion](#page-0-0) 3, [which](#page-0-0) can be read independently of Section 2, we prove our main result, which ex[plain](#page-20-6)s the similarities between the theorems stated above. Our result shows that, for most characteristics, the Theorems 1.2 and 1.3 about tilting modules are equivalent to the Theorems 1.5 and 1.6 about parity sheaves.

Recall [27, Prop. 13.1] that, for  $\lambda \in \Lambda^+$  a dominant weight, the Weyl module  $\Delta_\lambda$  (resp.  $\nabla_\lambda$ ) goes under the geometric Satake equivalence to the standard sheaf  ${}^p\mathcal{J}_1(\lambda) := {}^p j_{\lambda!} \underline{k}_{\lambda} [d_{\lambda}]$ (resp. costandard sheaf  ${}^p\mathscr{J}_*(\lambda) := {}^p j_{\lambda *} \underline{k}_{\lambda}[d_{\lambda}]$ ) where  $j_{\lambda} : {}^g r^{\lambda} \to {}^g r$  denotes the inclusion,  $\underline{k}_{\lambda}$  the c[on](#page-4-1)stant sheaf on  $\mathcal{G}r^{\lambda}$  and  $d_{\lambda}$  the dimension of  $\mathcal{G}r^{\lambda}$ .

We say  $\mathcal{F} \in P(\mathcal{G}_r)$  is a tilting sheaf if it corresponds to a tilting module for G. This is equivalent to admitting two filtrations — one with standard successive quotients and the other with costandard successive quotients.<sup>(4)</sup> We denote by  $\mathcal{F}(\lambda)$  the tilting sheaf corresponding to the indecomposable tilting module  $T(\lambda)$ .

<span id="page-4-0"></span>Our main theorem is the following geometric characterization of the tilting sheaves on the affine Grassmannian. We will need to assume that the characteristic  $p$  is bigger than some bound depending only on the root system  $\Phi$  of G.

<span id="page-4-1"></span><sup>(2)</sup> Unless stated otherwise, in this paper parity complexes are defined with respect to the constant pariversity  $\sharp$ .

<sup>(3)</sup> This restriction can be removed by working in the non-equivariant setting.

<sup>(4)</sup> Warning: this definition of tilting sheaf is more general than that of [2], which does not apply to this setting.

DEFINITION 1.7. – If the root system  $\Phi$  is irreducible, let  $b(\Phi)$  be given by the following table:



<span id="page-5-0"></span>

In the general case, let  $\Phi = \bigcup_{i=1}^{s} \Phi_i$  be the decomposition into irreducible components. Then we set  $b(\Phi) := \max_{1 \leq i \leq s} b(\Phi_i)$ .

THEOREM 1.8. – *If*  $p > b(\Phi)$ *, then the group* **G** *satisfies* 

$$
(*)\qquad\qquad\forall\lambda\in\Lambda^+,\quad\mathcal{E}(\lambda)=\mathcal{T}(\lambda).
$$

In particular, for p as in the Theorem, every  $\mathcal{E}(\lambda)$  is perverse. Note that we know examples where  $\mathcal{E}(\lambda)$  fails to be perverse, for bad primes (see Lemma 3.7). However, Lemma 3.7(4) suggests that the following may always be true:

CONJECTURE 1.9. – *In arbitrary character[istic](#page-0-0), for every dominant weight*  $\lambda \in \Lambda^+$ *, the perverse sheaf*  ${}^p{\mathcal{H}}^0{\mathcal{E}}(\lambda)$  *is tilting.* 

On the other hand, note that by Proposition 3.3, the property  $(*)$  is actually equivalent to all  $\mathcal{E}(\lambda)$  being perverse.

#### **1.4. Applications**

One motivation for this work was a conjecture of Mirković and Vilonen [27, Conjecture 13.3]. They conjectured that the stalks of standard sheaves with Z-coefficients on the affine Grassmannian are torsion free. [Th](#page-19-4)is conjecture is equivalent to the standard sheaves being ∗-parity for all fields (i.e. their non-zero stalks should be concentrated in one parity). Actually the minimal nilpotent orbit singularities provide counterexamples to this conjecture, in all types but in type  $A_n$ : see [16], where the conjecture is modified to exclude bad primes. We get the following reformulation:

CONJECTURE 1.10. – [If](#page-0-0) p is a good prime for  $\bf G$ , then the standard sheaves with coefficients *in a field of characteristic* p *are* ∗*-parity.*

Note that if Conjecture 1.10 is true, then it implies that **G** satisfies ( $*$ ) whenever p is a good prime.

Conversely, since an earlier draft of the current paper was circul[ated,](#page-0-0) Ach[ar-R](#page-0-0)ider [1] proved that if G satisfies (∗), then the Mirkovic-Vilonen conjecture is correct. In partic[ular,](#page-0-0) ´ using our Theorem 1.8, [the](#page-0-0)y set[tle th](#page-0-0)e conjecture for [all bu](#page-0-0)t a [hand](#page-0-0)ful of cases.

Our results also may be use[d to](#page-0-0) obtain new proofs of Theorems 1.2 and 1.3 [in](#page-0-0) [most](#page-0-0) characteristics. To see this note that for all groups for which the hypothesis of Theorem 1.8 applies, our Theorems 1.5 and 1.6 imply Theorems 1.2 and 1.3. The careful reader will observe that in proving Theorem 1.8 we do not use any results that rely on Theorem 1.2 or 1.3.

Lastly, in Section 4, we observe that our main result implies the existence (for most characteristics) of q-characters for tilting modules, meaning a natural q-analogue of the characters of tilting modules.

#### **1.5. Acknowledgments**

<span id="page-6-0"></span>We would like to thank Steve Donkin and Jens Carsten Jantzen for substantial assistance with tilting modules and Jim Humphreys and the referees for some comments and minor corrections. The second and third authors are also very grateful to the Max Planck Institute for wonderful working conditions.

#### **2. Hy[perb](#page-7-0)olic localization**

<span id="page-6-2"></span>[The](#page-7-0) goal of this section is to prove Theorem 1.6. In [2.1,](#page-0-0) we recall the notion of hyperbolic localization and Braden's Theorem. In 2.2, we introduce some simplifying assumptions and study the hyperbolic localization of parity sheaves in this setting. In 2.3, we apply the result of 2.2 to the affine Grassmannian and prove Theorem 1.6.

#### **2.1. Bra[den'](#page-20-7)[s Th](#page-19-5)eorem**

<span id="page-6-3"></span>Let  $T$  be a complex torus and  $X$  a complex  $T$ -variety.

We make the following assumption, which is automatic if  $X$  is normal by Sumihiro's theorem [31, 19]:

 $(X)$  has a covering by T-stable affine open subvarieties.

Let  $\chi : \mathbb{G}_m \to T$  be a cocharacter of T. Our goal is to understand the hyperbolic localization of certain parity complexes on X with respect to the  $\mathbb{G}_m$ -action defined by  $\chi$ . We begin by recalling Braden's definition.

Let  $Z \subset X$  denote the variety of  $\chi$ -fixed points and  $Z_1, \ldots, Z_m$  its connected components. Consider the attracting and repelling varieties for each component  $Z_i$ ,  $1 \le i \le m$ :

$$
Z_i^+ = \{ x \in X | \lim_{s \to 0} \chi(s) \cdot x \in Z_i \},
$$
  

$$
Z_i^- = \{ x \in X | \lim_{s \to \infty} \chi(s) \cdot x \in Z_i \}.
$$

Let  $Z^+$  (respectively  $Z^-$ ) [be th](#page-19-6)e disjoint, disco[nne](#page-6-1)cted union of the  $Z_i^+$  (respectively  $Z_i^-$ ) and the maps  $f^{\pm}: Z \to Z^{\pm}$  and  $g^{\pm}: Z^{\pm} \to X$  be the component-wise inclusions. Define projection maps  $p^{\pm}$  :  $Z^{\pm} \to Z$  by  $p^+(x) = \lim_{t \to 0} \chi(t) \cdot x$  and  $p^-(x) = \lim_{t \to \infty} \chi(t) \cdot x$ . These are algebraic maps by [13, Proposition 4.2].<sup>(5)</sup>

For this section, we let  $D(X)$  denote the constructible derived category of sheaves of  $k$ -vector spaces on  $X$ .

<span id="page-6-1"></span>Recall that the hyperbolic localization functors  $(-)^{!*}, (-)^{**}$ :  $D(X) \rightarrow D(Z)$  for the character  $\chi$  are defined by

$$
\mathcal{G}^{!*} := (f^+)^!(g^+)^*\mathcal{G},
$$

<sup>&</sup>lt;sup>(5)</sup> In this article we only consider attractive sets, which corresponds to the speed  $m = 1$  case in the setting of [13]. One can reduce to the affine case using  $[13,$  Lemma 4.4] because of our standing assumption that X admits a covering by T-stable open affine subvarieties.

$$
\mathcal{G}^{*!} := (f^-)^*(g^-)^! \mathcal{G}.
$$

We will use the following results of Braden:

THEOREM 2.1 ([7], Theorem 1). – *For any*  $\mathcal{F} \in D(X)$ , there is a natural morphism  $\iota_{\mathcal{G}}:(\mathcal{F})^{*!}\to (\mathcal{F})^{!*}$ *. If*  $\mathcal F$  *is weakly equivariant (e.g., comes from an object in the equivariant derived category), then*

- (i) *there are natural isomorphisms*  $\mathcal{F}^{!*} \cong (p^+)_!(g^+)^* \mathcal{F}$  *and*  $\mathcal{F}^{*!} \cong (p^-)_*(g^-)^! \mathcal{F}$  *and*
- (ii) the morphism  $\iota_{\mathcal{F}} : \mathcal{F}^{*!} \to \mathcal{F}^{!*}$  is an isomorphism.

<span id="page-7-0"></span>Using this result, Braden proves that for  $k = \mathbb{Q}$ , the hyperbolic localization of the intersection cohomology complex  $\mathbf{IC}(X; \mathbb{Q})$  is a direct sum of shifted intersection cohomology complexes. Our goal here is [to pr](#page-6-2)ove a similar result for certain parity complexes.

#### **2.2. Parity of stalks at** T**-fixed points**

We keep the notation of  $\S 2.1$ .

Let Y be a smooth projective T-variety and  $\pi : Y \to X$  a T-equivariant proper morphism. Furthermore, we will assume that:

- 1. The sets of T-fixed points,  $X^T$  and  $Y^T$ , are finite.
- 2. For any T-fixed point  $x \in X^T$ , there exists a cocharacter  $\mathbb{G}_m \to T$  such that  $\mathbb{G}_m$  acts attractively on a neighborhood of x.
- 3. There exists a  $T$ -module  $V$ , for which  $Y$  admits a closed  $T$ -equivariant embedding into the projective space  $\mathbb{P}(V)$ .

**PROPOSITION** 2.2. – *The cohomology of the stalk of*  $(\pi_* \underline{k}_Y)^{!*}$  *at any*  $z \in X^T$  *is con[cen](#page-0-0)trated in even degrees.*

*Proof.* – The push-forward  $\pi_* \underline{k}_Y$  is weakly equivariant and thus by (i) of Theorem 2.1,  $(\pi_* \underline{k}_Y)^{!*} \cong (p^+)_! (g^+)^* \pi_* \underline{k}_Y.$ 

Let  $C^+$  be defined by the Cartesian square



In other words, it is the disjoint union of the preimages  $\pi^{-1}(Z_i^+)$ ,  $1 \le i \le m$ . We abuse notation and also write  $\pi$  for the induced map  $\pi : C^+ \to Z^+$ . The map  $\pi$  is proper and so by base change we have

$$
(\pi_*\underline{k}_Y)^{!*} = (p^+)_! \pi_! \underline{k}_{C^+}.
$$

The stalk of  $(\pi_* \underline{k}_Y)^{!*}$  at z thus has cohomology  $H_c^*((p^+ \circ \pi)^{-1}(z))$ . Using the following lemma, we will construct a Białynicki-Birula decomposition on Y which restricts to one on  $(p^+ \circ \pi)^{-1}(z)$ .

LEMMA 2.3. – *For any*  $z \in X^T$ , consider the direct sum decomposition of the Zariski *tangent space*  $T_z X \cong V^+ \oplus V^0 \oplus V^-$ , such that  $\chi$  acts on  $V^+$ ,  $V^0$ , and  $V^-$  with positive, *zero, and negative weights respectively. There exists a cocharacter*  $\zeta : \mathbb{G}_m \to T$  *whose action on*  $T_z X$  *is attractive on*  $V^+$  *and repulsive on*  $V^0 \oplus V^-$ *, and such that*  $Y^{\zeta} = Y^T$ *.* 

*Proof.* – Let  $Y(T)$  denote the cocharacters of T. The set of cocharacters which act on  $T_zX$  with negative weights is the intersection of  $\mathbf{Y}(T)$  with an open cone in  $\mathbf{Y}(T) \otimes \mathbb{R}$ . By assumption [2.2\(](#page-7-0)2), this intersection is nonempty. In other words, there exists a  $\rho \in Y(T)$ that acts on  $T_zX$  with negative weights.

At each fixed point  $y \in Y^T$ , the action of T on  $T_y Y$  splits as a direct sum of characters. By assumption 2.2(1), the set { $\beta_i$ } of all characters of T obtained in this way is finite. As Y is smooth, for any cocharacter  $\sigma \in \mathbf{Y}(T)$ ,  $Y^{\sigma} = Y^{T}$  if and only if  $\sigma$  is not contained in one of the (finitely many) hyperplanes  $\langle \sigma, \beta_i \rangle = 0$ . Thus we can choose  $\rho$  above such that it acts on  $T_z X$  with negative weights and  $Y^\rho = Y^T$ .

Let  $\{\alpha_k^+\}$  denote the set of characters of T occurring in  $V^+$ . Thus  $\langle \chi, \alpha_k^+ \rangle$  is positive and  $\langle \rho, \alpha_k^+ \rangle$  is negative. For  $m \in \mathbb{Z}$  sufficiently large,  $\langle m \chi + \rho, \alpha_k^+ \rangle = m \langle \chi, \alpha_k^+ \rangle + \langle \rho, \alpha_k^+ \rangle$  is positive for all k. Thus for m large enough the cocharacter  $m\chi + \rho$  acts attractively on  $V^+$ . On the other hand, for any  $m > 0$ ,  $m\chi + \rho$  has strictly negative weights on  $V^0 \oplus V^-$ .

Lastly, as  $\rho$  has been chosen such that  $\langle \rho, \beta_i \rangle \neq 0$  for all i,  $\langle m\chi + \rho, \beta_i \rangle$  will also be nonzero for *m* sufficiently large.

 $\Box$ For such [an](#page-6-3) m, we may [defin](#page-6-2)e  $\zeta$  to be  $m\chi + \rho$ .

Consider the attracting Białynicki-Birula decomposition of Y with respect to  $\zeta$ . By the assumption (C) in Section 2.1, a T-stable neighborhood of  $z$  embeds T-equivariantly into the Zariski tange[nt sp](#page-7-0)ace  $T_z X$ . By the construction of  $\zeta$ ,  $(p^+)^{-1}(z)$  $(p^+)^{-1}(z)$  $(p^+)^{-1}(z)$  is thus the attracting set of z for the action of  $\zeta$ . If  $y \in Y$ , then  $\lim_{t \to 0} \zeta(t)y \in \pi^{-1}(z)$  if and only if  $y \in (p^+ \circ \pi)^{-1}(z)$ . It follows that the space  $(p^+ \circ \pi)^{-1}(z)$  is a [unio](#page-0-0)n of cells in the decomposition.

By assumption 2.2(3), the cell decomposition is filtrable [5], hence the fundamental classes of the cells give a basis for  $H_c^*((p^+\circ \pi)^{-1}(z))$ , which is therefore concentrated in even degrees. This concludes the proof of Proposition 2.2.  $\Box$ 

**PROPOSITION** 2.4. – *The cohomology of the costalk of*  $(\pi_* \underline{k}_Y)^{!*}$  *at any*  $z \in X^T$  *is concentrated in even degrees.*

*Proof.* – It suffices to show that the stalk of the Verdier dual  $\mathbb{D}((\pi_* \underline{k}_Y)^{!*})$  is concentrated in even degrees. For the duration of this proof, let us write  $(-)^{!*, x}$  for the hyperbolic localization with respect to  $\chi$  to emphasize the depe[nden](#page-0-0)ce on  $\chi$ . Then we have:

$$
\mathbb{D}((\pi_*\underline{k}_Y)^{!*},X)\cong \mathbb{D}((\pi_*\underline{k}_Y)^{*\cdot!},X)=(\mathbb{D}(\pi_*\underline{k}_Y))^{!*,-X}=(\pi_*\underline{k}_Y)^{!*,-X}[2\dim Y],
$$

<span id="page-8-0"></span>where the first isomorphism is given by Theorem 2.1(ii), the second by the definition of hyperbolic localization and the third because  $\pi$  is proper and Y is smooth. The cohomology of the stalk at  $z \in X^T$  of the right hand side is concentrated in even degrees by Proposition 2.2.  $\Box$ 

#### **2.3. Hyperbolic localization on the affine Grassmannian**

We now specialize to the case: X is the affine Grassmannian  $\mathcal{G}_r$  and T is the maximal torus  $\check{T} \subset \check{G}$ .

Recall that the connected components of  $\mathcal{G}_r$  are parametrized by the group  $Z(\mathbf{G})^\vee$ of characters of the center  $Z(G) \subset G$ . For any  $\zeta \in Z(G)^{\vee}$ , let  $\mathcal{G}r_{\zeta}$  denote the corresponding connected component. For any  $\mathcal{F} \in D(\mathcal{G}_r)$ , we write  $\mathcal{F} = \bigoplus_{\zeta \in Z(\mathbf{G})^{\vee}} \mathcal{F}_{\zeta}$  where  $\mathcal{F}_{\zeta}$  is supported on  $\mathcal{G}_{r_{\zeta}}$ .

Fix a Levi subgroup  $L \subset G$  containing the maximal torus T. Correspondingly, there is a Levi subgroup L of G containing T<sup>'</sup> whose roots are dual to those of **L**. Let  $\chi$  be the cocharacter of  $\check{T}$  defined by  $2\rho_{\mathbf{G}} - 2\rho_{\mathbf{L}}$ , where  $\rho_{\mathbf{G}}$  (resp.  $\rho_{\mathbf{L}}$ ) denotes the half-sum of the positive roots of G (resp. L). The set of  $\chi$ -fixed points of  $\mathcal{G}r$  is  $\mathcal{G}r_L^*$ . We denote by  $R_{\check{L}}^{\check{G}}:D(\mathcal{G}r)\to D(\mathcal{G}r_{\check{L}})$  the shifted hyperbolic localization functor:

$$
R_{\check{L}}^{\check{G}}(\mathcal{F}) = \bigoplus_{\zeta \in Z(\mathbf{L})^{\vee}} (\mathcal{F}^{*!})_{\zeta} [\langle \zeta, 2\rho_{\check{L}} - 2\rho_{\check{G}} \rangle].
$$

As shown in [3, 5.3.27-31]<sup>(6)</sup>,  $R_{\tilde{L}}^{\tilde{G}}$  $R_{\tilde{L}}^{\tilde{G}}$  is t-exact and cor[resp](#page-0-0)onds under geometric Satake to the restriction functor  $Rep(G) \rightarrow Rep(L)$ [.](#page-19-2) (This generalizes the Mirkovic-Vilonen weight functors which are t[he ca](#page-0-0)se when  $\check{L} = \check{T}$ .)

We can now use Proposition 2.2 to prove Theorem 1.6.

*Proof of Theorem 1.6.* – Recall from [17, Theorem 4.6] and its proof, that every indecomposable parity complex is a direct summand of the push forward of the constant sheaf from a generalized Bott-Samelson resolution  $f : BS \to \mathcal{G}r^{\lambda}$ [. Th](#page-7-0)us it suffices to show that  $R_{\check{L}}^{\check{G}}(f_*\underline{k}_{\text{BS}})$  [is a](#page-0-0) parit[y com](#page-0-0)plex.

Let  $\check{T} \times \mathbb{C}^*$  act on BS and  $\mathcal{G}r^{\lambda}$ , where  $\mathbb{C}^*$  acts by 'loop-rotation'. We will now show that  $\check{T} \times \mathbb{C}^*$  and  $f : BS \to \mathcal{G}r^{\lambda}$  satisfy the assumptions of 2.1 and 2.2 on T, X, and Y. Then by Proposition 2.2 (resp. 2.4),  $((f_* \underline{k}_{BS})^*)$ <sub> $\zeta$ </sub> has  $*$ -even stalks (resp. costalks) at the  $\check{T} \times \mathbb{C}^*$ -fixed points (equivalently the T-fixed points[\) o](#page-6-3)[f](#page-6-2)  $\mathcal{G}r_L$ . But  $(f_*\underline{k}_{\text{BS}})^*$  is also  $\check{L}(\theta)$ [-equ](#page-7-0)ivariant and every  $\check{L}(\mathcal{O})$ -orbit contains a  $\check{T}$ -fixed point, thus  $(f_*\underline{k}_{\text{BS}})^{*!}$  is an even complex and  $R_{\tilde{L}}\check{G}(f_*\underline{k}_{\rm BS})$  is parity.

It remains to check the assumptions  $(C)$  of Section 2.1 and  $(1-3)$  of Section 2.2. We will outline why they a[re va](#page-19-8)lid for any Schubert variety in any partial flag variety of a Kac-Moody group (of which the varieties  $\mathcal{G}r^{\lambda}$  are a special case).

Assume that  $\mathcal G$  is a Kac-Moody group with maximal torus  $\mathcal T$ , Weyl group W and simple reflections S. In [20, Chapter VII] it is shown that any Schubert variety in a partial flag variety for  $\mathcal G$  embeds into the projectivization of a finite dimensional  $\mathcal F$ -representation. A[lso,](#page-19-9) the T-fixed points on any Schubert variety are parametrized by an ideal in the Bruhat order on  $W/W_I$ , where W denotes the Weyl group and  $W_I \subset W$  is a sta[ndar](#page-6-2)[d p](#page-6-3)arab[olic](#page-7-0) subgroup. In particular, the  $\mathcal{I}\text{-fixed points}$  on any Schubert variety are finite. Recall (see e.g., [12, §7, Def.-Prop. 1]) that generalized Bott-Samelson resolutions may be embedded as closed subvarieties of products of Schubert varieties for  $\mathcal{C}$ . It follows that 2.1 (C) and 2.2 (1) and (3) hold for Schubert varieties and their Bott-Samelson resolutions.

Finally, the T-fixed point corresponding to  $w \in W/W_I$  in any Schubert variety is attractive. Indeed, all weights in the tangent space belong to the set  $-w(R^+)$ , [whe](#page-7-0)re  $R^+ \subset \mathbf{X}(\mathcal{T})$  denotes the positive real roots. Hence if  $\chi \in \mathbf{Y}(\mathcal{T})$  is a cocharacter which is negative on all simple roots (which exists because the simple roots are linearly independent in  $\mathbf{X}(\mathcal{T})$  $\mathbf{X}(\mathcal{T})$  $\mathbf{X}(\mathcal{T})$ ) then  $w \cdot \chi$  acts attractively at the fixed point corresponding to w. Hence 2.2(2) is satisfied for any Kac-Moody Schubert variety.  $\Box$ 

<span id="page-9-0"></span> $(6)$  While [3] is written in the setting of  $\mathcal D$ -modules, the proofs of these statements hold without modification in our setting.

#### **3. Tilting modules**

The aim of this section, which may be read independently of the previous one, is to prove our main result, Theorem 1.8.

#### **3.1. Tilting objects in highest weight categories**

Let  $\mathcal C$  be a highest weight category with poset  $\Lambda$  and standard (resp. costandard) objects  $\Delta_{\lambda}$  (resp.  $\nabla_{\lambda}$ ) for each  $\lambda \in \Lambda$ .

Let  $\mathcal{F}(\Delta) \subset \mathcal{C}$  (resp.  $\mathcal{F}(\nabla) \subset \mathcal{C}$ ) de[not](#page-20-0)e the full subcategory of (co)standard filtered objects, meaning  $X \in \mathcal{C}$  and X has a filtration whose successive quotients are (co)standard objects. Thus  $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) \subset \mathcal{C}$  is the full subcategory of tilting objects.

The following theorem is due to Ringel [30, Theorem 4 and 4\*] and gives a useful criterion for determining if an object is tilting.

THEOREM 3.1. – An object  $X \in \mathcal{F}(\Delta)$  if and o[nly](#page-19-11) if  $\text{Ext}^1(X, \nabla_\lambda) = 0$  for all  $\lambda \in \Lambda$ . *Dually,*  $X \in \mathcal{F}(\nabla)$  *if and only if*  $\text{Ext}^1(\Delta_\lambda, X) = 0$  *for all*  $\lambda \in \Lambda$ *.* 

We also mention the following result of Donkin [11, Proposition A4.4] that will be used in the appendix:

PROPOSITION 3.2. – An object X of  $\mathcal C$  is in  $\mathcal F(\nabla)$  (resp.  $\mathcal F(\Delta)$ ) if and only if it admits a *finite left (resp. right) resolution by tilting modules.*

#### **3.2. Parity and Tilti[ng](#page-19-2)**

In this section we apply the tilting criterion, Theorem 3.1, to parity sheaves. We first briefly recall the setting of [17].

Let  $H$  denote a connected linear complex algebraic group. Let  $X$  be a complex algebraic variety (resp. H-variety) together with an algebraic stratification  $X = \bigsqcup_{\lambda \in \Lambda} X_{\lambda}$  into smooth locally closed (resp. H-stable) subsets. We let  $D(X)$ , or  $D(X; k)$ , denote the bounded (equivariant) constructible derived category of k-sheaves on X.

Let  $P(X)$  or  $P(X; k)$  denote the abelian subcategory of  $D(X)$  obtained as the heart of the perverse t-structure for the middle perversity. The objects in this category are (equivariant) perverse sheaves and we denote the simple objects, which are parametrized by strata  $X_\lambda$  and irreducible (equivariant) local systems  $\mathcal{L}$ , by  $\mathbf{IC}(\lambda, \mathcal{L})$ , or simply  $\mathbf{IC}(\lambda)$  when  $\mathcal{L} = \underline{k}_{X_{\lambda}}$  is the constant sheaf.

Recall that for any choice of a function  $\dagger : \Lambda \to \mathbb{Z}/2$ , which we refer to as a pariversity, there are notions of (∗- or !-) even, odd and parity complexes. Unless stated otherwise, the pariversity is assumed to be the trivial function  $\sharp : \Lambda \to \mathbb{Z}/2$ ,  $\sharp(\lambda) = 0$ . Recall that the dimension (or diamond) pariversity  $\Diamond : \Lambda \to \mathbb{Z}/2$  is the function for which  $\Diamond(\lambda)$  is given by the parity of the dimension of the stratum  $X_\lambda$ . Note that if all of the strata are even dimensional, it is equal to the  $\mu$ -pariversity.

We assume that for each stratum  $X_\lambda$  and each (H-equivariant) local system  $\mathcal L$  on  $X_\lambda$ , the (equivariant) cohomology  $H^i(X_\lambda, \mathcal{L}) = 0$  vanishes for all i odd.

The following criterion provides a technique for showing that parity sheaves are tilting.

P 3.3. – *Let* X *be as above. Assume moreover that the strata of* X *are simply connected and that* P(X) *is a highest weight category with highest weight poset equal to the closure ordering on the set of strata and whose standard (resp. costandard) objects are given by the perverse extension sheaves,*  ${}^p \mathcal{J}_1(\lambda) := {}^p j_{\lambda!} \underline{k}_{\lambda} [d_{\lambda}]$  *(resp.*  ${}^p \mathcal{J}_*(\lambda) := {}^p j_{\lambda*} \underline{k}_{\lambda} [d_{\lambda}]$ ).

*If a complex*  $\mathcal E$  *on*  $X$  *is pe[rvers](#page-0-0)e and parity with respect to the dimension pariversity*  $\Diamond$ *, then it is tilting.*

*Proof.* – By Proposition 3.1, it suffices to demonstrate for any  $\lambda \in \Lambda$  the vanishing

 $\mathrm{Ext}^{1}(\mathcal{E}, \mathcal{Y}_{*}(\lambda)) = 0 = \mathrm{Ext}^{1}(\mathcal{Y}_{1}(\lambda), \mathcal{E}).$ 

We prove the first equality. The second follows by duality.

Consider the distinguished triangle

$$
{}^p {\mathcal J}_*(\lambda) \to j_{\lambda*} \underline{k}_{\lambda}[d_{\lambda}] \to A \to
$$

where  $A = {}^{p}\tau_{>0}j_{\lambda*}\underline{k}_{\lambda}[d_{\lambda}] \in {}^{p}D^{>0}$ . By applying  $\text{Ext}^i(\mathcal{E},-)$  to this distinguished triangle in the constructible or equivariant derived category, one obtains a long exact sequence

$$
\cdots \to \text{Hom}(\mathcal{E}, A) \to \text{Ext}^1(\mathcal{E}, {^p\mathcal{J}}_*(\lambda)) \to \text{Ext}^1(\mathcal{E}, j_{\lambda *}\underline{k}_{\lambda}[d_{\lambda}]) \to \cdots.
$$

The term  $Hom(\mathcal{E}, A) = 0$  because  $\mathcal{E}$  is perverse and  $A \in \mathcal{P}D^{>0}$ . By adjunction  $\text{Ext}^1(\mathcal{E}, j_{\lambda *}\underline{k}_{\lambda}[d_{\lambda}]) = \text{Ext}^1(j_{\lambda}^*\mathcal{E}, \underline{k}_{\lambda}[d_{\lambda}]).$  By the parity assumptions and the fact that  $X_\lambda$  is a single stratum,  $j^*_\lambda \mathcal{E}$  is  $\Diamond$ -even. On the other hand  $\underline{k}_\lambda[d_\lambda]$  is also  $\Diamond$ -even, which implies that the  $Ext<sup>1</sup>$  between them vanishes. We have shown that the left and right terms in the sequence above vanish and therefore the middle term does too.  $\Box$ 

REMARK 3.4. – The assumption that the strata be simply connected is made purely for the sake of exposition. The obvious analogue with that assumption removed is true and proven by the same method.

#### **3.3. A [key o](#page-3-1)bservation**

We now restrict our attention to the affine Grassmannian for  $\tilde{G}$ . Recall the notation from Section 1.2. [As m](#page-0-0)entioned there, the affine Grassm[annia](#page-0-0)n satisfies the conditions needed to define parity sheaves. On the other hand,  $P(\mathcal{G}_r)$  is also a highest weight category by the geometric Satake theorem. Thus, we can use the previous proposition in this setting. Together with Theorem 1.5 it gives a weak version of Theorem 1.2.

LEMMA 3.5. – *Suppose*  $T_1$  *and*  $T_2$  *are tilting modules for* G *such that the corresponding tilting sheaves,*  $\mathcal{T}_1$  *and*  $\mathcal{T}_2$ *, on*  $\mathcal{G}_T$  *are parity. Then* 

- 1. *the convolution product*  $\mathcal{T}_1 \star \mathcal{T}_2$  *is perverse and parity, and;*
- 2. *the tensor product*  $T_1 \otimes T_2$  *is tilting.*

*Proof.* – (1) The convolution [produ](#page-0-0)ct  $\mathcal{T}_1 \star \mathcal{T}_2$  is both parity by Theorem 1.5 and perverse as  $\star$  is t-exact.

(2) The tensor product  $T_1 \otimes T_2$  corresponds under geometric Satake to the convolution product  $\mathcal{T}_1 \star \mathcal{T}_2$ . By Proposition 3.3 and (1), it is therefore tilting.  $\Box$ 

#### **3.4. Reduction to simple simply-connected groups**

Our goal is to prove Theorem 1.8. We begin with the following reduction.

- LEMMA 3.6. 1. **G** *satisfies* (\*) *if and only if its derived group*  $\mathcal{D}(\mathbf{G})$  *does.*
- 2. Let  $Z \subset \mathbf{G}$  be a finite central subgroup and  $\mathbf{H} = \mathbf{G}/Z$  be the quotient. If  $\mathbf{G}$  satisfies  $(*)$ , *then so does* H*.*
- 3. If  $G = G_1 \times \cdots \times G_k$  is a product of connected reductive groups  $G_i$  and each  $G_i$ *satisfies* (∗)*, then* G *satisfies* (∗)*.*

*Proof.* – (1) The short exact sequence  $1 \rightarrow \mathcal{D}(\mathbf{G}) \rightarrow \mathbf{G} \rightarrow \mathbf{G}/\mathcal{D}(\mathbf{G}) \rightarrow 1$  is Langlands dual to the short exact sequence  $1 \to Z(\check{G})^0 \to \check{G} \to \check{G}/Z(\check{G})^0 \to 1$ . The latter gives rise to maps

$$
gr_{Z(\check{G})^0} \to \mathcal{G}r \to \mathcal{G}r_{\check{G}/Z(\check{G})^0},
$$

which express  $\mathcal{G}r$  as a trivial cover of  $\mathcal{G}r_{\check G/Z(\check G)^0}$  with fiber  $\mathcal{G}r_{Z(\check G)^0}$ . Thus every orbit closure in  $\mathcal{G}r$  is isomorphic to an orbit closure in  $\mathcal{G}r_{\check{G}/Z(\check{G})^0}$  and vice versa.

(2) Let  $\check{H}$  denote the Langlands dual of **H**. The map  $\mathbf{G} \to \mathbf{H}$  is dual to a finite covering map  $\check{H} \to \check{G}$ . The latter induces an inclusion of connected components  $\mathcal{G}r_{\check{H}} \hookrightarrow \mathcal{G}r$ . Thus every orbit closure in  $\mathcal{G}r_{\tilde{H}}$  is isomorphic to an orbit closure in  $\mathcal{G}r$ .

(3) A dominant weight  $\lambda$  of G is a tuple  $(\lambda_1, \dots, \lambda_k)$  of dominant weights for each  $G_i$ . The closure of  $\Im r^{\lambda}$  in  $\Im r$  is the product of the closures of  $\Im r^{\lambda_i}$  in each  $\Im r_{\check G_i}$ . Consider the box product  $\mathcal{E} = \mathcal{E}(\lambda_1) \boxtimes \cdots \boxtimes \mathcal{E}(\lambda_k)$  of parity sheaves. It is parity, indecomposable, supported on the closure of  $\mathcal{G}r^{\lambda}$  and  $\mathcal{E}|_{\mathcal{G}r^{\lambda}} = \underline{k}_{\mathcal{G}r^{\lambda}}[\dim \mathcal{G}r^{\lambda}]$  $\mathcal{E}|_{\mathcal{G}r^{\lambda}} = \underline{k}_{\mathcal{G}r^{\lambda}}[\dim \mathcal{G}r^{\lambda}]$  $\mathcal{E}|_{\mathcal{G}r^{\lambda}} = \underline{k}_{\mathcal{G}r^{\lambda}}[\dim \mathcal{G}r^{\lambda}]$ . Thus  $\mathcal{E} = \mathcal{E}(\lambda)$ . By assumption, the  $\mathcal{E}(\lambda_i)$ are each perverse and tilting. Thus  $\mathcal{E} = \mathcal{E}(\lambda)$  is also perverse and hence tilting.  $\Box$ 

By part (1) of Lemma 3.6 we can replace G by its derived subgroup  $\mathcal{D}(G)$ , which is semisimple. Any semisimple group is a quotient of a product of simple simply connected groups by a finite central subgroup. Thus by parts (2) and (3), it suffices to determine for each simple simply-connected group if (∗) is satisfied.

#### **3.5. Minuscule weights and the highest short root**

We now assume that **G** is simple and simply-connected. We first check the theorem in two special cases.

LEMMA 3.7. – Let  $\mu$  be a minuscule highest weight and  $\alpha_0$  denote the highest short root *of* G*. Then*

- 1.  $\mathcal{E}(\mu) = \mathcal{T}(\mu) = \mathbf{IC}(\mu)$ ;
- 2. *if* p *is a good prime for* **G**, then  $\mathcal{E}(\alpha_0)$  *is perverse and*  $\mathcal{E}(\alpha_0) = \mathcal{F}(\alpha_0)$ *;*
- 3. *if* p *is a good prime for* G *and moreover*  $p \nmid n+1$  *in type*  $A_n$ *, resp.*  $p \nmid n$  *in type*  $C_n$ *, then*  $\mathcal{E}(\alpha_0) = \mathcal{F}(\alpha_0) = \mathbf{IC}(\alpha_0).$

*Although it is not necessary for what follows, we also note:*

(4) In any characteristic,  ${}^p\mathcal{H}^0\mathcal{E}(\alpha_0)$  is tilting.

*Proof.* – ([1\) T](#page-20-8)he  $\check{G}(\theta)$ -orbit in  $\mathcal{G}_r$  corresponding to the minuscule highest weight  $\mu$  is closed, thus  $IC(\mu) = \nu \mathcal{J}_1(\mu) = \nu \mathcal{J}_*(\mu) = \underline{k}_{\mu}[d_{\mu}]$ , which implies  $\mathcal{E}(\mu) = \mathcal{J}(\mu) = \underline{k}_{\mu}[d_{\mu}]$ .

(2) Recall [24, 2.3.3] that the orbit closure  $\overline{Gr^{\alpha_0}}$  consists of two strata, a point  $\overline{Gr}^0$  an[d it](#page-19-2)s complement  $\mathcal{G}r^{\alpha_0}$ , and the singularity is equivalent to that of the orbit closure  $\overline{\mathcal{O}_{\min}}$  of the minimal orbit  $\mathcal{O}_{\text{min}}$  of the corresponding nilpotent cone of  $\check{\mathfrak{g}} = \text{Lie}(\check{G})$ . More pr[ecis](#page-19-2)ely there is an open embedding  $j: \overline{\mathcal{O}_{\min}} \to \overline{\mathcal{G}r^{\alpha_0}}$ [. W](#page-19-2)e can thus apply the results of Section 4.3 of [17], for the group  $\check{G}$ .

When  $p$  is a good prime for G, it is in particular not one of the primes listed in [17, 4.22(5)]. It follows from Proposition 4.22 [of \[](#page-0-0)17], that there is an indecomposable perverse parity extension  $\mathscr E$  of the constant sheaf on  $\mathscr O_{\rm min}$  to  $\overline{\mathscr O_{\rm min}}$ . Consider  $j_{!*}\mathscr E$ . As  $\overline{\mathscr G^{x^{\alpha_0}}}$  is smooth away from  $gr^0$ , the restriction of  $j_{!*}E$  $j_{!*}E$  to  $gr^{\alpha_0}$  is the constant sheaf. We conclude that  $j_{!*}E = E(\alpha_0)$ and is perverse, so by Proposition 3.3,  $\mathcal{E}(\alpha_0) = \mathcal{T}(\alpha_0)$ .

(3) As a consequence of [17, Proposition 4.23], we have a short exact sequence:

 $0 \longrightarrow i_*(k \otimes_{\mathbb{Z}} H) \longrightarrow {^p\mathscr{J}}_! (\alpha_0) \longrightarrow \textbf{IC}(\alpha_0) \longrightarrow 0$ 

(here *i* denotes the inclusion  $\mathcal{G}r^0 \to \overline{\mathcal{G}r^{\alpha_0}}$  and H is defined as the fundamental group of the root system consisting of the long roots of  $\check{G}$ , see [17, Proposition 4.23].) So  ${}^p\mathscr{J}_1(\alpha_0) \simeq$  **as soon as p does not divide H. Assuming that p is good, we** only need to add the conditions stated for  $A_n$  and  $C_n$ .

(4) Let us regard  $\mathcal{G}_r$  as a flag variety for the affine Kac-Moody group  $\mathcal G$  associated to  $\check{G}$ (see e.g., [17, Example 4.2]). Let us parametrize the simple roots of  $\mathcal{G}$  by  $\{0, \ldots, \ell\}$  so that  $\Delta = \{1, \ldots, \ell\}$  corresponds to the simple roots of  $\check{G}$ . For any subset  $I \subset \{0, \ldots, \ell\}$  one has a standard parabolic subgroup  $\mathcal{P}_I \subset \mathcal{G}$ . Let J denote the subset of  $\Delta$  of simple roots which are orthogonal to the highest root of  $\tilde{G}$  (i.e. those simple roots corresponding to nodes which are not connected to the exceptional node in the affine Dynkin diagram of  $\tilde{G}$ ). Now consider the Bott-Samelson space

$$
BS:=\mathcal{P}_\Delta\times_{\mathcal{P}_J}\mathcal{P}_{J\cup\{\alpha_0\}}\times_{\mathcal{P}_J}\mathcal{P}_\Delta/\mathcal{P}_\Delta.
$$

Then BS is a  $\mathscr{P}_{J \cup \{\alpha_0\}}/\mathscr{P}_J \cong \mathbb{P}^1$  bundle over  $\mathscr{P}_{\Delta}/\mathscr{P}_J = \check{G}/P$  (where  $P \subset \check{G}$  denotes the standard parabolic subgroup corresponding to  $J \subset \Delta$ ). From this one can deduce that dim<sub>C</sub>  $BS = \dim \check{G}/P + 1$  and that the  $\check{T}$ -fixed points on BS are parametrized by  $W/W_J \times \{\text{id}, s_0\}$ , where W denotes the Weyl group of  $\check{G}$  and  $s_0$  denotes the affine simple reflection. Now consider the map induced by multiplication:

$$
m:BS\to \mathcal{G}/\mathcal{P}_\Delta=\mathcal{G}r.
$$

We leave it as an exercise for the reader to show that m is a resolution of  $\overline{Gr^{\alpha_0}}$ .

Let  $\mathcal{P} := f_* k_{\text{BS}}$ [dim BS]. By [17, Lemma 4.21(1)] we have a short exact sequence:

(1) 
$$
0 \to {}^p \mathcal{J}_1(\alpha_0) \to {}^p \mathcal{H}^0 \mathcal{P} \to {}^p \mathcal{H}^0 j_{0*} j_0^* \mathcal{P} \to 0.
$$

This is a filtration by standard sheaves and, as  ${}^p\mathcal{H}^0\mathcal{P}$  is self-dual ( ${}^p\mathcal{H}^0$  is preserved by duality), duality gives a filtration by costandard sheaves. Thus  ${}^p{\mathcal{H}}^0{\mathcal{P}}$  is tilting.  $\Box$ 

#### **3.6. Fund[am](#page-5-0)ental weights**

PROPOSITION 3.8. – Let **G** be simple and simply connected with root system  $\Phi$ . If  $p > b(\Phi)$ *(see Table 1), then for each fundamental weight*  $\varpi_i$ ,

$$
\mathscr{E}(\varpi_i)=\mathscr{T}(\varpi_i).
$$

*Proof*. – We use the f[ollow](#page-0-0)ing method: first we express the Weyl modules with fundamental highest weights in characteristic zero as direct summands of tensor products of Weyl modules corresponding to minuscule weights or the highest short root.

By part (1) of Lemma 3.7, we have that  $\mathscr{E}(\mu) = \mathscr{T}(\mu) = {^p\mathscr{J}_1(\mu)}$  for any minuscule weight  $\mu$ , and by part (3), if p is good and  $\Phi$  not of type  $A_n$  or  $C_n$ , then  $\mathcal{E}(\alpha_0) = \mathcal{I}(\alpha_0) = \mathcal{I}_1(\alpha_0)$ . [T](#page-0-0)hus, i[n ch](#page-0-0)aracteristic p, the analogous tensor product of Weyl modules corresponds under the geometric Satake theorem to a perverse sheaf  $\mathcal E$  obtained as a convolution product of parity sheaves. The perverse sheaf  $\mathcal E$  is therefore parity by Theorem 1.5 and tilting by Lemma 3.5. In particular, any summand of  $\mathscr E$  is perverse, parity and tilting.

On the other hand, if we know that the Weyl modules appearing as direct summands in the tensor products in characteristic  $\theta$  remain simple in characteristic  $p$  (and hence are indecomposable tilting modules), then by compar[ing](#page-19-12) t[he c](#page-19-13)h[ara](#page-20-9)cter[s w](#page-20-10)e can conclude that the same decomposition occurs as in characteristic 0.

Thus, we need to know for which primes the Weyl modules remain simple. This has already been done for us and the answers may be found [in](#page-19-14) [15], [14], [22] or [26]. The careful reader should observe that these results are logically independent of Theorems 1.2 and 1.3. In particular, they are obtained via Jantzen's sum formula.

In what follows, we use Bourbaki's notation [6, Planches] for roots, simple roots, fundamental weights, etc. For  $\lambda \in \Lambda^+$ , we denote by  $V(\lambda)$  the Weyl module of highest weight  $\lambda$ over Q.

3.6.1*.* Type  $A_n$ . – All fundamental weights are minuscule, so there is nothing to prove.

3.6.2*. Type*  $B_n$ . – The weight  $\varpi_n$  is minuscule, and we have

 $V(\varpi_n)^{\otimes 2} \simeq V(2\varpi_n) \oplus V(\varpi_{n-1}) \oplus \cdots \oplus V(\varpi_1) \oplus V(0),$  $V(\varpi_n)^{\otimes 2} \simeq V(2\varpi_n) \oplus V(\varpi_{n-1}) \oplus \cdots \oplus V(\varpi_1) \oplus V(0),$  $V(\varpi_n)^{\otimes 2} \simeq V(2\varpi_n) \oplus V(\varpi_{n-1}) \oplus \cdots \oplus V(\varpi_1) \oplus V(0),$ 

all these Weyl modules being simple modulo p as soon as  $p > 2$  (see [15, II.8.21] and the references therein, particularly [26, Remark 3.4]). So we can generate all fundamental tilting modules when  $p > 2$ , as claimed.

3.6.3*. Type*  $C_n$ . – The weight  $\varpi_1$  is minuscule, and for  $1 \le i \le n$ , we have

$$
\Lambda^i V(\varpi_1) \simeq V(\varpi_i) \oplus V(\varpi_{i-2}) \oplus \cdots
$$

where for convenience we set  $\varpi_0 = 0$ . All these Weyl modules remain simple modulo p as soon as  $p > n$  (again, see [15, II.8.21] and [26, Remark 3.4]). Moreover, the *i*-th exterior power splits as a summand of the *i*-th tensor product for  $p > i$ , so  $p > n$  is always sufficient. It follows that we can generate all fundamental tilting modules when  $p > n$ .

3.6.4*. Type*  $D_n$ . – The weights  $\varpi_{n-1}$  and  $\varpi_n$  are minuscule, and we have

$$
V(\varpi_n)^{\otimes 2} \simeq V(2\varpi_n) \oplus V(\varpi_{n-2}) \oplus V(\varpi_{n-4}) \oplus \cdots
$$

$$
V(\varpi_n) \otimes V(\varpi_{n-1}) \simeq V(\varpi_n + \varpi_{n-1}) \oplus V(\varpi_{n-3}) \oplus V(\varpi_{n-5}) \oplus \cdots
$$

all these Weyl modules remaining simple modulo p as soon as  $p > 2$  (again, see [15, II.8.21] and [26, Remark 3.4]). Hence we can generate all fundamental tilting modules when  $p > 2$ .

3.6.5*. Type*  $E_6$ . – The minuscule weights are  $\varpi_1$  and  $\varpi_6$ , and the highest (short) root is  $\varpi_2$ . Moreover, we have

$$
\Lambda^2 V(\varpi_1) \simeq V(\varpi_3)
$$
  
\n
$$
\Lambda^2 V(\varpi_6) \simeq V(\varpi_5)
$$
  
\n
$$
\Lambda^2 V(\varpi_2) \simeq V(\varpi_4) \oplus V(\varpi_2)
$$

and all these Weyl modules remain simple modulo p as soon as  $p > 3$  [14, 6.4]. Hence we can generate all fundamental tilting modules when  $p > 3$ .

3.6.6*. Type*  $E_7$ . – The weight  $\varpi_7$  is minuscule, and the highest (short) root is  $\varpi_1$ . Moreover, we have

$$
V(\varpi_1)^{\otimes 2} \simeq V(2\varpi_1) \oplus V(\varpi_1) \oplus V(\varpi_3) \oplus V(\varpi_6) \oplus V(0)
$$
  

$$
V(\varpi_6) \otimes V(\varpi_7) \simeq V(\varpi_6 + \varpi_7) \oplus V(\varpi_1 + \varpi_7) \oplus V(\varpi_2) \oplus V(\varpi_5) \oplus V(\varpi_7)
$$
  

$$
V(\varpi_5) \otimes V(\varpi_7) \simeq V(\varpi_5 + \varpi_7) \oplus V(\varpi_2 + \varpi_7) \oplus V(\varpi_1 + \varpi_6) \oplus V(\varpi_3) \oplus V(\varpi_6) \oplus V(\varpi_4)
$$

and all these Weyl modules remain simple modulo  $p$  as soon as  $p > 19$ , because then all the weights involved lie in the fundamental alcove.<sup>(7)</sup> Thus we can generate all fundamental tilting modules when  $p > 19$ .

3.6.7*. Type*  $E_8$ . – There is no minuscule weight. The highest (short) root is  $\varpi_8$ . We have

$$
V(\varpi_8)^{\otimes 2} \simeq V(2\varpi_8) \oplus V(\varpi_7) \oplus V(\varpi_1) \oplus V(\varpi_8) \oplus V(0)
$$
  
\n
$$
V(\varpi_7) \otimes V(\varpi_8) \simeq V(\varpi_7 + \varpi_8) \oplus V(\varpi_1 + \varpi_8) \oplus V(2\varpi_8)
$$
  
\n
$$
\oplus V(\varpi_8) \oplus V(\varpi_7) \oplus V(\varpi_6) \oplus V(\varpi_2) \oplus V(\varpi_1)
$$
  
\n
$$
V(\varpi_6) \otimes V(\varpi_8) \simeq V(\varpi_6 + \varpi_8) \oplus V(\varpi_7 + \varpi_8) \oplus V(\varpi_2 + \varpi_8)
$$
  
\n
$$
\oplus V(\varpi_1 + \varpi_8) \oplus V(\varpi_1 + \varpi_7) \oplus V(\varpi_7)
$$
  
\n
$$
\oplus V(\varpi_6) \oplus V(\varpi_5) \oplus V(\varpi_3) \oplus V(\varpi_2)
$$
  
\n
$$
V(\varpi_5) \otimes V(\varpi_8) \simeq V(\varpi_5 + \varpi_8) \oplus V(\varpi_1 + \varpi_6) \oplus V(\varpi_2 + \varpi_8)
$$
  
\n
$$
\oplus V(\varpi_6 + \varpi_8) \oplus V(\varpi_1 + \varpi_2) \oplus V(\varpi_3 + \varpi_8)
$$
  
\n
$$
\oplus V(\varpi_2 + \varpi_7) \oplus V(\varpi_7) \oplus V(\varpi_6) \oplus V(\varpi_5) \oplus V(\varpi_4) \oplus V(\varpi_3)
$$

<span id="page-15-1"></span><span id="page-15-0"></span>and all these Weyl modules remain simple modulo p as soon as  $p > 31$ , because then all the weights involved lie in the fundamental alcove.(8) Th[us w](#page-20-9)e can generate all fundamental tilting modules when  $p > 31$ .

<sup>(7)</sup> We remark that  $V(\varpi_1 + \varpi_7)$  is reducible modulo 19, according to [22].

<sup>&</sup>lt;sup>(8)</sup> We remark that  $V(2\varpi_8)$  is reducible modulo 31, according to [22].

<sup>4</sup> <sup>e</sup> SÉRIE – TOME 49 – 2016 – N<sup>o</sup> 2

3.6.8*. Type*  $F_4$ . – The short dominant root is  $\varpi_4$ , and we have

$$
\Lambda^2 V(\varpi_4) \simeq V(\varpi_1) \oplus V(\varpi_3)
$$
  

$$
\Lambda^3 V(\varpi_4) \simeq V(\varpi_2) \oplus V(\varpi_1 + \varpi_4) \oplus V(\varpi_3)
$$

and all these Weyl modules remain simple modulo  $p$  as soon as  $p > 3$  [14, 6.4], [22].

So, for  $p > 3$ , we can get  $T(\varpi_1)$  and  $T(\varpi_3)$  as direct summands of  $T(\varpi_4)^{\otimes 2}$ , and  $T(\varpi_2)$ as a direct summand of  $T(\varpi_4)^{\otimes 3}$ .

3.6.9. Type  $G_2$ . – The short dominant root is  $\varpi_1$ . We have

$$
\Lambda^2 V(\varpi_1) \simeq V(\varpi_1) \oplus V(\varpi_2),
$$

and these Weyl modules remain simple modulo p for  $p > 3$  [14, 6.4]. Thus we can generate all fundamental tilting modules when  $p > 3$ .  $\Box$ 

#### **3.7. Arbitrary weigh[ts](#page-0-0)**

We can now complete the proof of our main theorem.

*Proof of Theorem 1.8.* – Recall that any domi[nan](#page-0-0)t weight  $\lambda \in \Lambda^+$  can be expressed as

$$
\lambda=\sum_i a_i\varpi_i
$$

for non-negative integers  $a_i$ . By Proposition 3.8, for every fundamental weight  $\varpi_i$ ,  $\mathcal{T}(\varpi_i) = \mathcal{E}(\varpi_i)$ . Thus the tensor product  $\bigotimes T(\varpi_i)^{\otimes a_i}$  corresponds to a perverse sheaf that is also parity (as it is a convolution of parity sheaves) and tilting (by Lemma 3.5). The tensor product is therefore tilting and as it is of highest weight  $\lambda$ , contains  $T(\lambda)$  as a direct summand. We conclude that  $\mathcal{T}(\lambda)$  is a summand of the corresponding (parity) convolution product and thus  $\mathcal{T}(\lambda) = \mathcal{E}(\lambda)$ .  $\Box$ 

#### **3.8.** A remark on [the bo](#page-0-0)und in type  $C_n$

We do not know in all cases the exact bound on p for which the property  $(*)$  holds. Recall that if Conjecture 1.10 is true, then  $(*)$  holds whenever p is a good prime.

Stephen Donkin has pointed out to us that in type  $C_n$  there exist p with  $2 < p \le n$  such that not all indecomposable tilting modules can be obtained as direct summands of tensor products of the minuscule tilting module  $T(\varpi_1)$  and  $T(\alpha_0)$ . Thus, while it is possible that (∗) may hold in type  $C_n$  for all  $p > 2$ , a different method of proof will be required. Let us explain Donkin's argument.

So consider  $G = Sp_{2n}$ , and assume  $p > 2$ . The only minuscule fundamental weight is  $\varpi_1$ , and  $E := L(\varpi_1) = T(\varpi_1)$  is the natural G-module, of dimension 2n. Now the highest short root is  $\varpi_2$ , and  $T(\varpi_2)$  is a direct summand of  $\Lambda^2 E$  which itself is a direct summand of  $E^{\otimes 2}$  since  $p > 2$ . Hence the question is whether every indecomposable tilting module is a direct summand of a tensor power of E.

Now assume p divides n. Thus p divides dim E. By [4, Proposition 2.2] (which is stated for representations of finite groups but is valid with the same proof here), any indecomposable summand of a tensor power  $E^{\otimes a}$  (with  $a \ge 1$ ) also has dimension divisible by p. Hence an affirmative answer would imply that all indecomposable tilting modules except  $T(0) = k$ have dimension divisible by p.

From this it would follow by induction that the dimension of each standard module  $\nabla_{\lambda}$ , where  $\lambda$  is not in the principal block, would be divisible by p. Indeed, there is a short exact sequence

$$
0 \longrightarrow R \longrightarrow T(\lambda) \longrightarrow \nabla_{\lambda} \longrightarrow 0
$$

where R is filtered by  $\nabla_{\mu}$ 's with  $\mu \leq \lambda$ . Hence by induction, p divides dim R, and by hypothesis also dim  $T(\lambda)$ , hence also dim  $\nabla_{\lambda}$ . The case where  $R = 0$  is the base of the induction: then  $\nabla_{\lambda} = T(\lambda)$ , and  $\lambda \neq 0$ , since  $\lambda$  is not in the principal block.

Now by Weyl's dimension formula, for any  $m \in \mathbb{N}$  we have

$$
\dim \nabla_{(m-1)\rho_{\mathbf{G}}} = m^{n^2}
$$

(in general type, one gets  $m^{|\Phi^+|}$ ). Taking m prime to p, this would force  $(m-1)\rho_{\mathbf{G}}$  to be in the principal block. In particular we could take  $m = 2$  (since  $p > 2$ ), and so  $\rho_{\mathbf{G}}$  itself should be in the principal block. However this is not true when n is congruent to 1 or 2 modulo 4, since in the basis of roots the coefficient of  $\rho$  along  $\alpha_n$  is  $\frac{1}{4}m(m+1)$ .

Since we have reached a contradiction, we cannot sharpen the bound of the Proposition in type  $C_n$ .

#### **4.** q**-Characters for tilting modules**

When  $p$  satisfies the conditions of Theorem 1.8 we are able to deduce that the character of a tilting module has a natural graded refinement. More precisely,

COROLLARY 4.1. – *Suppose that*  $\mathcal{T}(\lambda) = \mathcal{E}(\lambda)$ *. Then the total cohomology of the stalk of* the tilting sheaf  $\mathcal{T}(\lambda)$  at [a poi](#page-8-0)nt in  $\mathcal{G}r^\nu$  has the same dimension as the weight space  $T(\lambda)^\nu$ . Thus *the dimension of the weight space has a natural graded refinement.*

*Proof.* – Recall from 2.3 that the weight space functor  $F_\nu$  corresponds under geometric Satake to the summand  $(\mathcal{F}^{*1})_{\nu}[\langle \nu, 2\rho \rangle]$  of  $R_{\tilde{T}}^{\tilde{G}}$  supported at the  $\tilde{T}$ -fixed point  $t^{\nu} \in \mathcal{G}r^{\nu}$ .

As explained in [7, Prop. 3], the local Euler characteristic of a sheaf  $\mathcal F$  at a torus fixed point x is equal to the local Euler characteristic of any hyperbolic localization of  $\mathcal F$ . Therefore, the stalk of a perverse sheaf in the Satake category at the point  $t^{\nu}$  has an Euler characteristic of absolute value equal to the dimension of the  $\nu$ -th weight space of the corresponding representation of  $\check{G}$ . On the other hand, the cohomology of the stalk of the parity sheaf  $\mathcal{T}(\lambda)$ is concentrated in even or odd degree, thus its total dimension is equal to the dimension of the weight space  $T(\lambda)^{\nu}$ . The total cohomology of the stalk is graded and thus the dimen[sion](#page-20-11) of the weight space inherits a natural grading.  $\Box$ 

REMARK 4.2. – In characteristic zero (where the indecomposable tilting modules and simple modules coincide) the above  $q$ -analogue of weight multiplicity is due to Lusztig [23]. Lusztig shows that the  $q$ -characters of simple modules in characteristic zero are given by certain Kazhdan-Lusztig polynomials associated to the (extended) affine Weyl group. In fact, equipped with Lusztig's results, the  $q$ -characters of tilting modules can be deduced from the ordinary characters of the indecomposable tilting modules and Lustig's q-characters of simple modules in characteristic zero. Indeed, one can lift  $\mathscr{E}(\lambda)$  to a parity sheaf  $\mathscr{E}(\lambda, \mathbb{O})$  with

<span id="page-18-0"></span>coefficients in  $\mathbb{O}$ , a complete local ring with residue field k. If K denotes the fraction field of  $\mathbb{O}$ one has an isomorphism

(2) 
$$
\mathscr{E}(\lambda, \mathbb{O}) \otimes_{\mathbb{O}} \mathbb{K} \cong \bigoplus \mathbf{IC}(\overline{\mathcal{G}r^{\mu}}; \mathbb{K})^{\oplus m_{\mu,\lambda}}
$$

where  $m_{\mu,\lambda}$  denotes the multiplicity of  $\Delta_{\mu}$  i[n a](#page-18-0)  $\Delta$ -flag on  $T(\lambda)$ . (We use that the parity sheaves with coefficients in  $K$  on the affine Grassmannian are the intersection cohomology complexes and that  $P(\mathcal{G}_r; \mathbb{K})$  is semi-simple). The q-character of  $\mathcal{E}(\lambda)$  agrees with the q-character of  $\mathcal{E}(\lambda, \mathbb{O})$ , which in turn [ag](#page-19-16)rees with that of (2). Hence on can deduce the q-character of  $\mathcal{E}(\lambda)$ once one knows the multiplicities  $m_{\mu,\lambda}$  and the q-characters in characteristic zero.

REMARK 4.3. – In [8] R. Brylinski has shown that Lusztig's q-analogue of weight multiplicity can be interpreted in terms of a filtration on each weight space coming from the action of a principal nilpotent element. It would be interesting to find a similar interpretation for the q-character of tilting modules.

#### **5. Appendix**

Recall that a G-module V is said to admit a good filtration if it admits a filtration with successive quotients isomorphic to the induced modules  $\nabla_{\lambda}$ .

In the references, the Theorems 1.2 and 1.3 are formulated as follows.

THEOREM 5.1. – If V and V' are G-modules admitting a good filtration, then so is  $V \otimes V'$ .

THEOREM 5.2. - Let **L** be a Levi subgroup of **G**. If V is a **G**-module with a good filtration, *then* V *has also a good filtratio[n wh](#page-0-0)en considered as an* L*-m[odu](#page-0-0)le.*

The aim of this appendix is [to s](#page-0-0)how that these formulat[ions](#page-0-0) are equivalent:

T 5.3. – *Theorem 1.2 [is e](#page-0-0)quiv[alen](#page-0-0)t to Theorem 5.1.*

THEOREM 5.4. – *Theorem 1.3 is equivalent to Theorem 5.2.* 

*Proof.* – Suppose Theorems 5.1 and 5.2 are true. If  $T$  and  $T'$  are tilting G-modules, then in particular they admit good filtrations. Thus  $T \otimes T'$  also admits a good filtration, as does  $\text{Res}_{\mathbf{L}}^{\mathbf{G}} T$ . On the other hand, t[enso](#page-0-0)r p[rodu](#page-0-0)ct and  $\text{Res}_{\mathbf{L}}^{\mathbf{G}}$  both commute with duality. We conclude that the  $T \otimes T'$  $T \otimes T'$  and  $\operatorname{Res}_{\mathbf{L}}^{\mathbf{G}} T$  are tilting.

Conversely, suppose that Th[eore](#page-0-0)ms 1.2 and 1.3 are true. Let  $V$  and  $V'$  be  $G$ -modules that admit good filtrations. By Proposition 3.2, they both admit finite left resolutions by tilting modules. The tensor product of these resolutions is a finite left resolution of  $V \otimes V'$  by tilting modules. Applying Proposition 3.2 again, we conclude that  $V \otimes V'$  admits a good filtration.

Similarly, as  $\operatorname{Res}_{\mathbf{L}}^{\mathbf{G}}$  is exact and by assumption takes tilting modules to tilting modules,  $\mathrm{Res}_{\mathbf{L}}^{\mathbf{G}}$  applied to a finite left resolution of V by tilting modules is a finite left resolution of  $\operatorname{Res}_{\mathbf{L}}^{\mathbf{G}} V$  by tilting modules. We conclude that  $\operatorname{Res}_{\mathbf{L}}^{\mathbf{G}} V$  admits a good filtration.

#### [BIBLI](http://arxiv.org/abs/1305.1684)OGRAPHY

- <span id="page-19-10"></span>[1] P. ACHAR, L. RIDER, Parity sheaves on the affine Grassmannian and the Mirkovic-Vilonen conjecture, preprint arXiv:1305.1684, to appear in *Acta Math*.
- [2] A. BEILINSON, R. BEZRUKAVNIKOV, I. MIRKOVIĆ[, Tilting exercises,](http://www.math.uchicago.edu/~mitya/langlands/hitchin/BD-hitchin.pdf) *Mosc. Math. J.* 4 (2004), [547–557, 782.](http://www.math.uchicago.edu/~mitya/langlands/hitchin/BD-hitchin.pdf)
- <span id="page-19-15"></span><span id="page-19-7"></span>[\[3\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#4) A. BEILINSON, V. DRINFEL'D, Quantization of Hitchin integrable system and Hecke eigensheaves, preprint http://www.math.uchicago.edu/~mitya/langlands/hitchin/ BD-hitchin.pdf, 2000.
- [4] D. J. BENSON, J. F. CARLSON, Nilpotent elements in the Green ring, *J. Algebra* 104 (1986), 329–350.
- <span id="page-19-14"></span>[\[5\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#6) A. BIALYNICKI-BIRULA, Some properties of the decompositions of algebraic varieties determined by actions of a torus, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **24** (1976), 667–674.
- <span id="page-19-3"></span>[\[6\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#7) N. B, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV : Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI : Systèmes de racines*, Actualités Scientifiques et Industrielles **1337**, Hermann, Paris, 1968 ; réédition Springer, 2002.
- <span id="page-19-16"></span>[7] T. BRADEN, Hyperbolic localization of intersection cohomology, *Transform. Groups* **8** (2003), 209–216.
- <span id="page-19-0"></span>[8] R. K. BRYLINSKI, Limits of weight spaces, Lusztig's  $q$ -analogs, and fiberings of adjoint orbits, *J. Amer. Math. Soc.* **2** (1989), 517–533.
- <span id="page-19-11"></span>[\[9\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#11) S. DONKIN, *Rational representations of algebraic groups*, Lecture Notes in Math. 1140, Springer, Berlin, 1985.
- <span id="page-19-9"></span>[\[10\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#12) S. DONKIN, On tilting modules for algebraic groups, *Math. Z.* **212** (1993), 39–60.
- <span id="page-19-6"></span>[11] S. DONKIN, *The q-Schur algebra*, London Math. Soc. Lecture Note Series 253, Cambridge Univ. Press, Cambridge, 1998.
- [12] S. GAUSSENT, P. LITTELMANN, LS galleries, the path model, and MV cycles, *Duke Math. J.* **127** (2005), 35–88.
- <span id="page-19-13"></span>[\[13\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#14) W. H. HESSELINK, Concentration under actions of algebraic groups, in *Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 33rd Year (Paris, 1980)*, Lecture Notes in Math. **867**, Springer, Berlin, 1981, 55–89.
- <span id="page-19-12"></span>[\[14\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#15) J. C. JANTZEN, First cohomology groups for classical Lie algebras, in *Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991)*, Progr. Math. **95**, Birkhäuser, 1991, 289–315.
- <span id="page-19-4"></span><span id="page-19-2"></span>[15] J. C. JANTZEN, *R[epresentations of](http://arxiv.org/abs/0804.2041) algebraic groups*, second ed., Mathematical Surveys and Monographs **107**, Amer. Math. Soc., Providence, RI, 2003.
- <span id="page-19-1"></span>[16] D. JUTEAU, Modular representations of reductive groups and geometry of affine Grassmannians, preprint arXiv:0804.2041.
- <span id="page-19-5"></span>[17] D. JUTEAU, C. MAUTNER, G. WILLIAMSON, Parity sheaves, *J. Amer. Math. Soc.* **27** (2014), 1169–1212.
- [18] M. K, Based modules and good filtrations in algebraic groups, *Hiroshima Math. J.* **28** (1998), 337–344.
- <span id="page-19-8"></span>[\[19\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#20) F. KNOP, H. KRAFT, D. LUNA, T. VUST, Local properties of algebraic group actions, in *Algebraische Transformationsgruppen und Invariantentheorie*, DMV Sem. **13**, Birkhäuser, 1989, 63–75.
- [20] S. KUMAR, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Math. **204**, Birkhäuser, 2002.

- <span id="page-20-11"></span><span id="page-20-9"></span><span id="page-20-3"></span>[\[21\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#23) P. LITTELMANN, Good filtrations and decomposition rules for representations with standard monomial theory, *J. reine angew. Math.* **433** (1992), 161–180.
- [22] F. LÜBECK, Small degree representations of finite Chevalley groups in defining characteristic, *LMS J. Comput. Math.* **4** (2001), 135–169.
- <span id="page-20-8"></span>[\[23\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#24) G. LUSZTIG, Singularities, character formulas, and a  $q$ -analog of weight multiplicities, in *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, Astérisque **101**, Soc. Math. France, Paris, 1983, 208–229.
- <span id="page-20-10"></span><span id="page-20-2"></span> $[24]$  A. MALKIN, V. OSTRIK, M. VYBORNOV, The minimal degeneration singularities in the affine Grassmannians, *Duke Math. J.* **126** (2005), 233–249.
- <span id="page-20-6"></span>[\[25\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#27) O. MATHIEU, Filtrations of G-modules, *Ann. Sci. École Norm. Sup.* **23** (1990), 625-644.
- <span id="page-20-5"></span>[\[26\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#28) G. J. MCNINCH, Semisimplicity of exterior powers of semisimple representations of groups, *J. Algebra* **225** (2000), 646–666.
- [27] I. MIRKOVIĆ, K. VILONEN, Geometric Langlands duality and representations of algebraic groups over commutative rings, *Ann. of Math.* **166** (2007), 95–143.
- <span id="page-20-4"></span>[\[28\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#29) J. PARADOWSKI, Filtrations of modules over the quantum algebra, in *Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991)*, Proc. Sympos. Pure Math. **56**, Amer. Math. Soc., Providence, RI, 1994, 93–108.
- <span id="page-20-7"></span><span id="page-20-0"></span>[\[29\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#31) P. POLO, Modules associés aux variétés de Schubert, in *Proceedings of the Indo-French Conference on Geometry (Bombay, 1989)*, Hindustan Book Agency, Delhi, 1993, 155–171.
- <span id="page-20-1"></span>[\[30\]](http://smf.emath.fr/Publications/AnnalesENS/4_49/html/ens_ann-sc_49_2.html#32) C. M. RINGEL, The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences, *Math. Z.* **208** (1991), 209–223.
- [31] H. SUMIHIRO, Equivariant completion, *J. Math. Kyoto Univ.* **14** (1974), 1–28.
- [32] J. P. W, Sheaf cohomology on G/B and tensor products of Weyl modules, *J. Algebra* **77** (1982), 162–185.

(Manuscrit reçu le 8 mars 2014 ; accepté, après révision, le 27 janvier 2015.)

Daniel J LMNO, Université de Caen Basse-Normandie **CNRS** BP 5186 14032 Caen, France E-mail: daniel.juteau@unicaen.fr URL: http://www.math.unicaen.fr/~juteau

Carl MAUTNER Max Planck Institut für Mathematik Vivatsgasse 7 53111 Bonn, Germany E-mail: cmautner@mpim-bonn.mpg.de URL: http://people.mpim-bonn.mpg.de/cmautner

Geordie W Max Planck Institut für Mathematik Vivatsgasse 7 53111 Bonn, Germany E-mail: geordie@mpim-bonn.mpg.de URL: http://people.mpim-bonn.mpg.de/geordie