ISSN 0012-9593

quatrième série - tome 49 fascicule 2 mars-avril 2016

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Nhan NGUYEN & Guillaume VALETTE

Lipschitz stratifications in o-minimal structures

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition / *Publication*

Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : smf@smf.univ-mrs.fr

Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Marc Peigné
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 49, 2016, p. 399 à 421

LIPSCHITZ STRATIFICATIONS IN O-MINIMAL STRUCTURES

BY Nhan NGUYEN AND Guillaume VALETTE

ABSTRACT. – This paper establishes existence of Lipschitz stratifications in the sense of Mostowski for sets which are definable in a polynomially bounded o-minimal structure. We also improve L. van den Dries and P. Speissegger's preparation theorem for definable functions.

RÉSUMÉ. – Cet article établit l'existence des stratifications lipschitziennes au sens de Mostowski pour les ensembles définissables dans une structure o-minimale polynomialement bornée. On améliore aussi le théorème de préparation de L. van den Dries et P. Speissegger.

Introduction

Stratifications naturally appear in many contexts of modern geometry. They are needed to perform differential geometry on singular sets, to prove stability theorems, or to establish finiteness properties. Recall that a stratification of a set $X \subset \mathbb{R}^n$ is a locally finite partition of X into smooth submanifolds of $\mathbb{R}^n$, called strata. We often generally require some extra conditions on the strata in order to describe the way these sets glue together. The most famous regularity conditions for stratifications are the Whitney's conditions (a) and (b). One can prove that many sets occurring in algebraic or analytic geometry, such as semi-algebraic or subanalytic sets, do admit Whitney stratifications [16, 2, 7]. Whitney's (b) condition turned out to have many properties. It was used by R. Thom and then J. Mather to establish the now famous isotopy lemmas.

The first Thom-Mather isotopy lemma ensures that if X has a Whitney (b) regular stratification and if $f : X \to \mathbb{R}^p$ is a proper continuous map which induces a submersion on every stratum then f is a topologically trivial fibration. This is a generalization of Ehresmann's theorem to singular sets.

The topological equivalence considered in Thom-Mather isotopy lemma is often too weak to investigate the geometry of singular sets. It was also observed that C^1 equivalence is too strong to investigate the stability of singularities since it admits continuous moduli even in

This work was supported by the NCN grant 2011/01/B/ST1/03875.

0012-9593/02/

the algebraic category. People therefore set about investigating an intermediate equivalence relation: the bi-Lipschitz equivalence. Metric stability naturally appeared as an intermediate notion between C^1 and topological equivalence. Bi-Lipschitz equivalence provides a much more accurate information than its topological counterpart. For instance, bi-Lipschitz maps preserve the Hausdorff dimension and negligible sets.

In order to study the singularities from the metric point of view, T. Mostowski introduced *the Lipschitz stratifications* [19]. These stratifications satisfy a bi-Lipschitz version of first Thom-Mather isotopy lemma (Theorem 2.8). T. Mostowski also established that every complex analytic set can be stratified in this way. Existence of Lipschitz stratifications was then extended to the (real) semi-analytic and subanalytic sets by A. Parusiński ([20] [22]). We show in this paper that every set which is definable in a polynomially bounded o-minimal structure admits a Lipschitz stratification (Theorem 2.6). This generalizes Parusiński's theorem to a much wider class of sets enclosing, for instance, all the sets which are definable in the quasi-analytic Denjoy-Carleman classes [25]. O-minimal structures have recently been proved to have many applications to analysis. Their study from the metric point of view is hence definitely of interest and valuable for applications.

Existence of Lipschitz stratifications for globally subanalytic sets was used for instance in [3] in order to establish that the set of parameters at which the fibers of a globally subanalytic family have finite volume is globally subanalytic. The argument used in the latter article indeed also applies to any o-minimal structure that admits Lipschitz stratifications. Theorem 2.6 therefore makes it possible to extend this result to the o-minimal framework (polynomially bounded). For families of surfaces, this result was obtained by T. Kaiser [8] without using Lipschitz stratifications.

Bi-Lipschitz triviality of families that are definable in a polynomially bounded o-minimal structure was proved by the second author without using integration of vector fields [26].

The polynomially bounded o-minimal structures are those which satisfy the so-called Łojasiewicz inequality. These categories of sets can thus be considered as generalizations of the semi-algebraic and subanalytic sets.

If it is well known that sets which are definable in an o-minimal structure (polynomially bounded or not) admit Whitney regular stratifications [14, 15], it was however unclear whether they admit Lipschitz stratifications. If the structure is not polynomially bounded then it is possible to show that there is a definable set for which the bi-Lipschitz version of Thom-Mather isotopy lemma (Theorem 2.8) does not hold (for any stratification of this set, see Example 2.9). Consequently, Lipschitz stratifications do not always exist for definable sets if the o-minimal structure is not required to be polynomially bounded. This is the reason why this work definitely settles the issue of the existence of Lipschitz stratifications for sets which are definable in an o-minimal structure expanding the real field.

The main ingredient of A. Parusiński's proof of existence of Lipschitz stratifications for subanalytic sets [22] is the Preparation Theorem (see also [13]). This theorem offers a nice description of subanalytic functions (up to a partition) in terms of convergent series. This statement unfortunately no longer holds true in the o-minimal framework. It is even actually unclear whether the quasi-analytic o-minimal structures [25] satisfy a preparation theorem as in [13, 22]. In [7], the authors have proved an o-minimal version of the preparation theorem but this statement does not provide estimates on the partial derivatives of the unit (see

Definition 3.2, see also Remark (1) after Theorem 2.1 of [7]). One of the major difficulties of the proof was therefore to achieve an adequate version of the preparation theorem. In this article, we improve L. van den Dries and P. Speissegger's preparation theorem by showing that the unit can be expressed as a composite of a map with bounded derivative together with a map of the same form as in the subanalytic setting (see [13, 22]). This result which is of its own interest is one the key ingredients of the proof of existence of Lipschitz stratifications for definable sets.

Acknowledgment. – This work was carried out in Cracow, while the first author was a guest of the Polish Academy of Science. We thank this institution for its support and hospitality. It is also our pleasure to thank David Trotman who asked the question which led us to write this article.

1. Lipschitz stratifications in o-minimal structures

We start by recalling the notion of o-minimal structure. For a more detailed introduction on the subject, we refer the reader to [6, 4].

1.1. O-minimal structures

A *structure* on an ordered field $(\mathbb{R}, +, .)$ is a family $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ such that for each n the following properties hold

(1) $\mathcal{D}_n$ is a Boolean algebra of subsets of $\mathbb{R}^n$.
(2) If $A \in \mathcal{D}_n$ then $\mathbb{R} \times A$ and $A \times \mathbb{R}$ belong to $\mathcal{D}_{n+1}$.
(3) $\mathcal{D}_n$ contains $\{x \in \mathbb{R}^n : P(x) = 0\}$, where $P \in \mathbb{R}[X_1, \dots, X_n]$.
(4) If $A \in \mathcal{D}_n$ then $\pi(A)$ belongs to $\mathcal{D}_{n-1}$, where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the standard projection onto the first $(n-1)$ coordinates.

Such a family $\mathcal{D}$ is said to be *o-minimal* if in addition:

(5) Any set $A \in \mathcal{D}_1$ is a finite union of intervals and points.

A set belonging to the structure $\mathcal{D}$ is called a *$\mathcal{D}$-set* (or a *definable set*) and a map whose graph is in the structure $\mathcal{D}$ is called a *$\mathcal{D}$-map* (or a *definable map*).

A structure $\mathcal{D}$ is said to be *polynomially bounded* if for each $\mathcal{D}$-function $f : \mathbb{R} \to \mathbb{R}$, there exists a positive number a and an $n \in \mathbb{N}$ such that $|f(x)| < x^n$ for all $x > a$.

Examples of polynomially bounded o-minimal structures are the semi-algebraic sets, the globally subanalytic sets [5, 13] but also the so called x^λ-sets [18, 13] as well as the structures defined by the Denjoy-Carleman classes of functions [25].

Let $p \in \mathbb{N}$. We say that a subset C of $\mathbb{R}^n$ is a C^p *$\mathcal{D}$-cell* if

$n = 1 :$ C is either a point or an open interval.
$n > 1 :$ C is of one of the following forms

$$\begin{aligned} \Gamma_\xi &:= \{(x,y) \in B \times \mathbb{R} : y = \xi(x)\}, \\ (\xi_1, \xi_2) &:= \{(x,y) \in B \times \mathbb{R} : \xi_1(x) < y < \xi_2(x)\}, \\ (-\infty, \xi) &:= \{(x,y) \in B \times \mathbb{R} : y < \xi(x)\} \\ (\xi, +\infty) &:= \{(x,y) \in B \times \mathbb{R} : \xi(x) < y\}, \end{aligned}$$

where B is a C^p cell of $\mathbb{R}^{n-1}$, ξ, ξ_1, ξ_2 are $\mathcal{D}$-functions of class C^p on B and $\xi_1(x) < \xi_2(x), \forall x \in B$. The cell B is called the *basis* of C.

It is obvious that a C^p $\mathcal{D}$-cell in $\mathbb{R}^n$ is a connected submanifold of $\mathbb{R}^n$.

A C^p *cylindrical $\mathcal{D}$-cell decomposition* (C^p cdcd for short) of $\mathbb{R}^n$ is defined by induction as follows

(i) A C^p cdcd of $\mathbb{R}$ is a finite collection of points and intervals
$$(a_0, a_1), \dots, (a_k, a_{k+1}), \{a_1\}, \dots, \{a_k\},$$
where $-\infty = a_0 < a_1 < a_2 < \dots < a_k < a_{k+1} = \infty$.

(ii) A C^p cdcd of $\mathbb{R}^n$ is a partition $\mathbb{C}$ of $\mathbb{R}^n$ into C^p cells such that the collection of all images of these cells under the natural projection onto the first $(n-1)$ coordinates $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ forms a C^p cdcd of $\mathbb{R}^{n-1}$ (that will be denoted $\pi(\mathbb{C})$).

We say that a C^p cdcd of $\mathbb{R}^n$ is *compatible* with $\mathcal{X} = \{X_1, \dots, X_k\}$, a family of $\mathcal{D}$-subsets of $\mathbb{R}^n$, if each X_i is the union of some C^p $\mathcal{D}$-cells of the decomposition.

THEOREM 1.1 (Cell decomposition). – *Let $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ be an o-minimal structure and fix $p \in \mathbb{N}$.*

(1) *Given any family $\mathcal{X} = \{X_1, \dots X_k\}$ of $\mathcal{D}$-subsets of $\mathbb{R}^n$, there exists a C^p cdcd of $\mathbb{R}^n$ compatible with $\mathcal{X}$.*

(2) *Let $f : X \to \mathbb{R}$ be a $\mathcal{D}$-function. There exists a C^p cdcd of $\mathbb{R}^n$ compatible with X such that the restriction of f to each cell of the cdcd is of class C^p.*

In the sequel, $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ *will stand for a fixed polynomially bounded o-minimal structure on* $\mathbb{R}$ and Λ for the set of all $r \in \mathbb{R}$ such that the function $x \mapsto x^r$ is a $\mathcal{D}$-function. Note that such a Λ is a subfield of $\mathbb{R}$. We will generally not mention C^p, but all the considered cdcd will be C^p, with $p \geq 2$.

2. Lipschitz stratifications

We recall the definition of the Lipschitz stratifications in the sense of Mostowski [19] and state the Main Theorem of the article.

Let X be a $\mathcal{D}$-subset of $\mathbb{R}^n$. A *$\mathcal{D}$-stratification* of X is a partition of X into finitely many connected C^2 $\mathcal{D}$-manifolds[(1)], called *strata*. Let $X_1, \dots, X_m$ be a family of $\mathcal{D}$-subsets of X. We say that a $\mathcal{D}$-stratification of X is *compatible* with $X_1, \dots, X_m$ if each X_i is a union of some strata of this stratification.

Given a $\mathcal{D}$-stratification Σ of X, denote by X^i the *i-th skeleton* of the stratification Σ, which is union of all the strata of dimension less than or equal to i. We thus get a sequence $\mathcal{X} = \{X^i\}_{i=l}^d$ satisfying

(2.1)
$$X = X^d \supset X^{d-1} \supset \cdots \supset X^l$$

and such that each difference $\mathring{X}^i = X^i \setminus X^{i-1}$ is an i-dimensional $\mathcal{D}$-submanifold of $\mathbb{R}^n$ or empty. The strata coincide with the connected components of $\mathring{X}^i$. We will sometimes abusively regard the sequence $\mathcal{X}$ as a stratification of X.

(1) A $\mathcal{D}$-manifold in $\mathbb{R}^n$ is a submanifold which is also a $\mathcal{D}$-subset of $\mathbb{R}^n$.

In the above definition, we said $\mathcal{D}$-stratification in order to emphasize that strata are definable. Nevertheless, as we will only deal with definable strata, we will generally shorten it into "stratification".

DEFINITION 2.1. – Let $c > 1$ be a fixed constant. A *c-chain* of $q \in \mathring{X}^j$ is a strictly decreasing sequence of indices

$$j = j_1 > j_2 > \cdots > j_r = l$$

and a corresponding sequence of points $q_{j_s} \in \mathring{X}^{j_s}$ such that $q_{j_1} = q$ and each j_s is the greatest integer for which

$$d(q, X^k) \geq 2c^2 d(q, X^{j_s}) \ \ \forall k < j_s \text{ and } |q - q_{j_s}| \leq c d(q, X^{j_s}), \tag{2.2}$$

where $d(.,.)$ denotes the usual distance function.

Roughly speaking, a c-chain is characterized by a sequence $(j_s)_{s \leq r}$ such that the distances from q to the subsequent skeletons X^{j_s} increase rapidly. The successive q_{j_s} are then points close to the points realizing these distances.

For each $q \in \mathring{X}^j$, let $P_q : \mathbb{R}^n \to T_q\mathring{X}^j$ and $P_q^\perp = Id - P_q : \mathbb{R}^n \to T_q^\perp \mathring{X}^j$ respectively denote the orthogonal projections from $\mathbb{R}^n$ onto the tangent and normal spaces to $\mathring{X}^j$.

DEFINITION 2.2 (Mostowski, [19]). – A stratification $\mathcal{X} = \{X^i\}_{i=l}^d$ is said to be a *Lipschitz stratification* if for every $c > 1$ there is some $C > 0$ such that for every c-chain $\{q = q_{j_1}, \dots, q_{j_r}\}$, we have for every $1 \leq k \leq r$,

$$|P_{q_{j_1}}^\perp P_{q_{j_2}} \dots P_{q_{j_k}}| \leq C \frac{|q - q_{j_2}|}{d(q, X^{j_k - 1})};$$

and, for $q' \in \mathring{X}^{j_1}$ and $|q - q'| \leq \frac{1}{2c} d(q, X^{j_1 - 1})$ then

$$|(P_q - P_{q'}) P_{q_{j_2}} \dots P_{q_{j_k}}| \leq C \frac{|q - q'|}{d(q, X^{j_k - 1})};$$

in particular,

$$|P_q - P_{q'}| \leq C \frac{|q - q'|}{d(q, X^{j_1 - 1})},$$

(set $d(q, X^{l-1}) = 1, \forall q \in X$).

REMARK 2.3. – A Lipschitz stratification of a locally closed set X always satisfies the frontier condition, in the sense that the closure in X of one stratum is the union of some strata of the stratification. Indeed, the first inequality of the above definition entails Verdier (w)-condition, which, in the framework of o-minimal structures, implies Whitney's (b) condition [15]. It is well known that Whitney's (b) condition implies the frontier condition [17].

Let Σ be a stratification of X. A vector field v defined on a subset of X is said to be Σ-*compatible* if $v(x) \in T_x S$ for all $S \in \Sigma$ and for all $x \in S$.

The following proposition gives a geometric interpretation of Definition 2.2.

PROPOSITION 2.4 ([19, 20, 22]). – *The following condition is equivalent to the definition of Lipschitz stratifications:*

($\star$) *There exists $C > 0$ such that for every $W \subset X$ such that $X^{j-1} \subseteq W \subset X^j$ for some $j = l, \dots, d$, each Lipschitz Σ-compatible vector field on W with Lipschitz constant L and bounded on $W \cap X^l$ by K can be extended to a Lipschitz Σ-compatible vector field on X^j with Lipschitz constant $C(L + K)$.*

REMARK 2.5. – (i) It is indeed enough to check the above property ($\star$) for definable Lipschitz Σ-compatible vector fields (see [20, 21]).

(ii) It is well known that every Lipschitz function defined on a subset of $\mathbb{R}^n$ can be extended to a Lipschitz function on $\mathbb{R}^n$ (with the same Lipschitz constant, see [1] (7.5) p. 121). It is therefore easily derived from the above proposition that if $X \in \mathcal{D}_n$ is n-dimensional then any Lipschitz stratification of X_{sing} (the set of points at which X fails to be a C^2 manifold of dimension n) gives rise to a stratification of X (whose maximal dimension strata are the connected components of $X \setminus X_{sing}$).

In [22] (Theorem 1.4), Parusiński proved that every compact subanalytic subset of $\mathbb{R}^n$ admits a Lipschitz stratification. In this paper we show that this result still holds for $\mathcal{D}$-sets in $\mathbb{R}^n$ (we recall that $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ is a fixed polynomially bounded o-minimal structure on $\mathbb{R}$).

THEOREM 2.6 (Main Theorem). – *Let X be a compact $\mathcal{D}$-subset of $\mathbb{R}^n$. There exists a Lipschitz $\mathcal{D}$-stratification of X.*

REMARK 2.7. – In the definition of stratifications, we required the strata to be C^2 manifolds. The stratifications that we will construct in this article could indeed be required to have C^p regular strata, for any given $p \in \mathbb{N}$. In the framework of o-minimal structures, it is however not possible to demand the strata to be C^∞ [12].

The main feature of Lipschitz stratifications is that, by integrating Lipschitz vector fields on the strata of a Lipschitz stratification, one can prove a Lipschitz version of Thom-Mather Isotopy Lemma.

THEOREM 2.8 ([19] Lipschitz Isotopy Lemma). – *Let Σ be a Lipschitz stratification of a $\mathcal{D}$-set $X \subset \mathbb{R}^n$.*

(i) *Let Y be a C^2 $\mathcal{D}$-submanifold and let $f : \mathbb{R}^n \to Y$ be a C^2 mapping. Assume that $f_{|S}$ is submersive for each stratum S of Σ and $f|_X$ is proper. Then $f_{|X}$ is locally bi-Lipschitz trivial over Y, i.e., for each $y \in Y$ there are a neighborhood U_y of y in Y and a bi-Lipschitz homeomorphism*

$$f^{-1}(U_y) \cap X \xrightarrow{h} U_y \times (f^{-1}(y) \cap X),$$

such that $\pi(h(x)) = f(x)$, for all $x \in f^{-1}(U_y)$, where $\pi : U_y \times (f^{-1}(y) \cap X) \to U_y$ is the projection onto the first factor.

(ii) *The set X is locally bi-Lipschitz trivial along each stratum, that is for each stratum S of Σ, for each $p \in S$ there are a neighborhood U_p of p in $\mathbb{R}^n$, a $\mathcal{D}$-submanifold N_p of U_p transverse to S at p and of dimension $(n - \dim S)$, and a bi-Lipschitz homeomorphism*

$$U_p \cap X \xrightarrow{h} (U_p \cap S) \times (N_p \cap X).$$

Theorem 2.6 is not true for $\mathcal{D}$-sets definable in arbitrary o-minimal structures (not necessarily polynomially bounded). Below is a counterexample given by Parusiński.

EXAMPLE 2.9. – Let $X(t) = \{(x, y) \in \mathbb{R}^2 : |y| = x^t, x \geq 0\}$. The Lipschitz types of $X(t)$ at the origin, $t > 1$, are all pairwise distinct.

Proof. – Let $1 < t_2 < t_1$. We are going to show that the two set-germs $X(t_1)$ and $X(t_2)$ are not bi-Lipschitz equivalent at $(0, 0)$. Assume otherwise, i.e., assume that there exists a germ of bi-Lipschitz homeomorphism $h : (X(t_1), 0) \to (X(t_2), 0)$. In particular, there is a constant C such that $\frac{|y|}{C} \leq |h(y)| \leq C|y|$, for all $y \in X(t_1)$ near the origin. Let $z_1 = (x_1, x_1^{t_1}) \in X(t_1)$, $x_1 > 0$, be close to the origin. We have:

$$|z_1| = \sqrt{x_1^2 + x_1^{2t_1}} \leq \sqrt{2}|x_1|. \tag{2.3}$$

Set $z_2 := h(z_1) := (x_2, y_2)$. Since $\frac{|z_1|}{C} \leq |z_2| \leq C|z_1|$, we easily see (with a similar computation like in (2.3)) that:

$$\frac{|x_1|}{\sqrt{2}C} \leq |x_2| \leq C\sqrt{2}|x_1|. \tag{2.4}$$

Let $z_1' = (x_1, -x_1^{t_1}) \in X(t_1)$ as well as $z_2' = h(z_1')$. Observe that

$$|z_1 - z_1'| = 2x_1^{t_1},$$

and, since z_2 and z_1 lie on different branches, there is $\varepsilon > 0$ such that (using (2.4))

$$|z_2 - z_2'| \geq \varepsilon x_1^{t_2}.$$

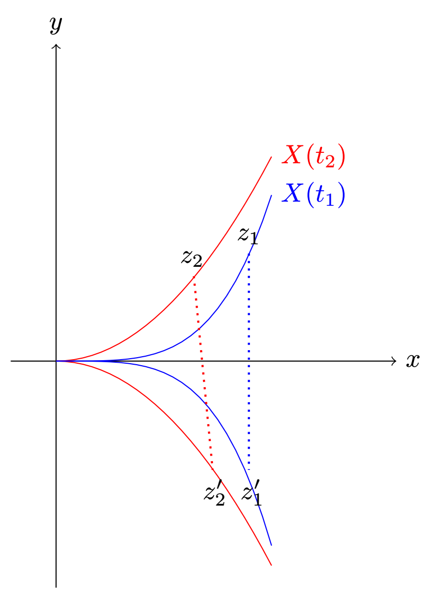

FIGURE 1.

Therefore,

$$\frac{|z_1 - z_1'|}{|h(z_1) - h(z_1')|} = \frac{|z_1 - z_1'|}{|z_2 - z_2'|} \leq \frac{2}{\varepsilon} x_1^{(t_1 - t_2)}.$$

Because of $t_1 - t_2 > 0$, the right hand side of the latter inequality tends to 0 as x_1 tends to 0, contradicting that h is bi-Lipschitz. □

Thanks to Miller's dichotomy [18] (which ensures that every non polynomially bounded o-minimal structure expanding the real field must contain the graph of the function $x \mapsto e^x$), the set $X := \{(q,t) : q \in X(t)\}$ is definable in every non polynomially bounded o-minimal structure on $\mathbb{R}$. Theorem 2.8 is still valid even if the o-minimal structure is not polynomially bounded (even definability is not really needed in fact). As the computations of the above example contradict the conclusion of Theorem 2.8, this means in particular that X does not admit a Lipschitz stratification. In other words, Theorem 2.6 fails on every non polynomially bounded o-minimal structure expanding $\mathbb{R}$.

3. The Preparation Theorem

The preparation theorem originates in [22] (see also [13]) where it was established for the subanalytic functions. It is so-called because it bears some resemblance with the Weierstrass Preparation Theorem. A few years later, van den Dries and Speissegger achieved an o-minimal version of this theorem which can be stated as follows.

We recall that $\mathcal{D} = (\mathcal{D}_n)_{n \in \mathbb{N}}$ is a fixed polynomially bounded o-minimal structure on $\mathbb{R}$ and Λ is the set of $r \in \mathbb{R}$ such that $x \mapsto x^r$ is a $\mathcal{D}$-function.

THEOREM 3.1 ([7], Theorem 2.1). – *Let $X \in \mathcal{D}_n$ and let $f_1, \ldots, f_k : X \to \mathbb{R}$ be $\mathcal{D}$-functions. There is a cdcd Σ of $\mathbb{R}^n$ compatible with X and such that for each cell $C \in \Sigma$ there are $r_1, \ldots, r_k$ in Λ and $\mathcal{D}$-functions $\theta, a_1, \ldots, a_k : B \to \mathbb{R}$, $u_1, \ldots, u_k : C \to \mathbb{R}$, where B is the basis of C, such that for each $(x,y) = (x_1, \ldots, x_{n-1}, y) \in C$, we have for all $i \leq k$:*

$$f_i(x,y) = a_i(x)|y - \theta(x)|^{r_i} u_i(x,y), \quad |u_i(x,y) - 1| < \frac{1}{2}. \tag{3.5}$$

This theorem will be used in the proof of Theorem 3.5. We shall also improve this result by giving a more precise description of the functions u_i (called $\mathcal{D}$-units, see Definition 3.2). Parusiński's (or Lion and Rolin's) Preparation Theorem actually provides an explicit description of the unit as a composite of a suitable mapping with an analytic function. We shall establish a result of the same nature on polynomially bounded o-minimal structures in Theorem 3.5 where the analyticity is replaced with a boundedness assumption on the derivative (and subanalyticity with definability, see Definitions 3.2 and 3.3 below). This improvement will be needed in the proof of the Main Theorem.

DEFINITION 3.2. – Let $C \subset \mathbb{R}^n$ be a cell of basis B and let $\theta : B \to \mathbb{R}$ be a $\mathcal{D}$-function. A *$\mathcal{D}$-unit in the variable $(y - \theta)$* on C is a $\mathcal{D}$-function $u : C \to \mathbb{R}$ bounded away from zero and infinity that can be written $u = \psi \circ V$, where for $(x,y) \in C \subset \mathbb{R}^{n-1} \times \mathbb{R}$

$$V(x,y) = (a_1(x), \ldots, a_s(x), b_1(x)|y - \theta(x)|^{\frac{1}{p_1}}, \ldots, b_k(x)|y - \theta(x)|^{\frac{1}{p_k}}),$$

for some $s, k \in \mathbb{N}$, $p_i \in \Lambda$, $1 \le i \le k$, with $a_1, \dots, a_s, b_1, \dots, b_k$ $\mathcal{D}$-functions on B, and where ψ is a C^2 $\mathcal{D}$-function on $V(C)$, with $D\psi$ bounded(2) and $V(C)$ relatively compact.

DEFINITION 3.3. – Let $C \subset \mathbb{R}^n$ be a cell of basis B and let $\theta : B \to \mathbb{R}$ be a $\mathcal{D}$-function. A $\mathcal{D}$-function $\xi : C \to \mathbb{R}$ is said to be *reduced with $\mathcal{D}$-translation θ* if

$$\xi(x,y) = a(x)|y - \theta(x)|^r u(x,y), \qquad (x,y) \in C \subset \mathbb{R}^{n-1} \times \mathbb{R},$$

where $r \in \Lambda$, $u(x,y)$ is a $\mathcal{D}$-unit in the variable $(y - \theta)$ on C and $a : B \to \mathbb{R}$ is a $\mathcal{D}$-function.

Let X be a $\mathcal{D}$-subset of $\mathbb{R}^n$. A $\mathcal{D}$-function $f : X \to \mathbb{R}$ is said to be *reducible* if there is a cdcd of $\mathbb{R}^n$ compatible with X such that the restriction of f to each cell in X of the cdcd is reduced.

REMARK 3.4. – Let $\xi : C \to \mathbb{R}$ be a reduced function with $\mathcal{D}$-translation θ. It easily follows from the above definitions that if $(y - \theta)$ is reduced on C with $\mathcal{D}$-translation θ', then so is ξ.

This section is devoted to the proof of the following theorem.

THEOREM 3.5. – *Every $\mathcal{D}$-function $f : X \to \mathbb{R}$ is reducible.*

The proof of Theorem 3.5 will be given in the end of the section, after some preliminary lemmas. The first three lemmas below list facts that were already used by J.-M. Lion and J.-P. Rolin in their proof of the preparation theorem for globally subanalytic functions [13] (see Section 1.1 of the latter article) and that we establish in our framework.

Given $x \in \mathbb{R}$, let $sign(x) := 1$ if x is positive, $sign(0) := 0$, and $sign(x) := -1$ whenever x is negative. We say that a function $\xi : A \to \mathbb{R}$ *has constant sign* on $B \subset A$ if the function $sign(\xi(x))$ is constant on B (thus, nonnegative functions will not necessarily have constant sign, they can vanish).

LEMMA 3.6. – *Let $\xi_1, \dots, \xi_k : C \to \mathbb{R}$ be reduced functions with $\mathcal{D}$-translations $\theta_1, \dots, \theta_k$ respectively, $C \in \mathcal{D}_n$. There exists a cdcd of $\mathbb{R}^n$ compatible with C such that on each cell $D \subset C$, $\xi_1, \dots, \xi_k$ are reduced with the same $\mathcal{D}$-translation.*

Proof. – As a consequence of Theorem 1.1, there is a cdcd of $\mathbb{R}^n$ compatible with C such that on each cell $D \subset C$, the functions $|y - \theta_i|, |\theta_i - \theta_j|$, $i < j \le k$, are comparable with each other (for relation $\le$) and the functions $(y - \theta_i)$, $(\theta_i - \theta_j)$, $i < j \le k$, are of constant signs. Fix a cell D and choose j such that for all i:

$$|y - \theta_j| \le |y - \theta_i|. \tag{3.6}$$

We are going to show that for all i, the function $(y - \theta_i)$ is reduced on D with $\mathcal{D}$-translation θ_j. The statement of the lemma will then follow from Remark 3.4. Fix $i \le k$. The proof now breaks down into two cases.

(2) Here and in the sequel we say that a mapping $F : A \to \mathbb{R}^p$ is bounded to express that $\sup_{x \in A} |F(x)| < \infty$.

Case 1. – Assume $|y-\theta_j| \leq |\theta_j - \theta_i|$ on D.

If there is x in the basis of D such that $\theta_j(x) = \theta_i(x)$ then $y \equiv \theta_j \equiv \theta_i$ on D, so the result is trivial. Otherwise, either $(y-\theta_j)$ and $(\theta_j - \theta_i)$ have the same sign or $\left|\frac{y-\theta_j}{\theta_j-\theta_i}\right| \leq \frac{1}{2}$ (by (3.6)). This means that $(1+\frac{y-\theta_j}{\theta_j-\theta_i})$ is a $\mathcal{D}$-unit in the variable $(y-\theta_j)$ and hence, the function $(y-\theta_i)$ can be reduced by

$$y-\theta_i = (\theta_j - \theta_i)(1+\frac{y-\theta_j}{\theta_j-\theta_i}).$$

Case 2. – Assume $|y-\theta_j| \geq |\theta_j - \theta_i|$ on D.

If there is $(x,y) \in D$ such that $y = \theta_j(x)$ then $y \equiv \theta_j$ on D and it is clear that all the respective $\mathcal{D}$-translations of the reductions of the θ_i can be chosen identically zero. Otherwise, by (3.6), $(y-\theta_j)$ and $(\theta_j - \theta_i)$ are of the same sign, and hence the function $(1+\frac{\theta_j-\theta_i}{y-\theta_j})$ is a $\mathcal{D}$-unit. As a matter of fact, the function $(y-\theta_i)$ is reduced by writing

$$y-\theta_i = (y-\theta_j)(1+\frac{\theta_j-\theta_i}{y-\theta_j}). \qquad \square$$

REMARK 3.7. – A direct consequence of Lemma 3.6 is that the product of two reducible functions $\xi_1 : X \to \mathbb{R}$ and $\xi_2 : X \to \mathbb{R}$ is reducible. So is the quotient if it is well defined.

Given two functions $f, g : B \to \mathbb{R}$. We write $f \sim g$ if there exist positive numbers C_1, C_2 such that $C_1 g \leq f \leq C_2 g$.

LEMMA 3.8. – *Let X be a $\mathcal{D}$-subset of $\mathbb{R}^n$ and let $\xi : X \to \mathbb{R}$ be a reducible function. There exists a cdcd of $\mathbb{R}^n$ compatible with X such that on every cell $E \subset X$, ξ is reduced with $\mathcal{D}$-translation θ_E satisfying on E either $y \sim \theta_E(x)$ or $\theta_E(x) \equiv 0$.*

Proof. – By assumption, there is a cell decomposition compatible with X such that ξ is reduced on every cell included in X. Fix a cell $C \subset X$ of this cdcd and denote by θ the corresponding $\mathcal{D}$-translation. By Remark 3.4, it is enough to show, up to a refinement, that either $y \sim \theta$ or $(y-\theta)$ is a reduced function with $\mathcal{D}$-translation 0.

There is a refinement of our cell decomposition such that on each cell $E \subset C$ either $|y| \sim |\theta|$ or $|y| > 2|\theta|$ or $2|y| < |\theta|$. We can also assume that y and θ are of constant signs on each cell $E \subset C$ of this refinement.

On a cell such that $|y| > 2|\theta|$, write

$$y - \theta(x) = y\big(1 - \frac{\theta(x)}{y}\big). \tag{3.7}$$

Since $(1-\frac{\theta(x)}{y})$ is a $\mathcal{D}$-unit (in the variable y), this yields that $(y-\theta)$ is a reduced function with $\mathcal{D}$-translation 0.

Similarly, if on a cell we have $2|y| < |\theta|$, write then

$$y - \theta(x) = -\theta(x)\big(1 - \frac{y}{\theta(x)}\big).$$

Since $(1-\frac{y}{\theta(x)})$ is a $\mathcal{D}$-unit, this establishes that $(y-\theta)$ is a reduced function with $\mathcal{D}$-translation 0.

Finally, assume that $|y| \sim |\theta|$. If the two functions have the same sign, we are done. Otherwise, by (3.7), we again can identify the translation function with 0. $\square$

LEMMA 3.9. – *Let $C \in \mathbb{R}^n$ be a cell of basis B and let $H : C \to \mathbb{R}^n$ be a mapping of type*

$$H(x,y) = (x, \alpha(x)|y-\beta(x)|^{\frac{1}{p}}), \tag{3.8}$$

where $p \in \mathbb{Z}$ and $\alpha(x)$ and $\beta(x)$ are $\mathcal{D}$-functions on B. If $v : H(C) \to \mathbb{R}$ is a reducible function, then the function $u := v \circ H$ is reducible.

Proof. – It is clear from the definitions that a translation $(x,y) \mapsto (x, y+\beta(x))$ preserves reducible functions. We thus can suppose that $\beta \equiv 0$. Up to a cell decomposition, we can assume that y and α are of constant (nonzero) sign on C (we will assume that they are positive for simplicity). Note that $H(C)$ is a cell of basis B and the inverse image under the mapping H of a cell included in $H(C)$ is also a cell. We also assume that v is a reduced function in the variable $(y-\theta(x))$ on $H(C)$, where $\theta : B \to \mathbb{R}$ is a $\mathcal{D}$-function on the basis of C.

Since the inverse image under H of a cell is a cell and because H preserves the $(n-1)$ first coordinates, it directly follows from the definition of reduced functions that, in order to show that $u(x,y) = v \circ H(x,y)$ is reducible, it suffices to show that the function $y_1 := |\alpha(x)y^{\frac{1}{p}} - \theta(x)| = |y-\theta| \circ H$ is a reducible function.

There is a refinement of our cdcd such that θ has constant sign on every cell E that is contained in C. Fix such a cell E. If $\theta \equiv 0$, then the function y_1 is obviously reduced on this cell. Otherwise, thanks to Lemma 3.8, we can suppose that $y \sim \theta(x)$ on $H(E)$. It means that, on E,

$$\alpha(x) \cdot y^{\frac{1}{p}} \sim \theta(x). \tag{3.9}$$

Consequently, as α was assumed to vanish nowhere, if we set $\theta'(x) := \frac{\theta(x)}{\alpha(x)}$, we have $y \sim \theta'(x)^p$ on E, and we get $y_1 = \alpha(x)|y^{\frac{1}{p}} - \theta'(x)|$.

Write then,

$$y_1 = \alpha(x)|y^{\frac{1}{p}} - \theta'(x)| = \left| \frac{\alpha(x)(y-\theta'(x)^p)}{y^{\frac{p-1}{p}} + y^{\frac{p-2}{p}}\theta'(x) + \cdots + \theta'(x)^{p-1}} \right|. \tag{3.10}$$

Denote by $F(x,y)$ the denominator of the fraction (3.10). Since F is a finite sum of nonnegative functions which are all $\sim$ to $\theta'(x)^{p-1}$, we have $F(x,y) \sim \theta'(x)^{p-1}$. Hence, the function $W(x,y) := \theta'(x)^{1-p}F(x,y)$ is bounded away from zero and infinity. It is therefore a $\mathcal{D}$-unit. This is enough to conclude that y_1 is a reduced function. □

We shall also need the following theorem:

THEOREM 3.10 ([9], Proposition 1). – *Let $\xi : B \times [a,b] \to \mathbb{R}$ be a bounded $\mathcal{D}$-function, where B is a bounded $\mathcal{D}$-subset of $\mathbb{R}^{n-1}$. Assume that for every $x \in B$ the function $\xi_x : [a,b] \to \mathbb{R}$, $\xi_x(y) := \xi(x,y)$, is Lipschitz with Lipschitz constant L independent of x. Then there exist a bounded $\mathcal{D}$-subset S of $\mathbb{R}^{n-1}$ and a Lipschitz $\mathcal{D}$-bijection $\phi : S \to B$ such that $\xi \circ (\phi \times \mathrm{id}_{[a,b]}) : S \times [a,b] \to \mathbb{R}$, $(s,y) \mapsto \xi(\phi(s),y)$ is Lipschitz.*

This theorem has the following consequence:

LEMMA 3.11. – *Let $u(x,y)$ be a C^2 $\mathcal{D}$-function on a bounded cell C of $\mathbb{R}^n$ with $\varepsilon < |u| < \frac{1}{\varepsilon}$, ε positive real number. If $|\frac{\partial u}{\partial y}|$ is bounded then there is a cdcd of $\mathbb{R}^n$ compatible with C such that on each cell $E \subset C$, the restriction of the function u coincides with a $\mathcal{D}$-unit of E.*

Proof. – Write C as (ξ_1, ξ_2) where $\xi_1 < \xi_2$ are $\mathcal{D}$-functions on the basis B of C (if the cell C is a graph there is nothing to prove). Let $H(x, y) := (x, \frac{y-\xi_1}{\xi_2(x)-\xi_1(x)})$. By Lemma 3.9, it is enough to show that $v := u \circ H^{-1}$ is a $\mathcal{D}$-unit (as $\varepsilon < u < \frac{1}{\varepsilon}$, if u is reducible then it is a $\mathcal{D}$-unit on the cells of a cdcd).

Apply Theorem 3.10 to the function $v : B \times (0, 1) \to \mathbb{R}$ (observe that $|\frac{\partial v}{\partial y}|$ is bounded since so are $\frac{\partial u}{\partial y}$, ξ_1, and ξ_2). This provides a definable bijection $\phi : S \to B$ such that $\psi(x, y) := v(\phi(x), y)$ has bounded first derivative (up to a cell decomposition, we may assume ϕ to be C^2). But then $v = \psi \circ W$, where $W(x, y) := (\phi^{-1}(x), y)$. This shows that v is a $\mathcal{D}$-unit on $H(C)$. □

Proof of Theorem 3.5. – By Theorem 3.1, there exists a cdcd of $\mathbb{R}^n$ compatible with X such that for each cell $E \subset X$ there are $\mathcal{D}$-functions $a, \theta : D \to \mathbb{R}$, where D is the basis of E, such that on E

$$f(x, y) = a(x)|y - \theta(x)|^\lambda u(x, y), \qquad \text{for some } \lambda \in \Lambda,$$

where $u : E \to \mathbb{R}$ is a $\mathcal{D}$-function bounded away from zero and infinity.

Fix such a cell E. We are going to show that there is a cell decomposition compatible with E such that u induces a $\mathcal{D}$-unit on every cell $C \subset E$.

Applying now Theorem 3.1 to the partial derivative $\partial u/\partial y$, we see that there exists a cdcd compatible with E such that on each cell $C \subset E$, there are $\mathcal{D}$-functions α and $\tilde{\theta}$ on the basis B of C such that

$$\frac{\partial u}{\partial y}(x, y) \sim \alpha(x)|y - \tilde{\theta}(x)|^s, \text{ for some } s \in \Lambda. \tag{3.11}$$

Refining the cell decomposition, we can assume that on C y has constant sign (say positive) and that either $y \le 1$ or $y \ge 1$ (for all $(x, y) \in C$). If $y \ge 1$ then after a change $(x, y) \mapsto (x, \frac{1}{y})$, we see that we are reduced to $y \le 1$ (by Lemma 3.9, we can argue up to such a map). Up to a definable homeomorphism of $\mathbb{R}^{n-1}$ (that extends vertically to a definable homeomorphism of $\mathbb{R}^n$) we can assume that the basis of C is bounded. We will thus assume that $0 < y \le 1$ and that C is bounded without changing notations.

Write now C as (ξ_1, ξ_2), where $\xi_1, \xi_2 : B \to \mathbb{R}$ are $\mathcal{D}$-functions with $0 < \xi_1 < \xi_2$ (in the case where C is the graph of a $\mathcal{D}$-function on B, the result is trivial). Refining the cell decomposition if necessary, we can assume that $\frac{\partial u}{\partial y}$ is of constant (nonzero) sign on B. For the same reason, we can assume that y and $\tilde{\theta}$ are comparable with each other (for relation $\le$). We will assume for simplicity that $y \ge \tilde{\theta}(x)$ (which amounts to $\tilde{\theta} \le \xi_1$).

Let now $H(x, y) := (x, y - \tilde{\theta}(x))$. By Lemma 3.9, it is enough to check that $v = u \circ H^{-1}_{|H(C)}$ is a $\mathcal{D}$-unit. It means that, possibly changing u with v and C with $H(C)$, we can assume that $\tilde{\theta} \equiv 0$ (observe that $H(C)$ is still a bounded cell since y is bounded on C and, by Lemma 3.8, we can assume that either $y \sim \tilde{\theta}$ or $\tilde{\theta} \equiv 0$). We will thus assume $\tilde{\theta} \equiv 0$.

We are now ready to show that u is a $\mathcal{D}$-unit. The idea is that, in every of the cases we will distinguish below, we will find a mapping H of type (3.8) such that the partial derivative with respect to y of the function $u' := u \circ H^{-1}$ is bounded. That the function u' coincides with $\mathcal{D}$-units on the cells of a suitable cdcd will then follow from Lemma 3.11. Lemma 3.9 implies in turn that so does u. In every of the cases we will distinguish below, we will denote by C' the cell $(\xi_1', \xi_2') := H(C)$.

Refining one more time the cell decomposition, we may assume that on our cell one of the following situations occurs:

Case 1. – Assume $\xi_2 \geq 2\xi_1$ and $s < -1$.

In this case, we first check that $\xi_1(x) > 0$ for all $x \in B$. Indeed, if $\xi_1(x)$ vanished for some $x \in B$ ($\xi_1 \geq 0$ because $\xi_1 \geq \tilde{\theta} \equiv 0$) then for all $0 < y < y_0 < \xi_2(x)$ we would have:

$$|u(x, y_0) - u(x, y)| = \left| \int_y^{y_0} \frac{\partial u}{\partial y}(x, y) dy \right| \sim |\alpha(x)| \cdot |y_0^{s+1} - y^{s+1}|,$$

which means that $y \mapsto u(x, y)$ would be unbounded on $(0, \xi_2(x))$ (since $s < -1$), a contradiction.

We thus may define a homeomorphism on C by:

$$H(x, y) := \big(x, (\frac{\xi_1(x)}{y})^{\frac{1}{p}}\big),$$

for some $p > \frac{-1}{s+1}$, $p \in \mathbb{N}$. Then, $H^{-1}(x, y) = \big(x, \frac{\xi_1(x)}{y^p}\big)$ and for $(x, y) \in C' = (\xi_1', \xi_2') = \big((\frac{\xi_1}{\xi_2})^{\frac{1}{p}}, 1\big)$ we have

$$\left|\frac{\partial u'}{\partial y}(x, y)\right| \sim \left|\alpha(x)\xi_1^{s+1}(x) \cdot \frac{1}{y^{ps+p+1}}\right| \sim \beta(x) y^q, \tag{3.12}$$

where $\beta(x) := |\alpha(x)\xi_1^{s+1}(x)|$ and $q := -(ps + p + 1) > 0$.

Note that $0 < \xi_1' \leq (\frac{1}{2})^{\frac{1}{p}}$ and hence that for all $x \in B$:

$$\left| \int_{(\frac{1}{2})^{\frac{1}{p}}}^1 \frac{\partial u'}{\partial y} dy \right| \sim \int_{(\frac{1}{2})^{\frac{1}{p}}}^1 \beta(x) y^q dy \sim \beta(x).$$

Since u is bounded on C, the left-hand integral is bounded. Hence, $\beta(x)$ is bounded. By (3.12), this shows $\frac{\partial u'}{\partial y}(x, y)$ is bounded on C', as required.

Case 2. – Assume $\xi_2 \geq 2\xi_1$ and $s > -1$.

In this case we set:

$$H(x, y) := \big(x, (\frac{y}{\xi_2(x)})^{\frac{1}{p}}\big),$$

$p \in \mathbb{Z}$, $p > \frac{1}{s+1}$, so that, carrying out the same computation as in case 1, we get:

$$|\frac{\partial u'}{\partial y}(x, y)| \sim |-\alpha(x)\xi_2^{s+1}(x)| \cdot y^{ps+p-1} \sim \beta(x) y^q,$$

where $\beta(x) := |\alpha(x)\xi_2^{s+1}(x)|$ and $q := ps + p - 1 > 0$. Again (see case 1), we have $\xi_1' = (\frac{\xi_1}{\xi_2})^{\frac{1}{p}} \leq 2^{-\frac{1}{p}}$ and $\xi_2' \equiv 1$. The same computation of integration as in case 1 then shows that $\frac{\partial u'}{\partial y}(x, y)$ is bounded on C'.

Before dealing with the third case, we establish the following fact:

CLAIM. – if $s = -1$ then there is a real number t such that $\xi_2(x) < t\xi_1(x)$ for all $x \in B$.

Assume otherwise. By Curve Selection Lemma, there exists a $\mathcal{D}$-curve $\gamma : (0,1] \to B$ such that $\lim_{\nu\to 0} \frac{\xi_2(\gamma(\nu))}{\xi_1(\gamma(\nu))} = \infty$.

Since $s = -1$ we have $\frac{\partial u}{\partial y}(x,y) \sim \frac{\alpha(x)}{y}$, so that, integrating with respect to y over $[\xi_1(\gamma(\nu)), t\xi_1(\gamma(\nu))]$, we get that for $t \in (2, \frac{\xi_2(\gamma(\nu))}{\xi_1(\gamma(\nu))})$ we have $g_\nu(t) \sim \ln t$ (with constants that are independent of ν), where

$$g_\nu(t) =: \left| \frac{u\big(\gamma(\nu), t\xi_1(\gamma(\nu))\big) - u\big(\gamma(\nu), 2\xi_1(\gamma(\nu))\big)}{\alpha\big(\gamma(\nu)\big)} \right|.$$

Applying Theorem 3.1 to the two-variable function $g_\nu(t)$, we see that there are $\mathcal{D}$-functions w, c on a right-hand-side neighborhood of zero such that for t large enough

$$g_\nu(t) \sim w(\nu)|t - c(\nu)|^r \sim \ln t,$$

for any $\nu \in (0, \mu(t)]$, where $\mu : (M, \infty) \to \mathbb{R}$ is a positive $\mathcal{D}$-function, $M > 0$ real number.

If $c(\nu)$ tends to $\pm\infty$ as ν tends to 0 then $|c(\nu)|^r \sim |t - c(\nu)|^r$, for ν small, so that (for $\nu \in (0, \mu(t)]$ small):

$$w(\nu)|c(\nu)|^r \sim w(\nu)|t - c(\nu)|^r \sim \ln t.$$

Denote by l the limit of the left-hand-side as ν tends to zero (for fixed $t > M$). This limit has to be finite since $\ln t$ is constant (for fixed t). Passing to the limit as ν goes to zero, we get $\ln t \sim l$ for $t \in [M, \infty)$, a contradiction.

In the case where $c(\nu)$ has finite limit then, taking a smaller positive $\mathcal{D}$-function μ if necessary, we can assume that for $\nu \in (0, \mu(t))$, t large enough, we have $t^r \sim |t - c(\nu)|^r$, and

$$w(\nu)t^r \sim w(\nu)(t - c(\nu))^r \sim \ln t.$$

Making ν going to zero, we again see that it is impossible.

Case 3. – Consider $\xi_2 < 2\xi_1$ or $s = -1$.

Observe that if $s = -1$ then, thanks to the above claim, we know that there is a real number $t > 1$ such that

$$\xi_2(x) \leq t\xi_1(x), \tag{3.13}$$

for all x in the basis of our cell C. By assumption, this remains true if $s \neq -1$ (with $t = 2$).

Inequality (3.13) clearly entails that $y \sim \xi_1(x)$ on C. By (3.11) (recall that $\tilde{\theta} \equiv 0$), this implies that there is a $\mathcal{D}$-function β on D such that we have on C:

$$\frac{\partial u}{\partial y}(x,y) \sim \beta(x). \tag{3.14}$$

We now set

$$H(x,y) := (x, \frac{y - \xi_1(x)}{\xi_2(x) - \xi_1(x)}).$$

By (3.14), a straightforward computation of partial derivative yields that there is a $\mathcal{D}$-function $\gamma : D \to \mathbb{R}$ such that $\frac{\partial u'}{\partial y}(x,y) \sim \gamma(x)$ on C'. Integrating with respect to y, we get:

$$\gamma(x) \sim \int_0^1 \frac{\partial u'}{\partial y}(x,y)dy < \infty.$$

This proves that the function γ is bounded on D, which entails that $\frac{\partial u'}{\partial y}(x,y) \sim \gamma(x)$ is bounded on C', as required. □

4. Metric properties of $\mathcal{D}$-sets

In this section we recall some basic results about metric properties of definable sets.

4.1. On definable Lipschitz functions

So far, we have regarded definable sets as metric spaces because we endowed these sets with the Euclidean metric. Given a $\mathcal{D}$-set $X \subset \mathbb{R}^n$ we also can define the distance between two points of X as the infimum of the lengths of the continuous definable curves joining these two points ($\mathcal{D}$-curves are piecewise-smooth), with the convention that this distance is infinite if these two poins are not in the same connected component. This gives another metric on X that is generally called *the inner metric* of X. By analogy, we sometimes refer the restriction to X of the Euclidean metric as *the outer metric* of X.

A continuous $\mathcal{D}$-map $f : X \to \mathbb{R}$ which has bounded first derivative (that exists almost everywhere) is Lipschitz with respect to the inner metric. It may be derived (for instance) from the existence of Whitney stratifications for definable mappings (and this is no longer true if we drop the definability assumption on the map f).

The two metrics are not equivalent as it is shown by the simple example of a cusp. Hence, a function which is Lipschitz with respect to the inner metric may fail to be Lipschitz for the outer metric. This is the motivation of the following definition.

DEFINITION 4.1. – Let us define the *L-regular cells C of* $\mathbb{R}^n$ as follows:

(i) $\dim C = 0$ then C is a point.
(ii) If $\dim C = n$ then C is a set of the form:
$$\{(x,y) \in D \times \mathbb{R} : \phi_1(x) < y < \phi_2(x)\},$$
where ϕ_1 and ϕ_2 are either $\pm\infty$ or $\mathcal{D}$-functions on an L-regular cell D of $\mathbb{R}^{n-1}$, C^2, of bounded first derivative, and satisfying $\phi_1 < \phi_2$, on D.
(iii) If $\dim C = k < n$ then C is the graph Γ_ϕ of a $\mathcal{D}$-mapping $\phi : D \to \mathbb{R}^{n-k}$ on an L-regular cell D of $\mathbb{R}^k$, C^2 and of bounded derivative on D.

We then also say that $E := \mathrm{cl}(C)$ is an *L-regular closed cell of* $\mathbb{R}^n$ and define ∂E as the set $\mathrm{cl}(C) \setminus C$. If $\dim E = n$, then E is said to be *thick*.

On an L-regular cell the inner and outer metrics are obviously equivalent. Every $\mathcal{D}$-set can be decomposed into finitely many sets which are L-regular cells, after a suitable change of coordinates. A proof of this fact can be found in [10, 22] for subanalytic sets and [11, 24] for sets which are definable in an o-minimal structure. This entails the following useful result:

PROPOSITION 4.2. – *Let X be a $\mathcal{D}$-set. There exists a covering of X by finitely many definable manifolds $Y_1, \dots, Y_k$ such that for every i, the inner metric and the outer metric of Y_i are equivalent.*

A useful consequence of the above proposition is the following fact:

PROPOSITION 4.3. – *Let $\xi : X \to \mathbb{R}$ be a C^1 $\mathcal{D}$-function with $X \subset \mathbb{R}^n$ submanifold. If ξ has bounded derivative then there is a stratification of X such that ξ is Lipschitz on every stratum.*

Proof. – The function ξ is Lipschitz with respect to the inner metric. Any stratification of X compatible with the subsets given by Proposition 4.2 thus has the required properties. □

The vector $v \in S^{n-1}$ is *regular* for a set X if there exists a positive constant ε such that for any nonsingular point x of X we have:

$$d(v, T_x X) \geq \varepsilon.$$

We denote by π the orthogonal projection along e_n, the last vector of the canonical basis of $\mathbb{R}^n$. Proposition 4.3 entails:

PROPOSITION 4.4. – *For any $X \in \mathcal{D}_n$, the following statements are equivalent:*

(1) *e_n is regular for X;*
(2) *there is a stratification of $\pi(X)$ such that X is the union of the graphs of some C^2 Lipschitz functions defined on the strata.*

Proof. – A Lipschitz C^2 function has bounded derivative. Hence, if X is the union of some graphs of Lipschitz functions then e_n is regular for X. This shows that $(2) \Rightarrow (1)$. To show the converse, take a cdcd of $\mathbb{R}^n$ compatible with X. This provides a decomposition of X into graphs of definable C^2 functions. As e_n is regular, these functions must have bounded first derivative. The result thus follows from Proposition 4.3. □

4.2. Separated sets

This section introduces the notion of separated sets which will help us to glue Lipschitz vector fields into Lipschitz vector fields. This notion was already used in [22, 23].

DEFINITION 4.5. – Let $Y \subset \mathbb{R}^n$ be an L-regular closed cell and let $Z \in \mathcal{D}_n$. We say that Z is *L-separated* from Y if there exists $C > 0$ such that for every $q \in Y$

$$d(q, \partial Y) \leq C d(q, Z). \tag{4.15}$$

Two L-regular closed cells are *L-bi-separated* if they are L-separated from each other.

The usefulness of this notion lies in the following proposition:

PROPOSITION 4.6. –

(i) *Let Y be an L-regular closed cell of $\mathbb{R}^n$, $Z \subset \mathbb{R}^n$ be a $\mathcal{D}$-set L-separated from Y, and set $X := Y \cup Z$. Let $f : X \to \mathbb{R}$ be a continuous function. If f induces Lipschitz functions on Y and $\partial Y \cup Z$ then it is a Lipschitz function on X.*
(ii) *Let Y and Z be two L-regular L-bi-separated closed cells of $\mathbb{R}^n$ and set $X := Y \cup Z$. Let $f : X \to \mathbb{R}$ be a continuous function. If f induces Lipschitz functions on Y, Z, and $\partial Y \cup \partial Z$ then it is a Lipschitz function on X.*

Proof. – We have to check the Lipschitz condition for two points q and q' with q in Y and q' in Z. Let y be the point of ∂Y that realizes the distance $d(q, \partial Y)$.

By definition of separated sets we have $|y-q| \leq C|q-q'|$, for some constant C independent of q and q'. This clearly entails that $|y-q'| \leq (C+1)|q-q'|$. It thus suffices to write:

$$|f(q) - f(q')| \leq |f(q) - f(y)| + |f(y) - f(q')| \leq M|q - q'|,$$

for some positive constant M independent of q and q' (thanks to the Lipschitzness assumptions on the respective restrictions of f to the sets Y and $\partial Y \cup Z$).

Point (ii) may be proved by applying twice the point (i). It can also be established independently by a completely similar argument. □

DEFINITION 4.7. – A *tower of L-regular k-dimensional leaves* is a set that can be written, after a possible linear change of coordinates, as the union of the respective graphs of finitely many Lipschitz $\mathcal{D}$-mappings $\xi_1, \dots, \xi_m : B \to \mathbb{R}^{n-k}$, C^2 on the interior of B, with $B \in \mathcal{D}_k$ thick L-regular closed cell.

A $\mathcal{D}$-set Z is *L-separated from a tower* if it is L-separated from all the L-regular closed cells constituting this tower.

Let us recall the following theorem (Λ_p-decomposition theorem of [23]) that we translated in the terminology of the present article. This theorem was actually originally proved in [22] (Proposition 2.13) in the subanalytic category.

THEOREM 4.8. – *Let $X \in \mathcal{D}_n$ be a closed set and let $k := \dim X$. There exists a finite decomposition*

$$X = A \cup M_1 \cup \cdots \cup M_s$$

such that each M_i is a tower of L-regular k-dimensional leaves, A is a closed definable subset of $\dim < k$ *and, for any $i, j \in \{1, ..., s\}$ $(i \neq j)$, M_j is L-separated from M_i, and A is L-separated from M_i.*

5. Existence of Lipschitz stratifications

In this section we prove the main result of this article. It is worthy of notice that this will lead to achieve another improvement of van den Dries and Speissegger's Preparation Theorem. We will show that, up to a linear change of coordinates, the $\mathcal{D}$-translation of the reduction provided by Theorem 3.5 can be chosen in such a way that its first derivative (exists and) is bounded (Proposition 5.2 below). This result seems to be of its own interest.

5.1. Back to the preparation theorem

We will need the following result to prove the above proposition.

PROPOSITION 5.1 ([26], Proposition 3). – *There exist $L_1, \dots, L_N$ in S^{n-1} such that for any family of $\mathcal{D}$-subsets $X_1, \dots, X_k$ of $\mathbb{R}^n$, there is a cdcd of $\mathbb{R}^n$ compatible with $X_1, \dots, X_k$ such that for each cell C of this cdcd, we can find $i \in \{1, \dots, N\}$ such that L_i is regular for* $\mathrm{cl}(C) \setminus \mathrm{int}(C)$.

A stronger result was indeed also proved in [24] where it is shown that the family $\{L_1, \dots, L_N\}$ can be chosen as the canonical basis of $\mathbb{R}^n$.

Proposition 5.2. – *Let X be a $\mathcal{D}$-set and $\xi : X \to \mathbb{R}$ be a $\mathcal{D}$-function. There is a stratification of X such that on every stratum S, up to a linear change of coordinates, the function ξ is reduced with some $\mathcal{D}$-translation θ_S (see Definition 3.3) which has bounded first derivative.*

Proof of Proposition 5.2. – Let $L_1, \dots, L_N$ be as Proposition 5.1. We will denote by π the orthogonal projection onto $\mathbb{R}^{n-1}$.

For each i, applying Theorem 3.5 to $\xi \circ A_i : A_i^{-1}(X) \to \mathbb{R}$, where A_i is an orthogonal linear mapping of $\mathbb{R}^n$ sending e_n onto L_i (e_n being the last vector of the canonical basis of $\mathbb{R}^n$) we obtain a partition of $\mathbb{R}^n$ compatible with $A_i^{-1}(X)$. The images of all the elements of this partition that are subsets of $A_i^{-1}(X)$ under the map A_i provide a partition of X, denoted by Σ_i. Let $\mathcal{V} = (V_j)_{j\in J}$ be a decomposition of $\mathbb{R}^n$ compatible with all the elements of the Σ_i, $i = 1, \dots, N$. Then, for each $j \in J$ and each $i \leq N$ there is a $\mathcal{D}$-function $\theta_{i,j} : \pi(A_i^{-1}(V_j)) \to \mathbb{R}$ such that $\xi \circ A_i$ is reduced on $A_i^{-1}(V_j)$ with $\mathcal{D}$-translation $\theta_{i,j}$.

Applying Proposition 5.1 to the family consisting of all the elements of $\mathcal{V}$ and the $A_i(\Gamma_{\theta_{i,j}})$, $i \leq N$, $j \in J$, where $\Gamma_{\theta_{i,j}}$ stands for the graph of the function $\theta_{i,j}$, we obtain a cdcd compatible with X which gives rise to a stratification Σ of X.

Let E be an element of Σ of dimension n. By the construction and Proposition 5.1, there is $i \leq N$ such that L_i is regular for $\mathrm{cl}(E) \setminus \mathrm{int}(E)$. This means that e_n is regular for $A_i^{-1}(\mathrm{cl}(E) \setminus \mathrm{int}(E))$. Hence (see Proposition 4.4), there is a partition of $A_i^{-1}(E)$ into cells, such that each element C of this partition which is of dimension n is of the form

$$C = \{(x, y) \in B \times \mathbb{R} : \xi_1(x) < y < \xi_2(x)\},$$

with $B \in \mathcal{D}_{n-1}$, $\xi_1 < \xi_2$ C^1 $\mathcal{D}$-functions of bounded first derivative.

Fix such a cell $C \subset A_i^{-1}(E)$. There is j such that $C \subset A_i^{-1}(V_j)$. By Remark 3.4, it suffices to show that $(y - \theta_{i,j}(x))$ is reducible with some $\mathcal{D}$-translation θ'_C having first derivative bounded.

Since the cdcd Σ is compatible with $A_i(\Gamma_{\theta_{i,j}})$, we know that the graph of $\theta_{i,j}$ must lie outside the cell C. Refining our cell decomposition, we can assume that $\theta_{i,j}$ is continuous. This means that we can assume that $\theta_{i,j} \leq \xi_1$ or $\theta_{i,j} \geq \xi_2$. We will suppose (for simplicity) that $\theta_{i,j} \leq \xi_1$. Up to an extra refinement, we can assume that on C either $y \geq 2\xi_1 - \theta_{i,j}$ or $y \leq 2\xi_1 - \theta_{i,j}$.

If $y \geq 2\xi_1 - \theta_{i,j}$ (or $y - \xi_1 \geq \xi_1 - \theta_{i,j}$), then $(1 + \frac{\xi_1 - \theta_{i,j}}{y - \xi_1})$ is a $\mathcal{D}$-unit. Therefore, $(y - \theta_{i,j})$ is reduced by

$$y - \theta_{i,j} = (y - \xi_1)\big(1 + \frac{\xi_1 - \theta_{i,j}}{y - \xi_1}\big).$$

If $y \leq 2\xi_1 - \theta_{i,j}$ (or $y - \xi_1 \leq \xi_1 - \theta_{i,j}$), then $(1 + \frac{y - \xi_1}{\xi_1 - \theta_{i,j}})$ is a $\mathcal{D}$-unit. Hence, $(y - \theta_{i,j})$ is reduced by

$$y - \theta_{i,j} = (\xi_1 - \theta_{i,j})\big(1 + \frac{y - \xi_1}{\xi_1 - \theta_{i,j}}\big).$$

In both cases, $(y - \theta_{i,j})$ is reduced with $\mathcal{D}$-translation $\theta'_C = \xi_1$ which has bounded first derivative.

On a cell which is of positive codimension the result is trivial. □

5.2. Extension of Lipschitz vector fields

We end the proof following a similar method as in [22]. We first establish o-minimal counterparts of lemmas 3.1 and 3.2 of [22] (Proposition 5.3 and Lemma 5.4 below). We give full details. We would like to emphasize that, contrarily to [22], we assume the vector field to be definable in Lemma 5.4.

PROPOSITION 5.3. – *Let X be a $\mathcal{D}$-subset of $\mathbb{R}^n$ and let $f : X \to \mathbb{R}$ be a $\mathcal{D}$-function. There exist a stratification Σ of $\mathbb{R}^n$ compatible with X and a constant $C > 0$ such that f is C^2 on strata and such that we have for every L-Lipschitz Σ-compatible vector field v:*

$$|Df(x)v(x)| \leq CL|f(x)|, \quad \forall x \in X. \tag{5.16}$$

In particular, any refinement of Σ also satisfies this property.

Proof. – We proceed by induction on n (starting with the case $n = 0$ which is vacuous). By Proposition 5.2, there exists a stratification of X such that for every stratum S, there is a linear change of coordinates A such that $f \circ A$ is reduced on $C := A^{-1}(S)$ with a $\mathcal{D}$-translation θ_S that has bounded first derivative. As A is a linear automorphism, it is enough to establish the required statement for the restriction of $f \circ A$ to C. It means that, without loss of generality, we can assume that f is a smooth function on a cell C such that for $x = (x', x_n) \in C$:

$$f(x', x_n) = a(x') \cdot |x_n - \theta_S(x')|^r \cdot U(x', x_n), \quad r \in \Lambda, \tag{5.17}$$

where $U(x', x_n) = \psi \circ W(x', x_n)$ with ψ bounded away from zero and infinity and such that $|D\psi(x)|$ is bounded, where W is a bounded $\mathcal{D}$-mapping of type

$$W(x, y) = \big(u_1(x'), \dots, u_s(x'), b_1(x')|x_n - \theta_S(x')|^{\frac{1}{p_1}}, \dots, b_k(x')|x_n - \theta_S(x')|^{\frac{1}{p_k}}\big),$$

with $a, u_1, \dots, u_s, b_1, \dots, b_k$ $\mathcal{D}$-functions on the basis of C.

By the inductive assumption, there is a stratification $\mathcal{B}$ of $\mathbb{R}^{n-1}$ compatible with the basis of C such that $a, u_1, \dots, u_s, b_1, \dots, b_k$ satisfy (5.16). The pre-images by π of the strata of $\mathcal{B}$ give rise to a stratification Σ of $\mathbb{R}^n$ compatible with C (taking the intersection of these pre-images with C) and satisfying (5.16) for the $\mathcal{D}$-functions $a, u_1, \dots, u_s, b_1, \dots, b_k$ (regarding now these functions as n-variable functions). Refining this stratification, we can assume it to be compatible with the graph Γ_{θ_S} of θ_S.

Notice that a product of functions satisfying (5.16) satisfies this inequality as well. We now are going check that the $\mathcal{D}$-function $\mu : (x', x_n) \mapsto \mu(x', x_n) := x_n - \theta_S(x')$ satisfies (5.16).

For this purpose, fix a Lipschitz Σ-compatible vector field $v = (v', v_n)$, with Lipschitz constant $L \in \mathbb{R}$. Since v is tangent to Σ, $D\mu(x)v(x) = 0$, for all $x \in \Gamma_{\theta_S}$, which is equivalent to $v_n(x', \theta_S(x')) - D\theta_S(x')v'(x', \theta_S(x')) = 0$. If $M = \sup |D\theta_S|$ we thus have

$$\begin{aligned} |D\mu(x)v(x)| &= |v_n(x) - D\theta_S(x')v'(x)| \\ &\leq |v_n(x) - v_n(x', \theta_S(x'))| + |D\theta_S(x')\big[v'(x) - v'(x', \theta_S(x'))\big]| \\ &\leq (M+1)L|x_n - \theta_S(x')| = (M+1)L|\mu(x)|. \end{aligned}$$

This shows that μ satisfies (5.16).

Observe that it easily follows from the chain rule that for every $s \in \Lambda$, the function $|x_n - \theta_S(x')|^s$ satisfies this inequality as well, which entails that so do the components of the mapping W. We claim that so does the function U. Indeed, as (5.16) holds for W, every L-Lipschitz Σ-compatible vector field v satisfies:

$$|DU(x)v(x)| = |D\psi(W(x))DW(x)v(x)| \le CNL|W(x)|,$$

where $N := \sup_{x\in W(C)} |D\psi(x)|$. As W is a bounded mapping and since U is bounded away from zero, this yields (5.16) for U. That f also fulfills this inequality now follows from (5.17). □

LEMMA 5.4. – *Let B be a thick L-regular cell of $\mathbb{R}^k$ and let $\xi : B \to \mathbb{R}^{n-k}$ ($n > k$) be a Lipschitz $\mathcal{D}$-map. Assume that ξ is C^2 on B. Let $\mathcal{B}$ be a stratification of $\mathbb{R}^k$ compatible with B and satisfying the statement of Proposition 5.3 for all the components of the partial derivatives of ξ. Then there exists a constant $C > 0$ such that if $v'(x)$ is a Lipschitz $\mathcal{B}$-compatible $\mathcal{D}$-vector field on B with Lipschitz constant L then $v(x, \xi(x)) := D\xi(x)v'(x)$, for $x \in B$, is a Lipschitz vector field with Lipschitz constant CL on the graph of ξ.*

Proof. – As B is a thick L-regular cell, it is sufficient to show (see section 4.1) that $D\xi(x)v'(x)$ has first order partial derivatives (which exist almost everywhere) bounded by CL, for some constant C independent of L. Applying Proposition 5.3 to the components of all the mappings $D_i\xi$ (denoting the i^{th}-partial derivative of ξ) $i = 1, \dots, k$, we get for every $x \in B$

$$|D_iD\xi(x)v'(x)| = |DD_i\xi(x)v'(x)| \le CL|D_i\xi(x)|,$$

which, since $|D_i\xi(x)|$ is bounded, provides the desired inequality. □

Proof of Theorem 2.6. – It suffices to show that there exists a stratification for which condition $(\star)$ of Proposition 2.4 holds for $K = 1$. We proceed by induction on $k = \dim X$. For $k = 0$ the statement is obvious. Take some $k > 0$. We may assume that $k < n$ for if $k = n$ any Lipschitz stratification of X_{sing} (which is of positive codimension) gives rise to a Lipschitz stratification of X (see Remark 2.5 (ii)). We shall prove the following statement: given finitely many $\mathcal{D}$-subsets $X_1, \dots, X_l$ of X, we are going to prove that there is a Lipschitz stratification of X which is compatible with all the X_i.

Given a stratification $\mathcal{S}$, we will denote by $\mathcal{S}_i$ the collection of the strata of $\mathcal{S}$ whose dimension does not exceed i.

First case. – We assume that X is a tower of L-regular leaves, i.e., that there exist finitely many Lipschitz $\mathcal{D}$-mappings $\xi_i : B \to \mathbb{R}^{n-k}$, $i = 1, \dots, m$, where B is an L-regular thick closed cell of $\mathbb{R}^k$, such that $X = \cup_{i=1}^m \Gamma_{\xi_i}$ (where Γ_{ξ_i} is the graph of ξ_i).

Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ denote the canonical projection. Take a C^2 cdcd $\mathbb{C}$ of $\mathbb{R}^n$ compatible with the X_i and the Γ_{ξ_i}. Let $\mathcal{B}$ be a stratification (not necessarily Lipschitz) of $\mathbb{R}^k$ compatible with all the elements of $\pi(\mathbb{C})$ and satisfying the statement of Proposition 5.3 for all the components of the mappings $(\xi_i - \xi_j)$, for all $i < j$, as well as for all the components of the partial derivatives of the ξ_i (these functions are C^2 on the cells of $\pi(\mathbb{C})$ since $\mathbb{C}$ is compatible with the graphs Γ_{ξ_i}).

Let $\mathcal{B}'$ be the stratification of X constituted by the respective graphs of the functions $\xi_{i|S}$, $i \le m$, $S \subset B$, $S \in \mathcal{B}$. By induction on k, there is a refinement $\mathcal{B}''$ of $\mathcal{B}'_{k-1}$ which is a Lipschitz stratification. Let now $\mathcal{S}$ denote the stratification constituted by the elements of $\mathcal{B}''$ together with the strata of $\mathcal{B}'$ of dimension k.

We claim that $\mathcal{S}$ is a Lipschitz stratification of X. To see this, denote by X^j the union of the elements of $\mathcal{S}_j$, take W such that $X^{j-1} \subseteq W \subseteq X^j$, and let v be a Lipschitz $\mathcal{S}$-compatible $\mathcal{D}$-vector field on W with Lipschitz constant L. If $j < k$, the result is clear, since $\mathcal{B}''$ is a Lipschitz stratification. So, we just have to address the case $j = k$. To complete the proof, we have to extend v to a Lipschitz $\mathcal{S}$-compatible $\mathcal{D}$-vector field on X (with a proportional Lipschitz constant).

Let us write $v(x)$ as $(v'(x), v''(x))$ in $\mathbb{R}^k \times \mathbb{R}^{n-k}$ and extend the mapping $v' : W \to \mathbb{R}^k$ to an L-Lipschitz mapping on the whole of X, keeping the notation v' for this extension. Fix $S \in \mathcal{S}$ and choose $\alpha \leq m$ such that $S \subset \Gamma_{\xi_\alpha}$. For $x = (x', x'') \in S \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, we define

$$w(x) = (v'(x), D\xi_{\alpha|\pi(S)}(x')v'(x)).$$

It is easily checked that since the cell decomposition $\mathbb{C}$ (from which we constructed our stratification) was required to be compatible with the graphs of the ξ_i, $w(x)$ is independent of the choice of α. Moreover, as $\mathcal{B}$ satisfies the assumptions of Lemma 5.4 for the functions ξ_i, w induces a Lipschitz vector field on $\Gamma_{\xi_{\alpha|\mathrm{int}(B)}}$ of Lipschitz constant CL, where C is some positive constant (independent of v). Because ξ_α is C^1 at almost every boundary point of B, we see that the vector field w is indeed Lipschitz on the whole of Γ_{ξ_α}, for each α.

To finish the proof of the first case we only need to check the Lipschitz condition of w on the couples of points (p, q) with $p \in \Gamma_{\xi_\alpha}$ and $q \in \Gamma_{\xi_\beta}$, $\alpha \neq \beta$.

Let $p = (x, \xi_\alpha(x))$ and $q = (x', \xi_\beta(x'))$ and set $\tilde{p} := (x', \xi_\alpha(x'))$. It follows from Proposition 5.3 above that

$$|w(\tilde{p}) - w(q)| = |(D\xi_\alpha(x') - D\xi_\beta(x'))v'(x)| \leq CL|\xi_\alpha(x') - \xi_\beta(x')| = CL|\tilde{p} - q|.$$

Let L_α denote the Lipschitz constant of ξ_α. We conclude

$$\begin{aligned} |w(p) - w(q)| &\leq |w(p) - w(\tilde{p})| + |w(\tilde{p}) - w(q)| \\ &\leq CL\big(|p - \tilde{p}| + |\tilde{p} - q|\big) \\ &\leq CL\big(2|p - \tilde{p}| + |p - q|\big) \\ &\leq CL(2L_\alpha + 1)|p - q|. \end{aligned}$$

This completes our *first case*. We now turn to the general case.

By Theorem 4.8, there is a finite decomposition of X as

$$X = A \cup Y_1 \cup \cdots \cup Y_s,$$

where for every i, Y_i is a tower of L-regular k-dimensional leaves, $\dim A < k$, A is L-separated from Y_i, and, for each j, Y_i is L-bi-separated from Y_j. Since every Y_i is a tower, by the *first case*, we know that Y_i has a Lipschitz stratification, say Σ^i. Moreover, this stratification may be required to be compatible with the sets $X_j \cap Y_i$, $j = 1, \ldots, l$. Let X' denote the union of A together with all the strata of dimension less than k of all the Σ^i. Since X' has dimension less than k, by induction, it admits a Lipschitz stratification Σ' compatible with the sets $X_j \cap A$, $j = 1, \ldots, l$, as well as with all the strata of the Σ^i_{k-1}, $i \leq s$.

Let now $\mathcal{S}$ be the stratification of X constituted by the strata of Σ'_{k-1} together with the connected components of $X \setminus |\Sigma'_{k-1}|$. We claim that $\mathcal{S}$ is a Lipschitz stratification of X. By the construction, it is clear that $\mathcal{S}_{k-1}$ is a Lipschitz stratification (since so is Σ' and $\Sigma'_{k-1} = \mathcal{S}_{k-1}$). It is thus enough to show that any Lipschitz $\mathcal{S}$-compatible $\mathcal{D}$-vector field on $|\mathcal{S}_{k-1}| \subset W \subset X$ may be extended to a Lipschitz $\mathcal{S}$-compatible $\mathcal{D}$-vector field (with a proportional Lipschitz constant).

Take such a vector field $v : W \to \mathbb{R}^n$ and let $\mathcal{S}^i$ denote the stratification of Y_i induced by $\mathcal{S}$ (it is easily checked that $\mathcal{S}$ is compatible with all the Y_i). As, by the construction, $\mathcal{S}^i$ is a refinement of Σ^i, the vector field v is tangent to the strata of Σ^i_{k-1}. It thus can be extended to a Σ^i-compatible Lipschitz $\mathcal{D}$-vector field on Y_i. Doing this for every i we get a continuous vector field on X (still denoted v) Lipschitz on every Y_i (with a proportional Lipschitz constant). Since the Y_i are bi-separated from each other, by Proposition 4.6 (ii), we conclude that v is a Lipschitz $\mathcal{S}$-compatible vector field on $\cup_{i=1}^s Y_i$. By Proposition 4.6 (i), we also see that v is Lipschitz on $A \cup Y_i$, for all i. □

BIBLIOGRAPHY

[1] S. BANACH, *Wstęp do teorii funkcji rzeczywistych*, Monografie Matematyczne **17**, Polskie Towarzystwo Matematyczne, Warszawa-Wrocław, 1951.

[2] J. BOCHNAK, M. COSTE, M.-F. ROY, *Géométrie algébrique réelle*, Ergebn. Math. Grenzg. **12**, Springer, Berlin, 1987.

[3] G. COMTE, J.-M. LION, J.-P. ROLIN, Nature log-analytique du volume des sous-analytiques, *Illinois J. Math.* **44** (2000), 884–888.

[4] M. COSTE, An introduction to o-minimal geometry, Dottorato di Ricerca in Matematica, Università degli Studi di Pisa, 2000.

[5] J. DENEF, L. VAN DEN DRIES, p-adic and real subanalytic sets, *Ann. of Math.* **128** (1988), 79–138.

[6] L. VAN DEN DRIES, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series **248**, Cambridge Univ. Press, Cambridge, 1998.

[7] L. VAN DEN DRIES, P. SPEISSEGGER, O-minimal preparation theorems, in *Model theory and applications*, Quad. Mat. **11**, Aracne, Rome, 2002, 87–116.

[8] T. KAISER, On convergence of integrals in o-minimal structures on Archimedean real closed fields, *Ann. Polon. Math.* **87** (2005), 175–192.

[9] B. KOCEL-CYNK, W. PAWŁUCKI, A. VALETTE, A short geometric proof that Hausdorff limits are definable in any o-minimal structure, *Adv. Geom.* **14** (2014), 49–58.

[10] K. KURDYKA, On a subanalytic stratification satisfying a Whitney property with exponent 1, in *Real algebraic geometry (Rennes, 1991)*, Lecture Notes in Math. **1524**, Springer, Berlin, 1992, 316–322.

[11] K. KURDYKA, A. PARUSIŃSKI, Quasi-convex decomposition in o-minimal structures. Application to the gradient conjecture, in *Singularity theory and its applications*, Adv. Stud. Pure Math. **43**, Math. Soc. Japan, Tokyo, 2006, 137–177.

[12] O. LE GAL, J.-P. ROLIN, An o-minimal structure which does not admit C^∞ cellular decomposition, *Ann. Inst. Fourier (Grenoble)* **59** (2009), 543–562.

[13] J.-M. LION, J.-P. ROLIN, Théorème de préparation pour les fonctions logarithmico-exponentielles, *Ann. Inst. Fourier (Grenoble)* **47** (1997), 859–884.

[14] T. L. LOI, Whitney stratification of sets definable in the structure $\mathbf{R}_{\exp}$, in *Singularities and differential equations (Warsaw, 1993)*, Banach Center Publ. **33**, Polish Acad. Sci., Warsaw, 1996, 401–409.

[15] T. L. LOI, Verdier and strict Thom stratifications in o-minimal structures, *Illinois J. Math.* **42** (1998), 347–356.

[16] S. ŁOJASIEWICZ, J. STASICA, K. WACHTA, Stratifications sous-analytiques. Condition de Verdier, *Bull. Polish Acad. Sci. Math.* **34** (1986), 531–539.

[17] J. MATHER, Notes on topological stability, *Bull. Amer. Math. Soc. (N.S.)* **49** (2012), 475–506.

[18] C. MILLER, Expansions of the real field with power functions, *Ann. Pure Appl. Logic* **68** (1994), 79–94.

[19] T. MOSTOWSKI, Lipschitz equisingularity, *Dissertationes Math. (Rozprawy Mat.)* **243** (1985), 46.

[20] A. PARUSIŃSKI, Lipschitz stratification of real analytic sets, in *Singularities (Warsaw, 1985)*, Banach Center Publ. **20**, PWN, Warsaw, 1988, 323–333.

[21] A. PARUSIŃSKI, Lipschitz stratification, in *Global analysis in modern mathematics (Orono, ME, 1991; Waltham, MA, 1992)*, Publish or Perish, Houston, TX, 1993, 73–89.

[22] A. PARUSIŃSKI, Lipschitz stratification of subanalytic sets, *Ann. Sci. École Norm. Sup.* **27** (1994), 661–696.

[23] W. PAWŁUCKI, A linear extension operator for Whitney fields on closed o-minimal sets, *Ann. Inst. Fourier (Grenoble)* **58** (2008), 383–404.

[24] W. PAWŁUCKI, Lipschitz cell decomposition in o-minimal structures. I, *Illinois J. Math.* **52** (2008), 1045–1063.

[25] J.-P. ROLIN, P. SPEISSEGGER, A. J. WILKIE, Quasianalytic Denjoy-Carleman classes and o-minimality, *J. Amer. Math. Soc.* **16** (2003), 751–777.

[26] G. VALETTE, Lipschitz triangulations, *Illinois J. Math.* **49** (2005), 953–979.

(Manuscrit reçu le 22 mai 2014 ;
accepté, après révision, le 1er décembre 2014.)

Nhan NGUYEN
LATP (UMR 7353),
Centre de Mathématiques et Informatique
Université d'Aix-Marseille
39 rue Joliot-Curie
13453 Marseille Cedex 13, France
E-mail: nguyenxuanvietnhan@gmail.com

Guillaume VALETTE
Instytut Matematyczny PAN
ul. Św. Tomasza 30
31-027 Kraków, Poland
E-mail: gvalette@impan.pl