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*Quantitative stochastic homogenization of convex integral functionals*

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## QUANTITATIVE STOCHASTIC HOMOGENIZATION OF CONVEX INTEGRAL FUNCTIONALS

#### BY SCOTT N. ARMSTRONG AND CHARLES K. SMART

ABSTRACT. – We present quantitative results for the homogenization of uniformly convex integral functionals with random coefficients under independence assumptions. The main result is an error estimate for the Dirichlet problem which is algebraic (but sub-optimal) in the size of the error, but optimal in stochastic integrability. As an application, we obtain quenched  $C^{0,1}$  estimates for local minimizers of such energy functionals.

Résumé. – Nous présentons des résultats quantitatifs pour l'homogénéisation de fonctionnelles intégrales uniformément convexes avec coefficients aléatoires sous hypothèses d'indépendance. Le résultat principal est une estimation d'erreur pour le problème de Dirichlet qui est algébrique (mais sousoptimale) en la taille de l'erreur, mais optimale en intégrabilité stochastique. Comme application, nous obtenons des estimées  $C^{0,1}$  pour les minimiseurs locaux de telles fonctionnelles d'énergie.

#### **1. Introduction**

#### **1.1. Informal summary of results**

<span id="page-2-0"></span>We consider stochastic homogenization of the variational problem

(1.1) minimize 
$$
\int_U L(Du(x), \frac{x}{\varepsilon}) dx
$$
 subject to  $u \in g + H_0^1(U)$ .

Here  $0 < \varepsilon \ll 1$  is a small parameter,  $U \subseteq \mathbb{R}^d$  is a smooth bounded domain and  $g \in H^1(U)$ is given. The precise hypotheses on the Lagrangian  $L$  are given below; here we mention that  $L(p, x)$  is uniformly convex in p and that L is a random field sampled by a given probability measure P. The crucial hypothesis on the statistics of L is a *finite range of dependence* condition: roughly, for all Borel sets  $U, V \subseteq \mathbb{R}^d$ , the families  $\{L(p, x) : p \in \mathbb{R}^d, x \in U\}$  and  $\{L(p,x): p \in \mathbb{R}^d, x \in V\}$  of random variables are assumed to be P-independent provided that dist $(U, V) \geq 1$ .

<span id="page-3-0"></span>An important special case of the model occurs if the Lagrangian is the quadratic form  $L(p, x) = p \cdot A(x)p$ . The corresponding Euler-Lagrange equation is then linear and the problem is equivalent to the stochastic homogenization of the equation

(1.2) 
$$
-\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)Du^{\varepsilon}\right)=0.
$$

This is also a continuum ver[sio](#page-59-0)[n o](#page-59-1)f [what](#page-2-0) is known in the probability literature as the *random conductance model*.

[Dal](#page-2-0) Maso and Modica [8, 9] proved, in a somewhat more general setting, the basic *qualitative homogenization r*esult for (1.1): there exists a (deterministic) function  $\overline{L}:\mathbb{R}^d\to\mathbb{R}$ called the *effective Lagrangian* such that, with probability one, the unique minimizer  $u^{\varepsilon}$ of (1.1) converges, as  $\varepsilon \to 0$ , to the unique minimizer of the variational problem

(1.3) minimize 
$$
\int_U \overline{L}(Du(x)) dx
$$
 subject to  $u \in g + H_0^1(U)$ .

This result was a generalization to the nonlinear setting of earlier qualitative results for linear elliptic partial differential equations in divergence form due to Kozlov [18], Papanicolaou and Varadhan [26] and Yurinkskii [28], using new v[ariat](#page-3-0)ional ideas based on subadditivity that were not present in earlier [wor](#page-59-3)[ks.](#page-59-4)

An intense focus has recently emerged on building a quantitative theory of st[och](#page-59-5)[asti](#page-59-6)c homogenization in the case of the linear Equation (1.2). This escalated significantly with the work of Gloria and Otto [15, 16], who proved optimal quantitative bo[und](#page-59-7)s for the energy de[nsit](#page-59-8)[y o](#page-59-9)f modified correctors a[nd](#page-59-10) then that of Gloria Ne[uk](#page-59-11)amm and Otto [14, 13], who proved [opti](#page-60-0)mal bounds for the err[or i](#page-60-1)n homogenization. These [res](#page-59-12)ults were pr[ove](#page-59-13)d for discrete elliptic equations, but have been extended to the continuum s[ettin](#page-60-2)[g i](#page-59-14)[n \[1](#page-60-3)[7\].](#page-60-4) [See](#page-59-15) also Mourrat [20, 21], Marahrens and Otto [19], Conlon and Spencer [7] as well as ear[lie](#page-58-0)r works of Yurinskii [28], Naddaf and Spencer [24], Bourgeat and Piatnitski [6] and Boivin [5]. For some recent work on limit theorems for the stochastic fluctuations, see [23, 22, 25, 27, 4]. The analysis in the present paper was informed by some ideas from our previous work [1], w[hich](#page-2-0) contained similar results for equations in nondivergence form.

In this paper, we present the first *quantitative* results for the homogenization of (1.1) which are also the first such results for divergence-form elliptic equations outside of the linear setting. We prove two main results: estimates for the  $L^2$  [and](#page-2-0)  $L^\infty$  error in homogenization of the Dirichlet problem, which is algebraic (yet sub-optimal) in its estimate of the typical size of the error, and essentially optimal in stochastic integrability; and a "stochastic higher regularity" result which states that local minimizers of (1.1), for a typical realization of the coefficients, satisfy the same *a priori*  $C^{0,1}$  and  $C^{1,\beta}$  regularity estimates as local minimizers of constant-coefficien[t en](#page-0-0)ergy functionals, down to microscopic and mesoscopic scales, respectively.

The first main result (Theorem 1.1) gives a sub-optimal algebraic error estimate in homogenization with strong stochastic integrability: it asserts roughly that, for any  $s < d$ , there exists an exponent  $\alpha > 0$ , depending on s, the dimension d and the constants controlling <span id="page-4-0"></span>the uniform convexity of L and a constant  $C \geq 1$ , depending additionally on the given data, such that, for every  $\delta \in (0, 1]$ ,

(1.4) 
$$
\mathbb{P}\left[\exists \varepsilon \in (0,\delta], \int_U |u^{\varepsilon}(x) - u_{\text{hom}}(x)|^2 dx \geq C\varepsilon^{\alpha}\right] \leq C \exp\left(-\delta^{-s}\right),
$$

where  $u^{\varepsilon}$  and  $u_{\text{hom}}$  denote the unique minimizers in  $g + H_0^1(U)$  of (1.1) and (1.3), respectively. Depending on the smoothness of the given Dirichlet boundary data  $g$ , this  $L^2$  estimate may be upgraded to  $L^{\infty}$  b[y int](#page-0-0)erpolating the latter between  $L^2$  and  $C^{0,\gamma}$  and using the nonlinear De Giorgi-Nash-Moser estimate. There is no loss in stochastic integrability in this interpolation and essentially no loss in the size of the error, since the exponent  $\alpha$  is already sub-optimal. (See Corollary 4.2.) We remark that, at this stage in the development of the theory, we are less concerned with the sub-optimality of the size of the error than with the strength of the stochastic integrabilit[y; th](#page-0-0)e former will be improved later. In (1.4) we have obtained [the](#page-2-0) best possible stochastic integrability in the sense that such an estimate is false for  $s > d$ .

<span id="page-4-1"></span>The second main result (Theorem 1.2) asserts that local minimizers of the energy functional in (1.1) are much smoother than minimizers for general functionals with measurable coefficients: it states roughly that any local minimizer  $u^{\varepsilon}$  of the energy functional satisfies the estimate

(1.5) 
$$
\sup_{x \in B_{1/2} \backslash B_{\varepsilon}} \frac{|u^{\varepsilon}(x) - u^{\varepsilon}(0)|}{|x|} \leq \mathcal{Y} \left( 1 + \|u^{\varepsilon}\|_{L^{2}(B_{1})} \right),
$$

where  $\mathcal Y$  is a random variable (i.e, it depends on the coefficients but not on  $u^{\varepsilon}$ ) which, for any  $s < d$  $s < d$ , can be chosen to satisfy

$$
\mathbb{E}\left[\exp(\mathcal{Y}^s)\right]<\infty.
$$

This is a quenc[hed L](#page-4-1)ipschitz estimate "down to the microscopic scale" since the left side of (1.5) is a finite difference approximation of  $|Du^{\epsilon}(0)|$ .

<span id="page-4-2"></span>The estimate (1.5) can be written in other forms, such as

$$
(1.6) \qquad \int_{B_r} |Du^{\varepsilon}(x)|^2 \ dx \le C\left(1 + \|u^{\varepsilon}\|_{L^2(B_1)}^2\right) \quad \text{for every } r \in \left[\varepsilon \mathcal{Y}, \frac{1}{2}\right].
$$

The latter gives very good control of the spatial averages of the energy density of  $u^{\varepsilon}$ . As was shown by Gloria and Otto [15] in the linear setting, if the probability measure  $\mathbb P$  satisfies a spectral ga[p hyp](#page-0-0)othesis, then an estimate like (1.6) implies optimal bounds on the variance of the energy of, e.g., minimizers with periodic boundary conditions. In a future work, we will [prove](#page-8-0) this and other optimal quantitative estimates from higher regularity estimates.

Theorem 1.2 also asserts that local minimizers behave ev[en](#page-59-0) [mo](#page-59-1)re smoothly on *meso*scopic scales (those of order  $\varepsilon^{\beta}$  for some  $\beta \in (0,1)$ ) by giving an improvement of flatness estimate: see (1.16).

The proof of the error estimates, like the arguments of [8, 9], is variational and centers on the analysis of certain subadditive and superadditive energy quantities. However, the methods here differ substantially from those of [8, 9], as quantitative results present difficulties which do not appear in the qualitative theory and which require not just a harder analysis but also a new approach to the problem. The qualitative theory is based on the observation

that the energy of a minimizer with respect to affine Dirichlet conditions is subadditive with respect to the domain. This monotonicity allows one to obtain a [det](#page-0-0)erministic limit for this energy, as the domain becomes large, via a relatively soft argument based on the ergodic theorem.

To obtain a convergence rate f[or](#page-9-0) this limit (see Theorem 3.1), we introduce a new *super*additive energy quantity by removing the boundary condition and adding a linear term to the energy functional. This is a kind of convex dual of the subadditive quantity, as we explain in more detail in Section 2. The main part of the analysis is to show that minimizers of the dual quantity are close to affin[e f](#page-9-0)unc[tio](#page-17-0)ns in a suitable sense, whi[ch i](#page-0-0)mplies that the subadditive [and](#page-39-0) superadditive quantities are close to each other, up to a small error. Thus the quantities are in fact additive, up to a suitably small error, which gives t[he](#page-58-1) [de](#page-58-2)sired rate for the limits. This is the focus of Sections 2 and 3, and the proof of Theorem 1.1 is then completed in Section 4 with the help of an oscillating test function argument.

The proof of the quenched Lipschitz estimate is inspired by Avellaneda and Lin [2, 3] who showed, using a perturbati[on ar](#page-0-0)gument in the context of periodic media and linear equations, that solutions of a heterogeneous equation inherit higher regularity from the homogenized equation. While we cannot make use of compactness arguments in the stochastic setting, the error estimates in Theorem 1.1 are st[rong](#page-59-10) enough to implement a quantitative version of this technique.

The closest previous result to the quenched Lipschitz estimate is a quenched Hölder estimate due to Marahrens and Otto [19] (recently extended to the continuum case by Gloria and Marahrens [12]). They proved a  $C^{0,1-\delta}$  estimate for linear equations, for every  $\delta > 0$ , with somewhat weaker stochastic integrability (in our notation, they obtained that all finite moments of  $\mathcal Y$  are bounded). The methods of proof in all of these works are completely different from the one here and based on logarithmic Sobolev or spectral gap inequalities. Here we also apply concentr[ati](#page-58-1)[on](#page-58-2) of measure, but we use it in a more elementary form and in a modular way. We therefore believe our results will extend in a straightforward way to coefficients satisfying only much weaker mixing conditions. Moreover, Hölder estimates, as noted by Avellaneda and Lin [2, 3] in the periodic case, are significantly easier to obtain than a Lipschitz estimate, which is the critical estimate for this problem.

While the results in this paper are completely new in the nonlinear setting, we emphasize that even for linear equations we prove new results: compared to previous works, our error estimates exhibit stronger stochastic integrability and the quenched Lipschitz estimate is new. Moreover, while we prove our results under an independence assumption, we believe that the arguments can be modified in a simple way to handle quantitative ergodicity assumptions in other forms (such as weaker mixing conditions or spectral gap-type assumptions) and to generalize easily to systems of equations under strong ellipticity assumptions.

In the next three subsections, we present the precise hypotheses and the statements of the main results.

#### **1.2. Modeling assumptions**

We take  $d \in \mathbb{N}$ ,  $d \geq 2$  and  $\Lambda \geq 1$  to be parameters which are fixed throughout the paper. We require the integrands  $L$  of our energy functionals to satisfy the following conditions:

- (L1)  $L: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$  is a *Carathéodory function*, that is,  $L(p, x)$  is measurable in x and continuous in p.
- (L2) L is uniformly convex in p: for every  $p_1, p_2, x \in \mathbb{R}^d$ ,

$$
\frac{1}{4}|p_1-p_2|^2 \leq \frac{1}{2}L(p_1,x) + \frac{1}{2}L(p_2,x) - L\left(\frac{1}{2}p_1 + \frac{1}{2}p_2,x\right) \leq \frac{\Lambda}{4}|p_1-p_2|^2.
$$

Note that (L2) implies  $L(\cdot, x)$  is  $C^1$ , for each  $x \in \mathbb{R}^d$ , and  $[D_p L(\cdot, x)]_{C^{0,1}(\mathbb{R}^d)} \leq \Lambda$ , where  $D_pL$  denotes the gradient of L with respect to the first variable.

We define  $\Omega$  to be the set of all such functions:

$$
\Omega := \{ L : L \text{ satisfies (L1) and (L2)} \}.
$$

Note that  $\Omega$  depends on the fixed parameter  $\Lambda > 1$ . We endow  $\Omega$  with the following family of  $\sigma$ -algebras: for each Borel  $U \subseteq \mathbb{R}^d$ , define

 $\mathcal{F}(U) :=$  the  $\sigma$ -algebra generated by the family of random variables

$$
L \mapsto \int_U L(p,x)\phi(x) dx, \quad p \in \mathbb{R}^d, \ \phi \in C_c^{\infty}(\mathbb{R}^d).
$$

The largest of these is denoted by  $\mathcal{F} := \mathcal{F}(\mathbb{R}^d)$ . It is also convenient to define a subset of  $\Omega$ consisting of Lagrangians L such that L and  $D_pL$  are locally bounded in p, uniformly in x. For each  $K \geq 0$ , We set

$$
\Omega(K) := \left\{ L \in \Omega \, : \, \forall p, x \in \mathbb{R}^d, \, |p|^2 - K(1+|p|) \le L(p,x) \le \Lambda |p|^2 + K(1+|p|) \right\}.
$$

The *random environment* is modeled by a given probability measure  $\mathbb P$  on  $(\Omega, \mathcal F)$ . The expectation with respect to  $\mathbb P$  is denoted by  $\mathbb E$ . We require  $\mathbb P$  to satisfy the following three assumptions:

(P1) P has a unit range of dependence: for all Borel subsets  $U, V \subseteq \mathbb{R}^d$  such that  $dist(U, V) \geq 1$ ,

 $\mathcal{F}(U)$  and  $\mathcal{F}(V)$  are P-independent.

(P2)  $\mathbb P$  is stationary with respect to  $\mathbb Z^d$ -translations: for every  $z \in \mathbb Z^d$  and  $E \in \mathcal F$ ,

$$
\mathbb{P}[E] = \mathbb{P}[T_z E],
$$

where the translation group  ${T_z}_{z \in \mathbb{Z}^d}$  acts on  $\Omega$  by  $(T_zL)(p, x) = L(p, x + z)$ .

(P3) L and  $D_pL$  are bounded locally uniformly in p and uniformly on the support of  $\mathbb P$ : there exists  $K_0 \geq 1$  such that

$$
\mathbb{P}\Big[L \in \Omega(K_0)\Big] = 1.
$$

These hypotheses are stronger than those of [9] in several respects. First, for the sake of simplicity, we consider only the case of quadratic growth. This assumption is probably not essential, and we speculate that adaptations of our arguments should give results, for example, in the case of Lagrangians growing like  $|p|^m$  for  $m > 1$ . Second, in (L2) we have strengthened the assumption of convexity to uniform convexity. While this assumption can probably be relaxed, some form of strict convexity is essential to our method. Third, the assumption of ergodicity has been strengthened to the independence condition (P1). Quantitative ergodicity assumptions are of course required for quantitative results, although

our methods yield quantitative homogenization results under, for example, much weaker *uniform mixing* conditions as well.

#### **1.3. A sub-optimal error estimate for the Dirichlet problem**

The first main result of the paper is an estimate for the error in homogenization of the Dirichlet problem. It gives a sub-optimal algebraic estimate for the size of the error but with essentially optimal stochastic integrability.

In the following statement and throughout the paper, we denote the Lebesgue measure of a set  $E \subseteq \mathbb{R}^d$  by  $|E|$  and set  $\oint_U f(x) dx := |U|^{-1} \int_U f(x) dx$ .

<span id="page-7-3"></span>THEOREM 1.1. – Let  $U \subseteq \mathbb{R}^d$  be a bounded Lipschitz domain,  $M \geq 1$ ,  $t > 2$  and  $s \in (0,d)$ . *There exist*  $\alpha(d, \Lambda, t) > 0$ ,  $C(d, \Lambda, s, t, U) > 1$  *and a nonnegative random variable*  $\mathcal{X}$  *on*  $(\Omega, \mathcal{F})$ *, depending on* (d,Λ, M, t, s) *and satisfying*

<span id="page-7-1"></span>
$$
\mathbb{E}\big[\exp(\mathcal{X})\big] \leq CM^d,
$$

*such that the following holds: for every*  $L \in \Omega$ ,  $\varepsilon \in (0,1]$  *and*  $g \in W^{1,t}(U)$  *such that* 

<span id="page-7-2"></span>(1.8) 
$$
K_0 + \left(\int_U |Dg(x)|^t dx\right)^{1/t} \leq M,
$$

*the unique functions*  $u^{\varepsilon}$ ,  $u_{\text{hom}} \in g + H_0^1(U)$  *for which* 

(1.9) 
$$
\int_{U} L\left(Du^{\varepsilon}(x), \frac{x}{\varepsilon}\right) dx \le \int_{U} L\left(Dw(x), \frac{x}{\varepsilon}\right) dx \text{ for every } w \in g + H_{0}^{1}(U),
$$

*and*

<span id="page-7-0"></span>(1.10) 
$$
\int_U \overline{L}(Du_{\text{hom}}(x)) dx \leq \int_U \overline{L}(Dw(x)) dx \text{ for every } w \in g + H_0^1(U),
$$

*satisfy the estimate*

(1.11) 
$$
\int_{U} |u^{\varepsilon}(x) - u_{\text{hom}}(x)|^{2} dx \leq CM^{2} (1 + \mathcal{X} \varepsilon^{s}) \varepsilon^{\alpha(d-s)}.
$$

By Chebyshev's inequality, the conclusion of Theorem 1.1 implies, in the notation of the theorem, that for some  $C(d, \Lambda, s, t, U) \geq 1$  and every  $s \in (0, d)$  and  $\delta \in (0, 1]$ ,

$$
\mathbb{P}\left[\exists \varepsilon \in (0,\delta], \int_U |u^{\varepsilon}(x) - u_{\text{hom}}(x)|^2 dx \geq CM^2 \varepsilon^{\alpha(d-s)}\right] \leq CM^d \exp\left(-\delta^{-s}\right).
$$

This is an algebraic estimate for the size of the homogenization error with very strong bounds on the probability of deviations. However, there is more information in (1.11) than the latter, and the former is often more convenient to work with since it is in the form of an *a priori* estimate (i.e.,  $\chi$  is independent of  $\varepsilon$ ,  $g$ , etc). Note the tradeoff between our control on the error threshold and the probability of deviations: as the exponent loses power, we gain more stochastic in[tegra](#page-0-0)bility, and vice versa. The dependence of  $\mathcal X$  on  $M$  can of course be removed in the linear case (i.e.,  $L$  a quadratic form), but in the general nonlinear setting the integrability of  $X$  necess[arily](#page-0-0) exhibits some mild dependence on  $M$ .

While Theorem 1.1 measures the error only in  $L^2$  in space, we also obtain interior error estimates in  $L^{\infty}$  by interpolating between the  $L^2$  error and the De Giorgi-Nash-Moser  $C^{0,\gamma}$ estimates. See Corollary 4.2.

#### **1.4. Stochastic higher regularity**

It is well known that minimizers of variable-coefficient energy functionals do not, in general, have better Hölder regularity than that provided by the De Giorgi-Nash-Moser estimate ( $C^{0,\gamma}$  for a small  $\gamma > 0$ ) or Sobolev regularity than that provided by the Meyers estimate (which is  $W^{1,2+\delta}$  for a small  $\delta > 0$ ). Nevertheless, the following theorems asserts that the regularity is typically much better for energy functionals sampled by a probability measure  $\mathbb P$  with a finite range of dependence: minimizers have  $C^{0,1} = W^{1,\infty}$  regularity, down to microscopic scales, and even  $C^{1,\beta}$  regularity down to mesoscopic scales.

THEOREM 1.2. – *Fix*  $M \ge 1$  *and*  $s \in (0,d)$ *. Then there exists*  $C(d, \Lambda, s) \ge 1$  *and a nonnegative random variable* Y *on* (Ω, F )*, depending on* (d,Λ, M, s)*, satisfying*

<span id="page-8-2"></span>
$$
\mathbb{E}\big[\exp\left(\mathcal{Y}^s\right)\big] \le CM^d
$$

*and the following: for every*  $L \in \Omega$ ,  $R \geq 2$  *and*  $u \in H^1(B_R)$  *satisfying* 

<span id="page-8-3"></span>(1.13) 
$$
K_0 + \frac{1}{R} \left( \int_{B_R} |u(x)|^2 dx \right)^{1/2} \leq M
$$

*and*

<span id="page-8-1"></span>
$$
(1.14) \qquad \int_{B_R} L\left(Du(x),x\right) dx \le \int_{B_R} L\left(Dw(x),x\right) dx \quad \text{for every } w \in u + H_0^1(B_R),
$$

*we have the estimate*

(1.15) 
$$
\frac{1}{r} \operatorname*{osc}_{B_r} u \le CM \quad \text{for every } \mathcal{Y} \le r \le \frac{1}{2}R.
$$

<span id="page-8-0"></span>*Moreover, there exist*  $\beta(d,\Lambda) > 0$  *and*  $c(d,\Lambda,s) > 0$  *such that, for every*  $\gamma \in (0,\beta]$  *and*  $r \in [R^{c\gamma}, R/2]$ ,

(1.16) 
$$
\inf_{p \in \mathbb{R}^d} \frac{1}{r} \operatorname*{osc}_{x \in B_r} (u(x) + p \cdot x) \le C \mathcal{Y} M \left(\frac{r}{R}\right)^{\gamma}.
$$

It is appropriate to consider a coarsening of the  $C^{0,1}$  seminorm by removing the effect of microscopic oscillations, as in the left side of (1.15), because the regularizing effect is [due to](#page-8-1) the correlation structure of the coefficients: which of course cannot have influence on scales smaller than the correlation length scale. Here we coarsen down to a *random* scale *Y*. Of course, we may also coarsen at the unit scale if we allow the right side to be ra[ndom](#page-8-1), as (1.15) implies

$$
\frac{1}{r} \operatorname*{osc}_{B_r} u \le C \mathcal{Y} M \quad \text{for every } 1 \le r \le \frac{1}{2} R.
$$

On the other hand, if the L's sampled by  $\mathbb P$  [are un](#page-8-1)iformly smooth, then (1.15) implies a full  $C^{0,1}$  estimate without the coarsening because in this case we may control the smaller scales by applying local Schauder estimates.

By the Caccioppoli inequality, the estimate (1.15) also implies

$$
\sup_{\mathcal{Y}\le r\le R/4}\int_{B_r}|Du(x)|^2\ dx\le CM^2.
$$

This gives very strong control of the energy density of local minimizers down to the unit scale. In the linear setting, special cases of this kind of estimate (applied to modified correctors and the Green's functions) lie at the heart of the quantitative theory of Gloria and Otto [15, 16] and Gloria, Neukamm an[d Ot](#page-0-0)to [14, 13]. Theorem 1.2 can therefore be used, together with spectral gap-ty[pe](#page-58-1) [co](#page-58-2)ncentration inequalities, to obtain an alternative proof of the optimal quantitative estimates obtained in these papers.

The proof of Theorem 1.2 is a nonlinear and quantitative version of an i[dea](#page-58-1) [o](#page-58-2)f Avellaneda and Lin [2, 3]: since the heterogeneous energy functional is close to the homogenized functional on large scales, we can obtain higher regularity by treating minimizers of the former as a perturbations of those [of th](#page-0-0)e latter. This idea was formalized in [2, 3] in the context of periodic media, via compactness arguments. In the stochastic setting here, we do not have compactness [and](#page-0-0) therefore the perturbation argument must [be m](#page-0-0)ore quantitative. The perfect tool is actually Theorem 1.1: what is needed is an algebraic rate of convergence in homogenization and strong control of the stochastic fluctuations. Note that the algebraic exponent in Theorem 1.1 disappears "into the constant" in Theorem 1.2 (so it is irrelevant that the exponent is sub-optimal), but the strong stochastic integrability  $\mathcal{Y}$  is inherited from that of  $\chi$ .

#### **1.5. Outline of the paper**

<span id="page-9-0"></span>Section 3 is the heart of the paper, wherein we state and prove the core resu[lt,](#page-39-0) Theorem 3.1, on the conv[erge](#page-0-0)nce of the subadditive and superadditive energies. In Section 2 [w](#page-43-0)e introduce the key concepts and make some preliminary observations in preparation for the proof of this result. We derive the error estimate for the Dirichlet problem in Section 4 by reducing it to Theorem 3.1, and from it we obtain the stochastic  $C^{0,1}$  estimate in Section 5.

#### **2. Subadditive and [su](#page-59-0)[per](#page-59-1)additive energies**

The analysis in the first part of this paper is centered on two monotone quantities involving the energy. Up to normalizing factors, one is *subadditive* and the other is *superadditive*. The former was considered already in [8, 9] and was the basis of the qualitative proof of homogenization there, while the latter is considered for the first time here. In this section we define these quantities and review some of their elementary properties, explain why they are convex duals of each other. We begin by reviewing some notation.

#### **2.1. Convention for constants**

Throughout the paper, unless otherwise indicated, the symbols  $C$  and  $c$  denote constants which depend on the dimension d and the parameter  $\Lambda$  in (L2) and may vary in each occurrence.

#### **2.2. Suppressing the dependence on** L

Throughout, the pr[obab](#page-0-0)ility space is  $(\Omega, \mathcal{F}, \mathbb{P})$  and L denotes the canonical element of  $\Omega$ that is sampled according to  $\mathbb P$ . Since it is cumbersome to display dependence on L in each of our quantities, we often suppress this dependence in our notation (for example, in the statement of Theorem 1.1, each of  $u^{\varepsilon}$ , u and  $\mathcal X$  depend on L). However, the reader should keep in mind that any quantity implicitly defined in terms of L is random, and the symbols  $\mathbb P$ and  $E$  should always be understood with respect to this randomness.

#### **2.3. Definition of the energy quantities**  $\mu$  and  $\nu$

For each  $p, q \in \mathbb{R}^d$ , bounded open subset  $U \subseteq \mathbb{R}^d$  and  $L \in \Omega$ , we define the two quantities

$$
\mu(U, q, L) := \min \left\{ \oint_U \left( L(Dw(x), x) - q \cdot Dw(x) \right) dx \, : \, w \in H^1(U) \right\}
$$

and

<span id="page-10-2"></span>
$$
\nu(U, p, L) := \min \left\{ \int_U L(Dw(x), x) \, dx \, : \, w - \ell_p \in H_0^1(U) \right\},
$$

where  $\ell_p$  denotes the plane  $\ell_p(x) := p \cdot x$ . The quantity  $\nu$  was introduced by Dal Maso and Modica [8, 9] and was central to their proof of qualitative homogenization. Note that, for every  $p \in \mathbb{R}^d$ ,  $U \subseteq \mathbb{R}^d$  and  $L \in \Omega$ ,

(2.1) 
$$
\sup_{q\in\mathbb{R}^d} (q \cdot p + \mu(U, q, L)) \leq \nu(U, p, L).
$$

To build some intuition for  $\mu$  and  $\nu$ , and to see that they are convex dual to each other in some sense, we examine the case that L has no spatial dependence, i.e.,  $L(p, x) = L(p)$ . The values of  $\mu$  and  $\nu$  are then easy to compute, as there is no dependence on U and the integrals may be removed. We obtain

$$
\mu(U, q, L) = \mu(q, L) = \min_{p \in \mathbb{R}^d} (L(p) - q \cdot p) \quad \text{and} \quad \nu(U, p, L) = \nu(p, L) = L(p).
$$

The Legendre transform  $L^*$  of  $L$  may therefore be written as

$$
L^*(q) = \sup_{p \in \mathbb{R}^d} (p \cdot q - L(p)) = -\mu(q, L).
$$

We thus observe that  $\nu$  identifies L while  $\mu$  naturally identifies  $L^*$ . We may also write L in terms of  $\mu$  [by](#page-37-0) duality:

(2.2) 
$$
L(p) = \sup_{q \in \mathbb{R}^d} (p \cdot q - L^*(q)) = \sup_{q \in \mathbb{R}^d} (p \cdot q + \mu(q, L)).
$$

In Section 3.4, we generalize (2.2) to the stochastic case.

As mentioned in the previous subsection, we usually suppress the dependence of  $\mu$  and  $\nu$ on L, unless required for clarity, by writing  $\mu(U, q)$  and  $\nu(U, p)$ . However, the reader should keep in mind that these quantities are random variables.

<span id="page-10-0"></span>We note that  $\mu(U, q)$  is well-defined and finite. Indeed, by (P3) we have

 $(2.3)$ [for every boun[ded,](#page-10-0) open  $U \subseteq \mathbb{R}^d$ ,  $-2(K_0 + |q|)^2 \leq \mu(U, q) \leq K_0$ ] = 1.

The first inequality holds because the left side (rather crudely) bounds  $L(p, x) - q \cdot p$  from below, uniformly in p, on the support of  $\mathbb{P}$ , which we see from the first inequality in (P3). We get the second inequality in (2.4) by taking zero as a test function in the definition of  $\mu$  and using the second inequality in (P3)[. In](#page-10-0) particular, as  $K_0 \geq 1$ , we obtain

<span id="page-10-1"></span>(2.4) 
$$
|\mu(U,q)| \le 2(K_0 + |q|)^2 \quad \mathbb{P}\text{-a.s.}
$$

A similar argument as the one for (2.4) leads to the bound

$$
(2.5) \t\t |p|^2 - K_0 (1+|p|) \le \nu(U, p) \le \Lambda |p|^2 + K_0 (1+|p|) \quad \mathbb{P}\text{-a.s.}
$$

Here we used Jensen's inequality in the expression for the energy of a minimizer to get the left side, and test with the zero function to get the right side.

#### **2.4.** Th[e mi](#page-0-0)nimizers  $u$  and  $v$

Up to an additive constant, the minimization problem in the definition of  $\mu$  has a unique minimizer in  $H^1(U)$ . (The uniqueness of the minimizer follows from uniform convexity, see Lemma 2.1 below.) We denote the unique minimizer which has mean zero on every connected component of U by  $u(\cdot, U, q)$ . In other words,  $u(\cdot, U, q)$  is the unique element of  $H^1(U)$ satisfying

<span id="page-11-1"></span>
$$
\begin{cases}\n\oint_{U} \left( L(Du(x, U, q), x) - q \cdot Du(x, U, q) \right) dx \\
\leq \oint_{U} \left( L(Dw(x), x) - q \cdot Dw(x) \right) dx \quad \forall w \in H^{1}(U), \\
\oint_{V} u(x, U, q) dx = 0 \quad \forall V \subseteq U \text{ with } V \text{ and } U \setminus V \text{ open.} \n\end{cases}
$$

We denote the spatial average of  $Du(\cdot, U, q)$  by

$$
(2.6) \t P(U,q) := \int_U Du(x,U,q) dx.
$$

We let  $v(\cdot, U, p)$  denote the minimizer for  $v(U, p)$ , that is, the unique function in  $H^1(U)$ satisfying

$$
\begin{cases}\n\oint_U L(Dv(x,U,p),x) dx \leq \oint_U L(Dv(x,U,p) + Dw(x),x) dx & \forall w \in H_0^1(U), \\
v(\cdot, U, p) - \ell_p \in H_0^1(U).\n\end{cases}
$$

<span id="page-11-0"></span>We stress once again that  $u(\cdot, U, q)$ ,  $v(\cdot, U, p)$  and  $P(U, q)$  are random elements, as they depend on L.

#### **2.5. Notation for cubes**

For each  $x \in \mathbb{R}^d$  and  $n \in \mathbb{N}_*$ , we define the triadic cube

$$
Q_n(x) := 3^n \lfloor 3^{-n} x \rfloor + \left( -\frac{1}{2}3^n, \frac{1}{2}3^n \right)^d.
$$

Here |r| denotes, for  $r \in \mathbb{R}$ , the largest integer not greater than r and, for a point  $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ , we set  $\lfloor x \rfloor := (\lfloor x_1 \rfloor, \ldots, \lfloor x_d \rfloor)$ . We have centered each cube  $Q_n(x)$ at a point on the lattice  $3^n \mathbb{Z}^d$  in order to keep them disjoint. In particular, we note that  $Q_n(x)$  is not necessarily centered at x and, neglecting a subset of Lebesgue measure zero (the boundary of the open cubes), we see that  $y \in Q_n(x)$  if and only if  $Q_n(x) = Q_n(y)$ . For  $m, n \in \mathbb{N}_*$ ,  $Q_{n+m}$  is the disjoint union, up to a zero measure set, of  $3^{dm}$  cubes of the form  $Q_n(x)$ . We write  $Q_n := Q_n(0)$ .

We also define the *trimmed* triadic cube by

$$
Q_n^{\circ}(x) := 3^n \lfloor 3^{-n} x \rfloor + \left( -\frac{1}{2} \left( 3^n - 1 \right), \frac{1}{2} \left( 3^n - 1 \right) \right)^d.
$$

The trimmed cube  $Q_n^{\circ}(x)$  is obtained from  $Q_n(x)$  by removing a layer of thickness 1/2 from each face. The reason we have trimmed this layer near the boundary is that it ensures that the trimmed cubes are separated by a unit distance from each other, which is convenient when we work with the independence assumption (P2). We set  $Q_n^{\circ} := Q_n^{\circ}(0)$ .

For future reference we note that the proportion of volume occupied by the trimmed region is of order  $3^{-n}$  since, for any  $n, m \in \mathbb{N}$ ,

<span id="page-12-3"></span>
$$
|Q_{n+m}| - 3^{dm} |Q_n^{\circ}| = 3^{d(n+m)} - 3^{dm} (3^n - 1)^d \leq C3^{d(n+m)-n}
$$

which implies

(2.7) 
$$
\frac{3^{dm} |Q_n^{\circ}|}{|Q_{n+m}|} \ge 1 - C3^{-n}
$$

We introduce a third family of cubes  $\widetilde{Q}_{n+1}$  defined for  $n \in \mathbb{N}_*$  by

$$
\widetilde{Q}_{n+1}(x) := 3^n \left[ 3^{-n} x \right] + Q_{n+1}.
$$

Thus  $\widetilde{Q}_{n+1}(x)$  is the cube centered at the same point as  $Q_n(x)$ , but with side lengths three times larger. These cubes are not disjoint and each cube of the form  $\tilde{Q}_{n+1}(x)$  int[erse](#page-0-0)cts  $5^d - 1$  others. Note that  $\widetilde{Q}_{n+1}(x)$  is the translation by an element of  $3^n \mathbb{Z}^d$  of the cube  $Q_{n+1}$ , and thus  $\mathbb P$  has the same statistical properties in these cubes by the stationarity assumption. We use this family of cubes when we compare the energy at different scales in Lemma 3.3, as the argument there requires some overlapping cubes in the construction.

#### **2.6.** Monotonicity of  $\mu$  and  $\nu$  and the definition of  $\overline{L}$

The quantity  $\mu$  has a monotonicity property which is immediate from its definition, obtained by simply restricting the minimizers of larger regions to smaller ones. Namely, the map  $U \mapsto |U|\mu(U, L, q)$  is *superadditive*, by which we mean that, for all collections of pairwise disjoint bounded open subsets  $U_1, \ldots, U_k \subseteq \mathbb{R}^d$  and every open set  $U \subseteq \mathbb{R}^d$  such that

$$
(2.8) \tU_1 \cup \cdots \cup U_k \subseteq U \text{ and } |U \setminus (U_1 \cup \cdots \cup U_k)| = 0,
$$

we have

(2.9) 
$$
\mu(U,q) \geq \sum_{j=1}^k \frac{|U_j|}{|U|} \mu(U_j,q).
$$

In other words,  $\mu(U, q)$  is bounded below by a weighted average of  $\{\mu(U_j, q)\}_{j=1}^k$ . To obtain (2.9), we note that, for each  $j$ ,

<span id="page-12-2"></span><span id="page-12-1"></span><span id="page-12-0"></span>
$$
\int_{U_j} \left( L\left(Du(x, U, q), x\right) - q \cdot Du(x, U, q) \right) dx \geq |U_j| \mu(U_j, q)
$$

and then sum over  $j \in \{1, \ldots, k\}$  $j \in \{1, \ldots, k\}$ .

The superadditivity of  $\mu$  implies that  $\mathbb{E}[\mu(Q_n, q)]$  is a monotone nondecreasing sequence in n. Indeed, recall that  $Q_{n+m}$  is the disjoint union of  $3^{dm}$  cubes of the form  $Q_n(x)$ , up to a zero measure set, and therefore (2.9) gives

(2.10) 
$$
\mu(Q_{n+m}, q) \geq 3^{-dm} \sum_{Q_n(x) \subseteq Q_{n+m}} \mu(Q_n(x), q).
$$

Taking expectations gives

$$
\mathbb{E}\left[\mu(Q_{n+m},q)\right] \geq \mathbb{E}\left[\mu(Q_n,q)\right].
$$

In view of  $(2.4)$ , the q[uantit](#page-12-1)y

<span id="page-13-2"></span>(2.12) 
$$
\overline{\mu}(q) := \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \mu(Q_n, q) \right]
$$

is finite and hence, by ([2.1](#page-59-1)1), we have the limit

(2.13) 
$$
\lim_{n \to \infty} \mathbb{E} \left[ \mu(Q_n, q) \right] = \overline{\mu}(q).
$$

<span id="page-13-1"></span>We next recall from [9] that, for each  $p \in \mathbb{R}^d$ , the quantity  $U \mapsto |U| \nu(U, p)$  is *subadditive*, i.e., for all collections of pairwise disjoint bounded open subsets  $U_1, \ldots, U_k \subseteq \mathbb{R}^d$  and open  $U \subseteq \mathbb{R}^d$  satisfying (2.8), we have

(2.14) 
$$
\nu(U, p) \leq \sum_{j=1}^{k} \frac{|U_j|}{|U|} \nu(U_j, p).
$$

This holds because a candidate for a minimizer for  $\nu(\cdot, p)$  in U can be obtained by assembling the minimizers in each of the  $U_j$ 's. Similarly to (2.10) and (2.11), specializing to the triadic cubes we deduce that, for every  $m, n \in \mathbb{N}$  and  $p \in \mathbb{R}^d$ ,

(2.15) 
$$
\nu(Q_{n+m}, p) \leq 3^{-dm} \sum_{Q_n(x) \subseteq Q_{n+m}} \nu(Q_n(x), p)
$$

and taking expectations and applying stationarity yields

$$
(2.16) \t\t\t\t\mathbb{E}\left[\nu(Q_{n+m},p)\right] \leq \mathbb{E}\left[\nu(Q_n,p)\right].
$$

We define the *effective Lagrangian*  $\overline{L}$  :  $\mathbb{R}^d \to \mathbb{R}$  by

<span id="page-13-3"></span>(2.17) 
$$
\overline{L}(p) := \inf_{n \in \mathbb{N}} \mathbb{E} \left[ \nu(Q_n, p) \right].
$$

Note that this is the same definition for  $\overline{L}$  given in [8, 9]. By (2.16), we have

(2.18) 
$$
\lim_{n \to \infty} \mathbb{E} \left[ \nu(Q_n, p) \right] = \overline{L}(p).
$$

### **2.7.** Comparisons between  $\mathbb{E}[\mu(Q_n, q)]$  and  $\mathbb{E}[\mu(Q_n^{\circ}, q)]$

<span id="page-13-0"></span>For our reference, we record here a few observations concerning the expectation of  $\mu$  in the trimmed and untrimmed dyadic cubes. We first note that  $\mu$  is also monotone with respect to the trim[med c](#page-13-0)ubes, up to a small error. We have:

$$
(2.19) \qquad \mu(Q_{n+m}^{\circ}, q) \ge 3^{-dm} \sum_{Q_n^{\circ}(x) \subseteq Q_{n+m}} \mu(Q_n^{\circ}(x), q) - C\left(K_0 + |q|\right)^2 3^{-n} \quad \mathbb{P}\text{-a.s.}
$$

<span id="page-13-4"></span>To obtain (2.19), we note that in view of the remarks in Section 2.5, we may write  $Q_{m+n}^{\circ}$ , up to a zero measure set, as the union of  $3^{dm}$  cubes of the form  $Q_n^{\circ}(x) \subseteq Q_{m+n}^{\circ}$  and an open set of measure at most  $C3^m$ . We then deduce (2.19) from (2.4), (2.7) and (2.9). Taking expectations and using stationarity gives

(2.20) 
$$
\mathbb{E}\left[\mu(Q_{m+n}^{\circ},q)\right] \geq \mathbb{E}\left[\mu(Q_{n}^{\circ},q)\right] - C(K_0+|q|)^2 3^{-n}.
$$

By a similar argument, we obtain

$$
(2.21) \qquad \nu(Q_{n+m}^{\circ}, p) \leq 3^{-dm} \sum_{Q_n^{\circ}(x) \subseteq Q_{n+m}} \nu(Q_n^{\circ}(x), q) + C\left(K_0 + |p|\right)^2 3^{-n} \quad \mathbb{P}\text{-a.s.}
$$

<span id="page-14-2"></span>It is also useful to have some comparison between  $\mu$  and  $\nu$  in the trimmed and untrimmed cubes. By  $|Q_n \setminus \overline Q^{\circ}_n$  $|Q_n \setminus \overline Q^{\circ}_n$  $|Q_n \setminus \overline Q^{\circ}_n$  $|n| \leq 3^{-n} |Q_n|$ , (2.4) and (2.9), we have

(2.22) 
$$
\mu(Q_n^{\circ}, q) \leq \mu(Q_n, q) + C(K_0 + |q|)^2 3^{-n} \quad \mathbb{P}\text{-a.s.}
$$

Similarly, by  $(2.5)$  and  $(2.14)$ , we have

<span id="page-14-0"></span>(2.23) 
$$
\nu(Q_n^{\circ}, p) \ge \nu(Q_n, p) - C(K_0 + |p|)^2 3^{-n} \quad \mathbb{P}\text{-a.s.}
$$

We also need an inequality bounding  $\mathbb{E}[\mu(Q_n, q)]$  from above by  $\mathbb{E}[\mu(Q_n^{\circ}, q)]$ :

$$
(2.24) \mathbb{E}[\mu(Q_n, q)] \leq \mathbb{E}[\mu(Q_n^{\circ}, q)] + 3^d \left( \mathbb{E}[\mu(Q_{n+1}^{\circ}, q)] - \mathbb{E}[\mu(Q_n^{\circ}, q)] + C(K_0 + |q|)^2 3^{-n} \right).
$$

To get this, observe that the cube  $Q_{n+1}^{\circ}$  is the disjoint union (up to a set of measure zero) of the untrimmed cube  $Q_n$ ,  $3^d - 1$  trimmed cubes of the form  $Q_n^{\circ}(x)$  and a set of measure at most  $C3^{n(d-1)}$ . We obtain (2.24) after applying superadditivity, stationarity and (2.4) to this partition.

#### **2.8. Basic energy estimates**

In this subsection we record tw[o sim](#page-59-16)ple consequences of the uniform convexity assumption (L2) which are used repeatedly in the paper. The first is the following lemma, which gives gradient estimates for functions which are close to minimizers. The lemma is classical, e.g., it is almost the same as Giaquinta [10, Chapter IX, Lemma 4.1].

LEMMA 2.1. – *For every*  $L \in \Omega$ ,  $q \in \mathbb{R}^d$ , bounded open  $U \subseteq \mathbb{R}^d$  and  $w, \xi \in H^1(U)$ ,

$$
\int_{U} |Dw(x) - D\xi(x)|^{2} dx
$$
\n
$$
\leq 2 \Biggl( \int_{U} \bigl( L(Dw(x), x) - q \cdot Dw(x) \bigr) dx + \int_{U} \bigl( L(D\xi(x), x) - q \cdot D\xi(x) \bigr) dx - 2\mu(U, q, L) \Biggr).
$$

*Proof.* – Set  $\zeta := \frac{1}{2}w + \frac{1}{2}\xi \in H^1(U)$ . Using (L2), we compute

$$
\mu(U, q, L) \le \int_U \left( L(D\zeta(x), x) - q \cdot D\zeta(x) \right) dx
$$
  
\n
$$
\le \frac{1}{2} \int_U \left( L(Dw(x), x) - q \cdot Dw(x) \right) dx
$$
  
\n
$$
+ \frac{1}{2} \int_U \left( L(D\xi(x), x) - q \cdot D\xi(x) \right) dx - \frac{1}{4} \int_U |Dw(x) - D\xi(x)|^2 dx.
$$
  
\n
$$
\text{earrangement of the previous inequality gives the lemma.} \qquad \Box
$$

A rearrangement of the previous inequality gives the lemma.

<span id="page-14-1"></span>In some arguments in the next section, we apply Lemma 2.1 in the case that  $w = u(\cdot, U, q)$ and  $\xi = u(\cdot, V, q)$  for open sets U, V satisfying  $U \subset V$  and  $|V \setminus U| = 0$  (i.e., V is the interior of the closure of the disconnected set  $U$ ). The conclusion of the lemma gives

(2.25) 
$$
\int_{U} |Du(x, U, q) - Du(x, V, q)|^{2} dx \leq 2 (\mu(V, q) - \mu(U, q)).
$$

A particular case of (2.25) allows us to compare the gradients of minimizers in the trimmed and untrimmed cubes. Write  $U := Q_n^{\circ} \cup (Q_n \setminus Q_n^{\circ})$ , apply the inequality with  $V = Q_n$  and then use (2.4) (2.7), (2.9) and to get

$$
\int_{U} |Du(x, U, q) - Du(x, Q_n, q)|^2 dx
$$
\n
$$
\leq C (\mu(Q_n, q) - \mu(U, q))
$$
\n
$$
\leq C (\mu(Q_n, q) - \frac{|Q_n^{\circ}|}{|Q_n|} \mu(Q_n^{\circ}, q) - \frac{|Q_n \setminus Q_n^{\circ}|}{|Q_n|} \mu(Q_n \setminus Q_n^{\circ}, q))
$$
\n
$$
\leq C (\mu(Q_n, q) - \mu(Q_n^{\circ}, q)) + C (K_0^2 + |q|^2) 3^{-n}.
$$

Taking ex[pect](#page-10-0)ations [and](#page-12-3) using that

$$
u(\cdot, U, q)|_{Q_n^{\circ}} \equiv u(\cdot, Q_n^{\circ}, q) \quad \text{and} \quad u(\cdot, U, q)|_{Q_n \setminus Q_n^{\circ}} \equiv u(\cdot, Q_n \setminus Q_n^{\circ}, q),
$$

<span id="page-15-2"></span>as well as  $(2.4)$  and  $(2.7)$  again, we get

$$
(2.26) \quad \frac{1}{|Q_n|} \int_{Q_n^{\circ}} |Du(x, Q_n^{\circ}, q) - Du(x, Q_n, q)|^2 \, dx + \frac{1}{|Q_n|} \int_{Q_n \setminus Q_n^{\circ}} |Du(x, Q_n, q)|^2 \, dx
$$

$$
\leq C \left( \mathbb{E} \left[ \mu(Q_n, q) \right] - \mathbb{E} \left[ \mu(Q_n^{\circ}, q) \right] \right) + C \left( K_0^2 + |q|^2 \right) 3^{-n}.
$$

<span id="page-15-1"></span>Another consequence of Lemma 2.1 is the gradient bound

(2.27) 
$$
\int_{U} |Du(x, U, q)|^{2} dx \leq 6(K_{0} + |q|)^{2} \quad \mathbb{P}\text{-a.s.}
$$

This we get by comparing  $u(\cdot, U, q)$  to the zero function, using (P3) and (2.4).

<span id="page-15-0"></span>We also have the following variation of Lemma 2.1 from a nearly identical argument: for every  $w, \xi \in H^1(U)$  such that  $\frac{1}{2}w + \frac{1}{2}\xi - \ell_p \in H^1_0(U)$ , we have

$$
(2.28) \int_U |Dw(x) - D\xi(x)|^2 dx \le 2 \left( \int_U (L(Dw(x), x) + L(D\xi(x), x)) dx - 2\nu(U, p) \right).
$$

<span id="page-15-3"></span>(Recall that  $\ell_p$  is the plane  $\ell_p(x) = p \cdot x$ .) Comparing  $v(\cdot, U, p)$  to  $\ell_p$ , applying (P3) and (2.28) and using the triangle inequality, we get

(2.29) 
$$
\int_{U} |Dv(x, U, p)|^{2} dx \leq C (K_{0} + |p|)^{2} \quad \mathbb{P}\text{-a.s.}
$$

The second consequence of uniform convexity is kind of converse of Lemma 2.1 which allows us to perturb minimizers without increasing the energy too much.

LEMMA 2.2. – For every 
$$
L \in \Omega
$$
,  $q \in \mathbb{R}^d$ , bounded open  $U \subseteq \mathbb{R}^d$  and  $w, \xi \in H^1(U)$ ,

$$
\int_{U} \left( L(Dw(x), x) - q \cdot Dw(x) \right) dx
$$
\n
$$
\leq 2 \int_{U} \left( L(D\xi(x), x) - q \cdot D\xi(x) \right) dx - \mu(U, q, L) + 2\Lambda \int_{U} \left| Dw(x) - D\xi(x) \right|^{2} dx.
$$

Proof. – We set 
$$
\zeta := 2\xi - w \in H^1(U)
$$
 so that  $\xi = \frac{1}{2}w + \frac{1}{2}\zeta$  and then use (L2) to get  
\n
$$
\oint_U (L(D\xi(x), x) - q \cdot D\xi(x)) dx
$$
\n
$$
\geq \frac{1}{2} \oint_U (L(Dw(x), x) - q \cdot Dw(x)) dx + \frac{1}{2} \oint_U (L(D\zeta(x), x) - q \cdot D\zeta(x)) dx
$$
\n
$$
- \frac{\Lambda}{4} \oint_U |Dw(x) - D\zeta(x)|^2 dx
$$
\n
$$
\geq \frac{1}{2} \oint_U (L(Dw(x), x) - q \cdot Dw(x)) dx + \frac{1}{2}\mu(U, q, L)
$$
\n
$$
- \Lambda \oint_U |Dw(x) - D\xi(x)|^2 dx.
$$

A rearrangement now yields the lemma.

Similar to Lemma 2.1, we often use Lemma 2.2 in the case that  $\xi = u(\cdot, U, q)$  is the minimizer for  $\mu(U, q)$ , in which case [the](#page-0-0) conclusion gives, for every  $w \in H^1(U)$ ,

 $\Box$ 

$$
(2.30) \quad \int_U \left( L(Dw(x),x) - q \cdot Dw(x) \right) dx \leq \mu(U,q) + 2\Lambda \int_U \left| Dw(x) - Du(x,U) \right|^2 dx.
$$

<span id="page-16-0"></span>We also obtain a version of Lemma 2.2 for planar boundary conditions, which states that, for every  $p \in \mathbb{R}^d$  and  $w, \xi \in H^1(U)$  such that  $2\xi - w - \ell_p \in H_0^1(U)$ , we have

$$
(2.31) \quad \int_{U} L(Dw(x), x) dx
$$
  

$$
\leq 2 \int_{U} L(D\xi(x), x) dx - \nu(U, p, L) + 2\Lambda \int_{U} |Dw(x) - D\xi(x)|^{2} dx.
$$

#### **2.9.** Further properties of  $\mu$ ,  $\nu$  and  $\overline{L}$  $\overline{L}$  $\overline{L}$

For our reference, we record here some [prop](#page-16-0)erties of  $\mu$  and  $\nu$  and their minimizers, particularly regarding their dependence on  $p$  and  $q$ .

<span id="page-16-1"></span>An immediate consequence of (2.28) and (2.31) is that  $p \mapsto \nu(x, U, p)$  is uniformly convex in p. Precisely, we claim that

$$
(2.32) \qquad \frac{1}{4}|p_1-p_2|^2 \leq \frac{1}{2}\nu(U,p_1) + \frac{1}{2}\nu(U,p_2) - \nu(U,\tfrac{1}{2}p_1+\tfrac{1}{2}p_2) \leq \frac{\Lambda}{4}|p_1-p_2|^2.
$$

<span id="page-16-2"></span>To get the first inequality of (2.32), apply (2.28) with  $w = v(\cdot, U, p_1)$  and  $\xi = v(\cdot, U, p_2)$ ; to get the second inequality, apply (2.31) with  $p = p_1$ ,  $w = v(\cdot, U, p_2)$  and  $\xi = v(\cdot, U, \frac{1}{2}p_1 + \frac{1}{2}p_2)$ . A further [conse](#page-15-0)quence of (2.5) and (2.32) is the continuity of  $\nu$  in  $p$ [: fo](#page-16-1)r every  $p_1, p_2 \in \mathbb{R}^d$ , we have

$$
(2.33) \t |\nu(U, p_1) - \nu(U, p_2)| \le C (K_0 + |p_1| + |p_2|) |p_1 - p_2| \quad \mathbb{P}\text{-a.s.}
$$

Applying (2.28) to  $w := v(\cdot, U, p_1)$  and  $\xi := v(\cdot, U, p_2)$  and using (2.32), we get

<span id="page-16-3"></span>(2.34) 
$$
\int_{U} |Dv(x, U, p_1) - Dv(x, U, p_2)|^2 dx \leq \Lambda |p_1 - p_2|^2.
$$

Specializing to  $U = Q_n$  and applying the Poincaré inequality, we get

$$
(2.35) \t\t 3^{-2n} \int_{Q_n} \left( v(x, Q_n, p_1) - v(x, Q_n, p_2) \right)^2 dx \leq C \left| p_1 - p_2 \right|^2.
$$

The effective Lagrangian  $\overline{L}$  defined in (2.17) satisfies the same growth and uniform convexity conditions as L: for every  $p, p_1, p_2 \in \mathbb{R}^d$ ,

<span id="page-17-2"></span>
$$
(2.36) \t\t |p|^2 - K_0(1+|p|) \le \overline{L}(p) \le \Lambda |p|^2 + K_0(1+|p|),
$$

<span id="page-17-4"></span>
$$
(2.37) \qquad \frac{1}{4}|p_1 - p_2|^2 \le \frac{1}{2}\overline{L}(p_1) + \frac{1}{2}\overline{L}(p_2) - \overline{L}\left(\frac{1}{2}p_1 + \frac{1}{2}p_2\right) \le \frac{\Lambda}{4}|p_1 - p_2|^2
$$

and

(2.38) 
$$
\left|\overline{L}(p_1) - \overline{L}(p_2)\right| \leq C \left(K_0 + |p_1| + |p_2|\right) |p_1 - p_2|.
$$

<span id="page-17-3"></span>These are immediate consequences of  $(2.5)$ ,  $(2.32)$  and  $(2.33)$ . Observe that  $(2.37)$  implies that  $\overline{L}$  is differentiable at every  $p \in \mathbb{R}^d$  and  $D\overline{L}$  is Lipschitz continuous. In fact, for every  $p, p_1, p_2 \in \mathbb{R}^d$ , we have

$$
(2.39) \qquad |D\overline{L}(p)| \le C(K_0+|p|) \quad \text{and} \quad |D\overline{L}(p_1)-D\overline{L}(p_2)| \le 2\Lambda|p_1-p_2|.
$$

Finally, we record the continuity of the maps  $q \mapsto \mu(U, q)$  and  $q \mapsto u(\cdot, U, q)$ . Using  $u(\cdot, U, q_1)$  as a minimizer candidate for  $\mu(U, q_2)$  and the estimate (2.27), we discover that

<span id="page-17-5"></span>
$$
\mu(U, q_2) \le \mu(U, q_1) + C|q_1 - q_2| (K_0 + |q_1|) \quad \mathbb{P}\text{-a.s.}
$$

We ded[uce t](#page-0-0)hat, for every  $q_1, q_2 \in \mathbb{R}^d$  and bounded open  $U \subseteq \mathbb{R}^d$ ,

$$
(2.40) \qquad |\mu(U, q_1) - \mu(U, q_2)| \leq C (K_0 + |q_1| + |q_2|) |q_1 - q_2| \quad \mathbb{P}\text{-a.s.}
$$

Lemma 2.1 and a similar computation yield that

$$
(2.41) \qquad \int_{U} \left| Du(x, U, q_1) - Du(x, U, q_2) \right|^2 dx \le C \left( K_0 + |q_1| + |q_2| \right) |q_1 - q_2| \quad \mathbb{P}\text{-a.s.}
$$

Specializing to the case  $U = Q_n$  and applying the Poincaré inequality, we get

<span id="page-17-0"></span>
$$
(2.42) \ \ 3^{-2n} \int_{Q_n} \left( u(x, Q_n, q_1) - u(x, Q_n, q_2) \right)^2 dx \le C \left( K_0 + |q_1| + |q_2| \right) |q_1 - q_2| \quad \mathbb{P}\text{-a.s.}
$$

#### **3. Convergence of the e[nergy](#page-13-2) and [flatnes](#page-13-3)s of minimizers**

In this section we prove our first quantitative result: a sub-optimal algebraic estimate for the rate of convergence in the limits (2.13) and (2.18) as well as for the flatness of the respective minimizers. It is the main step toward the results stated in the introduction.

**THEOREM** 3.1. – *Fix*  $q \in \mathbb{R}^d$ . *There exist*  $\alpha(d, \Lambda) > 0$ ,  $c(d, \Lambda) > 0$ ,  $C(d, \Lambda) \geq 1$  and a unique  $\overline{P}(q) \in \mathbb{R}^d$  such that

<span id="page-17-1"></span>(3.1) 
$$
\overline{\mu}(q) + \overline{P} \cdot q = \overline{L}(\overline{P})
$$

*and, for every*  $s \in (0, d)$ *,*  $R \geq 1$ *,*  $n \in \mathbb{N}^*$  *and*  $t \geq 1$ *,* 

$$
(3.2) \quad \mathbb{P}\bigg[\sup_{y\in B_R} \left( |\mu(y+Q_n, q) - \overline{\mu}(q)| + |\nu(y+Q_n, \overline{P}) - \overline{L}(\overline{P})| \right) \bigg] \ge C(K_0 + |q|)^2 3^{-n\alpha(d-s)} t \bigg] \le CR^d \exp\left(-c3^{sn} t\right)
$$

<span id="page-18-2"></span>*and*

(3.3)  
\n
$$
\mathbb{P}\left[\sup_{y\in B_R} 3^{-2n} \int_{y+Q_n} \left( u(x, y+Q_n, q) - \overline{P} \cdot (x-y) \right)^2 + \left( v(x, y+Q_n, \overline{P}) - \overline{P} \cdot x \right)^2 dx \right]
$$
\n
$$
\geq C(K_0 + |q|)^2 3^{-n\alpha(d-s)} t \right] \leq CR^d \exp\left(-c3^{sn} t\right).
$$

We prove Theorem 3.1 in Section 3.3 using the flatness theory we construct in Section 3.2 plus an elementary concentration argument. The final subsection contains extensions of Theorem 3.1 and a demonstration of the fact that  $-\overline{\mu}$  and  $\overline{L}$  are convex dual functions.

#### **3.1. Reduction to the case**  $q = 0$

It suffices to prove Theorem 3.1 in the case that  $q = 0$ . To see this, suppose that the statement of the theorem holds in this special case and fix  $q \in \mathbb{R}^d$ . Consider the probability measure  $\mathbb{P}_q$  on  $(\Omega, \mathcal{F})$  which is the pushforward of  $\mathbb{P}$  under the map  $L \mapsto L_q$ , where the latter is defined by

(3.4) 
$$
L_q(p,x) := L(p,x) - q \cdot p.
$$

(Recall that if  $\pi : \Omega \to \Omega$  is an *F*-measurable map, then the *pushforward of*  $\mathbb P$  *[und](#page-0-0)er*  $\pi$  is the probability measure denoted by  $\pi_{\#} \mathbb{P}$  and defined for  $E \in \mathcal{F}$  by  $\pi_{\#} \mathbb{P} [E] := \mathbb{P} [\pi^{-1}(E)].$ Then it is easy to check that  $\mathbb{P}_q$  satisfies the assumptions (P1), (P2) and (P3) after we replace the constant  $K_0$  in (P3) by  $K_0 + |q|$ . Applying the special case of Theorem 3.1 with  $\mathbb{P}_q$  in place of  $\mathbb P$  and rewriting [the s](#page-18-0)tate[men](#page-30-0)t in terms of  $\mathbb P$  itself, we obtain the general statement of the theorem.

<span id="page-18-0"></span>Therefore, in Sections 3.2 and 3.3 we assume  $q = 0$  and drop q from our notation by writing, for example,  $\mu(U)$ ,  $u(\cdot, U)$  and  $P(U)$  in place of  $\mu(U, 0)$ ,  $u(\cdot, U, 0)$  and  $P(U, 0)$ . The variable  $q$  is reintroduced in Section 3.4 once the proof of Theorem 3.1 is complete.

#### **3.2. The flatness of minimizers**

The primary task in the proof of Theorem 3.1 is to quantify the limit (2.13). The main step, which is the focus of this subsection, is to show that, for  $n \gg 1$ , the minimizer  $u(\cdot, Q_n)$ is close to a plane. This allows us to compare  $\mu$  to  $\nu(\cdot, p)$  for an a[ppro](#page-11-1)priate choice of  $p \in \mathbb{R}^d$ .

In the first step, we use the finite range of dependence assumption to show that, unless the expectation of  $\mu$  increases significantly when passing to a larger scale, the variance of the average slope vector P must be small (recall that P is defined in  $(2.6)$ ). Since the argument is based on independence, we work with the trimmed cubes.

<span id="page-18-1"></span>**LEMMA** 3.2. – *There exists*  $C(d, \Lambda) \ge 1$  *such that, for every*  $n \in \mathbb{N}$ *,* 

(3.5) 
$$
\operatorname{var}[P(Q_n^{\circ})] \leq C \left( \mathbb{E}\left[\mu(Q_{n+1}^{\circ})\right] - \mathbb{E}\left[\mu(Q_n^{\circ})\right] + C K_0^2 3^{-n} \right).
$$

*Proof.* – Fix  $n \in \mathbb{N}$  and a unit vector  $e \in \partial B_1$ . Select a smooth vector field

$$
B:Q_{n+1}^\circ\to \mathbb{R}^d
$$

with

$$
\operatorname{div} B = 0 \quad \text{in } Q_{n+1}^{\circ}
$$

such that B has compact support in  $Q_{n+1}^{\circ}$  and satisfies  $B \equiv e$  in  $Q_n^{\circ}$  and the estimate  $|B| \leq C$ in  $Q_{n+1}^{\circ}$ . To see that such a solenoidal vector field exists, consider without loss of generality the case  $n = 0$  the take  $h \in H^1((Q_0 + B_{1/2}) \setminus (Q_0 + B_{1/4}))$  to be the solution of the Neumann problem

$$
\begin{cases}\n-\Delta h = 0 & \text{in } (Q_0 + B_{1/2}) \setminus (Q_0 + B_{1/4}), \\
\partial_{\nu} h = e \cdot \nu & \text{on } \partial(Q_0 + B_{1/2}), \\
\partial_{\nu} h = 0 & \text{on } \partial(Q_0 + B_{1/4}),\n\end{cases}
$$

where (unlike in the rest of the paper) in the previous line  $\nu$  denotes the outer unit normal vector. Then we define a vector field  $\widetilde{B}: \mathbb{R}^d \to \mathbb{R}^d$  by

<span id="page-19-0"></span>
$$
\widetilde{B}(x) := \begin{cases}\n e & \text{if } x \in Q_0 + B_{1/4}, \\
 e - Dh(x) & \text{if } x \in (Q_0 + B_{1/2}) \setminus (Q_0 + B_{1/4}), \\
 0 & \text{if } x \in \mathbb{R}^d \setminus (Q_0 + B_{1/2}).\n\end{cases}
$$

It is clear from the construction that  $\widetilde{B} \in L^{\infty}(\mathbb{R}^d; \mathbb{R}^d)$  is solenoidal in  $\mathbb{R}^d$  and has support in  $Q_0 + B_{1/2}$ . We may then obtain B as above by mollifying  $\widetilde{B}$ .

Observe that

(3.6) 
$$
\int_{Q_{n+1}^{\circ}} B(x) \cdot Du(x, Q_{n+1}^{\circ}) dx = 0.
$$

Let U be the union of the trimmed subcubes of  $Q_{n+1}^{\circ}$  of the form  $Q_n^{\circ}(x) \subseteq Q_{n+1}^{\circ}$  and set  $V := Q_{n+1}^{\circ} \setminus \overline{U}$ . Since U and V are disjoint, we have

$$
u(\cdot, U \cup V)|_U = u(\cdot, U)
$$
 and  $u(\cdot, U \cup V)|_V = u(\cdot, V)$ .

Similarl[y, fo](#page-19-0)r each  $x \in Q_{n+1}^{\circ}$ , we have

$$
u(\cdot, U \cup V)|_{Q_n^{\circ}(x)} = u(\cdot, Q_n^{\circ}(x)).
$$

By previous two lines and stationarity, we have  $\mathbb{E}[\mu(U)] = \mathbb{E}[\mu(Q_n^{\circ})]$ . Thus we may rewrite (3.6) as

$$
\int_{U} B(x) \cdot Du(x, U) dx
$$
  
= 
$$
\int_{U \cup V} B(x) \cdot (Du(x, U \cup V) - Du(x, Q_{n+1}^{\circ})) dx - \int_{V} B(x) \cdot Du(x, V) dx.
$$

Applying (2.25) [an](#page-12-0)d (2.27), using  $|B| \leq C$ ,  $|V| \leq C3^{-n} |Q_{n+1}|$  $|V| \leq C3^{-n} |Q_{n+1}|$  $|V| \leq C3^{-n} |Q_{n+1}|$  and (2.4), we obtain

(3.7) 
$$
\left(\int_{U} B(x) \cdot Du(x, U) dx\right)^{2} \leq C \left(\mu(Q_{n+1}^{\circ}) - \mu(U \cup V) + C K_{0}^{2} 3^{-n}\right).
$$

Using again that  $|V| \leq C3^{-n} |Q_{n+1}|$  with (2.4), we see from (2.9) that

$$
\mathbb{E}[\mu(U \cup V)] \ge \frac{|U|}{|U \cup V|} \mathbb{E}[\mu(U)] - C K_0^2 3^{-n} \ge \mathbb{E}[\mu(Q_n^{\circ})] - C K_0^2 3^{-n}.
$$

<span id="page-20-0"></span>Taking the expectation of (3.7) and using the previous line, we get

$$
(3.8) \qquad \mathbb{E}\left[\left(\oint_{U} B(x) \cdot Du(x,U) \, dx\right)^2\right] \leq C \left(\mathbb{E}\left[\mu(Q_{n+1}^{\circ})\right] - \mathbb{E}\left[\mu(Q_n^{\circ})\right] + CK_0^2 3^{-n}\right).
$$

Finally, we note that  $dist(Q_n^{\circ}, U \setminus Q_n^{\circ}) \ge 1$  by construction and therefore, using (P2), we see that the random variables

$$
\oint_{Q_n^{\circ}} B(x) \cdot Du(x, Q_n^{\circ}) dx \quad \text{and} \quad \oint_{U \setminus Q_n^{\circ}} B(x) \cdot Du(x, U \setminus Q_n^{\circ}) dx
$$

are P-independent.

 $\Box$ 

Therefore, using th[a](#page-20-0)t  $B(x) = e$  in  $Q_n^{\circ}$  as well as

$$
u(\cdot, Q_n^{\circ}) = u(\cdot, U)|_{Q_n^{\circ}}
$$
 and  $u(\cdot, U \setminus Q_n^{\circ}) = u(\cdot, U)|_{U \setminus Q_n^{\circ}}$ ,

we obtain from independence and (3.8) that

$$
\begin{split}\n\text{var}\left[e \cdot P(Q_n^{\circ})\right] &= \text{var}\left[\int_{Q_n^{\circ}} e \cdot Du(x, Q_n^{\circ}) \, dx\right] \\
&\leq \text{var}\left[\int_{Q_n^{\circ}} e \cdot Du(x, Q_n^{\circ}) \, dx\right] + \text{var}\left[\int_{U \setminus Q_n^{\circ}} B(x) \cdot Du(x, U \setminus Q_n^{\circ}) \, dx\right] \\
&= \text{var}\left[\int_U B(x) \cdot Du(x, U) \, dx\right] \\
&\leq \mathbb{E}\left[\left(\int_U B(x) \cdot Du(x, U) \, dx\right)^2\right] \\
&\leq C \left(\mathbb{E}\left[\mu(Q_{n+1}^{\circ})\right] - \mathbb{E}\left[\mu(Q_n^{\circ})\right] + CK_0^2 3^{-n}\right).\n\end{split}
$$

Summing over e in the standard basis for  $\mathbb{R}^d$  yields the lemma.

Motivated by the previous [lemm](#page-15-1)a, we define, for each  $n \in \mathbb{N}$ , the deterministic slope  $\overline{P}_n \in \mathbb{R}^d$  at the *n*th scale by

<span id="page-20-2"></span>
$$
\overline{P}_n := \mathbb{E}\left[ P(Q_n^{\circ}) \right].
$$

We note fo[r fut](#page-18-1)ure reference that  $(2.27)$  implies

<span id="page-20-1"></span>
$$
|\overline{P}_n| \le 3K_0.
$$

We may formulate the previous lemma in terms of the untrimmed cubes. Indeed, by  $(2.11)$ , (2.26) and (3.5), we have

$$
(3.10) \qquad \mathbb{E}\left[\left|P(Q_n)-\overline{P}_n\right|^2\right] \leq C\left(\mathbb{E}\left[\mu(Q_{n+1})\right]-\mathbb{E}\left[\mu(Q_n^\circ)\right]+CK_0^23^{-n}\right).
$$

We next present the key assertion regarding the flatness of minimizers. It states that, if  $\mathbb{E}[\mu(Q_{n+1})]-\mathbb{E}[\mu(Q_n^{\circ})]$  is small, then we can construct a candidate for the minimizer of  $\mu$  on an arbitrarily large scale which is very close to a plane of slope  $\overline{P}_n$  and [has](#page-0-0) expected energy not much more than  $\mathbb{E}[\mu(Q_n)]$ . By modifying the latter a little so that it has affine boundary conditions, we get a minimizer candidate for  $\nu(Q_{2n}^{\circ}, \overline{P}_n)$ , which gives an upper bound for  $\mathbb{E}[\nu(Q_{2n}^{\circ}, \overline{P}_n)]$  in terms of  $\mathbb{E}[\mu(Q_n)]$ . The argument uses Lemma 3.2 and a patching construction.

LEMMA 3.3. – *There exists a constant*  $C(d, \Lambda) > 0$  *and a universal*  $\alpha > 0$  *such that, for every*  $n \in \mathbb{N}$ *,* 

$$
(3.11) \qquad \mathbb{E}\left[\nu(Q_{2n}^{\circ}, \overline{P}_n)\right] \leq \mathbb{E}\left[\mu(Q_n)\right] + C\left(\mathbb{E}\left[\mu(Q_{n+1})\right] - \mathbb{E}\left[\mu(Q_n^{\circ})\right] + CK_0^2 3^{-\alpha n}\right).
$$

*Proof.* – Throughout, we fix  $n \in \mathbb{N}$ , let C denote a positive constant depending only on  $(d, \Lambda)$  which may vary in each occurrence, and denote

<span id="page-21-0"></span>
$$
\tau_n := \left(\mathbb{E}\left[\mu(Q_{n+1})\right] - \mathbb{E}\left[\mu(Q_n^{\circ})\right] + CK_0^2 3^{-n/60}\right).
$$

To prove the lemma, it suffices to construct  $v \in H_0^1(Q_{2n}^\circ)$  satisfying

(3.12) 
$$
\mathbb{E}\left[\int_{Q_{2n}^{\circ}} L\left(\overline{P}_n + Dv(x), x\right) dx\right] \leq \mathbb{E}\left[\mu(Q_n)\right] + C\tau_n.
$$

In the first step we give the construction of  $v$ . The rest of the argument is concerned with the proof of (3.12).

*Step 1.* – We construct the candidate minimizer  $v \in H_0^1(Q_{2n}^{\circ})$ . We build v by patching the minimizers for  $\mu$  on the family  $\{z + Q_{n+1} : z \in 3^n \mathbb{Z}^d\}$  of overlapping triadic cubes. We consider a partition of unity subordinate to this family by denoting, for each  $z \in 3^n \mathbb{Z}^d$ ,

<span id="page-21-2"></span>
$$
\psi(y) := \int_{Q_n} \psi_0(y - x) \, dx,
$$

where  $\psi_0 \in C_c^{\infty}(\mathbb{R}^d)$  is a smooth function satisfying

$$
0 \le \psi_0 \le C3^{-dn}, \quad \int_{\mathbb{R}^d} \psi_0(x) \, dx = 1, \quad |D\psi_0| \le C3^{-(d+1)n}, \quad \text{supp } \psi_0 \subseteq Q_n.
$$

It follows then that  $\psi$  is smooth and supported in  $Q_{n+1}$ , satisfies  $0 \leq \psi \leq 1$ , and the translates of  $\psi$  form a partition of unity:

(3.13) 
$$
\sum_{z \in 3^n \mathbb{Z}^d} \psi(x - z) = 1.
$$

Moreover, we have

<span id="page-21-3"></span>
$$
\sup_{x \in \mathbb{R}^d} |D\psi(x)| \le C3^{-n}.
$$

We next two smooth cutoff functions  $\xi, \zeta \in C_c^{\infty}(Q_{2n})$  satisfying

<span id="page-21-1"></span>
$$
(3.15) \quad 0 \le \xi \le 1, \quad \xi \equiv 1 \text{ on } \left\{ x \in Q_{2n} : \text{dist}(x, \partial Q_{2n}) > 3^{2n/(1+\delta)} \right\},
$$

$$
\xi \equiv 0 \text{ on } \left\{ x \in z + Q_n : z \in 3^n \mathbb{Z}^d, \ z + Q_{n+1} \not\subseteq Q_{2n}^\circ \right\}, \quad |D\xi| \le C3^{-2n/(1+\delta)},
$$

where  $\delta \in (0, \beta]$  will be selected below in Step 6, and

$$
(3.16) \quad 0 \le \zeta \le 1, \quad \zeta \equiv 1 \text{ on } \{x \in z + Q_n : z \in 3^n \mathbb{Z}^d, \ z + Q_{n+3} \subseteq Q_{2n}\},\
$$

$$
\zeta \equiv 0 \text{ on } \{x \in z + Q_{n+1} : z \in 3^n \mathbb{Z}^d, \ z + Q_{n+1} \not\subseteq Q_{2n}\}, \quad |D\zeta| \le C3^{-n}.
$$

To construct v, we first define a vector field  $\mathbf{f} \in L^2(\mathbb{R}^d; \mathbb{R}^d)$  by

(3.17) 
$$
\mathbf{f}(x) := \zeta(x) \sum_{z \in 3^n \mathbb{Z}^d} \psi(x - z) \left( Du(x, z + Q_{n+1}) - \overline{P}_n \right).
$$

Since  $f$  is not necessarily the gradient of an  $H<sup>1</sup>$  function, due to the errors made by introducing the partition of unity and the cutoff function, we need to take its Helmholtz-Hodge projection. We may write

(3.18) 
$$
\mathbf{f} = \overline{\mathbf{f}} + Dw - \text{div } \mathbf{S} \quad \text{in } Q_{2n},
$$

where

<span id="page-22-0"></span>
$$
\overline{\mathbf{f}} := \int_{Q_{2n}} \mathbf{f}(x) \, dx,
$$

 $w \in H^1_{loc}(\mathbb{R}^d)$  is defined as the unique solution of

$$
\begin{cases}\n-\Delta w = -\operatorname{div} \mathbf{f} & \text{in } Q_{2n}, \\
\oint_{Q_{2n}} w(x) dx = 0, \\
w \text{ is } Q_{2n}\text{-periodic},\n\end{cases}
$$

and **S** is a skew-symmetric matrix with entries  $S_{ij} \in H_{loc}^1(\mathbb{R}^d)$  uniquely determined (up to a constant) by

$$
\begin{cases}\n-\Delta S_{ij} = \partial_j f_i - \partial_i f_j & \text{in } Q_{2n}, \\
S_{ij} \text{ is } Q_{2n}\text{-periodic.} \n\end{cases}
$$

Here  $f_i$  is the *i*th entry of **f** and div **S** is the vector field [with e](#page-22-0)ntries  $\sum_{j=1}^d \partial_j S_{ij}$ . Indeed, one may check via a straightforward computation that each component of the vector field  $\mathbf{f} - Dw + \text{div } \mathbf{S}$  is harmonic and therefore constant by periodicity. This constant must be  $\mathbf{\bar{f}}$ since  $Dw$  and div **S** have zero mean in  $Q_{2n}$ . This confirms (3.18).

We define  $v \in H_0^1(Q_{2n}^{\circ})$  by cutting off w:

$$
v(x) := \xi(x)w(x), \quad x \in Q_{2n}.
$$

The cutoff function  $\xi$  is supported in  $Q_{2n}^{\circ}$  and thus  $v \in H_0^1(Q_{2n}^{\circ})$ .

The rest of the proof concerns the derivation of (3.12). This is accomplished by showing that Dv is expected to be close to **f** in  $L^2(Q_{2n}^{\circ})$  due to the fact that w,  $\bar{f}$  and div **S** each have a small expected  $L^2$  norm.

*Step 2.* – [We s](#page-0-0)how that, for every  $z \in 3^n \mathbb{Z}^d \cap Q_{2n}$ ,

$$
(3.19) \qquad \mathbb{E}\bigg[\oint_{z+Q_n}\Big(\big|\mathbf{f}(x)-\zeta(x)\,\big(Du(x,z+Q_{n+1})-\overline{P}_n\big)\big|^2\bigg]\leq C\tau_n.
$$

By Lemma 2.1 we have, for every  $z \in 3^n \mathbb{Z}^d$ ,

$$
\sum_{y \in \{-3^n, 0, 3^n\}^d} \int_{z+y+Q_n} |Du(x, z+Q_{n+1}) - Du(x, z+y+Q_n)|^2 dx
$$
\n
$$
\leq C \sum_{y \in \{-3^n, 0, 3^n\}^d} \left( \int_{z+y+Q_n} L \left( Du(x, z+Q_{n+1}), x \right) dx - \mu(y+z+Q_n) \right)
$$
\n
$$
= C \left( \mu(z+Q_{n+1}) - \sum_{y \in \{-3^n, 0, 3^n\}^d} \mu(z+y+Q_n) \right).
$$

<span id="page-23-2"></span>Taking expectations and using the triangle inequality, we get, for every  $z \in 3^n \mathbb{Z}^d$  and  $y \in \{-3^n, 0, 3^n\}^d$ ,

$$
(3.20) \qquad \mathbb{E}\left[\int_{z+Q_n} |Du(x,z+y+Q_{n+1}) - Du(x,z+Q_{n+1})|^2\ dx\right] \leq C\tau_n.
$$

Now observe that (3.19) follows from the previous inequality and the fact that, for every  $z \in 3^n \mathbb{Z}^d$  and  $x \in z + Q_n$ ,

<span id="page-23-1"></span>
$$
\mathbf{f}(x) - \zeta(x) \left( Du(x, z + Q_{n+1}) - \overline{P}_n \right) \n= \zeta(x) \sum_{y \in \{-3^n, 0, 3^n\}^d} \psi(x - y) \left( Du(x, z + y + Q_{n+1}) - Du(x, z + Q_{n+1}) \right).
$$

*Step 3*. – [We sh](#page-21-1)ow that

$$
(3.21) \t\t \mathbb{E}\left[\left|\overline{\mathbf{f}}\right|^2\right] \leq C\tau_n.
$$

In view of (3.16), it is convenient to denote

(3.22) 
$$
Z_n := \{ z \in 3^n \mathbb{Z}^d : z + Q_{n+3} \subseteq Q_{2n} \}.
$$

Observe that  $(3^n \mathbb{Z}^d \cap Q_{2n}) \setminus \mathbb{Z}_n$  has  $C3^{n(d-1)}$  elements. By (3.10), (3.19) and (2.27), we have

$$
\mathbb{E}\left[\left|\bar{\mathbf{f}}\right|^{2}\right] \leq \mathbb{E}\left[3^{-nd} \sum_{z \in 3^{n}\mathbb{Z}^{d} \cap Q_{2n}}\left|\int_{z+Q_{n}}\mathbf{f}(x) dx\right|^{2}\right]
$$
\n
$$
\leq 2\mathbb{E}\left[3^{-nd} \sum_{z \in 3^{n}\mathbb{Z}^{d} \cap Q_{2n}}\left|\int_{z+Q_{n}}\zeta(x)\left(Du(x,z+Q_{n+1}) dx - \overline{P}_{n}\right)\right|^{2}\right] + C\tau_{n}
$$
\n
$$
\leq 2\mathbb{E}\left[3^{-nd} \sum_{z \in \mathbb{Z}_{n}}\left|\int_{z+Q_{n}} Du(x,z+Q_{n+1}) dx - \overline{P}_{n}\right|^{2}\right] + CK_{0}^{2}3^{-n} + C\tau_{n}
$$
\n
$$
\leq C\tau_{n}.
$$

*Step 4*. – We show that

(3.23) 
$$
\mathbb{E}\left[3^{-4n}\int_{Q_{2n}}|w(x)|^2\ dx\right] \leq CK_0^2 3^{-n/2}.
$$

Let  $\phi \in H^2_{\text{loc}}(\mathbb{R}^d)$  denote the unique solution of

<span id="page-23-0"></span>
$$
\left\{ \begin{aligned} &-\Delta \phi = w \ \text{ in } \mathbb{R}^d, \\ &\oint_{Q_{2n}} \phi(x) \, dx = 0, \\ &\phi \ \text{ is } Q_{2n} \text{-periodic.} \end{aligned} \right.
$$

Integrating by parts, we have

$$
(3.24) \quad \int_{Q_{2n}} \left| D^2 \phi(x) \right|^2 dx = \int_{Q_{2n}} \left| w(x) \right|^2 dx = \int_{Q_{2n}} D\phi(x) \cdot \mathbf{f}(x) dx
$$

$$
= \int_{Q_{2n}} D\phi(x) \cdot \left( \mathbf{f}(x) - \mathbb{E} \left[ \int_{Q_n} \mathbf{f}(x) dx \right] \right) dx.
$$

We need a second mesoscale, given by an integer  $k \in (n, 2n)$  to be selected below. In what follows, we denote  $(D\phi)_z := \int_{z+Q_k} D\phi(x) dx$  and  $\sum_z = \sum_{z \in 3^k \mathbb{Z}^d \cap Q_{2n}}$  as well as  $\widetilde{\mathbf{f}} := \mathbf{f} - \mathbb{E}\left[\int_{Q_n} \mathbf{f}(x) \, dx\right].$  To estimate  $\|w\|_{L^2(Q_{2n})},$  we use the following splitting:

$$
\int_{Q_{2n}} |w(x)|^2 dx = \int_{Q_{2n}} D\phi(x) \cdot \tilde{\mathbf{f}}(x) dx
$$
  
= 
$$
\sum_{z} \left( \int_{z+Q_k} (D\phi(x) - (D\phi)_z) \cdot \tilde{\mathbf{f}}(x) dx + (D\phi)_z \cdot \int_{z+Q_k} \tilde{\mathbf{f}}(x) dx \right).
$$

To estimate the first term in the sum on the right side of the previous inequality, we use the Poincaré inequality, the discrete Hölder inequality, (3.24) and (2.27), and then Young's inequality:

$$
\sum_{z} \int_{z+Q_{k}} (D\phi(x) - (D\phi)_{z}) \cdot \tilde{\mathbf{f}}(x) dx
$$
\n
$$
\leq \sum_{z} 3^{k} \left( \int_{z+Q_{n}} |D^{2}\phi(x)|^{2} dx \right)^{\frac{1}{2}} \left( \int_{z+Q_{n}} \left| \tilde{\mathbf{f}}(x) \right|^{2} dx \right)^{\frac{1}{2}}
$$
\n
$$
\leq 3^{k} \left( \sum_{z} \int_{z+Q_{n}} |D^{2}\phi(x)|^{2} dx \right)^{\frac{1}{2}} \left( \sum_{z} \int_{z+Q_{n}} \left| \tilde{\mathbf{f}}(x) \right|^{2} dx \right)^{\frac{1}{2}}
$$
\n
$$
= 3^{k} \left( \int_{Q_{2n}} |D^{2}\phi(x)|^{2} dx \right)^{\frac{1}{2}} \left( \int_{Q_{2n}} \left| \tilde{\mathbf{f}}(x) \right|^{2} dx \right)^{\frac{1}{2}}
$$
\n
$$
\leq 3^{k} \left( \int_{Q_{2n}} |w(x)|^{2} dx \right)^{\frac{1}{2}} (CK_{0}^{2} |Q_{2n}|)^{\frac{1}{2}}
$$
\n
$$
\leq \frac{1}{4} \int_{Q_{2n}} |w(x)|^{2} dx + CK_{0}^{2} |Q_{2n}| 3^{2k}.
$$

We next estimate the expectation of the second term in the sum, using two different forms of Hölder's inequality:

(3.25) 
$$
\mathbb{E}\left[\sum_{z} (D\phi)_{z} \cdot \int_{z+Q_{k}} \tilde{\mathbf{f}}(x) dx\right]
$$
  
\n
$$
= \sum_{z} \mathbb{E}\left[(D\phi)_{z} \cdot \int_{z+Q_{k}} \tilde{\mathbf{f}}(x) dx\right]
$$
  
\n
$$
\leq \sum_{z} \mathbb{E}\left[|(D\phi)_{z}|^{2}\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_{z+Q_{k}} \tilde{\mathbf{f}}(x) dx\right)^{2}\right]^{\frac{1}{2}}
$$
  
\n
$$
\leq \left(\sum_{z} \mathbb{E}\left[(D\phi)_{z}|^{2}\right]\right)^{\frac{1}{2}} \left(\sum_{z} \mathbb{E}\left[\left(\int_{z+Q_{k}} \tilde{\mathbf{f}}(x) dx\right)^{2}\right]\right)^{\frac{1}{2}}.
$$

<span id="page-25-0"></span>For the first factor on the right side of the previous inequality, we have, by the Poincaré inequality and (3.24),

$$
(3.26) \qquad \sum_{z} \mathbb{E}\left[ |(D\phi)_z|^2 \right] = \mathbb{E}\left[ \sum_{z} |(D\phi)_z|^2 \right] \leq C3^{-kd} \mathbb{E}\left[ \int_{Q_{2n}} |D\phi(x)|^2 dx \right]
$$

$$
\leq C3^{-kd+4n} \mathbb{E}\left[ \int_{Q_{2n}} |D^2\phi(x)|^2 dx \right]
$$

$$
= C3^{-kd+4n} \mathbb{E}\left[ \int_{Q_{2n}} |w(x)|^2 dx \right].
$$

In preparation to estimate the second factor, we use independence to get

$$
\mathbb{E}\left[\left(\int_{Q_k} \widetilde{\mathbf{f}}(x) dx\right)^2\right] = \mathbb{E}\left[\sum_{y,y'\in 3^n\mathbb{Z}^d\cap Q_k} \int_{y+Q_n} \widetilde{\mathbf{f}}(x) dx \int_{y'+Q_n} \widetilde{\mathbf{f}}(x) dx\right]
$$
  

$$
\leq C \left(CK_0 |Q_n|\right)^2 \left|3^n \mathbb{Z}^d \cap Q_k\right| = C K_0^2 3^{d(k+n)}.
$$

By stationarity, the sam[e esti](#page-15-1)mate holds with  $z + Q_k$  in place of  $Q_k$  provided that the cube  $z + Q_k$  does not touch  $\partial Q_{2n}$ . For the cubes which do touch the boundary of the macroscopic cube (and thus intersect the support of  $\zeta$ ), we use the following cruder, deterministic bound given by (2.27):

<span id="page-25-1"></span>
$$
\left(\int_{Q_k} \widetilde{\mathbf{f}}(x) dx\right)^2 \leq C3^{dk} \int_{Q_k} \left|\widetilde{\mathbf{f}}(x)\right|^2 dx \leq C K_0^2 3^{2dk}.
$$

Combining these, using that there are at most  $C3^{(2n-k)(d-1)}$  cubes of the form  $z + Q_k$  which touch the boundary of  $Q_n$ , we get

$$
(3.27) \qquad \qquad \sum_{z} \mathbb{E}\left[\left(\int_{z+Q_k} \widetilde{\mathbf{f}}(x) \, dx\right)^2\right] \leq CK_0^2 3^{2dn+dk} \left(3^{-d(k-n)} + 3^{-(2n-k)}\right).
$$

We may now estimate the right side of (3.25) using applying (3.26), (3.27) and Young's inequality. The result is:

$$
(3.28) \mathbb{E}\left[\sum_{z} (D\phi)_z \cdot \int_{z+Q_k} \tilde{\mathbf{f}}(x) dx\right] \le \frac{1}{4} \mathbb{E}\left[\int_{Q_{2n}} |w(x)|^2 dx\right] + K_0^2 |Q_{2n}| 3^{4n} \left(C' 3^{-d(k-n)} + C 3^{-(2n-k)}\right)
$$

Combining the above inequalities now yields

<span id="page-25-2"></span>
$$
\mathbb{E}\left[\int_{Q_{2n}}|w(x)|^2\ dx\right]\leq K_0^23^{4n}\left|Q_{2n}\right|\left(C'3^{-d(k-n)}+C3^{-(2n-k)}\right).
$$

Taking finally  $k$  to be the nearest integer to  $3n/2$ , we obtain (3.23).

*Step 5.* – We estimate the expected size of  $|div \mathbf{S}|^2$ . The claim is

(3.29) 
$$
\mathbb{E}\left[\int_{Q_{2n}}\left|\mathrm{div}\,\mathbf{S}(x)\right|^2\,dx\right]\leq C\tau_n.
$$

As in the previous step, we use the abbreviations  $u_z := u(\cdot, z + Q_{n+1}), \psi_z := \psi(\cdot - z)$ ,  $\sum_z := \sum_{z \in 3^n \mathbb{Z}^d \cap Q_{2n}}$  and  $\int_z := \int_{z+Q_{n+1}}$  Observe that, in the sense of distributions, for every  $i, j \in \{1, \ldots, d\}$ , we have

$$
\partial_j f_i - \partial_i f_j = \sum_z \zeta \left( \partial_j \psi_z (\partial_i u_z - \overline{P}_{n,i}) - \partial_i \psi_z (\partial_j u_z - \overline{P}_{n,j}) \right)
$$

$$
+ \sum_z \psi_z \left( \partial_j \zeta (\partial_i u_z - \overline{P}_{n,i}) \right) - \partial_i \zeta (\partial_j u_z - \overline{P}_{n,j}) \right) \quad \text{in } Q_{2n}.
$$

The right side belongs to  $L^2(Q_{2n})$ , thus  $\partial_j f_i - \partial_i f_j \in L^2(Q_{2n})$  and  $S_{ij} \in H^2_{loc}(\mathbb{R}^d)$ . Using the fact that, for every  $x \in \mathbb{R}^d$ ,

(3.30) 
$$
\sum_{z} \zeta(x) D\psi_{z}(x) = 0,
$$

we may also express the previous identity slightly differently as

(3.31) 
$$
(\partial_j f_i - \partial_i f_j)(x) = \sum_z \zeta(x) \left[ D\psi_z(x), D u_z(x) - \overline{P}_n - \mathbf{f}(x) \right]_{ij} + \sum_z \psi_z \left[ D\zeta, D u_z - \overline{P}_n \right]_{ij} \text{ in } Q_{2n},
$$

where we henceforth use the notation

<span id="page-26-0"></span>
$$
[\mathbf{v},\mathbf{w}]_{ij}:=v_jw_i-v_iw_j
$$

for indices  $i, j \in \{1, ..., d\}$  and vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$  with entries  $(v_i)$  and  $(w_i)$ , respectively. Next we define, for each  $i \in \{1, \ldots, d\}$ ,

$$
\sigma_i := -(\text{div }\mathbf{S})_i = -\sum_{j=1}^d \partial_j S_{ij}.
$$

It is evident that  $\sigma_i \in H^1_{per}(Q_{2n})$  and  $\sigma_i$  is a solution of the equation

$$
-\Delta \sigma_i = -\sum_{j=1}^d \partial_j (\partial_j f_i - \partial_i f_j) \quad \text{in } Q_{2n}.
$$

Since  $\sigma_i$  has zero mean in  $Q_{2n}$ , there exists  $\rho_i \in H_{\text{loc}}^3(\mathbb{R}^d)$ , which is unique up to an additive constant, satisfying

$$
\begin{cases}\n-\Delta \rho_i = \sigma_i & \text{in } \mathbb{R}^d, \\
\rho_i & \text{is } Q_{2n}\text{-periodic.} \n\end{cases}
$$

We have the identities

<span id="page-26-1"></span>(3.32) 
$$
\int_{Q_{2n}} |D^2 \rho_i(x)|^2 dx = \int_{Q_{2n}} |\sigma_i(x)|^2 dx = \int_{Q_{2n}} D\rho_i(x) \cdot D\sigma_i(x) dx.
$$

Integrating by parts and using the equation for  $\sigma_i$ , we obtain

$$
\int_{Q_{2n}} |\sigma_i(x)|^2 dx = \int_{Q_{2n}} D\rho_i(x) \cdot D\sigma_i(x) dx = \sum_{j=1}^d \int_{Q_{2n}} \partial_j \rho_i(x) \left( \partial_j f_i - \partial_i f_j \right) dx.
$$

To further shorten the notation, in each of the following expressions we keep the sum over  $j$ implicit (note that  $i$  is not summed over even though it is repeated) and set

 $(\partial_j \rho_i)_z := \int_{z+Q_{n+1}} \partial_j \rho_i(x) dx$ . Continuing then the computation by substituting (3.31), we obtain

$$
\int_{Q_{2n}} \partial_j \rho_i(x) \left( \partial_j f_i - \partial_i f_j \right) dx
$$
\n
$$
= \sum_z \int_z \left( \partial_j \rho_i(x) - (\partial_j \rho_i)_z \right) \zeta(x) \left[ D\psi_z(x), Du_z(x) - \overline{P}_n - \mathbf{f}(x) \right]_{ij} dx
$$
\n
$$
+ \sum_z (\partial_j \rho_i)_z \int_z \zeta(x) \left[ D\psi_z(x), Du_z(x) - \overline{P}_n - \mathbf{f}(x) \right]_{ij} dx
$$
\n
$$
+ \sum_z \int_z \partial_j \rho_i(x) \psi_z(x) \left[ D\zeta(x), Du_z(x) - \overline{P}_n \right]_{ij} dx.
$$

We put the second sum on the right side into a more convenient form via (3.17), (3.13), integration by parts and (3.30):

$$
\sum_{z} \left(\partial_{j}\rho_{i}\right)_{z} \int_{z} \zeta(x) \left[D\psi_{z}(x), Du_{z}(x) - \overline{P}_{n} - \mathbf{f}(x)\right]_{ij} dx
$$
  
\n
$$
= \sum_{y,z} \left(\partial_{j}\rho_{i}\right)_{z} \int_{z} \zeta(x)\psi_{y}(x) \left[D\psi_{z}(x), Du_{z}(x) - \overline{P}_{n} - \zeta(x)(Du_{y}(x) - \overline{P}_{n})\right]_{ij} dx
$$
  
\n
$$
= \sum_{y,z} \left(\left(\partial_{j}\rho_{i}\right)_{z} - \left(\partial_{j}\rho_{i}\right)_{y}\right) \int_{z} - \left(\zeta(x)\right)^{2}\psi_{y}(x) \left[D\psi_{z}(x), Du_{y}(x) - \overline{P}_{n}\right]_{ij} dx
$$
  
\n
$$
+ \sum_{z} \left(\partial_{j}\rho_{i}\right)_{z} \int_{z} - \psi_{z}(x) \left[D\zeta(x), Du_{z}(x) - \overline{P}_{n}\right]_{ij} dx.
$$

<span id="page-27-0"></span>Combining this with the previous identity, we get

$$
(3.33) \quad \int_{Q_{2n}} |\sigma_i(x)|^2 dx
$$
  
=  $\sum_{z} \int_{z} (\partial_j \rho_i(x) - (\partial_j \rho_i)_z) \zeta(x) [D\psi_z(x), Du_z(x) - \overline{P}_n - \mathbf{f}(x)]_{ij} dx$   
+  $\sum_{y,z} ((\partial_j \rho_i)_y - (\partial_j \rho_i)_z) \int_{z} (\zeta(x))^2 \psi_y(x) [D\psi_z(x), Du_y(x) - \overline{P}_n]_{ij} dx$   
+  $\sum_{z} \int_{z} (\partial_j \rho_i(x) - (\partial_j \rho_i)_z) \psi_z(x) [D\zeta(x), Du_z(x) - \overline{P}_n]_{ij} dx.$ 

We now proceed to estimate the three terms on the right side of (3.33). For the first sum, we use (3.14), the Hölder, discrete Hölder and Poincaré inequalities, (3.32) and Young's inequality to get

$$
\sum_{z} \int_{z} \left( \partial_{j} \rho_{i}(x) - (\partial_{j} \rho_{i})_{z} \right) \zeta(x) \left[ D \psi_{z}(x), D u_{z}(x) - \overline{P}_{n} - \mathbf{f}(x) \right]_{ij} dx
$$
\n
$$
\leq C \left( \sum_{z} \int_{z} \left| D^{2} \rho_{i}(x) \right|^{2} dx \right)^{\frac{1}{2}} \left( \sum_{z} \int_{z} \left| D u_{z}(x) - \overline{P}_{n} - \mathbf{f}(x) \right|^{2} dx \right)^{\frac{1}{2}}
$$
\n
$$
\leq \frac{1}{4} \int_{Q_{2n}} \left| \sigma_{i}(x) \right|^{2} dx + C \sum_{z} \int_{z} \left| D u_{z}(x) - \overline{P}_{n} - \mathbf{f}(x) \right|^{2} dx.
$$

Taking expectations and using (3.19), we obtain

$$
(3.34) \mathbb{E}\left[\sum_{z}\int_{z}\left(\partial_{j}\rho_{i}(x)-\left(\partial_{j}\rho_{i}\right)_{z}\right)\zeta(x)\left[D\psi_{z}(x), Du_{z}(x)-\overline{P}_{n}-\mathbf{f}(x)\right]_{ij}dx\right]
$$
  

$$
\leq \frac{1}{4}\mathbb{E}\left[\int_{Q_{2n}}\left|\sigma_{i}(x)\right|^{2}dx\right]+C\left|Q_{2n}\right|\tau_{n}.
$$

For the second sum, we notice that each entry vanishes unless  $y \in z + Q_{n+2}$ , there are at most C such entries y in the sum for any given entry z, and for such y and z, the Poincaré inequality gi[ves](#page-21-2)

$$
\left| \left( \partial_j \rho_i \right)_y - \left( \partial_j \rho_i \right)_z \right|^2 \leq C 3^{2n} |Q_n|^{-1} \int_{z + Q_{n+3}} \left| D^2 \rho_i(x) \right|^2 dx.
$$

Using this, (3.14), (3.32) and (2.4), the Hölder and Young inequalities and the fact that  $3^n \mathbb{Z}^d \cap Q_{2n}$  has  $C3^{nd}$  elements, we get

$$
\sum_{y,z} \left( (\partial_j \rho_i)_y - (\partial_j \rho_i)_z \right) \int_z (\zeta(x))^2 \psi_y(x) \left[ D \psi_z(x), D u_y(x) - \overline{P}_n \right]_{ij} dx
$$
  
\n
$$
\leq C \left( 3^{2n} |Q_n|^{-1} \sum_z \int_{z + Q_{n+3}} |D^2 \rho_i(x)|^2 dx \right)^{\frac{1}{2}} \left( \sum_z 3^{-2n} C K_0^2 |Q_n| \right)^{\frac{1}{2}}
$$
  
\n
$$
\leq \frac{1}{4} \int_{Q_{2n}} |\sigma_i(x)|^2 dx + C K_0^2 3^{nd}.
$$

For the third sum on the right side of (3.33), we proceed in almost the same way as for the first two, except that rather than use (3.14) we use the estimate for  $D\zeta$  in (3.16) and the fact that D $\zeta$  vanishes except if  $z \notin \mathcal{Z}_n$  (recall that  $\mathcal{Z}_n$  is defined in (3.22)) and there are at most  $C3^{n(d-1)}$  such elements in the sum. We obtain:

$$
\sum_{z} \int_{z} \left( \partial_{j} \rho_{i}(x) - (\partial_{j} \rho_{i})_{z} \right) \psi_{z}(x) \left[ D\zeta(x), Du_{z}(x) - \overline{P}_{n} \right]_{ij} dx
$$
  

$$
\leq C \left( 3^{2n} \sum_{z} \int_{z} \left| D^{2} \rho_{i}(x) \right|^{2} dx \right)^{\frac{1}{2}} \left( \sum_{z \notin \mathcal{Z}_{n}} 3^{-2n} C K_{0}^{2} |Q_{n}| \right)^{\frac{1}{2}}
$$
  

$$
\leq \frac{1}{4} \int_{Q_{2n}} |\sigma_{i}(x)|^{2} dx + C K_{0}^{2} 3^{n} (2d - 1).
$$

Combining the previous [two i](#page-25-2)nequalities with (3.32), (3.33) and (3.34) yields

<span id="page-28-0"></span>
$$
\mathbb{E}\left[\int_{Q_{2n}}|\sigma_i(x)|^2\ dx\right] \leq CK_0^2 3^{n(2d-1)} + C\left|Q_{2n}\right|\tau_n.
$$

Dividing by  $|Q_{2n}|$  gives (3.29).

*Step 6.* – We show that the effect of the cutoff  $\xi$  in the definitions of v and **h** is expected to be small: precisely,

(3.35) 
$$
\mathbb{E}\left[\int_{Q_{2n}}\left|Dv(x)-Dw(x)\right|^2\,dx\right]\leq C\tau_n.
$$

<span id="page-29-0"></span>We [use the](#page-21-3) identity

$$
Dv(x) - Dw(x) = w(x)D\xi(x) + (\xi(x) - 1)Dw(x)
$$

and (3.15) to obtain

$$
(3.36) \int_{Q_{2n}} |Dv(x) - Dw(x)|^2 dx
$$
  

$$
\leq C3^{-4n/(1+\delta)} \int_{Q_{2n}} |w(x)|^2 dx + C \int_{Q_{2n}} |\xi(x) - 1|^2 |Dw(x)|^2 dx.
$$

The expectation of the first integral on the right side is controlled by (3.23):

$$
\mathbb{E}\left[3^{-4n/(1+\delta)}\int_{Q_{2n}}|w(x)|^2\ dx\right]\leq CK_0^2 3^{-4n/(1+\delta)+7n/2}\leq C3^{-n/60}\leq C\tau_n,
$$

where we have defined

$$
\delta:=\frac{1}{14}.
$$

For the expectation of the second integral on the right side of (3.36), we recall from (3.15) that  $\xi \equiv 1$  except in

$$
D := \left\{ x \in Q_{2n} : \text{dist}(x, \partial Q_{2n}) > C3^{2n/(1+\delta)} \right\}.
$$

Therefore, using that D intersects at most  $C3^{n(d-2\delta/(1+\delta))}$  subcubes of the form  $z + Q_{n+1}$ , with  $z \in 3^n \mathbb{Z}^d$ , and applying (2.27), (3.21) and (3.29), we obtain

$$
(3.37) \qquad \mathbb{E}\left[\int_{Q_{2n}} |\xi(x) - 1|^2 |Dw(x)|^2 dx\right] \le \frac{1}{|Q_{2n}|} \mathbb{E}\left[\int_D |Dw(x)|^2 dx\right] \\
\le \frac{C}{|Q_{2n}|} \mathbb{E}\left[\int_D |\mathbf{f}(x)|^2 dx + \int_{Q_{2n}} |\mathbf{f}(x) - Dw(x)|^2 dx\right] \\
\le \frac{C}{|Q_{2n}|} \left(CK_0^2 3^{n(d - 2\delta/(1+\delta))} |Q_n| + C|Q_{2n}|\tau_n\right) \le C\tau_n.
$$

Combining the previous inequality with (3.23) and (3.36), we obtain the desired estimate for the first term on the left of (3.35).

<span id="page-29-1"></span>*Step* 7. – We estimate the expected difference in  $L^2(z+Q_n)$  between  $Dv$  and  $Du(\cdot, z+Q_{n+1})$ for each  $z \in 3^n \mathbb{Z}^d \cap Q_{2n}^{\circ}$ . The claim is that

$$
(3.38) \qquad \mathbb{E}\left[3^{-dn}\sum_{z\in 3^n\mathbb{Z}^d\cap Q_{2n}^{\circ}}\int_{z+Q_n}\left|Dv(x)-Du(x,z+Q_{n+1})+\overline{P}_n\right|^2\,dx\right]\leq C\tau_n.
$$

Indeed, for each  $z \in 3^n \mathbb{Z}^d \cap Q_{2n}^{\circ}$  and  $x \in z + Q_n$ , we have

$$
Dv(x) - Du(x, z + Q_{n+1}) + \overline{P}_n
$$
  
=  $(Dv(x) - Dw(x)) + (\mathbf{f}(x) - Du(x, z + Q_{n+1}) + \overline{P}_n) - (\overline{\mathbf{f}} - div \mathbf{S}).$ 

The desired estimate is a consequence of the previous inequality, (3.19) (note that  $\zeta \equiv 1$ on  $z + Q_{n+1}$  for every  $z \in 3^n \mathbb{Z}^d \cap Q_{2n}^{\circ}$ , (3.21), (3.29) and (3.35).

*Step 8.* – We complete the argument by deriving (3.12). By Lemma 2.2, we have, for each  $z \in 3^n \mathbb{Z}^d \cap Q_{2n}^{\circ},$ 

$$
\int_{z+Q_n} L(\overline{P}_n + Dv(x), x) dx \le 2 \int_{z+Q_n} L(Du(x, z + Q_{n+1}), x) dx - \mu(z + Q_n) + C \int_{z+Q_n} |Dv(x) - Du(x, z + Q_{n+1}) + \overline{P}_n|^2 dx.
$$

In view of (3.15), it is convenient to denote  $\mathcal{Z}'_n := \{ z \in 3^n \mathbb{Z}^d : z + Q_{n+1} \nsubseteq Q_{2n}^{\circ} \}$  and  $U := \bigcup_{z \in \mathcal{Z}'_n} (z + Q_n)$ . Note that  $\xi$  vanishes on  $Q_{2n}^{\circ} \setminus U$  and thus v does as w[ell. O](#page-10-0)[bserv](#page-15-1)e [also t](#page-23-2)hat

$$
|Q_{2n}^{\circ} \setminus U| \leq C3^{-n} |Q_{2n}^{\circ}| \quad \text{and} \quad \left| |Z_n'||Q_n| - |Q_{2n}^{\circ}| \right| \leq C3^{-n} |Q_{2n}^{\circ}|.
$$

Now take the expectation of the previous inequality and sum over  $z \in \mathcal{Z}_n'$ , using (2.4), (2.27), (3.20), Lemma 2.2, (3.38), stationarity and the above to obtain

$$
\mathbb{E}\left[\int_{Q_{2n}^{\circ}} L(\overline{P}_n + Dv(x), x) dx\right]
$$
  
\n
$$
\leq \sum_{z \in Z'_n} \mathbb{E}\left[\int_{z+Q_n} L(\overline{P}_n + Dv(x), x) dx\right] + \mathbb{E}\left[\int_{Q_{2n}^{\circ} \backslash U} L(\overline{P}_n, x) dx\right]
$$
  
\n
$$
\leq |Q_{2n}^{\circ}| (\mathbb{E}\left[\mu(Q_n)\right] + C\tau_n).
$$

<span id="page-30-0"></span>Dividing by  $|Q_{2n}^{\circ}|$  [yield](#page-0-0)s (3.12) and completes the proof of the lemma.

 $\Box$ 

#### **3.3. The proof of Th[eore](#page-0-0)m 3.1**

We use Lemma 3.3 and a concentr[atio](#page-0-0)n argument to prove Theorem 3.1.

*Proof of Theorem 3.1.* – By the reduction explained in Section 3.1, we assume  $q = 0$  and drop the variable  $q$  from our notation, as w[e did](#page-17-1) in the previous subs[ectio](#page-18-2)n.

We first argue by iterating Lemma 3.3 that  $\mathbb{E}[\mu(Q_n)] \to \overline{\mu}$  as  $n \to \infty$  at a rate which is at most a power of the length scale,  $3<sup>n</sup>$ . We then use this result and a concentration argument to improve the stochastic co[nver](#page-0-0)gence to (3.2), and finally obtain (3.3) by from this and the flatness theory. Throughout, we allow  $C(d, \Lambda) \geq 1$  and  $\alpha(d, \Lambda) > 0$  to vary in each occurrence.

*Step 1.* – We iterate Lemma 3.3 to find  $C(d, \Lambda) \ge 1$  and  $\alpha(d, \Lambda) > 0$  [such](#page-14-2) that, f[or ev](#page-14-0)ery  $n \in \mathbb{N}_{*},$ 

$$
(3.39) \qquad |\overline{\mu} - \mathbb{E}\left[\mu(Q_n)\right]| \leq C K_0^2 3^{-n\alpha}.
$$

We first get an analogous estimate for the trimmed cubes and then use (2.22) and (2.24) to obtain the desired inequality for the untrimmed cubes. By (2.20), if the constant  $C(d,\Lambda) \geq 1$  $C(d,\Lambda) \geq 1$ is t[aken](#page-14-2) large [enou](#page-14-0)gh and we define

<span id="page-30-2"></span><span id="page-30-1"></span>
$$
\mu_n := \mathbb{E}\left[\mu(Q_n^{\circ})\right] - CK_0^2 3^{-n\alpha}, \quad n \in \mathbb{N},
$$

then  $\mu_n$  is an increasing sequence in n. Clearly  $\mu_n$  is bounded from above by (2.4). In view of (2.22) and (2.24), we have

$$
\lim_{n \to \infty} \mu_n = \overline{\mu}.
$$

Fix  $M \in \mathbb{N}$  with  $M \ge 2$  to be selected below. By the pigeonhole principle, the monotonicity of  $\{\mu_n\}_{n\in\mathbb{N}}$  and (3.40), we deduce, for each  $n \in \mathbb{N}$ , the existence of  $m \in \{0, \ldots, M-1\}$ such that

(3.41) 
$$
\mu_{n+m+2} - \mu_{n+m} \leq \frac{2}{M} (\overline{\mu} - \mu_n).
$$

We apply Lemma 3.3 to obtain

$$
\overline{\mu} - \mu_{n+m} = \overline{\mu} - \mathbb{E} \left[ \mu(Q_{n+m}^{\circ}) \right] + CK_0^2 3^{-n\alpha}
$$
  
\n
$$
\leq \overline{\mu} - \mathbb{E} \left[ \mu(Q_{n+m}) \right] + C \left( \mathbb{E} \left[ \mu(Q_{n+m+1}^{\circ}) \right] - \mathbb{E} \left[ \mu(Q_{n+m}^{\circ}) \right] + CK_0^2 3^{-n\alpha} \right)
$$
  
\n
$$
\leq C \left( \mathbb{E} \left[ \mu(Q_{n+m+1}) \right] - \mathbb{E} \left[ \mu(Q_{n+m}^{\circ}) \right] + CK_0^2 3^{-n\alpha} \right)
$$
  
\n
$$
\leq C \left( \mathbb{E} \left[ \mu(Q_{n+m+2}^{\circ}) \right] - \mathbb{E} \left[ \mu(Q_{n+m}^{\circ}) \right] + CK_0^2 3^{-n\alpha} \right)
$$
  
\n
$$
\leq C \left( \frac{1}{M} \left( \overline{\mu} - \mu_n \right) + K_0^2 3^{-n\alpha} \right).
$$

Here we used (2.24) to obtain the second line, Lemma 3.3 and (2.22) to get the third line, (2.20) and (2.24) to get the fourth line, and finally (3.41) in the fifth line.

By monotonicity and  $M \ge m \ge 0$ , we obtain

$$
\overline{\mu} - \mu_{n+M} \leq C \left( \frac{1}{M} \left( \overline{\mu} - \mu_n \right) + K_0^2 3^{-n\alpha} \right).
$$

Taking  $M := C(d, \Lambda)$  large enough, we obtain

$$
\overline{\mu} - \mu_{n+M} \le \frac{1}{3} (\overline{\mu} - \mu_n) + C K_0^2 3^{-n\alpha}
$$

.

Since  $M \ge 2$ , we also have, with the same constant  $C(d, \Lambda)$  on both sides,

$$
\overline{\mu} - \mu_{n+M} + C K_0^2 3^{-(n+M)\alpha} \le \frac{1}{3} \left( \overline{\mu} - \mu_n + C K_0^2 3^{-n\alpha} \right).
$$

Therefore, the sequence  $\beta_k := \overline{\mu} - \mu_{kM} + CK_0^2 3^{-kM\alpha}$  $\beta_k := \overline{\mu} - \mu_{kM} + CK_0^2 3^{-kM\alpha}$  $\beta_k := \overline{\mu} - \mu_{kM} + CK_0^2 3^{-kM\alpha}$  satisfies

$$
\beta_{k+1} \leq \frac{1}{3}\beta_k.
$$

By induction,  $\beta_k \leq 3^{-k}\beta_0$ . Since  $\beta_0 \leq CK_0^2$  by (2.4), we obtain in particular that

$$
\overline{\mu} - \mu_{kM} \le \beta_k \le CK_0^2 3^{-k}.
$$

By monotonicity we get, for every  $m \geq kM$ ,

<span id="page-31-0"></span>
$$
|\overline{\mu}-\mu_m|=\overline{\mu}-\mu_m\leq CK_0^23^{-k}.
$$

The previous line [yields](#page-14-2), for each  $n \in \mathbb{N}_{*}$ , the estimate

$$
(3.42) \qquad |\overline{\mu} - \mathbb{E}\left[\mu(Q_n^{\circ})\right]| \leq C K_0^2 3^{-n\alpha}.
$$

By mo[noton](#page-30-2)icity, (2.22) and the previous line, we get

$$
|\overline{\mu}-\mathbb{E}\left[\mu(Q_n)\right]|=\overline{\mu}-\mathbb{E}\left[\mu(Q_n)\right]\leq \overline{\mu}-\mathbb{E}\left[\mu(Q_n^{\circ})\right]+CK_0^23^{-n}\leq CK_0^23^{-n\alpha}.
$$

This is (3.39).

*Step 2.* – We deduce t[he ex](#page-14-1)istence of  $\overline{P} \in \mathbb{R}^d$  such that, for every  $n \in \mathbb{N}$ ,

$$
\left|\overline{P} - \overline{P}_n\right|^2 \leq C K_0^2 3^{-n\alpha}.
$$

It is immediate from (2.25) (taking  $V = Q_{n+1}$  and U to be the union of the  $3^d$  n-scale subcubes of  $Q_{n+1}$  $Q_{n+1}$ ) and (3.39) that

<span id="page-32-0"></span>
$$
|\mathbb{E}[P(Q_n)] - \mathbb{E}[P(Q_{n+1})]|^2 \leq CK_0^2 3^{-n\alpha}.
$$

Then from (3.10) we deduce that

<span id="page-32-2"></span><span id="page-32-1"></span>
$$
\left|\overline{P}_n - \overline{P}_{n+1}\right|^2 \leq CK_0 3^{-n\alpha}.
$$

Summing this over  $\{n, n+1, ...\}$  yields the existence of  $\overline{P} \in \mathbb{R}^d$  satisfying (3.43). For future reference we note that, by (3.9),

$$
\left|\overline{P}\right|^2 \leq CK_0^2.
$$

*Step 3.* – [After](#page-14-0) p[ossib](#page-16-2)ly [rede](#page-20-2)fining  $\alpha(d, \Lambda)$  $\alpha(d, \Lambda)$  $\alpha(d, \Lambda)$  [to be](#page-30-2) s[malle](#page-32-0)r, we [obtai](#page-32-1)n

$$
(3.45) \t\t\t\t\mathbb{E}\left[\nu(Q_n^{\circ}, \overline{P}) - \mu(Q_n^{\circ})\right] \leq CK_0^2 3^{-n\alpha}.
$$

By (2.20), (2.24), (2.33), (3.9), Lemma 3.3, (3.39), (3.43) and (3.44), we have

$$
\mathbb{E}\left[\nu(Q_{2n}^{\circ}, \overline{P}) - \mu(Q_{2n}^{\circ})\right] \n\leq \mathbb{E}\left[\nu(Q_{2n}^{\circ}, \overline{P}) - \nu(Q_{2n}^{\circ}, \overline{P}_n)\right] + \mathbb{E}\left[\nu(Q_{2n}^{\circ}, \overline{P}_n) - \mu(Q_n^{\circ})\right] + CK_0^2 3^{-n} \n\leq CK_0|\overline{P} - \overline{P}_n| + C\left(\mathbb{E}\left[\mu(Q_{n+2}^{\circ})\right] - \mathbb{E}\left[\mu(Q_n^{\circ})\right] + K_0^2 3^{-n}\right) \n\leq CK_0^2 3^{-\alpha n/2}.
$$

This yields (3.45) after we replace  $\alpha$  by  $\alpha/4$ .

Observe also that (3.39) and (3.45) imply that  $\overline{\mu} = \overline{L}(\overline{P})$  and

(3.46) 
$$
\left|\mathbb{E}\left[\nu(Q_n^{\circ},\overline{P})\right]-\overline{\mu}\right|\leq CK_0^2 3^{-n\alpha}.
$$

<span id="page-32-3"></span>*Step 4.* – We use independence to improve the convergence of the expectations from the previous step to convergence in  $L^1(\Omega, \mathbb{P})$ . The claim is that, after redefining  $\alpha(d, \Lambda) > 0$  to be smaller, we have

(3.47) 
$$
\mathbb{E}\left[\left|\nu(Q_n^{\circ}, \overline{P}) - \overline{\mu}\right| + |\mu(Q_n^{\circ}) - \overline{\mu}|\right] \leq CK_0^2 3^{-n\alpha}.
$$

Using the fact that  $\mu(Q_k^{\circ}) \leq \nu(Q_k^{\circ}, \overline{P})$ , we have

$$
|\nu(Q_n^\circ)-\overline{\mu}|\leq \nu(Q_n^\circ)-\mu(Q_n^\circ)+|\mu(Q_n^\circ)-\overline{\mu}|\,.
$$

Observe that

$$
\mathbb{E}\left[\left|\mu(Q_n^{\circ})-\overline{\mu}\right|\right] \leq \mathbb{E}\left[\left|\mu(Q_n^{\circ})-\mathbb{E}\left[\mu(Q_n^{\circ})\right]\right|\right] + CK_0^2 3^{-n\alpha}
$$

$$
= 2\mathbb{E}\left[\left(\mathbb{E}\left[\mu(Q_n^{\circ})\right] - \mu(Q_n^{\circ})\right)_+\right] + CK_0^2 3^{-n\alpha}.
$$

Moreover,

$$
\mathbb{E}\left[\left(\mathbb{E}\left[\mu(Q_{n+1}^{\circ})\right] - \mu(Q_{n+1}^{\circ})\right)_{+}^{2}\right] \n\leq \mathbb{E}\left[\left(\mathbb{E}\left[\mu(Q_{n}^{\circ})\right] - 3^{-d} \sum_{Q_{n}^{\circ}(x) \subseteq Q_{n+1}^{\circ}} \mu(Q_{n}^{\circ}(x))\right)_{+}^{2}\right] + CK_{0}^{4}3^{-n\alpha} \n\leq 3^{-2d} \sum_{Q_{n}^{\circ}(x) \subseteq Q_{n+1}^{\circ}} \mathbb{E}\left[\left(\mathbb{E}\left[\mu(Q_{n}^{\circ})\right] - \mu(Q_{n}^{\circ}(x))\right)_{+}^{2}\right] + CK_{0}^{4}3^{-n\alpha} \n= 3^{-d} \mathbb{E}\left[\left(\mathbb{E}\left[\mu(Q_{n}^{\circ})\right] - \mu(Q_{n}^{\circ})\right)_{+}^{2}\right] + CK_{0}^{4}3^{-n\alpha}.
$$

Here we used (2.4), (2.19) and (3.42) in the first line, independence and (3.45) in the second line and finally stationarity in the third line. Since  $\mathbb{E}\left[\left(\mathbb{E}\left[\mu(Q_1^{\circ})\right] - \mu(Q_1^{\circ})\right)_+^2\right] \leq CK_0^4$ , an iteration of the previous inequality yields

$$
\mathbb{E}\left[\left(\mathbb{E}\left[\mu(Q_n^{\circ})\right] - \mu(Q_n^{\circ})\right)_+^2\right] \leq CK_0^4 3^{-n\alpha}.
$$

Combining the inequalities above and using (3.45) yields (3.47) after a redefinition of  $\alpha$ .

*Step 5.* – We upgrade the stochastic integrability of (3.47), using an elementary concentration argument. The claim is th[at, fo](#page-13-0)r every  $m, n \in \mathbb{N}$  and  $t \ge C3^{-n\alpha}$ ,

$$
(3.48) \qquad \mathbb{P}\big[\left|\nu(Q_{n+m}^{\circ},\overline{P})-\overline{\mu}\right|+\left|\overline{\mu}-\mu(Q_{n+m}^{\circ})\right|\geq K_0^2t\big]\leq \exp\left(-c3^{dm}t\right).
$$

Fix  $s > 0$  and compute, using (2.19), independence, and stationary:

$$
\log \mathbb{E} \left[ \exp \left( s 3^{dm} \left( \overline{\mu} - \mu(Q_{n+m}^{\circ}) \right)_+ \right) \right]
$$
  
\n
$$
\leq \log \mathbb{E} \left[ \prod_{Q_n^{\circ}(x) \subseteq Q_{n+m}^{\circ}} \exp \left( s \left( \overline{\mu} - \mu(Q_n^{\circ}(x)) \right)_+ \right) \right] + C K_0^2 3^{dm-n}
$$
  
\n
$$
= \sum_{Q_n^{\circ}(x) \subseteq Q_{n+m}^{\circ}} \log \mathbb{E} \left[ \exp \left( s \left( \overline{\mu} - \mu(Q_n^{\circ}(x)) \right)_+ \right) \right] + C K_0^2 3^{dm-n}
$$
  
\n
$$
= 3^{dm} \log \mathbb{E} \left[ \exp \left( s \left( \overline{\mu} - \mu(Q_n^{\circ}) \right)_+ \right) \right] + C K_0^2 3^{dm-n}.
$$

 $By (2.4),$ 

$$
\left(\overline{\mu}-\mu(Q_n^\circ)\right)_+\leq 4K_0^2\quad \mathbb{P}\text{-a.s.}
$$

Therefore, using the elementary inequalities

$$
\begin{cases} \exp(t) \le 1 + 2t & \text{for every } 0 \le t \le 1, \\ \log(1 + t) \le t & \text{for every } t \ge 0, \end{cases}
$$

we deduce that, for each  $0 < s \leq (4K_0)^{-2}$  $0 < s \leq (4K_0)^{-2}$  $0 < s \leq (4K_0)^{-2}$ ,

(3.49) 
$$
\log \mathbb{E}\left[\exp\left(s3^{dm}\left(\overline{\mu}-\mu(Q_{n+m}^{\circ})\right)_{+}\right)\right] \leq 2s3^{dm}\mathbb{E}\left[\left(\overline{\mu}-\mu(Q_{n}^{\circ})\right)_{+}\right] + CsK_0^23^{dm-n}.
$$
 Now an application of (3.47) yields

$$
3^{-dm}\log\mathbb{E}\left[\exp\left(s3^{dm}\left(\overline{\mu}-\mu(Q_{n+m}^{\circ})\right)_{+}\right)\right]\leq CsK_0^23^{-n\alpha}.
$$

Take  $s := (4K_0)^{-2}$  and write the previous inequality in the form

$$
3^{-dm}\log \mathbb{E}\left[\exp\left(c3^{dm}K_0^{-2}\left(\overline{\mu}-\mu(Q_{n+m}^{\circ})\right)_+\right)\right] \leq C3^{-n\alpha}.
$$

By a similar argument, replacing  $(\overline{\mu} - \mu(Q_k^{\circ}))_+$  by  $(\nu(Q_k^{\circ}, \overline{P}) - \overline{\mu})_+$  and using (2.21) rather than (2.19), we also get

$$
3^{-dm}\log \mathbb{E}\left[\exp\left(c3^{dm}K_0^{-2}\left(\nu(Q_{n+m}^{\circ},\overline{P})-\overline{\mu}\right)_+\right)\right] \leq C3^{-n\alpha}.
$$

Define

$$
E(U) := |\nu(U, \overline{P}) - \overline{\mu}| + |\overline{\mu} - \mu(U)|
$$

and observe by  $\mu(U) \leq \nu(U, \overline{P})$  that

$$
(3.50) \tE(U) \leq 2\left(\nu(U,\overline{P}) - \overline{\mu}\right)_+ + 2\left(\overline{\mu} - \mu(U)\right)_+.
$$

Therefore we obtain

$$
3^{-dm}\log \mathbb{E}\left[\exp\left(c3^{dm}K_0^{-2}E(Q_{n+m}^{\circ})\right)\right] \leq C3^{-n\alpha}.
$$

An application of Chebyshev's inequality yields, for every  $m, n \in \mathbb{N}$  and  $t \geq C3^{-n\alpha}$ :

(3.51) 
$$
\mathbb{P}\left[K_0^{-2}E(Q_{n+m}^{\circ}) \ge t\right] = \mathbb{P}\left[\exp\left(c3^{dm}K_0^{-2}E(Q_{n+m}^{\circ})\right) \ge \exp\left(c3^{dm}t\right)\right] \\ \le \exp\left(-c3^{dm}t\right) \mathbb{E}\left[\exp\left(c3^{dm}K_0^{-2}E(Q_{n+m}^{\circ})\right)\right] \\ \le \exp\left(C3^{dm-\alpha n} - c3^{dm}t\right) \\ \le \exp\left(-c3^{dm}t\right).
$$

This is (3.48).

*Step 6*. – We complete the proof of (3.2). The main point still to be address[ed is](#page-14-2) to allow for arbitrary translations of the cubes, and this is handled by a union bound and a stationarity argument to get the desired estimate from (3.48). Recall that, for every  $y \in \mathbb{R}^d$ , there exists argument to get the desired estimate from (5.48). Recall that, for every  $y \in \mathbb{R}$ , there exists  $z \in \mathbb{Z}^d$  with  $|z - y| \le \sqrt{d}$  and  $z + Q_n^{\circ} \subseteq y + Q_n$  for every  $n \in \mathbb{N}$ . Thus by (2.22) we obtain, for every  $R > 0$  and  $n \in \mathbb{N}$ ,

$$
\sup_{y \in B_R} \mu(y + Q_n) \ge \max_{z \in \mathbb{Z}^d \cap B_{R+\sqrt{d}}} \mu(z + Q_n^{\circ}) - C K_0^2 3^{-n} \quad \mathbb{P}\text{-a.s.}
$$

Hence for all  $R \ge 1$  and  $n \in \mathbb{N}$ ,

$$
\sup_{y \in B_R} (\overline{\mu} - \mu (y + Q_n))_+ \le \max_{z \in \mathbb{Z}^d \cap B_{CR}} (\overline{\mu} - \mu (z + Q_n^{\circ}))_+ + C K_0^2 3^{-n} \quad \mathbb{P}\text{-a.s.}
$$

By a union bound, stationarity and (3.48), we obtain, for every  $n, m \in \mathbb{N}, R \ge 1$  and  $t \geq C3^{-n\alpha},$ 

$$
\mathbb{P}\left[K_0^{-2} \sup_{y \in B_R} (\overline{\mu} - \mu(y + Q_{n+m}))_+ \ge t\right]
$$
  
\n
$$
\le \sum_{z \in \mathbb{Z}^d \cap B_{CR}} \mathbb{P}\left[K_0^{-2} (\overline{\mu} - \mu(z + Q_{n+m}^{\circ}))_+ \ge t\right]
$$
  
\n
$$
\le CR^d \mathbb{P}\left[K_0^{-2} (\overline{\mu} - \mu(Q_{n+m}^{\circ}))_+ \ge t\right] \le CR^d \exp(-c3^{dm}t).
$$

By an analogous argument, using (2.23) instead of (2.22), we obtain, for  $t \geq C3^{-n\alpha}$ ,

$$
\mathbb{P}\left[K_0^{-2} \sup_{y \in B_R} \left(\nu(y + Q_{n+m}, \overline{P}) - \overline{\mu}\right)_+ \geq t\right] \leq CR^d \exp\left(-c3^{dm}t\right).
$$

Using (3.50) again and replacing t by  $C3^{-n\alpha}t$ , we obtain, for every  $n, m \in \mathbb{N}, R \ge 1$  and  $t \geq 1$ ,

$$
(3.52) \qquad \mathbb{P}\left[K_0^{-2} \sup_{y \in B_R} E(y + Q_{n+m}) \geq C3^{-n\alpha} t\right] \leq CR^d \exp\left(-c3^{dm-n\alpha} t\right).
$$

To see that this implies (3.2), fix  $s \in (0, d)$ . Choose  $m = m(n)$  to be the smallest positive integer such that

$$
s < \frac{dm - n\alpha}{n + m}.
$$

That is,  $m(n) := |(s + \alpha)n/(d - s)| \geq cn/(d - s)$ . Then (3.52) yields, for every  $t \geq 1$ ,

$$
\mathbb{P}\left[K_0^{-2} \sup_{y \in B_R} E(y + Q_{n+m}) \ge C3^{-n\alpha} t\right] \le CR^d \exp\left(-c3^{s(n+m)}t\right).
$$

This implies (3.2) after a redefinition of  $\alpha$ .

*Step 7.* – We prove the flatness estimates (3.3). It is easier to work with the minimizers for  $\nu$ , so we handle them first and obtain the flatness of the  $\mu$  minimizers as a consequence. Fix  $y \in \mathbb{R}^d$  and denote

$$
v_n(x) := v(x, y + Q_n(x), \overline{P}), \quad x \in \mathbb{R}^d.
$$

In other words, for each  $n \in \mathbb{N}$ , the function  $v_n : \mathbb{R}^d \to \mathbb{R}$  is obtained by splicing together the minimizers for  $\nu(\cdot,\overline{P})$  in each triadic cube of the form  $y + Q_n(x) \subseteq \mathbb{R}^d$ . Observe that  $v_n \in H^1_{\text{loc}}(\mathbb{R}^d).$ 

Fix  $m, n \in \mathbb{N}^*$  and estimate the  $L^2$  difference between the scales n and  $n + m$  using the Poincaré inequality and (2.28):

$$
\int_{y+Q_{n+m}} (v_{n+m}(x) - v_n(x))^2 dx
$$
\n
$$
\leq C 3^{2(n+m)} \int_{y+Q_{n+m}} |Dv_n(x) - Dv_{n+m}(x)|^2 dx
$$
\n
$$
\leq C 3^{2(n+m)} \left( \int_{y+Q_{n+m}} L(Dv_n(x), x) dx - \nu(y + Q_{n+m}, \overline{P}) \right)
$$
\n
$$
= C 3^{2(n+m)} \left( \int_{y+Q_{n+m}} \nu(y + Q_n(x), \overline{P}) dx - \nu(y + Q_{n+m}, \overline{P}) \right).
$$

Next we observe that, viewed from a length scale much larger than  $3^n$ ,  $v_n$  is close to the plane  $\overline{P} \cdot x$ :

$$
\int_{y+Q_{n+m}} (v_n(x) - \overline{P} \cdot x)^2 dx
$$
\n
$$
= \int_{y+Q_{n+m}} \int_{y+Q_n(\xi)} (v_n(x) - \overline{P} \cdot x)^2 dx d\xi
$$
\n
$$
\leq \int_{y+Q_{n+m}} C 3^{2n} \int_{y+Q_n(\xi)} |Dv_n(x) - \overline{P}|^2 dx d\xi \qquad \text{(by Poincaré inequality)}
$$
\n
$$
\leq \int_{y+Q_{n+m}} C 3^{2n} \int_{y+Q_n(\xi)} (|Dv_n(x)|^2 + |\overline{P}|^2) dx d\xi
$$
\n
$$
\leq C 3^{2n} K_0^2 \qquad \text{(by (2.4), (3.44))}.
$$

Assembling these, we obtain

$$
\int_{y+Q_{n+m}} |v_{n+m}(x) - \overline{P} \cdot x|^2 dx
$$
\n
$$
\leq C 3^{2n} K_0^2 + C 3^{2(n+m)} \left( \int_{y+Q_{n+m}} \nu(y + Q_n(x), \overline{P}) dx - \nu(y + Q_{n+m}, \overline{P}) \right).
$$

Since  $y \in \mathbb{R}^d$  was arbitrary, the previous inequality yields, for each  $R \geq 1$ ,

$$
\sup_{y \in B_R} 3^{-2(n+m)} \int_{y+Q_{n+m}} \left| v_{n+m}(x) - \overline{P} \cdot x \right|^2 dx
$$
  

$$
\leq CK_0^2 \left( 3^{-2m} + \sup_{y \in B_R} \left( \left| \nu(y+Q_{n+m}, \overline{P}) - \overline{\mu} \right| + \sup_{x \in y+Q_{n+m}} \left| \nu(y+Q_n(x), \overline{P}) dx - \overline{\mu} \right| \right) \right).
$$

Fix  $s \in (3d/4, d)$  and apply (3.2) to obtain, for every  $n, m \in \mathbb{N}, R \ge 1$  and  $t \ge 1$ ,

$$
\mathbb{P}\left[\sup_{y\in B_R} K_0^{-2} 3^{-2(n+m)} \int_{y+Q_{n+m}} \left| v_{n+m}(x) - \overline{P} \cdot x \right|^2 dx \ge C \left(3^{-2m} + 3^{-n\alpha(d-s)}\right) t \right] \le C \left( R^d + 3^{d(n+m)} \right) \exp\left( -c 3^{sn} t \right).
$$

Take m to be the smallest integer larger than  $n\alpha(d - s)/2$ , replace  $n + m$  by n and s by  $s - c(d - s)$  and shrink  $\alpha$ , if necessary, to obtain, for every  $t \ge 1$ ,

$$
\mathbb{P}\left[\sup_{y\in B_R} K_0^{-2} 3^{-2n} \oint_{y+Q_n} \left| v_n(x) - \overline{P} \cdot x \right|^2 dx \ge C 3^{-n\alpha(d-s)} t \right] \le C\left(R^d + 3^{dn}\right) \exp\left(-c 3^{sn} t\right).
$$

<span id="page-36-0"></span>Replacing s by  $s - c(d - s)$  again, we obtain, for every  $s \in (0, d)$ ,  $n \in \mathbb{N}$ ,  $R \ge 1$  and  $t \ge 1$ ,

$$
(3.53) \quad \mathbb{P}\left[\sup_{y\in B_R} K_0^{-2} 3^{-2n} \int_{y+Q_n} \left| v_n(x) - \overline{P} \cdot x \right|^2 \, dx \ge C 3^{-n\alpha(d-s)} t \right] \le CR^d \exp\left(-c 3^{sn} t\right).
$$

We complete the proof of (3.3) by obtaining the flatness of minimizers for  $\mu$ . Fix  $y \in \mathbb{R}^d$ . Observe that, by Lemma 2.1,

$$
\int_{y+Q_n} \left| Du(x, y+Q_n) - Dv(x, y+Q_n, \overline{P}) \right|^2 dx \le \nu(y+Q_n, \overline{P}) - \mu(y+Q_n).
$$

<span id="page-37-2"></span>Hence

$$
(3.54) \quad \left| \oint_{y+Q_n} Du(x, y+Q_n) dx - \overline{P} \right|^2
$$

$$
= \left| \oint_{y+Q_n} \left( Du(x, y+Q_n) - Dv(x, y+Q_n, \overline{P}) \right) dx \right|^2 \le \nu(y+Q_n, \overline{P}) - \mu(y+Q_n).
$$

and so, by the Poincaré inequality,

$$
\int_{y+Q_n} \left( u(x, y+Q_n) - v(x, y+Q_n, \overline{P}) + y \cdot \overline{P} \right)^2 dx
$$
\n
$$
\leq C 3^{2n} \left( \int_{y+Q_n} \left| Du(x, y+Q_n) - Dv(x, y+Q_n, \overline{P}) \right|^2 dx + \nu(y+Q_n, \overline{P}) - \mu(y+Q_n) \right)
$$
\n
$$
\leq C 3^{2n} \left( \nu(y+Q_n, \overline{P}) - \mu(y+Q_n) \right).
$$

The previous inequality, (3.2) and (3.53) yield, for every  $s \in (0, d)$ ,  $n \in \mathbb{N}$ ,  $R \ge 1$  and  $t \ge 1$ ,

$$
\mathbb{P}\left[\sup_{y\in B_R} K_0^{-2} 3^{-2n} \int_{y+Q_n} \left| u(x, y+Q_n) - \overline{P} \cdot (x-y) \right|^2 dx \ge C 3^{-n\alpha(d-s)} t \right] \le CR^d \exp\left(-c 3^{sn} t\right).
$$
  
his completes the proof of (3.3).

<span id="page-37-0"></span>This completes the proof of (3.3).

#### <span id="page-37-1"></span>**3.4.** Convex duality between  $\overline{\mu}$  and  $\overline{L}$

An immediate consequence of (3.1) is the following formula for  $\overline{\mu}$  in terms of  $\overline{L}$ : for every  $q \in \mathbb{R}^d$ ,

(3.55) 
$$
\overline{\mu}(q) = - \sup_{p \in \mathbb{R}^d} (p \cdot q - \overline{L}(p)).
$$

Indeed, the difficult [half](#page-37-1) of  $(3.55)$  is implied by  $(3.1)$  and the other, easier half is a consequence of (2.1).

The expression (3.55) asserts that  $-\overline{\mu}$  is the Legendre-Fenchel transform of  $\overline{L}$ . Since the latter is uniformly convex by (2.37), it follows by convex duality that, for every  $p \in \mathbb{R}^d$ ,

(3.56) 
$$
\overline{L}(p) = \sup_{q \in \mathbb{R}^d} (p \cdot q + \overline{\mu}(q)).
$$

Since  $\overline{L}$  [is un](#page-0-0)iformly convex, its gradient  $D\overline{L}$  is a bijective Lipschitz map on  $\mathbb{R}^d$ . The Formula (3.56) implies that  $D\overline{L}(p)$  is the unique q achieving the supremum in (3.56). The inverse of the this map is evidently the function  $q \mapsto \overline{P}(q)$  given in the statement of Theorem 3.1. That is,  $p = \overline{P}(D\overline{L}(p))$  and moreover, for every  $p \in \mathbb{R}^d$ ,

$$
\overline{L}(p) = p \cdot D\overline{L}(p) + \overline{\mu}(D\overline{L}(p)).
$$

In particular, the map  $\overline{P}$  can be inverted, and this allows us to reformulate the statement of Theorem 3.1 so that the parameter  $p$  is given rather than  $q$ . It is convenient to gather all of the errors we wish to measure with respect to a bounded, connected domain  $U \subseteq \mathbb{R}^d$  and a given  $p \in \mathbb{R}^d$  into one random variable. Set

$$
(3.57) \quad \mathcal{E}(U,p) := \left| \overline{L}(p) - \mu\left(U, D\overline{L}(p)\right) - p \cdot D\overline{L}(p) \right| + \left| \overline{L}(p) - \nu(U,p) \right|
$$

$$
+ |U|^{-2/d} \int_{U} \left( \left( v(x,U,p) - p \cdot x \right)^{2} + \left( u(x,U,D\overline{L}(p)) - p \cdot (x - x_{U}) \right)^{2} \right) dx,
$$

where  $x_U := \oint_U x \, dx$  denotes the barycenter of U.

COROLLARY 3.4. – *With*  $\alpha(d,\Lambda) > 0$  *as in the statement of Theorem 3.1, there exist*  $C(d, \Lambda) \geq 1$  and  $c(d, \Lambda) > 0$  such that, for every  $s \in (0, d)$ ,  $p \in \mathbb{R}^d$ ,  $n \in \mathbb{N}$  and  $t \geq 1$ ,

$$
\mathbb{P}\Big[\exists y\in B_R,\ \mathcal{E}(y+Q_n,p)\geq C\left(K_0+|p|\right)^23^{-n\alpha(d-s)}t\Big]\leq CR^d\exp\left(-c3^{sn}t\right).
$$

*Proof.* – Apply Theorem 3.1 to  $q = D\overline{L}(p)$ . By the remarks preceding the statement of the corol[lary,](#page-0-0) we have  $\overline{P}(q) = p$ . From the first inequality of (2.39) we have

$$
\left|D\overline{L}(p)\right| \leq C(K_0+|p|).
$$

Theorem 3.1 thus yields the corollary.

We conclude this section with a further refinement of Theorem 3.1 which gives some uniformity in our estimates of  $\mathcal{E}(U, p)$  with respect to p. This is needed in the next section in the argument for the error in the Dirichlet problem.

COROLLARY 3.5. – *Fix*  $M, R, k \ge 1$  *and*  $s \in (0, d)$ *. There exist*  $\alpha(d, \Lambda) > 0$ ,  $c(d, \Lambda) > 0$ *and*  $C(d, \Lambda, s, k) \geq 1$  *such that, for every*  $n \in \mathbb{N}$  *and*  $t \geq 1$ *,* 

$$
\mathbb{P}\Big[\exists p \in B_{M3^{kn}}, \ \exists y \in B_{R3^{kn}}, \ \mathcal{E}(y + Q_n, p) \ge C\left(K_0 + |p|\right)^2 3^{-n\alpha(d-s)}t\Big] \le CM^d R^d \exp\left(-c3^{sn}t\right).
$$

<span id="page-38-0"></span>*Proof*. – We see from (2.33), (2.35), (2.38), (2.39), (2.40) and (2.42) that the error term is continuous in p, uniformly on the support of  $\mathbb P$ : that is, for every  $n \in \mathbb N_*$  and  $p_1, p_2 \in \mathbb R^d$ ,

$$
(3.58) \qquad \left| \mathcal{E}(Q_n, p_1) - \mathcal{E}(Q_n, p_2) \right| \le C \left( K_0 + |p_1| + |p_2| \right) |p_1 - p_2| \quad \mathbb{P}\text{-a.s.}
$$

Therefore, it is enough to check the error estimate for  $p$ 's on a discrete mesh with spacings  $h_n:=3^{-n\alpha d}.$  Denoting this mesh by  $G_n:=(h_n\mathbb{Z}^d)\cap B_{M3^{kn}}$  and letting  $\Omega'\subseteq \Omega$  be the event with  $\mathbb{P}[\Omega'] = 1$  on which (3.58) holds, we have, for each  $n \in \mathbb{N}$  and  $t \ge 1$ ,

$$
(3.59) \quad \{ \forall p \in G_n, \, \forall y \in B_{R3^{kn}} \mathcal{E}(y + Q_n, p) \le C \left(K_0 + |p|\right)^2 3^{-n\alpha(d-s)} t \} \cap \Omega'
$$
  

$$
\subseteq \{ \forall p \in B_{M3^{kn}}, \, \forall y \in B_{R3^{kn}}, \, \mathcal{E}(y + Q_n, p) \le C \left(K_0 + |p|\right)^2 3^{-n\alpha(d-s)} t \}.
$$

Here the  $C$  on the right side is larger than the one on the left to accommodate the discretization error of order  $C(K_0+|p|)h_n \lesssim C(K_0+|p|)^2 3^{-n\alpha d}$  coming from the right side of (3.58).

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 $\Box$ 

By (3.59), a union bound and an application of Corollary 3.4, we find that, for each  $n \in \mathbb{N}$ and  $t > 1$ ,

$$
\mathbb{P}\Big[\exists p \in B_{M3^{kn}}, y \in B_{R3^{kn}}, \ \mathcal{E}(y + Q_n, p) \ge C \left(K_0 + |p|\right)^2 3^{-n\alpha(d-s)}t\Big] \n\le \sum_{p \in G_n} \mathbb{P}\Big[\exists y \in B_{R3^{kn}}, \ \mathcal{E}(y + Q_n, p) \ge C \left(K_0 + |p|\right)^2 3^{-n\alpha(d-s)}t\Big] \n\le |G_n| \max_{p \in \mathbb{R}^d} \mathbb{P}\Big[\exists y \in B_{R3^{kn}}, \ \mathcal{E}(y + Q_n, p) \ge C \left(K_0 + |p|\right)^2 3^{-n\alpha(d-s)}t\Big] \n\le CR^d 3^{knd} |G_n| \exp\left(-c 3^{sn} t\right).
$$

<span id="page-39-1"></span>The number of elements of the set  $G_n$  is easy to compute:

$$
|G_n| \le Ch_n^{-d}|B_{M3^{kn}}| = CM^d 3^{dn(d\alpha+k)}.
$$

We thus deduce that, for every  $s \in (0, d)$ ,  $n \in \mathbb{N}$  and  $t \ge 1$ ,

$$
(3.60) \mathbb{P}\Big[\forall p \in B_{M3^{kn}}, \forall y \in B_{R3^{kn}}, \ \mathcal{E}(y + Q_n, p) \ge C \left(K_0 + |p|\right)^2 3^{-n\alpha(d-s)} t\Big] \le CM^d R^d 3^{dn(d\alpha + 2k)} \exp\left(-c3^{sn} t\right) \le CM^d R^d \exp\left(Cn - c3^{sn} t\right).
$$

Set  $s_1 := (s + d)/2$ , note that  $s_1 \in (s, d)$  depends only on d and s and apply (3.60) with  $s_1$ in place of s and use the fact that, for  $t \geq 1$ ,

$$
\exp\left(Cn - c3^{s_1 n}t\right) \le C \exp\left(-c3^{-s n}t\right),
$$

to obtain

$$
\mathbb{P}\Big[\forall p\in B_{M3^{kn}},\,\forall y\in B_{R3^{kn}},\,\,\mathcal{E}(y+Q_n,p)\geq C\left(K_0+|p|\right)^23^{-n\alpha(d-s_1)}t\Big] \leq CM^dR^d\exp\left(-c3^{sn}t\right).
$$

<span id="page-39-0"></span>Since  $d - s_1 = (d - s)/2$ , we get the desired conclusion after replacing  $\alpha$  by  $\alpha/2$ .  $\Box$ 

#### **4. The error estimate for the Dirichlet problem**

In this section we prove Theorem 1.1, obtaining error estimates in homogenization for Dirichlet problems in bounded Lipschitz domains with fairly general boundary conditions. The arguments her[e are](#page-0-0) mostly technical and completely deterministic: all of the heavy lifting was done in the previous section, where in particular we proved error estimates for the Dirichlet problem in cubes with planar boundary conditions. It turns out that this is enough to give us Theorem 1.1, as [we w](#page-0-0)ill see from fairly simple oscillating test function and energy comparison arguments.

#### **4.1. The proof of Theorem 1.1**

We begin with the statement of an abstract tool which provides control of the error for general Dirichlet problems in terms of the error for the Dirichlet problem in mesoscopic cubes with planar boundary conditions. This "black box" is oblivious to the randomness and to much of the precise structure of the problem. Although straightforward, its proof (given in the appendix) is unfortunately a rather technical and lengthy energy comparison argument relying on some classical interior regularity results.

**PROPOSITION** 4.1. – Let  $U \subseteq \mathbb{R}^d$  be a bounded Lipschitz domain,  $K_0, M \geq 1$  and  $t > 2$ . *Fix*  $\varepsilon \in (0,1], L \in \Omega(K_0), g \in W^{1,t}(U)$  *satisfying* (1.8) *and*  $u_{\varepsilon}, u_{\text{hom}} \in g + H_0^1(U)$ *satisfying* (1.9) *and* (1.10)*, respectively. Select*  $n \in \mathbb{N}$  *such that*  $3^{-n} < \varepsilon \leq 3^{-n+1}$  *and*  $fix m, l \in \mathbb{N}$  *such that*  $m \leq l \leq n$ . Then there exist constants  $C(d, \Lambda, t, U) \geq 1$  and  $\beta(d, \Lambda, t) \in (0, 1]$  *such that* 

$$
(4.1) \quad \left| \oint_{U} \left( L \left( Du_{\varepsilon}(x), \frac{x}{\varepsilon} \right) - \overline{L} (Du_{\text{hom}}(x)) \right) \, dx \right| + \oint_{U} (u_{\varepsilon}(x) - u_{\text{hom}}(x))^2 \, dx \leq C \mathcal{E}' + CM \sqrt{\mathcal{E}'} + CM^2 \left( 3^{-(l-m)} + 3^{-\beta(n-l)} \right),
$$

*where*

 $\mathscr{E}':=\sup\left\{\mathscr{E}(y+Q_m,p)+\mathscr{E}(y'+Q_{m+2},p')\,:\,y,y'\in B_{C3^n},\;p,p'\in B_{CM3^{(n-m)d/2}}\right\}.$ and  $\&$  *is defined in* (3.57).

The proof of Proposition 4.1 is presented in Appendix .

Assuming the pro[posi](#page-0-0)tion, numerological and bookkeeping details and the choices of the parameters  $m$  and  $l$  ar[e ess](#page-0-0)entially all that still stand between us and the demons[trati](#page-0-0)on of the first main result.

*Proof of Theorem 1.1.* – Fix  $\varepsilon \in (0,1]$  and  $s \in (0,d)$ . Take  $\alpha(d,\Lambda) > 0$  as in the statement of Corollary 3.5 and  $\beta(d, \Lambda, t) > 0$  as in the statement of Proposition 4.1. Also set  $s_1 := (2t + d)/3$  and  $s_2 := (t + 2d)/3$  so that  $s < s_1 < s_2 < d$ , with the gaps between these numbers bounded by  $c(d, \Lambda, s) > 0$ .

<span id="page-40-1"></span>Let  $n \in \mathbb{N}$  be such that  $3^{-n} < \varepsilon \leq 3^{-n+1}$  and select  $m = m(n) \in \mathbb{N}$  to be the smallest integer satisfying

(4.2) 
$$
3^{ns_1} \leq 3^{ms_2}
$$
 and  $2d(n-m) \leq m\alpha(d-s_2)$ .

Note that  $m \le n$  $m \le n$ . Pick  $l \in \mathbb{N}$  to be the smallest integer such that  $l \ge (m+n)/2$ . It is then evident that, for an exponent  $\gamma(d, \Lambda, t) > 0$ ,

$$
3^{-(l-m)}+3^{-\beta(n-l)}\leq C\varepsilon^\gamma.
$$

Let  $\mathcal{E}'_n$  be the [rand](#page-0-0)om variable  $\mathcal{E}'$  defined in the statement of Proposition 4.1, with respect to the choice of n and  $m(n)$ , above.

Proposition 4.1 gives the estimate

<span id="page-40-0"></span>
$$
\int_U (u_\varepsilon(x) - u_{\text{hom}}(x))^2 dx \le C \mathcal{E}'_n + CM \sqrt{\mathcal{E}'_n} + CM^2 \varepsilon^{\gamma}.
$$

To complete the proof of the theorem, it therefore suffices to demonstrate that there exists a random varia[ble](#page-40-0)  $\chi$  satisfying (1.7) and  $\gamma(d,\Lambda,s) > 0$  $\gamma(d,\Lambda,s) > 0$  $\gamma(d,\Lambda,s) > 0$  such that, for every  $n \in \mathbb{N}$ ,

(4.3) 
$$
\mathcal{E}'_n \leq CM^2 \left(1 + \chi 3^{-sn}\right) 3^{-n\gamma}.
$$

We arg[ue th](#page-40-1)at (4.3) is a consequence of Corollary 3.5. The latter implies that, for every  $t \ge 1$ ,

$$
\mathbb{P}\left[\mathcal{E}_n' \ge C\left(K_0^2 + M^2 3^{(n-m)d}\right) 3^{-m\alpha(d-s_2)}t\right] \le CM^d \exp\left(-c3^{s_2m}t\right).
$$

Using (4.2), this yields, for  $t \geq 1$ ,

$$
\mathbb{P}\left[\mathcal{E}'_n \geq CM^2 3^{-(n-m)d}t\right] \leq CM^d \exp\left(-c3^{s_1 n}t\right).
$$

Replacing t by  $1 + t$  and rearranging, we obtain, for all  $t > 0$ ,

$$
\mathbb{P}\left[\left(3^{(n-m)d}M^{-2}\mathcal{E}'_n - C\right) \geq Ct\right] \leq CM^d \exp\left(-c3^{s_1 n}t\right).
$$

Replacing t by  $3^{-sn}t$ , we obtain, for all  $t > 0$ ,

$$
(4.4) \qquad \mathbb{P}\left[3^{sn}\left(3^{(n-m)d}M^{-2}\mathcal{E}'_n - C\right)_+ \ge Ct\right] \le CM^d \exp\left(-c3^{(s_1-s)n}t\right).
$$

Let  $X$  be the random variable

$$
\mathcal{X} := \sup_{n \in \mathbb{N}} 3^{sn} \left( 3^{(n-m)d} M^{-2} \mathcal{E}'_n - C \right)_+,
$$

where  $m = m(n)$  is defined as above. As  $3^{-(n-m)d} \leq 3^{-n\gamma}$  for some  $\gamma(d, \Lambda, s) > 0$ , it is evident that (4.3) holds. By a union bound and summing th[e rig](#page-7-3)ht side of (4.4) over  $n \in \mathbb{N}$ , we obtain

$$
\mathbb{P}\left[\mathcal{X}\geq Ct\right]\leq CM^d\exp\left(-ct\right).
$$

Replacing  $X$  by  $cX$  and integrating the previous line yields (1.7). This completes the proof of Theorem 1.1.  $\Box$ 

#### **4.2. Uniform approximation in** L<sup>∞</sup>

By int[erpo](#page-0-0)lating  $L^{\infty}$  between  $L^2$  and  $C^{0,\gamma}$  and applying Theorem 1.1 and the (nonlinear version of the) De Giorgi-Nash-Moser estimates, we obtain estimates for the Dirichlet problem with the spatial error measured in  $L^{\infty}$  rather than  $L^2$ . Since the estimate in Theorem 1.1 is already suboptimal in the exponent, there is essentially no loss in passing from  $L^2$  to  $L^\infty$ . We present a model result in the following corollary, which, in view of its application in the next section, is stated as a local approximation result (rather than an error estimate for the Dirichlet problem) and scaled differently (the microscopic scale is of order one).

COROLLARY 4.2. – *For every*  $M \geq 1$ ,  $s \in (0,d)$ , there exist  $\alpha(d,\Lambda) > 0$ ,  $C(d,\Lambda,s) \geq 1$ *and a nonnegative random variable* X *on* (Ω, F ) *satisfying*

$$
\mathbb{E}\big[\exp(\mathcal{X})\big] \leq CM^d
$$

such that the following holds: for every  $L \in \Omega$   $R \geq 1$  and  $u \in H^1(B_R)$  satisfying

$$
K_0 + \frac{1}{R} \left( \int_{B_R} |u(x)|^2 \ dx \right)^{1/2} \le M
$$

*and*

(4.5) 
$$
\int_{B_R} L\left(Du(x),x\right) dx \leq \int_{B_R} L\left(Dw(x),x\right) dx \text{ for every } w \in u + H_0^1(B_R),
$$

*there exists*  $v \in H^1(B_{R/2})$  *such that* 

<span id="page-41-0"></span>
$$
\int_{B_{R/2}} \overline{L}(Dv(x)) dx \le \int_{B_{R/2}} \overline{L}(Dw(x)) dx \text{ for every } w \in v + H_0^1(B_{R/2}),
$$

*and*

(4.6) 
$$
\sup_{x \in B_{R/2}} |u(x) - v(x)| \le CM \left(1 + \mathcal{X}R^{-s}\right) R^{-\alpha(d-s)}.
$$

*Proof*. – By the interior Meyers estimate (c.f. [11, Theorem 6.7] and the remarks in Section A.1), there exists  $t(d, \Lambda) > 2$  and  $C(d, \Lambda) \ge 1$  such that  $u \in W^{1,t}(B_{3R/4})$  and

$$
\left(\int_{B_{3R/4}}|Du(x)|^t\,dx\right)^{2/t}\leq C\left(K_0^2+\int_{B_R}|Du(x)|^2\,dx\right).
$$

We let  $\mathcal X$  be the random variable in the statement of Theorem 1.1 for  $U = B_1$  and with t as in the previous sentence. We take  $v \in u + H_0^1(B_{3R/4})$  to be the unique minimizer of the Dirichlet problem

$$
\int_{B_{3R/4}} \overline{L}(Dv(x)) dx \le \int_{B_{3R/4}} \overline{L}(Dw(x)) dx \quad \text{for every } w \in u + H_0^1(B_{3R/4}).
$$

<span id="page-42-0"></span>By Theorem 1.1, for some  $\alpha(d,\Lambda) > 0$ , we have

(4.7) 
$$
\int_{B_R} |u(x) - v(x)|^2 dx
$$
  
\n
$$
\leq C \left(1 + R^{-s} \mathcal{X}\right) R^{2-\alpha(d-s)} \left(K_0^2 + \left(\int_{B_R} |Du(x)|^t dx\right)^{2/t}\right)
$$
  
\n
$$
\leq C \left(1 + R^{-s} \mathcal{X}\right) R^{-\alpha(d-s)} \left(K_0^2 R^2 + \int_{B_R} |u(x)|^2 dx\right).
$$

<span id="page-42-1"></span>By the De Giorgi-Nash-Moser estimate (see [10] or [11]), there exists  $\gamma(d,\Lambda) \in (0,1]$  such that

(4.8) 
$$
[u - v]_{C^{0,\gamma}(B_{3R/4})} \leq [u]_{C^{0,\gamma}(B_{3R/4})} + [v]_{C^{0,\gamma}(B_{3R/4})}
$$

$$
\leq CR^{-\gamma} \left(K_0 R + \left(\frac{1}{B_R} |u(x)|^2 dx\right)^{1/2}\right).
$$

Applying the interpolation inequality

$$
\sup_{x \in B_r} |\phi(x)| \le \left( \int_{B_r} |\phi(x)|^2 dx \right)^{\gamma/(2\gamma+d)} \left( \sup_{x,y \in B_r} \frac{\phi(x) - \phi(y)}{|x - y|^{\gamma}} \right)^{d/(2\gamma+d)} \n= Cr^{d\gamma/(2\gamma+d)} \left( \int_{B_r} |\phi(x)|^2 dx \right)^{\gamma/(2\gamma+d)} \left( \sup_{x,y \in B_r} \frac{\phi(x) - \phi(y)}{|x - y|^{\gamma}} \right)^{d/(2\gamma+d)}
$$

to  $\phi = u - v$  with  $r = R/2$ , and then using (4.7) and (4.8) to estimate the two terms on the right side, we obtain

$$
\sup_{x \in B_{R/2}} |u(x) - v(x)|
$$

$$
\leq C(1+R^{-s}\mathcal{X})R^{-\alpha(d-s)\gamma/(2\gamma+d)}\left(K_0R+\left(\int_{B_R}|u(x)|^2\,dx\right)^{1/2}\right).
$$

This yields the result, after we redefine  $\alpha$  to be  $\alpha\gamma/(2\gamma + d)$ .

The interpolation inequality is elementary and of course well-known, but for the convenience of the reader we indicate a short proof here. By homogeneity, we may assume that

 $|\phi(y)| = \phi(y) = \sup_{x \in B_r} |\phi(x)| = 1$ . Then the  $\|\phi\|_{L^2(B_r)}$  may be estimated from below by  $k := [\phi]_{C^{0,\gamma}(B_r)}$  by observing that

$$
\phi(x) \ge \left(1 - k^{-\gamma}|x - y|^{\gamma}\right)_+,
$$

<span id="page-43-0"></span>and directly computing the  $L^2$  norm of the function on the right. This gives the interpolation inequality.  $\Box$ 

#### **5. Higher regularity: the quenched Lipschitz e[sti](#page-58-1)[ma](#page-58-2)te**

In this section we prove Theorem 1.2. The argument is based on an idea which was first applied in the context of homogenization by Avellaneda and Lin [2, 3]: functions which can be appro[xima](#page-0-0)ted in  $L^{\infty}$  by function[s sat](#page-0-0)isfying an improvement of flatness property must inherit this property—at least on scales larger enough that the approximation is valid. We proceed by formalizing this idea in an elementary lemma, which makes it quite evident that Theorem 1.2 follows from Theorem 1.1. The lemma is a deterministic statement which is oblivious even to the PDE.

<span id="page-43-1"></span>LEMMA 5.1. – *For*  $r > 0$  *and*  $\theta \in (0, 1/2]$ *, let*  $\mathcal{C}(r, \theta) \subseteq L^{\infty}(B_r)$  *denote the set of functions*  $w \in L^{\infty}(B_r)$  *satisfying* 

$$
(5.1) \qquad \frac{1}{\theta r} \inf_{p \in \mathbb{R}^d} \operatorname*{osc}_{x \in B_{\theta r}} \left( w(x) - p \cdot x \right) \leq \frac{1}{2} \left( \frac{1}{r} \inf_{p \in \mathbb{R}^d} \operatorname*{osc}_{x \in B_r} \left( w(x) - p \cdot x \right) \right).
$$

*Suppose*  $\alpha > 0$ ,  $K \geq 0$ ,  $1 \leq r_0 \leq R/4$  *and*  $u \in L^{\infty}(B_R)$  *have the property that, for every*  $r \in [r_0, R/2]$ *, there exists*  $v \in \mathcal{C}(r, \theta)$  *satisfying* 

<span id="page-43-3"></span>(5.2) 
$$
\frac{1}{r} \sup_{x \in B_r} |u(x) - v(x)| \leq r^{-\alpha} \left( K + \frac{1}{2r} \operatorname*{osc}_{2r} u \right).
$$

*Then there exists*  $\beta(\theta) > 0$  *and*  $C(\alpha, \theta) \ge 1$  *such that, for every*  $s \in [r_0, R/2]$ *,* 

<span id="page-43-2"></span>(5.3) 
$$
\frac{1}{s} \underset{B_s}{\operatorname{osc}} u \leq C \left( \frac{1}{R} \underset{B_R}{\operatorname{osc}} u + \left( \frac{s}{R} \right)^{\alpha} K \right)
$$

*and*

$$
(5.4) \quad \frac{1}{s} \inf_{p \in \mathbb{R}^d} \operatorname*{osc}_{x \in B_s} \left( u(x) - p \cdot x \right) \leq C \Bigg( \left( \frac{s}{R} \right)^{\beta} \frac{1}{R} \inf_{p \in \mathbb{R}^d} \operatorname*{osc}_{x \in B_R} \left( u(x) - p \cdot x \right) + s^{-\alpha} \left( K + \frac{1}{R} \operatorname*{osc}_{B_R} u \right) \Bigg).
$$

The proof of the [lemm](#page-0-0)a is given below. We first apply it to show that [Theo](#page-8-2)rem 1.2 [is a](#page-8-3) consequence of Corollary 4.2.

*Proof of Theorem 1.2.* – Let  $M \geq 1$ ,  $R \geq 2$  and  $u \in H^1(B_R)$  satisfy (1.13) and (1.14). Note that by the interior De Giorgi-Nash-Moser estimate, we have

$$
\frac{2}{R} \operatorname*{osc}_{B_{R/2}} u \leq CM.
$$

Fix  $s \in (0, d)$  and let X be as in the statement of Corollary 4.2 with M replaced by  $C'M$ for  $C' \geq 1$  to be selected below. Then, according to (4.6), there exists  $C(d, \Lambda, s) \geq 1$  such that, for every  $r \in [\chi^{1/s} + C, R/4]$  such that

$$
K_0 + \frac{1}{2r} \operatorname*{osc}_{B_{2r}} u \le C'M,
$$

there exists a local minimizer  $v \in H^1(B_r) \cap L^\infty(B_r)$  of the homogenized functional satisfying

$$
\sup_{x \in B_r} |u(x) - v(x)| \le r^{-\alpha(d-s)} \left( Kr + \underset{B_{2r}}{\text{osc}} u \right).
$$

Define the random variable  $\mathcal{Y} := \mathcal{X}^{1/s} + C$  $\mathcal{Y} := \mathcal{X}^{1/s} + C$  $\mathcal{Y} := \mathcal{X}^{1/s} + C$ . Observe that  $\mathcal{Y}$  satisfies (1.12).

[We ne](#page-43-1)xt establish that local minimizers of the homogenized functional satisfy the improvement of flatness property. Since  $\overline{L}$  is uniformly convex by (2.37), there exists  $\theta(d,\Lambda) \in (0,1/2]$ such that every local minimizer  $w \in H^1(B_s)$  $w \in H^1(B_s)$  $w \in H^1(B_s)$  of the homogenized energy functional satisfies (5.1) for every  $r \leq s/2$ . This is a simple consequence of the interior  $C^{1,\beta}$  estimate for uniformly convex energy functionals, which can be found in Giaquinta [10], and a scaling argument. In the notation of Lemma 5.1, we have that  $w \in \bigcap_{0 \le r \le s/2} \mathcal{C}(r, \theta)$ .

We claim that, for every  $s \in [\mathcal{Y}, R/4]$ ,

$$
\frac{1}{s} \operatorname*{osc}_{B_s} u \le C\left(K + \frac{1}{R} \operatorname*{osc}_{B_R} u\right).
$$

We argue by induction: let  $n \in \mathbb{N}$  and suppose, for every  $r \in \{2^{-j}R : j = 2, \dots, n\}$ ,

(5.5) 
$$
K_0 + \frac{1}{2r} \underset{B_{2r}}{\csc} u \leq \frac{1}{2} C'M.
$$

This implies that, for e[very](#page-0-0)  $r \in [2^{-n}R, R/4]$ ,

$$
K_0 + \frac{1}{r} \operatorname*{osc}_{B_r} u \le C'M.
$$

If  $r \geq \mathcal{Y}$ , then Lemma 5.1 gives

$$
K_0 + \frac{1}{r} \operatorname*{osc}_{B_r} u \le CM \le \frac{1}{2}C'M,
$$

if we select  $C' = C(d, \Lambda, M, s)$  sufficiently large. Thus (5.5) holds for  $r = 2^{-(n+1)}R$ . By inducti[on, w](#page-8-1)e deduce [that,](#page-8-0) for every  $r \in [\mathcal{Y}, R/4]$ ,

$$
K_0 + \frac{1}{r} \operatorname*{osc}_{B_r} u \le C'M \le CM.
$$

This is (1.15). We get (1.16) after applying the second conclusion of Lemma 5.1.  $\Box$ 

We conclude thi[s sec](#page-0-0)tion with the proof of Lemma 5.1.

*Proof of Lemma 5.1*. – In this argument, C and c denote positive constants depending only on  $(\alpha, \theta)$  which may vary in each occurrence.

*Step 1.* – We setup the argument. Using the assumptions, we find that, for every  $s \in [r_0/2, R/4],$ 

$$
\frac{1}{\theta s}\inf_{p\in\mathbb{R}^d}\operatorname*{osc}_{x\in B_{\theta s}}\left(u(x)-p\cdot x\right)\leq\frac{1}{2}\left(\frac{1}{s}\inf_{p\in\mathbb{R}^d}\operatorname*{osc}_{B_s}\left(u(x)-p\cdot x\right)\right)+Cs^{-\alpha}\left(K+\frac{1}{2s}\operatorname*{osc}_{B_{2s}}u\right).
$$

Define  $s_0 := R$  and, for each  $j \in \mathbb{N}$ , set  $s_j := \theta^{j-1} R/4$ . Pick  $m \in \mathbb{N}$  such that  $s_{m+1} < r_0/2 \leq s_m$ . The previous inequality yields, for every  $j \in \{1, \ldots, m\}$ ,

(5.6) 
$$
F_{j+1} \leq \frac{1}{2} F_j + C s_j^{-\alpha} (K + H_{j-1}),
$$

where we have set, for each  $j \in \{0, \ldots, m\},\$ 

$$
F_j := \frac{1}{s_j} \inf_{p \in \mathbb{R}^d} \operatorname*{osc}_{x \in B_{s_j}} \left( u(x) - p \cdot x \right) \quad \text{and} \quad H_j := \frac{1}{s_j} \operatorname*{osc}_{B_{s_j}} u
$$

Select  $p_j \in \mathbb{R}^d$  such that

$$
\frac{1}{s_j}\operatorname*{osc}_{s_j}(u(x)-p_j\cdot x)=F_j.
$$

Observe that the triangle inequality gives the bounds, for every  $j \in \{0, \ldots, m\}$ :

$$
(5.7) \t\t\t F_j \leq H_j \leq 2|p_j| + F_j,
$$

(5.8) 
$$
|p_j| = \frac{1}{2} \frac{1}{s_j} \operatorname*{osc}_{s_j} (p \cdot x) \le \frac{1}{2} F_j + \frac{1}{2} H_j \le H_j
$$

and, for every *j* ∈ {0, . . . , *m* − 1},

(5.9) 
$$
|p_{j+1} - p_j| = \frac{1}{2} \frac{1}{s_{j+1}} \operatorname*{osc}_{s_{j+1}} (p_{j+1} - p_j) \cdot x
$$

$$
\leq \frac{1}{s_{j+1}} \operatorname*{osc}_{s_{j+1}} (u(x) - p_{j+1} \cdot x) + \frac{1}{s_{j+1}} \operatorname*{osc}_{s_{j}} (u(x) - p_j \cdot x)
$$

$$
= F_{j+1} + \frac{1}{\theta} F_j \leq C(F_{j+1} + F_j).
$$

<span id="page-45-2"></span>Note that (5.9) gives  $|p_{j+1}| \leq |p_j| + C(F_{j+1} + F_j)$  and hence, by induction and (5.8),

(5.10) 
$$
|p_j| \leq H_0 + C \sum_{i=0}^j F_i.
$$

By (5.6), (5.7) and (5.10), we obtain, for every  $j \in \{0, ..., m-1\}$ ,

<span id="page-45-1"></span>(5.11) 
$$
F_{j+1} \leq \frac{1}{2} F_j + C s_j^{-\alpha} \left( K + H_0 + \sum_{i=0}^j F_i \right).
$$

*Step 2.* – By iterating (5.11), we show that

<span id="page-45-0"></span>(5.12) 
$$
F_j \leq 2^{-j} F_0 + C s_j^{-\alpha} (K + H_0).
$$

Arguing by induction, we fix  $A \geq 1$  be a constant to be selected below and suppose that  $k \in \{0, ..., m - 1\}$  is such that, for every  $j \in \{0, ..., k\},$ 

(5.13) 
$$
F_j \leq 2^{-j} F_0 + A s_j^{-\alpha} (K + H_0).
$$

Using (5.11) and this assumption, we find that

$$
F_{k+1} \leq \frac{1}{2} F_k + C s_k^{-\alpha} \left( K + H_0 + \sum_{j=0}^k F_j \right)
$$
  

$$
\leq 2^{-(k+1)} F_0 + \frac{1}{2} A s_k^{-\alpha} (K + H_0)
$$
  

$$
+ C s_k^{-\alpha} \left( K + H_0 + \sum_{j=0}^k \left( 2^{-j} F_0 + A s_j^{-\alpha} (K + H_0) \right) \right)
$$
  

$$
\leq 2^{-(k+1)} F_0 + (K + H_0) s_{k+1}^{-\alpha} \left( \frac{1}{2} A + C + C A s_k^{-\alpha} \right).
$$

If  $k \le n$  where n is such that  $Cs_n^{-\alpha} \le \frac{1}{4}$ , then we may select  $A = C$  sufficiently large that

$$
\frac{1}{2}A + C + CAs_k^{-\alpha} \le \frac{3}{4}A + C \le A.
$$

In this c[ase, w](#page-45-0)e obtain

$$
F_{k+1} \le 2^{-(k+1)}F_0 + As_{k+1}^{-\alpha}(K + H_0).
$$

This is (5.13) for  $j = k + 1$ . Since (5.1[3\) triv](#page-45-1)ially hol[ds fo](#page-43-2)r  $j = 0$ , we obtain (5.12) for all  $j \in \{0, ..., n\}$  by induction. Since  $1 \leq s_j/s_n \leq C$  [and t](#page-45-2)hus  $F_j \leq CF_n$  for all  $j \in \{n+1, ..., m\}$ , we conclude that (5.12) holds for all  $j \in \{0, ..., m\}$ .

*Step 3.* – The conclusion. Notice that (5.12) implies (5.4) for  $\beta := (\log 2)/|\log \theta|$ . To obtain (5.3), we need to bound  $H_j$ , and this is obtained from (5.10) and (5.12) as follows:

$$
H_j \le F_j + 2|p_j| \le 2H_0 + C \sum_{i=0}^j F_i
$$
  
\n
$$
\le 2H_0 + \sum_{i=0}^j \left(2^{-i}F_0 + Cs_i^{-\alpha} (K + H_0)\right)
$$
  
\n
$$
\le 2H_0 + F_0 + Cs_j^{-\alpha} (K + H_0)
$$
  
\n
$$
\le CH_0 + Cs_j^{-\alpha} K.
$$

This implies (5.3).

#### **Appendix**

 $\Box$ 

#### **The proof of Proposition 4.1**

The argument for Proposition 4.1 requires some ingredients from the classical regularity theory in the calculus of variations: we need (i) an interior  $H<sup>2</sup>$  estimate for minimizers of the homogenized energy functional and (ii) a glo[bal](#page-59-17)  $W^{1,t}$  estimate (for Lipschitz domai[ns\),](#page-59-16) for [som](#page-59-17)e  $t(d, \Lambda) > 2$ , for minimizers of both the heterogeneous and homogenized energy functionals. The latter estimate is a generalization to the nonlinear setting of the Meyers estimate for linear equations and can be found in [11]. The former can be found in either [10] or [11].

We now fix some notation used in the rest of this subsection. It is convenient to rescale the functions in the hypotheses of Proposition 4.1 so that the microscopic scale is of unit size and the ratio  $\varepsilon > 0$  of the length scales is reflected in the size of the domain. We thus fix a (large) bounded Lipschitz domain  $U \subseteq \mathbb{R}^d$  and take  $n, m \in \mathbb{N}_*$  such that

(A.1) 
$$
3^{d(n+m)} < |U| \le 3^{d(n+m+1)}.
$$

Essentially, this means that  $3^{-(n+m)} \approx \varepsilon$ , i.e., the macroscopic scale is of order  $3^{n+m}$ . We will take the mesoscopic scale to be of order  $3^n$  rather than  $3^{n-m}$ , as in the statement of the proposition. We fix one more parameter  $l \in \mathbb{N}$  such that

<span id="page-47-2"></span><span id="page-47-0"></span>
$$
n \le l \quad \text{and} \quad 2l \le m+n,
$$

which describes the (mesoscopic) thickness of a boundary strip we need to remove in the approximation argument. (In practice, we typically choose  $m = [cn]$  for a small  $0 < c \ll 1$ , and l roughly equidistant between n and  $n + m$ .) We also denote the normalized domain by  $\hat{U} := U/|U|$ .

Define domains  $V^{\circ} \subseteq V \subseteq U$  by

$$
V := \left\{ x \in \mathbb{R}^d \, : \, x + Q_{l+2} \subseteq U \right\} \quad \text{and} \quad V^\circ := \left\{ x \in \mathbb{R}^d \, : \, x + Q_{l+4} \subseteq U \right\}.
$$

Since  $U$  is a Lipschitz domain, we have

<span id="page-47-1"></span>
$$
|U \setminus V^{\circ}| \leq C3^{l-m-n}|U|,
$$

where the constant C depends only on d and  $\hat{U}$ . We note also that  $dist(V^{\circ}, \partial V) \geq 3^l$ . Denote by  $\eta \in C_0^{\infty}(\mathbb{R}^d)$  a cutoff function satisfying

(A.3) 
$$
0 \le \eta \le 1
$$
,  $\eta \equiv 1$  on  $\overline{V^{\circ}}$ ,  $\eta \equiv 0$  in  $\mathbb{R}^d \setminus V$  and  $|D\eta| \le C3^{-l}$ .

Throughout we fix  $L \in \Omega(K_0)$ ,  $t > 2$ ,  $g \in W^{1,t}(U)$  and denote by  $u \in g + H_0^1(U)$  the unique minimizer of the heterogeneous energy functional: that is,  $u$  satisfies

$$
\int_{U} L\left(Du(x),x\right) dx \le \int_{U} L\left(Dw(x),x\right) dx \quad \text{for every } w \in g + H_0^1(U).
$$

Also take  $u_{\text{hom}} \in g + H_0^1(U)$  to be the unique minimizer of the constant-coefficient energy functional, i.e.,  $v$  satisfies

$$
\int_{U} \overline{L}(Du_{\text{hom}}(x)) dx \le \int_{U} \overline{L}(Dw(x)) dx \text{ for every } w \in g + H_0^1(U).
$$

For convenience, we denote

$$
M:=K_0+\left(\int_U|Dg(x)|^t\,dx\right)^{1/t}
$$

.

as well as

$$
\mathscr{E}' := \sup \left\{ \mathscr{E}(y + Q_n, p) + \mathscr{E}(y' + Q_{n+2}, p') \, : \, y, y' \in B_{C3^{n+m}}, \, p, p' \in B_{CMS^{md/2}} \right\},
$$

which is precisely the rescaled version of  $\mathcal{E}'$  defined in the statement of Proposition 4.1. The convention for the constants C and c in this appendix is that they depend on  $(d, \Lambda, t, \hat{U})$  and may vary in each occurrence.

<span id="page-48-1"></span>In view of the above notation and scaling convention, to obtain Proposition 4.1 it suffices to prove the estimate

(A.4) 
$$
\left| \oint_{U} \left( L(Du(x), x) - \overline{L}(Du_{\text{hom}}(x)) \right) dx \right| + 3^{-2(n+m)} \oint_{U} (u(x) - u_{\text{hom}}(x))^2 dx
$$
  

$$
\leq C \mathcal{E}' + CM^2 \left( 3^{n-l} + 3^{\beta(l-n-m)} \right).
$$

<span id="page-48-0"></span>The main step in the proof of  $(A.4)$  is to show that the (het[eroge](#page-0-0)ne[ous\) e](#page-0-0)nergy of u is very close to the (effective) energy of  $u_{\text{hom}}$ . The plan is to modify each minimizer in order to get a candidate for a local minimizer of the other's functional, and thus an upper bound for the energy of the other. These steps are su[mmar](#page-48-1)ized in Lemmas A.1 and A.2, below.

#### **A.1. Classical regularity estimates**

<span id="page-48-2"></span>Before proceeding with the proof of  $(A.4)$ , we record here the needed estimates from regularity theory. According to the Meyers estimate [11, Theorem 6.8], there exists  $r(d, \Lambda, t) \in (2, t]$ such that  $u, u_{\text{hom}} \in W^{1,r}(U)$  and

(A.5) 
$$
\left(\int_U |Du(x)|^r dx\right)^{1/r} + \left(\int_U |Du_{\text{hom}}(x)|^r dx\right)^{1/r} \leq CM.
$$

<span id="page-48-3"></span>We also need the interior  $H^2$  estimate [11, Theorem 8.1] for solutions of constant coefficient functionals which, together with an easy covering argument (using that  $dist(V, \partial U) \geq c3^l$ ) implies  $u_{\text{hom}} \in H^2(V)$  [and](#page-59-17) the gives the estimate

(A.6) 
$$
3^{l} \left( \int_{V} |D^{2} u_{\text{hom}}(x)|^{2} dx \right)^{1/2} \leq CM.
$$

We remark that while [11, Theorem 6.8] does not include the *a priori* estimate we need (rather just the statement that the functions belong to  $W^{1,r}(U)$ ), but it can be extracted from the proof there. Also, the hypotheses in [11] are slightly stronger, namely they demand that the integrand  $L(p, x)$  be  $C<sup>2</sup>$  in the p variable rather than just  $C<sup>1,1</sup>$ . How[ever](#page-59-17), by inspecting the arguments, one finds that the estimates do not depend on the modulus of continuity of  $D_p^2 L(\cdot, x)$ , rather only on an upper bound for  $|D^2 L(\cdot, x)| = [D_p L(\cdot, x)]_{C^{0,1}}$ , which in our case is bounded above by  $\Lambda$ . Therefore we obtain the results we need from [11] after a routine approximating argument (by smoothing the coefficients).

#### **A.2. Estimate for the (homogenized) energy of** uhom

This is the easier of the two directions. The idea is to remove the microscopic oscillations from  $u$ , and for this it is natural to consider a spatial average on a mesoscopic scale. We thus define

$$
\xi(y) := \int_{y+Q_n} u(x) \, dx, \quad y \in V.
$$

Notice that  $\xi \in H^1(V)$ . We next modify  $\xi$  in order to get an element of  $g + H_0^1(U)$  by setting

$$
\widetilde{u}(x) := \eta(x)\xi(x) + (1 - \eta(x))u(x).
$$

It is clear that  $\tilde{u} \in g + H_0^1(U)$ . We estimate the (homogenized) energy of  $\tilde{u}$  in terms of the heterogeneous energy of  $u$  and the error term  $\mathscr{E}'$  defined above.

LEMMA A.1. – *There exists*  $C(d, \Lambda, t, \hat{U}) \geq 1$  *and*  $r(d, \Lambda, t) \in (2, t]$  *such that* 

(A.7) 
$$
\int_{U} \overline{L}(D\widetilde{u}(x)) dx \leq \int_{U} L(Du(x), x) dx + C\mathscr{E}' + CM^{2}3^{(l-n-m)(1-2/r)}.
$$

*Proof*. – We divide the argument into several steps.

*Step 1*. – We show that  $\xi \in W^{1,\infty}(V)$ . D[enot](#page-47-0)e, for each  $y \in V$ ,

<span id="page-49-0"></span>
$$
p(y) := D\xi(y) = \int_{y+Q_n} Du(x) dx \text{ and } q(y) := D\overline{L}(p(y)).
$$

Observe that, by Hölder's inequality and (A.1), for each  $y \in V$ ,

(A.8) 
$$
|p(y)|^2 \le \left(\int_{y+Q_n} |Du(x)| dx\right)^2 \le \frac{|U|}{|Q_n|} \left(\int_U |Du(x)|^2 dx\right) \le C3^{md} \left(\int_U |Du(x)|^2 dx\right) \le C3^{md} M^2.
$$

*Step 2.* – We use (A.5) to show that

(A.9) 
$$
\int_{U} |D\widetilde{u}(x)|^{r} dx \leq C \int_{U} |Du(x)|^{r} dx.
$$

Differentiati[ng the](#page-47-1) formula above for  $\tilde{u}$ , we find

<span id="page-49-2"></span>
$$
D\widetilde{u}(x) = Du(x) + \eta(x) \left( D\xi(x) - Du(x) \right) + D\eta(x) \left( \xi(x) - u(x) \right)
$$

and thus, by  $(A.3)$ ,

$$
\left(\oint_{U} |D\widetilde{u}(x)|^{r} dx\right)^{1/r} \leq \left(\oint_{U} |Du(x)|^{r} dx\right)^{1/r} + \left(\oint_{V} |D\xi(x) - Du(x)|^{r} dx\right)^{1/r} + C3^{-l} \left(\oint_{V} |\xi(x) - u(x)|^{r} dx\right)^{1/r}.
$$

We obtain (A.9) from the previous inequality and the following (recall  $l \geq n$ ):

<span id="page-49-1"></span>(A.10) 
$$
\int_{V} |\xi(x) - u(x)|^{r} dx \leq C 3^{nr} \int_{U} |Du(x)|^{r} dx
$$

and

(A.11) 
$$
\int_{V} |D\xi(x) - Du(x)|^r dx \leq C \int_{U} |Du(x)|^r dx.
$$

The se[cond e](#page-49-2)stimate (A.11) follows from the triangle inequality and

$$
\int_{V} |D\xi(x)|^r dx = \int_{V} \left| \int_{x+Q_n} Du(y) dy \right|^r dx \le \frac{|U|}{|V|} \int_{U} |Du(x)|^r dx \le C \int_{U} |Du(x)|^r dx.
$$
\nSo get (A, 10), we use that for every  $z \in Q$ , i.e.

To get (A.10), we use that, for every  $z \in Q_{n+1}$ ,

$$
\left(\int_{Q_{n+1}(z)} |u(x) - \xi(x)|^r dx\right)^{1/r} \le \left(\int_{Q_{n+1}(z)} \left|u(x) - \int_{Q_n(x)} u(y) dy\right|^r dx\right)^{1/r} + \left(\int_{Q_{n+1}(z)} \left|\int_{Q_n(x)} u(y) dy - \xi(x)\right|^r dx\right)^{1/r}.
$$

The first term on the right is estimated by the Poincaré inequality:

$$
\int_{Q_{n+1}(z)} \left| u(x) - \int_{Q_n(x)} u(y) \, dy \right|^r \, dx = \int_{Q_{n+1}(z)} \int_{Q_n(\xi)} \left| u(x) - \int_{Q_n(\xi)} u(y) \, dy \right|^r \, dx \, d\xi
$$
\n
$$
\leq C 3^{nr} \int_{Q_{n+1}(z)} \int_{Q_n(\xi)} |Du(y)|^r \, dy \, d\xi
$$
\n
$$
= C 3^{nr} \int_{Q_{n+1}(z)} |Du(x)|^r \, dx,
$$

while, to estimate the second, we observe that, for every  $x + Q_n$ ,  $x' + Q_n \subseteq Q_{n+1}(z)$ ,

$$
\left| \oint_{x+Q_n} u(y) \, dy - \oint_{x'+Q_n} u(y) \, dy \right| = \left| \oint_{Q_n} \int_0^1 (x - x') \cdot Du(tx + (1 - t)x' + y) \, dt \, dy \right|
$$
  

$$
\leq 3^d \left| x - x' \right| \oint_{Q_{n+1}(z)} |Du(y)| \, dy
$$
  

$$
\leq C 3^n \oint_{Q_{n+1}(z)} |Du(y)| \, dy.
$$

This yields

$$
\int_{Q_{n+1}(z)} \left| \int_{Q_n(x)} u(y) dy - \xi(x) \right|^r dx \leq C 3^n \int_{Q_{n+1}(z)} |Du(y)|^r dy
$$

<span id="page-50-0"></span>and completes the proof of (A.9).

*Step 3.* We prove that

(A.12) 
$$
\oint_{U} L(Du(x), x) dx \ge \oint_{V^{\circ}} (\mu(y + Q_n, q(y)) + p(y) \cdot q(y)) dy - CM^{2} 3^{(l-n-m)(1-2/r)}.
$$

Taking u as a minimizer candidate in the definition of  $\mu(y+Q_n, q(y))$  and using that  $p(y)$  is dual to  $q(y)$ , we have

$$
\int_{V^{\circ}} L(Du(x), x) dx
$$
\n
$$
\geq \int_{V^{\circ}} \int_{y+Q_n} L(Du(x), x) dx dy - \frac{1}{|V^{\circ}|} \int_{U \setminus V^{\circ}} |L(Du(x), x)| dx
$$
\n
$$
\geq \int_{V^{\circ}} (\mu(y + Q_n, q(y)) + p(y) \cdot q(y)) dy - \frac{C}{|U|} \int_{U \setminus V^{\circ}} (K_0 + |Du|)^2 dx.
$$

We next handle the error arising from the boundary strip. Using (A.2), we have

$$
\int_{U} L(Du(x), x) dx - \int_{V^{\circ}} L(Du(x), x) dx
$$
\n
$$
\geq -C3^{l-n-m} \left| \int_{U} L(Du(x), x) dx \right| - \frac{1}{|U|} \left| \int_{U \setminus V^{\circ}} L(Du(x), x) dx \right|
$$
\n
$$
\geq -C3^{l-n-m} \int_{U} (K_0 + |Du(x)|)^2 dx - \frac{C}{|U|} \int_{U \setminus V^{\circ}} (K_0 + |Du(x)|)^2 dx.
$$

Assembling the previous two strings of inequalities gives (A.12), after we estimate the error terms in the following way: by (A.[5\), w](#page-48-2)e have

$$
\oint_U (K_0 + |Du(x)|)^2 dx \le CM^2
$$

and then Hölder's inequality and (A.5) give

$$
\frac{1}{|U|} \int_{U \setminus V^{\circ}} (K_0 + |Du(x)|)^2 dx \le \left(\frac{|U \setminus V^{\circ}|}{|U|}\right)^{1-2/r} \left(\int_U (K_0 + |Du(x)|)^r dx\right)^{2/r} \le C3^{(l-n-m)(1-2/r)} M^2.
$$

This completes the proof of (A.12).

*Step 4.* – The conclusion. According to (A.12),

$$
\int_{U} L(Du(x), x) dx - \int_{V^{\circ}} \overline{L}(p(y)) dy
$$
\n
$$
\geq - \int_{V^{\circ}} \mathcal{E}(y + Q_n, p(y)) dy - CM^2 3^{(l-n-m)(1-2/r)}
$$
\n
$$
\geq -\mathcal{E}' - CM^2 3^{(l-n-m)(1-2/r)}.
$$

It remains to use (A.5), (A.9) and Hölder's inequality to estimate the energy of  $\tilde{u}$  in the boundary strip, much as we did above for  $u$  in [Step](#page-49-0) 3. We have

$$
\oint_{U} \overline{L}(D\widetilde{u}(y)) dy - \oint_{V^{\circ}} \overline{L}(p(y)) dy = -\frac{|U \setminus V^{\circ}|}{|U|} \oint_{V^{\circ}} \overline{L}(p(y)) dy + \frac{1}{|U|} \int_{U \setminus V^{\circ}} \overline{L}(D\widetilde{u}(y)) dy
$$

and we esti[mate](#page-49-0) the error terms as follows: by (A.9),

$$
\frac{|U \setminus V^{\circ}|}{|U|} \left| \int_{V^{\circ}} \overline{L}(p(y)) dy \right| = \frac{|U \setminus V^{\circ}|}{|U|} \int_{V^{\circ}} \left| \overline{L}(D\widetilde{u}(y)) \right| dy \leq CM^{2} 3^{l-n-m}
$$

and, using (A.9) and Hölder's inequality again,

$$
\frac{1}{|U|} \int_{U \setminus V^{\circ}} \overline{L}(D\widetilde{u}(y)) dy \leq C M^2 3^{(l-n-m)(1-2/r)}.
$$

 $\Box$ 

Combining these gives the lemma.

#### **A.3. Estimate of the (heterogeneous) energy of** u

We next modify  $u_{\text{hom}}$  to obtain a minimizer candidate  $\widetilde{u}_{\text{hom}}$  for the heterogeneous energy functional by a stitching together mesoscopic minimizers on an overlapping grid, not unlike the patching construction in the proof of Lemma 3.3.

We begin the construction by defining an affine approximation to  $u_{\text{hom}}$  in the mesoscopic cube  $z + Q_{n+2}$  by setting, for each  $z \in V \cap 3^n \mathbb{Z}^d$ ,

$$
\ell_z(x) := p(z) \cdot (x - z) + \zeta(z),
$$

where

$$
\zeta(z) := \int_{z + Q_{n+2}} u_{\text{hom}}(x) \, dx \quad \text{and} \quad p(z) := D\zeta(z) = \int_{z + Q_{n+2}} Du_{\text{hom}}(x) \, dx.
$$

For each  $z \in V \cap 3^n \mathbb{Z}^d$ , we introduce mesoscopic minimizers in  $z + Q_{n+2}$  of the heterogeneous energy functional with Dirichlet boundary conditions given by  $l_z$ :

$$
v_z(x) := v(x, z + Q_{n+2}, D\zeta(z)) - D\zeta(z) \cdot z + \zeta(z), \quad x \in z + Q_{n+2}(z).
$$

Observe that, for each  $z \in V \cap 3^n \mathbb{Z}^d$ , we have  $v_z \in H^1(z + Q_{n+2})$ ,

$$
\int_{z+Q_{n+2}} (v_z(x) - u_{\text{hom}}(x)) dx = 0 \text{ and } \int_{z+Q_{n+2}} (Dv_z(x) - Du_{\text{hom}}(x)) dx = 0.
$$

Next we patch these functions together to get a function defined on  $V$ . Define a smooth periodic partition of unity by setting

<span id="page-52-0"></span>
$$
\phi(x) := \int_{Q_n} \eta(x - y) \, dy,
$$

where  $\eta \in C^{\infty}(\mathbb{R}^d)$  satisfies

$$
0 \leq \eta \leq 1, \quad \eta \equiv 0 \text{ in } \mathbb{R}^d \setminus Q_{n-1}, \quad \int_{\mathbb{R}^d} \eta(y) \, dy = 1, \quad \text{and} \quad |D\eta| \leq C 3^{-(d+1)n}.
$$

Here we have essentially mollified the characteristic function of the cube  $Q_n$  to obtain a function  $\phi$ , which is supported in  $Q_{n+1}$  and satisfies

(A.13) 
$$
\sup_{x \in Q_{n+1}} |D\phi(x)| \leq C3^{-(d+1)n} |Q_n| \leq C3^{-n}
$$

and, for every  $x \in \mathbb{R}^d$ ,

$$
\sum_{z \in 3^n \mathbb{Z}^d} \phi(x - z) = 1.
$$

The latter holds since the cubes  $\{z+Q_n : z \in 3^n \mathbb{Z}^d\}$  form a disjoint partition of  $\mathbb{R}^d$ , up to a set of Lebesgue measure zero. Now set

$$
\widetilde{v}(x) := \sum_{z \in V \cap 3^n \mathbb{Z}^d} \phi(x - z) v_z(x), \quad x \in V.
$$

Then  $\tilde{v} \in H^1(V)$ . Finally, we modify  $\tilde{v}$  to match the boundary condition. Take  $\eta \in C_0^{\infty}(\mathbb{R}^d)$ <br>to be the sutoff function extighing (A, 2), as above and set to be the cutoff function satisfying (A.3), as above, and set

$$
\widetilde{u}_{\text{hom}}(x) := \eta(x)\widetilde{v}(x) + (1 - \eta(x)) u_{\text{hom}}(x).
$$

Then  $\widetilde{u}_{\text{hom}} \in g + H^1(U)$  is the minimizer candidate for the heterogeneous energy functional.<br>We estimate its energy from above using an example similar to the one in Lamma 2.2 We estimate its energy from above using an argument similar to the one in Lemma 3.3 combined with some aspects of the proof of Lemma A.1 and relying on Theorem 3.1. This result is summarized in the following lemma.

LEMMA A.2. – There exists 
$$
C(d, \Lambda, t, \hat{U}) \ge 1
$$
 and  $r(d, \Lambda, t) \in (2, t]$  such that  
\n
$$
\oint_{U} L(D\tilde{u}_{hom}(x), x) dx \le \oint_{U} \overline{L}(Du_{hom}(x)) dx + C\mathcal{E}' + CM^{2} \left(3^{n-l} + 3^{(l-n-m)(1-2/r)}\right).
$$

*Proof*. – We divide the proof into several steps.

<span id="page-53-0"></span>*Step 1.* – We record some estimates on the mesoscopic affine approximations which are needed below. By Hölder's inequality and (A.1), we have, for each  $z \in V$ ,

(A.15) 
$$
|p(z)|^2 \le \left(\int_{z+Q_{n+2}} |Du_{\text{hom}}(x)| dx\right)^2 \le \frac{|U|}{|Q_{n+2}|} \left(\int_U |Du_{\text{hom}}(x)|^2 dx\right)
$$

$$
\le C3^{md} \left(\int_U |Du_{\text{hom}}(x)|^2 dx\right) \le C3^{md} M^2.
$$

In view of the definition of  $v_z$  and the previous inequality, we obtain, for  $z \in V \cap \mathbb{Z}^d$ ,

(A.16) 
$$
\left| \int_{z+Q_{n+2}} L(Dv_z(x),x) dx - \overline{L}(p(z)) \right| \leq \mathcal{E}'.
$$

We next show that overlapping mesoscopic affine approximations to  $u<sub>hom</sub>$  agree, up to a small error. Compute, for every  $z, z' \in V \cap 3^n \mathbb{Z}^d$  such that  $z' \in z + Q_{n+1}$ ,

$$
p(z) - p(z') = \int_{z + Q_{n+2}} (Du_{\text{hom}}(x) - Du_{\text{hom}}(x + z' - z)) dx
$$
  
= 
$$
\int_{z + Q_{n+2}} (z - z') \cdot \int_0^1 D^2 u_{\text{hom}}(x + (1 - t)(z' - z)) dt dx
$$

and, after changing the order of integration, applying Jensen's inequality and using  $|z - z'| \leq C3^n$ , obtain

(A.17) 
$$
|p(z) - p(z')|^2 \leq C3^{2n} \int_{z + Q_{n+3}} |D^2 u_{\text{hom}}(x)|^2 dx.
$$

A similar computation yields

<span id="page-53-2"></span>(A.18) 
$$
|\zeta(z) - \zeta(z')|^2 \leq C3^{4n} \int_{z + Q_{n+3}} |D^2 u_{\text{hom}}(x)|^2 dx.
$$

Combining these, we get control ove[r the d](#page-53-0)ifferences of the affine approximations:

(A.19) 
$$
\sup_{x \in z + Q_{n+3}} |\ell_z(x) - \ell_{z'}(x)|^2 \leq C3^{4n} \int_{z + Q_{n+3}} |D^2 u_{\text{hom}}(x)|^2 dx.
$$

<span id="page-53-3"></span>For our reference, we also return to (A.15) and finish the estimate differently, using Jensen's inequality to obtain, for all such  $z, z' \in V$ ,

(A.20) 
$$
|p(z')|^2 \leq C \int_{z'+Q_{n+2}} |Du_{\text{hom}}(x)|^2 dx \leq C \int_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 dx.
$$

<span id="page-53-1"></span>*Step 2.* – We show that two mesoscopic minimizers  $v_z$  and  $v_{z}$  with overlapping domains agree, up to a small error, on the overlap. The claim is that, for every  $z, z' \in V \cap 3^n \mathbb{Z}^d$  and  $y \in 3^n \mathbb{Z}^d$  such that  $y + Q_n \subseteq (z + Q_{n+2}) \cap (z' + Q_{n+2})$ , we have

(A.21) 
$$
3^{-2n} \int_{y+Q_n} (v_{z'}(x) - v_z(x))^2 dx + \int_{y+Q_n} |Dv_z(x) - Dv_{z'}(x)|^2 dx
$$
  
\n $\leq C\mathcal{E}' + C3^{n-l} \left(K_0^2 + \int_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 dx\right) + C3^{n+l} \int_{z+Q_{n+3}} |D^2u_{\text{hom}}(x)|^2 dx.$ 

The estimate of the first term on the left side of (A.21) follows from (A.19), the definition of  $\mathcal{E}'$ , the triangle inequality and the fact that  $n \leq l$ . Therefore we focus on estimating the difference of gradient overlaps.

We first show that the energy of each mesoscopic minimizer  $v<sub>z</sub>$  spreads evenly in the  $3^{2d}$  subcubes of  $z + Q_{n+2}$  which are proportional to  $Q_n$ . We may these enumerate these subcubes by  $y + Q_n$  for  $y \in z + J_n$  where  $J_n := 3^n \mathbb{Z}^d \cap Q_{n+2}$ . Now compute, for  $y \in z + J_n$ ,

$$
\int_{y+Q_n} (L(Dv_z(x), x) - q(y) \cdot Dv_z(x)) dx
$$
\n
$$
= 3^{2d} \int_{z+Q_{n+2}} L(Dv_z(x), x) dx - 3^{2d} q(y) \cdot p(z)
$$
\n
$$
- \sum_{y' \in z + J_n \setminus \{y\}} \int_{y' + Q_n} (L(Dv_z(x), x) - q(y) \cdot Dv_z(x)) dx
$$
\n
$$
\leq 3^{2d} \nu(z + Q_{n+2}, p(z)) - 3^{2d} q(y) \cdot p(z) - \sum_{y' \in z + J_n \setminus \{y\}} \mu(y' + Q_n, q(y)).
$$

Using (2.33) and the triangle inequality, we obtain

$$
\int_{y+Q_n} (L(Dv_z(x), x) - q(y) \cdot Dv_z(x)) dx
$$
\n
$$
\leq \mu(y+Q_n, q(y)) + C\mathcal{E}(z+Q_{n+2}, p(y)) + \sum_{y' \in z+J_n} \mathcal{E}(y' + Q_n, p(y))
$$
\n
$$
+ C (K_0 + |p(z)| + |p(y)|) |p(y) - p(z)| + C |q(y)| |p(y) - p(z)|.
$$

<span id="page-54-0"></span>From the previous inequality, (A.17) and (A.20), we get

$$
(A.22) \int_{y+Q_n} (L(Dv_z(x),x) - q(y) \cdot Dv_z(x)) dx - \mu(y+Q_n, q(y))
$$
  
 
$$
\leq C\mathcal{E}' + C3^n \left( K_0^2 + \int_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 dx \right)^{1/2} \left( \int_{z+Q_{n+3}} |D^2 u_{\text{hom}}(x)|^2 dx \right)^{1/2}.
$$

We now apply Lemma 2.1 to obtain, for any  $z, z' \in 3^n \mathbb{Z}^d \cap V$  and  $y \in 3^n \mathbb{Z}^d$  such that  $y + Q_n \subseteq (z + Q_{n+2}) \cap (z' + Q_{n+2}),$ 

$$
\int_{y+Q_n} |Dv_z(x) - Dv_{z'}(x)|^2 dx
$$
\n
$$
\leq C\mathcal{E}' + C3^n \left( K_0^2 + \int_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 dx \right)^{1/2} \left( \int_{z+Q_{n+3}} |D^2u_{\text{hom}}(x)|^2 dx \right)^{1/2}.
$$

Young's inequality gives the desired estimate for the  $L^2$  difference of the gradients and completes the proof of (A.21).

<span id="page-55-0"></span>For future reference, we note that

(A.23) 
$$
\left| \oint_{z+Q_n} Dv_z(x) dx - p(z) \right|^2
$$
  
\n
$$
\leq C \mathcal{E}' + C3^{n-l} \left( K_0^2 + \oint_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 dx \right) + C3^{n+l} \oint_{z+Q_{n+3}} |D^2 u_{\text{hom}}(x)|^2 dx.
$$

To see this, we use Lemma [2.1](#page-0-0) and [\(A.22](#page-53-1)) to compare the gradients of  $v<sub>z</sub>$  and the minimizer of  $\mu(z + Q_n, q(z))$  and then apply (3.54), before using Young's inequality as above.

<span id="page-55-2"></span>*Step 3*. – We use Lemma 2.2 and (A.21) to derive an upper bound for the (heterogeneous) energy of  $\tilde{v}$  in V. The claim is that

(A.24) 
$$
\oint_{V} L(D\widetilde{v}(x),x) dx \leq \oint_{V} \overline{L}(Du_{\text{hom}}(x)) dx + C\mathscr{E}' + CM\sqrt{\mathscr{E}'} + C3^{n-l}M^{2}.
$$

For each  $z \in W \cap 3^n \mathbb{Z}^d$  and  $x \in Q_n(z) \subseteq W$ , we have

$$
\widetilde{v}(x) - v_z(x) = \sum_{y \in z + J'_n} \phi(x - y) (v_y(x) - v_z(x))
$$

where we denote  $J'_n := \{ y \in 3^n \mathbb{Z}^d : (y + J_n) \cap J_n \neq \emptyset \}$ , which we observe has at most C elements. Differentiating this expression yields, for all such  $z$  and  $x$ ,

<span id="page-55-1"></span>
$$
D\widetilde{v}(x) - Dv_z(x) = \sum_{y \in z + J'_n} \left( D\phi(x - y) \left( v_y(x) - v_z(x) \right) + \phi(x - y) \left( Dv_y(x) - Dv_z(x) \right) \right),
$$

and then applying (A.21), using the bound  $|D\phi| \leq C3^{-n}$  from (A.13), we obtain

(A.25) 
$$
\int_{z+Q_n} |D\tilde{v}(x) - Dv_z(x)|^2 dx
$$
  
\n
$$
\leq C\mathcal{E}' + C3^{n-l} \left( K_0^2 + \int_{z+Q_{n+3}} |D u_{\text{hom}}(x)|^2 dx \right) + C3^{n+l} \int_{z+Q_{n+3}} |D^2 u_{\text{hom}}(x)|^2 dx.
$$

The previous inequality, (A.22) with  $y = z$  and Lemma 2.2 yield

$$
\int_{z+Q_n} (L(D\tilde{v}(x), x) - q(z) \cdot D\tilde{v}(x)) dx - \mu(z + Q_n, q(z))
$$
\n
$$
\leq C\mathcal{E}' + C3^{n-l} \left( K_0^2 + \int_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 dx \right) + C3^{n+l} \int_{z+Q_{n+3}} |D^2 u_{\text{hom}}(x)|^2 dx.
$$

By  $(A.23)$  and  $(A.25)$ , we have

$$
\left| \oint_{z+Q_n} D\widetilde{v}(x) dx - p(z) \right|^2
$$
  
\n
$$
\leq C\mathcal{E}' + C3^{n-l} \left( K_0^2 + \oint_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 dx \right) + C3^{n+l} \oint_{z+Q_{n+3}} |D^2 u_{\text{hom}}(x)|^2 dx.
$$

The previous two lines yield

$$
\int_{z+Q_n} L(D\tilde{v}(x), x) dx - q(z) \cdot p(z) - \mu(z + Q_n, q(z))
$$
\n
$$
\leq C\mathcal{E}' + C3^{n-l} \left( K_0^2 + \int_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 dx \right) + C3^{n+l} \int_{z+Q_{n+3}} |D^2 u_{\text{hom}}(x)|^2 dx
$$
\n
$$
+ C\sqrt{\mathcal{E}'} \left( K_0^2 + \int_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 dx \right)^{\frac{1}{2}}.
$$

The sum of the last two terms on the left side is  $-\overline{L}(p(z))$ , up to an error of  $\mathscr{E}'$ . Using this and (A.16), we get

$$
(A.26) \int_{z+Q_n} L(D\tilde{v}(x),x) dx - \overline{L}(p(z))
$$
  
\n
$$
\leq C\mathcal{E}' + C3^{n-l} \left( K_0^2 + \int_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 dx \right) + C3^{n+l} \int_{z+Q_{n+3}} |D^2 u_{\text{hom}}(x)|^2 dx
$$
  
\n
$$
+ C\sqrt{\mathcal{E}'} \left( K_0^2 + \int_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 dx \right)^{\frac{1}{2}}.
$$

By the Poincaré inequality,

$$
\int_{z+Q_{n+2}} |Du_{\text{hom}}(x) - p(z)|^2 dx \leq C3^{2n} \int_{z+Q_{n+2}} |D^2 u_{\text{hom}}(x)|^2 dx.
$$

Using this and

$$
\overline{L}(p) \ge \overline{L}(p(z)) + q(z) \cdot (p - p(z)) - C|p - p(z)|^2,
$$

we obtain from (A.26) that

$$
\int_{z+Q_n} L(D\tilde{v}(x),x) dx - \int_{z+Q_{n+2}} \overline{L}(Du_{\text{hom}}(x)) dx
$$
\n
$$
\leq C\mathcal{E}' + C3^{n-l} \left( K_0^2 + \int_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 dx \right) + C3^{n+l} \int_{z+Q_{n+3}} |D^2 u_{\text{hom}}(x)|^2 dx + C\sqrt{\mathcal{E}'} \left( K_0^2 + \int_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 dx \right)^{\frac{1}{2}}.
$$

Summing this over all  $z \in V \cap 3^n \mathbb{Z}^d$  such that  $z + Q_{n+2} \subseteq V$  and applying (A.5) and (A.6), we at last obtain (A.24).

<span id="page-56-0"></span>*Step 4.* – We estimate the contribution of the energy of  $\tilde{u}_{\text{hom}}$  in the boundary strip  $U \setminus V^{\circ}$ . The claim is that

(A.27) 
$$
\int_{U\setminus V^{\circ}} |D\widetilde{u}_{\text{hom}}(x)|^2 dx \leq C\mathcal{E}' + C3^{n-l}M^2 + C3^{(l-n-m)(1-2/r)}M^2.
$$

<span id="page-57-1"></span>By Hölder's inequality and (A.5),

<span id="page-57-0"></span>(A.28) 
$$
\frac{1}{|U|} \int_{U \setminus V} |Du_{\text{hom}}(x)|^2 dx \le \left(\frac{|U \setminus V|}{|U|}\right)^{1-2/r} \left(\int_U |Du_{\text{hom}}(x)|^r dx\right)^{2/r} \le C3^{(l-n-m)(1-2/r)} M^2.
$$

By the triangle inequality, (A.3) and the expression

(A.29)  $D\widetilde{u}_{\text{hom}}(x) = \eta(x)D\widetilde{v}(x) + (1 - \eta(x))Du_{\text{hom}}(x) + D\eta(x)(\widetilde{v}(x) - u_{\text{hom}}(x)),$ we obtain, for each  $x \in V \setminus V^{\circ}$  $x \in V \setminus V^{\circ}$  $x \in V \setminus V^{\circ}$ ,

$$
|D\widetilde{u}_{\text{hom}}(x)| \le |D\widetilde{v}(x)| + |Du_{\text{hom}}(x)| + C3^{-l} |\widetilde{v}(x) - u_{\text{hom}}(x)|.
$$

By the Poincaré inequality, (A.19) and (A.21), we get, for every  $z \in V \cap 3^n \mathbb{Z}^d$ ,

$$
3^{-2n} \n\int_{z+Q_n} |\widetilde{v}(x) - u_{\text{hom}}(x)|^2 \, dx
$$
  
\n
$$
\leq C \mathcal{E}' + C \left( K_0^2 + \int_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 \, dx \right) + C 3^{n+l} \int_{z+Q_{n+3}} |D^2 u_{\text{hom}}(x)|^2 \, dx.
$$

According to (2.29), (A.20) and (A.25), for every  $z \in V \cap 3^{n} \mathbb{Z}^{d}$ ,

$$
\int_{z+Q_n} |D\tilde{v}(x)|^2 dx
$$
\n
$$
\leq C\mathcal{E}' + C\left(K_0^2 + \int_{z+Q_{n+3}} |Du_{\text{hom}}(x)|^2 dx\right) + C3^{n+l} \int_{z+Q_{n+3}} |D^2 u_{\text{hom}}(x)|^2 dx.
$$

We now obtain (A.27) afte[r sum](#page-57-1)[ming](#page-55-2) the [previo](#page-56-0)us two inequalities over all  $z \in (V \setminus V^{\circ}) \cap 3^{n} \mathbb{Z}^{d}$ , combining the result [with](#page-0-0) (A.29) and using (A.2), (A.6), (A.28) and the fact that  $\tilde{u}_{\text{hom}} \equiv u_{\text{hom}}$  in  $U \setminus V$ .

 $\Box$ 

The lem[ma no](#page-48-1)w follows from ([A.28](#page-0-0)), ([A.24\)](#page-0-0) and (A.27).

We now complete the proof of Proposition 4.1.

Proof of (A.4). – By Lemmas A.1 and A.2,  
\n
$$
\oint_{U} L(Du(x), x) dx \leq \oint_{U} L(D\tilde{u}_{hom}(x), x) dx
$$
\n
$$
\leq \oint_{U} \overline{L}(Du_{hom}(x))) dx + C\mathcal{E}''
$$
\n
$$
\leq \oint_{U} \overline{L}(D\tilde{u}(x))) dx + C\mathcal{E}''
$$
\n
$$
\leq \oint_{U} L(Du(x), x) dx + C\mathcal{E}''
$$

where we have set

$$
\mathcal{E}^{\prime\prime}:=\mathcal{E}^\prime+CM\sqrt{\mathcal{E}^\prime}+CM^2\left(3^{n-l}+3^{\beta(l-n-m)}\right).
$$

In particular,

$$
\left| \oint_U L\left(Du(x),x\right) \, dx - \oint_U \overline{L}\left(Du_{\text{hom}}(x)\right) \right) \, dx \right| \leq C \mathcal{E}'',
$$

which verifies part of  $(A.4)$ . The above string of inequalities also gives

$$
\oint_{U} L\left(D\widetilde{u}_{\text{hom}}(x),x\right) dx \leq \oint_{U} L\left(Du(x),x\right) dx + C\mathscr{E}''
$$

and

$$
\oint_{U} \overline{L} \left( D \widetilde{u}(x) \right) \, dx \leq \oint_{U} \overline{L} \left( D u_{\text{hom}}(x) \right) \, dx + C \mathscr{E}''.
$$

Then uniform convexity (i.e., an argument analogous to that of Lemma 2.1) yields

$$
\oint_{U} |Du(x) - D\widetilde{u}_{\text{hom}}(x)|^{2} dx + \oint_{U} |Du_{\text{hom}}(x) - D\widetilde{u}(x)|^{2} dx \leq C\mathcal{E}''.
$$

The Poincaré inequality then gives

$$
3^{-2(n+m)}\int_U \left(|u(x)-\widetilde{u}_{\text{hom}}(x)|^2+|u_{\text{hom}}(x)-\widetilde{u}(x)|^2\right) dx \leq C\mathcal{E}''.
$$

Recall from the de[finiti](#page-48-2)on of  $\tilde{u}$  [that](#page-49-2)

$$
\widetilde{u}(x) - u(x) = \eta(x) \left( \xi(x) - u(x) \right)
$$

and, according to  $(A.5)$  and  $(A.10)$ , that

$$
\oint_{U} \eta^{2}(x) (\xi(x) - u(x))^{2} dx \le \oint_{V} (\xi(x) - u(x))^{2} dx \le C3^{2n} M^{2}.
$$

Therefore the triangle inequ[ality](#page-48-1) gives

$$
3^{-2(n+m)}\int_U |u(x) - u_{\text{hom}}(x)|^2 \ dx \leq C\mathscr{E}'' + CM3^{-2m} \leq C\mathscr{E}''.
$$

 $\Box$ 

This completes the proof of (A.4) and therefore [of P](#page-0-0)roposition 4.1.

REMARK  $A.3.$  – The argument above and those of Lemma A.1 and A.2 contain more information than what is stated in Proposition 4.1, namely a quantitative estimate for the difference between mesoscopic spatial averages of the (heterogeneous) energy of  $u^{\varepsilon}$  and  $\overline{L}(Du_{\rm hom}),$  and a quantitative statement concerning the weak convergence of  $Du^\varepsilon$  to  $Du_{\rm hom},$ which can be stated in terms of an estimate on  $||Du^{\epsilon}-Du_{\text{hom}}||_{H^{-1}}$ . We leave the formulation of this result and the details of the argument to the reader.

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