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THE OBSTACLE PROBLEM FOR THE TOTAL VARIATION FLOW

BY VERENA BÖGELEIN, FRANK DUZAAR AND CHRISTOPH SCHEVEN

ABSTRACT. – We prove existence results for the obstacle problem related to the total variation flow. For sufficiently regular obstacles the solutions are obtained via the method of minimizing movements. The results for more general obstacles are derived by approximation with regular obstacles in the sense of a stability property of solutions with respect to the obstacle. Finally, we present the treatment of the evolutionary counterpart of a classical stationary result concerning minimal surfaces with thin obstacles by means of the (n-1)-dimensional variational measure introduced by De Giorgi, Colombini and Piccinini.

RÉSUMÉ. – Nous démontrons des résultats d'existence pour le problème de l'obstacle lié au flot de variation totale. Pour les obstacles suffisamment réguliers, nous obtenons les solutions via le procédé de minimisation des mouvements. Les résultats pour les obstacles plus généraux sont dérivés par approximation avec des obstacles réguliers dans le sens d'une propriété de stabilité de solutions relative à l'obstacle. Enfin, nous présentons le traitement de la contrepartie parabolique d'un résultat classique concernant les surfaces minimales avec des obstacles minces au moyen de la mesure variationnelle (n-1)-dimensionnelle introduite par De Giorgi, Colombini et Piccinini.

1. Introduction and results

The total variation flow

(1.1)
$$\partial_t u - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0$$

is an important prototype example of a nonlinear parabolic equation. The equation is one of the borderline cases of the parabolic *p*-Laplacian equation, namely the case p = 1, and therefore (1.1) is often called the parabolic 1-harmonic flow. Formally, the equation can be interpreted as the L^2 -gradient flow associated to the 1-energy. As it is well known, problems with linear growth find their natural formulation in the framework of functions of bounded variation, for short BV-functions. The precise setup shall be given later in § 1.1.

There is a large interest concerned with this equation, and we refer to [3, 4, 5, 6, 7, 10] for the first generalized (weak) formulations of (1.1); see also the monograph [8]. These concepts

rely on the Anzellotti pairing [9] and the existence proofs are based on nonlinear semigroup theory, in particular on techniques of completely accretive operators and Crandall & Liggett's semigroup generation theorem. Another approach, introduced by Lichnewsky & Temam in [32], suggests the interpretation of (1.1) in terms of the generating 1-energy in the sense that solutions of the associated Dirichlet problem solve a variational inequality. Roughly speaking and on a purely formal level, a solution u = u(x,t) to the Dirichlet problem associated to (1.1) on a space-time cylinder $\Omega_T = \Omega \times (0,T)$ (Ω a bounded domain in \mathbb{R}^n and T > 0) for given initial values $u_o: \Omega \to \mathbb{R}$, can be interpreted as a solution of the variational inequality

(1.2)
$$\begin{aligned} \iint_{\Omega_T} |Du| \mathrm{d}x \mathrm{d}t &\leq \iint_{\Omega_T} \left[\partial_t v(v-u) + |Dv| \right] \mathrm{d}x \mathrm{d}t \\ &- \frac{1}{2} \| (v-u)(T) \|_{L^2(\Omega)}^2 + \frac{1}{2} \| v(0) - u_o \|_{L^2(\Omega)}^2 \end{aligned}$$

for any (sufficiently regular) comparison function $v: \Omega_T \to \mathbb{R}$ coinciding with u on the lateral boundary $\partial \Omega \times (0, T)$. According to [32] solutions of the variational inequality are termed *pseudo solutions* or *variational solutions*. The viewpoint of pseudo solutions to variational inequalities has been adopted in [13] for the treatment of gradient flows related to functionals arising in image restoration problems, for example the famous Rudin, Osher & Fatemi image denoising model [36]; see also [12] for flows related to convex, coercive variational integrands.

In this work, we are concerned with the so-called *obstacle problem* related to the total variation flow equation, meaning that we are interested in solutions of (1.1) subject to the additional pointwise *obstacle constraint* that they lie above a given obstacle function $\psi: \Omega_T \to \mathbb{R}$. For the obstacle function at the initial time one poses the compatibility condition $\psi(0) := \psi(\cdot, 0) \ge u_o$ on Ω . This leads to the variational inequality (1.2) in the sense that a function u solves the obstacle problem to the total variation flow if $u \ge \psi$ on Ω_T and the variational inequality (1.2) holds true for any comparison map v with the same boundary values as u on the lateral boundary and such that $v \ge \psi$; see Definition 1.1 below for the precise notion of solution. Classic references for the obstacle problem related to the parabolic p-Laplacian, respectively the porous medium equation are [1], the monograph [34], and more recent ones [14, 15, 16, 37]. An alternative approach to obstacle problems would be the construction of the smallest supersolution to the total variation flow equation staying above the obstacle function ψ . This point of view, which plays a fundamental role in any nonlinear potential theory, is applied for parabolic p-Laplacian (type) equations in [31, 33] and more recently for the porous medium equation in [30].

Our main concern in this paper is to build up a satisfactory existence theory for the obstacle problem for the total variation flow. The challenge here is to find the proper formulation of the obstacle problem, making possible a sufficiently general existence theory, which, for example, allows the treatment of obstacle functions modeling thin obstacles. Such a theory could also be one of the building blocks for the definition of a parabolic 1-capacity.

1.1. Formulation of the obstacle problem

The rigorous formulation takes place in the parabolic function space $L^p_{w*}(0, T; BV(\Omega))$, consisting of those maps $v: (0, T) \to BV(\Omega)$ which are weakly*-measurable and such that $t \mapsto \|Dv(t)\|(\Omega)$ is in $L^p(0, T)$; see §2 for the precise definition and the notion of the total variation $\|Dv(t)\|(\Omega)$. As it is well known, dealing with boundary values for functions of bounded variation is a delicate issue, since the trace operator is not continuous with respect to the weak* convergence in $BV(\Omega)$. To overcome this problem, we consider a slightly larger reference domain Ω^* compactly containing the bounded open set Ω . Then, given a reference function $u_o \in BV(\Omega^*)$, the Dirichlet boundary condition $u = u_o$ on $\partial\Omega$ for a function $u \in BV(\Omega^*)$ is defined by requiring that $u = u_o$ a.e. on $\Omega^* \setminus \overline{\Omega}$. For functions with this property we write $u \in BV_{u_o}(\Omega)$ for short. The space $L^p_{w*}(0,T; BV_{u_o}(\Omega))$ is defined as the space of functions $u \in L^p_{w*}(0,T; BV(\Omega^*))$ such that for almost all time slices t the map $u(t) := u(\cdot, t)$ belongs to $BV_{u_o}(\Omega)$. In terms of the described notion of boundary values the obstacle problem for the total variation flow can be formulated as follows. We consider initial data $u_o: \Omega^* \to \mathbb{R}$ with

(1.3)
$$u_o \in L^2(\Omega^*) \cap \mathrm{BV}(\Omega^*),$$

and obstacle functions $\psi \colon \Omega_T^* \to \mathbb{R}$ with

(1.4)
$$\psi \in L^2(\Omega_T^*) \cap L^1_{w*}(0,T; \mathrm{BV}_{u_o}(\Omega)).$$

Moreover, we postulate that ψ admits initial values $\psi(0)$ in the $L^2(\Omega^*)$ -sense, satisfying the compatibility condition $\psi(0) \leq u_o$ a.e. in Ω^* . Finally, we assume that there exists a sufficiently regular extension g of the initial datum u_o to Ω_T^* , more precisely a mapping $g: \Omega_T^* \to \mathbb{R}$ such that

(1.5)
$$\begin{cases} g \in L^1_{w*}(0,T; \mathrm{BV}_{u_o}(\Omega)) \text{ with } \partial_t g \in L^2(\Omega^*_T), \\ g(0) = u_o \text{ and } g \ge \psi \text{ a.e. on } \Omega_T. \end{cases}$$

The following definition gives the notion of *variational solution* to the obstacle problem for the total variation flow, that we will use in this paper. In a certain sense, the concept here seems to be the natural extension of the classical definition of pseudo solutions given by Lichnewsky & Temam in [32].

DEFINITION 1.1 (Variational Solutions). – Assume that the Cauchy-Dirichlet datum u_o and the obstacle ψ fulfill the hypotheses (1.3) and (1.4). Moreover, assume that the compatibility condition (1.5) holds true. We identify a measurable map $u: \Omega_T^* \to \mathbb{R}$ in the class

$$u \in L^{\infty}([0,T]; L^{2}(\Omega^{*})) \cap L^{1}_{w^{*}}(0,T; \mathrm{BV}_{u_{o}}(\Omega))$$
 with $u \geq \psi$ a.e. in Ω_{T}

as a *variational solution* to the obstacle problem for the total variation flow if and only if the variational inequality

(1.6)
$$\int_{0}^{\tau} \|Du\|(\Omega^{*})dt \leq \iint_{\Omega_{\tau}^{*}} \partial_{t}v(v-u)dxdt + \int_{0}^{\tau} \|Dv\|(\Omega^{*})dt - \frac{1}{2}\|(v-u)(\tau)\|_{L^{2}(\Omega^{*})}^{2} + \frac{1}{2}\|v(0) - u_{o}\|_{L^{2}(\Omega^{*})}^{2}$$

holds true, for a.e. $\tau \in [0,T]$ and any $v \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$ with $\partial_t v \in L^2(\Omega^*_T)$, $v(0) \in L^2(\Omega^*)$ and $v \ge \psi$ a.e. in Ω_T .

Observe, by assumption (1.5) the map g is an admissible comparison function in the variational inequality (1.6). This allows the testing of (1.6) by v = g, and leads to certain energy bounds. In particular one can conclude that variational solutions attain the initial

datum u_o in the $L^2(\Omega^*)$ -sense; see Lemma 2.5 below. Our first main result concerning the obstacle problem to the total variation flow is the following:

THEOREM 1.2. – Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with $\Omega \Subset \Omega^*$, and moreover that u_o, ψ satisfy (1.3), (1.4), (1.5) and the compatibility condition $\psi(0) \leq u_o$ a.e. in Ω^* . Then there exists a variational solution

$$u \in L^{\infty}(0,T;L^{2}(\Omega^{*})) \cap L^{1}_{w*}(0,T;\mathrm{BV}_{u_{o}}(\Omega))$$

with $u \ge \psi$ a.e. on Ω_T of the obstacle problem in the sense of Definition 1.1. Furthermore, the solution attains the initial datum u_o in the usual $L^2(\Omega^*)$ -sense.

A second main result deals with obstacles for which $\psi - u_o$ is lower semicontinuous. In particular, in the case $u_o = 0$ this means that the obstacle function itself is lower semicontinuous. We note that in contrast to our other two existence results, in this case we need not require any regularity of the boundary of the domain. The precise result reads as follows:

THEOREM 1.3. – Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\Omega \subseteq \Omega^*$, that the initial values satisfy $u_o \in L^2(\Omega^*) \cap W^{1,1}(\Omega^*)$ and the obstacle function $\psi : \Omega_T^* \to \mathbb{R}$ fulfills the requirement that

(1.7) $\psi - u_o$ is lower semicontinuous on Ω_T with $\operatorname{spt}(\psi - u_o) \in \Omega_T$.

Moreover, we assume that the compatibility condition (1.5) is satisfied. Then there exists a variational solution $u \in L^{\infty}(0,T;L^{2}(\Omega^{*})) \cap L^{1}_{w*}(0,T;BV_{u_{o}}(\Omega))$ with $u \geq \psi$ a.e. on Ω_{T} of the obstacle problem in the sense of Definition 1.1.

The third main result of the present paper is concerned with the class of obstacle functions for which $\psi - u_o$ is upper semicontinuous, so that in particular *thin obstacles* that are concentrated on lower-dimensional sets are included. For thin obstacles, the variational formulation of the obstacle problem described before in general has no solution. The classical elliptic case is analyzed in [17], with the result that it is necessary to pass to a relaxed version of the total variation functional. Roughly speaking, this functional penalizes the violation of the obstacle constraint on lower dimensional sets. The relaxed functional is defined via De Giorgi's measure σ (see § 3.4 and (3.39) for the rigorous definition of the De Giorgi measure) that was originally introduced for the study of obstacle problems in the setting of geometric measure theory, see [19, 20, 35], and also [28] for corresponding results in metric spaces. Our main result in this case is as follows:

THEOREM 1.4. – Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with $\Omega \Subset \Omega^*$ and that the initial values satisfy $u_o \in L^2(\Omega^*) \cap W^{1,1}(\Omega^*)$. For the obstacle function $\psi \colon \Omega_T^* \to \mathbb{R}$ suppose that

(1.8) $\psi - u_o$ is upper semicontinuous on Ω_T with $\operatorname{spt}(\psi - u_o) \in \Omega_T$.

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Then there exists a solution $u \in L^{\infty}(0,T;L^{2}(\Omega^{*})) \cap L^{1}_{w*}(0,T;BV_{u_{o}}(\Omega))$ of the relaxed obstacle problem in the sense that

(1.9)
$$\int_{0}^{\tau} \|Du\|(\Omega^{*})dt + \int_{0}^{\tau} \left[\int_{\Omega} (\psi - u^{+})_{+} d\sigma\right] dt$$
$$\leq \iint_{\Omega^{*}_{\tau}} \partial_{t} v(v - u) dx dt + \int_{0}^{\tau} \|Dv\|(\Omega^{*}) dt$$
$$- \frac{1}{2} \|(v - u)(\tau)\|_{L^{2}(\Omega^{*})}^{2} + \frac{1}{2} \|v(0) - u_{o}\|_{L^{2}(\Omega^{*})}^{2}$$

holds for a.e. $\tau \in [0,T]$ and every $v \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$ with $\partial_t v \in L^2(\Omega^*_T)$ and $v(0) \in L^2(\Omega^*)$ for which

(1.10) $v - u_o$ is lower semicontinuous on $\overline{\Omega}_T$

and $v \geq \psi$ holds everywhere in Ω_T .

As mentioned already, by σ we denote the De Giorgi measure, and $u^+ \colon \Omega^* \to \mathbb{R}$ stands for the *upper approximate limit of* $u \in BV_{loc}(\Omega^*)$ defined by

$$u^{+}(x) = \inf \left\{ \lambda \in \mathbb{R} : \limsup_{r \downarrow 0} \frac{\mathcal{L}^{n}(\{u > \lambda\} \cap B_{r}(x))}{r^{n}} = 0 \right\}.$$

Observe that u^+ equals the Lebesgue values of u in the approximate continuity points and the larger of the two jump values in the approximate jump points. The solution of the obstacle problem may violate the obstacle constraint $u \ge \psi$. This is penalized by the integral involving the De Giorgi measure. However, as a consequence of the variational inequality the exceptional set $E = \{u^+ < \psi\}$ is small, in the sense that

$$\mathcal{H}$$
-dim $(E \cap \mathbb{R}^n \times \{t\}) \leq n-1$ for a.e. $t \in [0,T]$.

For the proof we refer to Remark 3.9. Finally, we point out that since the De Giorgi measure is not σ -finite, Fubini's theorem does not apply to the double integral involving the De Giorgi measure, cf. Remark 3.8. It is therefore important to evaluate the double integral as indicated in (1.9) by first integrating with respect to σ over each time slice and then to integrate with respect to time.

REMARK 1.5. – The lower semicontinuity assumption (1.10) on the comparison map v in (1.9) is natural—even in contrast to the upper semicontinuity condition for the obstacle ψ —in order to prescribe the obstacle condition $v \ge \psi$ pointwise. This can be seen by the simple model example of boundary values $u_o \equiv 0$ and a time-independent thin obstacle $\psi(x,t) = \chi_M(x)$, where $M \Subset \Omega$ is a closed (n-1)-dimensional submanifold. In this situation, the usage of lower semicontinuous comparison maps allows us to compare the solution to the thin obstacle problem with characteristic functions $v(x,t) = \chi_U(x)$ of open neighborhoods $U \supset M$. This is also reminiscent of the definition of the De Giorgi measure, cf. Definition 3.7, where a thin set is approximated by open sets from outside. In particular, to assume (1.10) seems to be more natural than to prescribe *upper* semicontinuity for the comparison maps. In the model case described above this allows us to take comparison maps $v \in L^1_{w*}(0, T; BV(\Omega^*))$ by thickening the obstacle by open sets.

REMARK 1.6. – A few words concerning uniqueness of solutions are in order. Uniqueness is quite easy to obtain for strong variational solutions to problems with regular obstacles, i.e., obstacles ψ such that $\partial_t \psi \in L^2(\Omega^*_T)$ and (2.12) hold true. For strong variational solutions—introduced in Definition 2.7—we require the additional regularity property $u \in C^0([0,T]; L^2(\Omega^*))$. This is the main difference to the concept of solutions introduced above and is crucial for the proof of the uniqueness that follows from the comparison principle in Lemma 2.12. In Theorem 3.1 we give an existence result for unique strong variational solutions for obstacles that are sufficiently regular in the sense that $\partial_t \psi \in L^2(\Omega_T^*)$ and $\partial_t D\psi \in L^1(\Omega^*_T, \mathbb{R}^n)$ hold true. These solutions are the building block for the construction of solutions to more general obstacles by suitable approximation schemes, during which the uniqueness property possibly is lost. The proof of uniqueness for these solutions fails because we do not have the property $u \in C^0([0,T]; L^2(\Omega^*))$ at our disposal, which we clearly cannot expect if the obstacle function fails to have the same property. Anyway, we chose to treat obstacle functions that are irregular with respect to time because our aim was to include obstacles of the form $\psi = \chi_A$ for $A \subset \Omega_T$, since these constitute the building blocks for a *parabolic potential theory* for the total variation flow. This, however, forces us to weaken the notion of solution, i.e., to abandon the assumption $u \in C^0([0,T]; L^2(\Omega^*))$ and require instead only $u \in L^{\infty}(0,T; L^2(\Omega^*))$. This is exactly the technical point where the proof of the uniqueness of the solutions in Theorem 1.2 fails.

Our existence result Theorem 1.4 for the case of thin obstacles is again motivated by possible potential theoretic applications. We are interested in obstacle functions of the type $\psi = \chi_K$ for a compact subset $K \subset \Omega_T$ and initial datum $u_o \equiv 0$, which corresponds to our assumption on upper semicontinuity of ψ . In particular, we wish to include thin obstacles that are moving in time such as $\psi(x,t) = \chi_{M(t)}(x)$, where M(t) denotes a moving hypersurface. Such obstacles do not satisfy a regularity assumption such as $\partial_t \psi \in L^2(\Omega_T^*)$, which is the reason why the proof of uniqueness fails in this case.

However, our goal in the present work is to establish existence results in the most general cases that are accessible by our methods, even if the uniqueness property is not available. In any case, uniqueness of solutions in the case of irregular obstacles is a major open problem.

1.2. Methods of proof

The key ingredient in the proofs is a basic result concerning *regular obstacles* ψ . Regular means, that

$$\psi \in W^{1,1}(\Omega_T^*), \partial_t \psi \in L^2(\Omega_T^*), \text{ and } \partial_t D \psi \in L^1(\Omega_T^*, \mathbb{R}^n)$$

with $\psi = u_o$ on $(\Omega^* \setminus \Omega) \times (0, T)$. For such obstacles the concept of variational solutions allows a stronger version, which we call *strong variational solutions*. The difference to Definition 1.1 consists in the facts that on the one hand the variational solution is assumed to satisfy the additional regularity requirement $u \in C^0([0, T]; L^2(\Omega^*))$, and on the other hand that the variational inequality is only imposed on the whole time interval [0, T]. The latter is natural, since the $C^0 - L^2$ -continuity of u and the strong regularity assumptions on ψ allow the localization of the variational inequality to any sub cylinder Ω^*_{τ} ; see §2.7.1. The main existence result for strong variational solutions is contained in Theorem 3.1. It guarantees the existence of a unique strong variational solution to the obstacle problem to the total variation flow on Ω_T , the uniqueness being a consequence of the comparison principle for strong variational solutions and regular obstacles.

For the construction of the strong variational solution in § 3.1 we use a time discretization procedure, also called the method of *minimizing movements*. In our setting this works as follows: For a fixed integer $\ell \in \mathbb{N}$ we sub-divide the interval (0,T] into subintervals ((i-1)h, ih] with $i \in \{1, \ldots, \ell\}$, where $h := \frac{T}{\ell}$. We also let $\psi_i := \psi(ih)$. We then inductively construct a sequence $u_i \in L^2(\Omega^*) \cap BV_{u_o}(\Omega)$ of minimizers to the elliptic variational functionals

$$\mathbf{F}_{i}[v] := \|Dv\|(\Omega^{*}) + \frac{1}{2h} \int_{\Omega^{*}} |v - u_{i-1}|^{2} dx$$

in the non-empty class of functions $v \in L^2(\Omega^*) \cap BV_{u_o}(\Omega)$ with $v \ge \psi_i$ a.e. in Ω . Note that $g_i := g(ih)$ is admissible. For i = 0, then u_o will be the initial datum. The sequence of minimizers is glued together to a map $u^{(h)} \colon \Omega^* \times (-h, T] \to \mathbb{R}$ by

$$u^{(h)}(t) := u_i$$
 for $t \in ((i-1)h, ih]$ with $i \in \{0, \dots, \ell\}$.

Using the minimizing property of the maps u_i , it is possible to derive an energy estimate that ensures weak*-subconvergence of $u^{(h)}$ to a limit map $u \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$. This is the point where the regularity assumption on the obstacle is crucial, because it enables us to construct suitable comparison maps that satisfy the obstacle constraint. Moreover, the minimizing property of the u_i can be translated into a variational inequality for $u^{(h)}$, and passing to the limit, we infer that u is the desired strong solution.

For the proof of the existence result in Theorem 1.2, which deals with obstacles satisfying the much weaker regularity assumption $\psi \in L^2(\Omega_T^*) \cap L^1_{w*}(0,T; BV_{u_o}(\Omega))$, we employ a two-step approximation procedure. In a first step, we mollify the obstacle in time to reduce the problem to the case of obstacles with weak time derivatives $\partial_t \psi \in L^2(\Omega_T^*)$. The second and more intricate step consists of a mollification of the obstacle in space. We define a mollification operator $M_{\varepsilon}[\psi]$ (cf. (3.17)) in such a way that the mollifications converge strictly in the sense that $\int_0^T \|D(M_{\varepsilon}[\psi])\|(\overline{\Omega})dt \to \int_0^T \|D\psi\|(\overline{\Omega})dt$ holds in the limit $\varepsilon \downarrow 0$ and at the same time, $M_{\varepsilon}[\psi]$ agrees with the mollification of u_o outside of Ω . The derivation of these properties relies on some subtle properties of the traces of BV-functions, cf. Lemma 3.6. Having this mollification operator at hand, we can then solve the obstacle problems for the regular obstacles $M_{\varepsilon}[\psi]$ and pass to the limit $\varepsilon \downarrow 0$. Applying the same operator to a comparison map v with $v \ge \psi$ a.e. on Ω , we obtain maps $M_{\varepsilon}[v] \ge M_{\varepsilon}[\psi]$ a.e. on Ω that are admissible as comparison maps for the approximating solutions. Passing to the limit $\varepsilon \downarrow 0$, we can thereby show that the limit map is a solution to the obstacle problem with the irregular obstacle ψ .

The approximation argument for lower semicontinuous obstacles ψ is straightforward because such obstacles can be approximated monotonically from below by smooth obstacles $\psi_i \uparrow \psi$ as $i \to \infty$. The assumption (1.5) provides us with energy bounds for the solutions u_i to the corresponding obstacle problems, from which we infer weak*-convergence to a limit map $u \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$. By construction, every comparison map $v \ge \psi$ for ualso satisfies $v \ge \psi_i$ and is therefore admissible as comparison map for u_i . Passing to the limit $i \to \infty$, we can thereby deduce the claimed variational inequality for the limit map u.

The case of upper semicontinuous obstacles is much more intricate, because it includes the case of thin obstacles. For the exposition of the main ideas in this situation, which is considered in Theorem 1.4, we restrict ourselves to the case of zero boundary values $u_o \equiv 0$. Then, our assumptions on the obstacle are that ψ is upper semicontinuous with compact support in Ω_T . This includes in particular obstacles whose support is a lowerdimensional set. The upper semicontinuity of ψ makes it possible to approximate the obstacle monotonically from above by regular obstacles. This approximation procedure can be made explicit by means of the Yosida regularization $\widehat{\psi}_k$ of ψ for $k \ge k_o \in \mathbb{N}$, which is, roughly speaking, the smallest Lipschitz continuous function with Lipschitz constant $\leq k$ that lies above ψ (cf. § 3.4, proof of Theorem 1.4 for the precise definition). An additional mollification provides us with a sequence of regular obstacles ψ_k that converges monotonically to the given obstacle ψ . The obstacle problems for the ψ_k have solutions u_k by the existence result from Theorem 3.1 for regular obstacles mentioned before. At this stage, it is crucial that these solutions satisfy a comparison principle, which implies that also u_k converges monotonically almost everywhere to a limit map u. Moreover, since the first obstacle function $\psi_{k_{\alpha}}$ is an admissible comparison map for the solutions u_k , we easily infer an energy estimate ensuring $u \in L^{\infty}(0,T; L^{2}(\Omega)) \cap L^{1}_{w*}(0,T; BV_{u_{\alpha}}(\Omega))$. However, the limit map may violate the obstacle constraint on exceptional sets that are lower-dimensional in the sense of Remark 3.9. This phenomenon was already observed in the elliptic setting in [17]. Following the approach from [17], we add a penalization term to the total variation. More precisely, we penalize the violation of the obstacle constraint on lower-dimensional sets by means of the De Giorgi measure. Using the lower semicontinuity result from [17] separately on each time slice, we are able to derive a variational inequality for the limit map u which contains the additional penalization term, see (1.9). In a first step, this inequality is derived for Lipschitz continuous comparison maps $v \ge \psi$, because they automatically lie above the Yosida-approximations ψ_k by their definition. In a second step, we then generalize the variational inequality to lower semicontinuous comparison functions with $v \ge \psi$ everywhere on Ω_T with the help of the mollification operator $M_{\varepsilon}[v]$ mentioned above.

1.3. Plan of the paper

The article is organized as follows. In § 2 we first introduce some notation, parabolic function spaces and a mollification procedure with respect to time. As already mentioned, in § 2.5 we prove that variational solutions in the sense of Definition 1.1 attain the initial datum u_o in the $L^2(\Omega^*)$ -sense. In § 2.7 we introduce the concept of strong variational solutions, used throughout the existence proof, together with some properties of strong solutions. § 3 is devoted to the proof of the existence results from Theorems 1.2—1.4. We start in § 3.1 with the existence of strong variational solutions to obstacle problems with regular obstacles; see Theorem 3.1. This result is then used in the last subsections to prove the main results on existence of variational solutions by different approximation procedures. In § 3.2 we prove the existence result from Theorem 1.2 for obstacles $\psi \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$. The existence in the setting of lower semicontinuous obstacles as considered in Theorem 1.3 is achieved in § 3.3. Finally, the existence result for upper semicontinuous obstacles from Theorem 1.4 is proved in § 3.4. Acknowledgments. – V. Bögelein has been supported by the DFG-Project BO3598/1-1 "Evolutionsgleichungen mit p, q-Wachstum".

2. Preliminaries and notations

2.1. Notations

For $p \in [1, \infty]$ and an open set $\Omega \subset \mathbb{R}^n$, the spaces $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$ denote the usual Lebesgue, respectively Sobolev spaces. Moreover, by Ω_T , with $T \in (0, \infty)$ we denote the space-time cylinder $\Omega \times (0, T)$. By BV(Ω) we denote the space of functions $u \in L^1(\Omega)$ with finite *total variation*

(2.1)
$$\|Du\|(\Omega) := \sup\left\{\int_{\Omega} u\operatorname{div}\zeta \mathrm{d}x : \zeta \in C_0^1(\Omega, \mathbb{R}^n), \, \|\zeta\|_{L^{\infty}(\Omega)} \le 1\right\} < \infty.$$

The norm in $BV(\Omega)$ is defined by

$$||u||_{\mathrm{BV}(\Omega)} := ||u||_{L^1(\Omega)} + ||Du||(\Omega).$$

It is well known that boundary values for $BV(\Omega)$ -functions are a delicate issue, since the trace operator is not anymore continuous with respect to the weak^{*} convergence in $BV(\Omega)$. For instance, a sequence of characteristic functions of finite perimeter sets converging to the characteristic function χ_{Ω} demonstrates the occurring difficulties. One way out of these difficulties is to consider a slightly larger domain Ω^* containing Ω on which the boundary values can be extended, and then to formulate the boundary condition in terms of the extension u_o by requiring that $u = u_o$ on $\Omega^* \setminus \Omega$. To consider a larger reference domain is natural, since in general the total variation of minimizers will charge the boundary $\partial\Omega$ of Ω . Instead of using the approach via the reference set Ω^* , one could use an integral representation formula for the total variation containing a boundary penalty term. Such a formula is well known for Lipschitz domains Ω , and we could have followed also this path. For more precise statements concerning the traces of BV-functions we refer to § 3.2.1 below. Our precise set up is as follows: Let Ω and Ω^* be two bounded open subsets of \mathbb{R}^n such that $\Omega \Subset \Omega^*$ and let $u_o \in BV(\Omega^*)$ be given. Then we define the space $BV_{u_o}(\Omega)$ as the space of functions $u \in BV(\Omega^*)$ such that $u = u_o$ almost everywhere in $\Omega^* \setminus \Omega$.

2.2. Parabolic function spaces

We recall that a function $v: I \to X$ from an interval $I \subset \mathbb{R}$ to a Banach space X is called strongly measurable if there is a sequence of simple functions $v_k: I \to X$ with $||v_k(t) - v(t)||_X \to 0$ for a.e. $t \in I$ as $k \to \infty$. For $1 \leq p \leq \infty$, we write $L^p(I; X)$ for the space of strongly measurable functions $v: I \to X$ for which the function $I \ni t \mapsto ||v(t)||_X$ is contained in $L^p(I)$.

2.2.1. Weak*-measurability. – The theory becomes more intricate in the case $X = BV(\Omega)$, because the space $BV(\Omega)$ is not separable. On the other hand, it is well-known that $BV(\Omega)$ is the dual space of a separable Banach space X_0 , whose elements can be written as $g - \operatorname{div} G$ for $g \in C_0^0(\Omega)$ and $G \in C_0^0(\Omega, \mathbb{R}^n)$, see e.g., [2, Remark 3.12]. A function $v: I \to BV(\Omega) = X'_0$ is called weakly*-measurable if the mapping $I \ni t \mapsto \langle v(t), \varphi \rangle \in \mathbb{R}$ is measurable for every $\varphi \in X_0$, where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $BV(\Omega)$ and X_0 . We note that for every weakly*-measurable function $v: I \to BV(\Omega)$, the norm $||v(t)||_{BV(\Omega)}$ also depends measurably on $t \in I$, since it can be rewritten as

$$\|v(t)\|_{\mathrm{BV}(\Omega)} = \sup\left\{ \langle v(t), \varphi \rangle : \varphi \in X_0, \|\varphi\|_{X_0} \le 1 \right\}$$

by the definition of the dual norm of X'_0 . The right-hand side depends measurably on $t \in I$ since by definition, $t \mapsto \langle v(t), \varphi \rangle$ is measurable and X_0 is separable.

2.2.2. Weak* vs. strong measurability. – The weak*-version of Pettis' theorem [22, Teorema 2.2] tells us that a function $v: I \to BV(\Omega)$ is strongly measurable if and only if it is weakly*-measurable and almost separably valued. The latter means that there exists a negligible set $N \subset I$ so that $v(I \setminus N)$ is a separable subset of $BV(\Omega)$. This condition is already violated by such simple examples as a characteristic function of a body of revolution $v(x,t) = \chi_{B_{\varrho(t)}}(x)$ with a non-constant function $\varrho \in C^0(I, \mathbb{R}_{>0})$. Since we certainly do not want to exclude such functions as possible obstacles, we are forced to take weak*-measurable functions into account. For a brief account on the different notions of measurability and the related concepts of integration, we refer to [21, Chap. 2].

2.2.3. *Weak*-Lebesgue spaces.* – For $1 \le p \le \infty$ we define

$$L^p_{w*}(I;\mathrm{BV}(\Omega)) := \left\{ v \colon I \to \mathrm{BV}(\Omega) \; \middle| \; \begin{array}{l} v \text{ is weakly*-measurable with} \\ t \mapsto \|v(t)\|_{\mathrm{BV}(\Omega)} \in L^p(I) \end{array} \right\}.$$

Every $v \in L^1_{w*}(I; BV(\Omega))$ is Gel'fand integrable in the sense that for every measurable set $E \subset I$ there exists a $V_E \in BV(\Omega)$, called the *Gel'fand* or *weak*-integral of* v and denoted by $V_E = (G) - \int_E v(t) dt$, with the property

$$\langle V_E, \varphi
angle = \int_E \langle v(t), \varphi
angle \mathrm{d} t \qquad ext{for every } \varphi \in X_0.$$

This follows from the fact that the right-hand side defines a continuous linear functional on X_0 and $BV(\Omega) = X'_0$. We note that we have the embedding

$$L^1_{w*}(I; BV(\Omega)) \hookrightarrow L^1(I; L^1(\Omega)) \simeq L^1(\Omega \times I).$$

Here, the strong measurability of $u: I \to L^1(\Omega)$ is a consequence of Pettis' theorem and the separability of $L^1(\Omega)$.

Since $BV(\Omega) = X'_0$, we know from [27, Sect. VII.4] that

$$L^{\infty}_{w*}(I; \mathrm{BV}(\Omega)) = \left[L^1(I; X_0) \right]',$$

and therefore, in the space $L_{w*}^{\infty}(I; BV(\Omega))$ we have the usual notion of weak*-convergence

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at our disposal. By [22, Prop. 3.2] the convergence $v_k \xrightarrow{*} v$ weakly* in $L^{\infty}_{w*}(I; BV(\Omega))$ is equivalent to

$$\begin{cases} v_k \stackrel{*}{\rightharpoonup} v & \text{weakly* in } \left[L^1(I; C^0_0(\Omega))\right]' \text{ and} \\ D_i v_k \stackrel{*}{\rightharpoondown} D_i v \text{ weakly* in } \left[L^1(I; C^0_0(\Omega))\right]' \text{ for any } i \in \{1, \dots, n\} \end{cases}$$

2.3. Mollification in time

Variational solutions in the sense of Definition 1.1 are in general not admissible as comparison maps in (1.6), since they do not obey the necessary regularity with respect to time. To overcome this problem, one is forced to use a mollification procedure with respect to time, to construct testing functions with the correct regularity with respect to time. One standard possibility is the usage of Steklov averages. However, Steklov averages have the disadvantage, that they are not preserving the initial condition at t = 0, which is crucial in case of global arguments. Much more suitable is the construction of the mollification $[v_h]_h$, h > 0, to a given function v, such that it (formally) solves the ordinary differential equation

(2.2)
$$\partial_t [v]_h = -\frac{1}{h} \left([v]_h - v \right)$$

with initial condition $[v]_h(0) = v_o$. The precise construction is as follows. Let X be a separable Banach space and $v_o \in X$; in the applications we will have $X = L^q(\Omega)$ for $q \ge 1$ and the related parabolic space $L^r(0,T;L^q(\Omega))$. Later on, we need the non-separable space $X = BV(\Omega)$ and the corresponding parabolic space $L^1_{w*}(0,T;BV(\Omega))$; recall that functions $v \in L^1_{w*}(0,T;BV(\Omega))$ belong to $L^1(0,T;L^1(\Omega)) = L^1(\Omega_T)$. Now, we consider $v \in L^r(0,T;X)$ for some $1 \le r \le \infty$, and define for $h \in (0,T]$ and $t \in [0,T]$ the mollification in time by

(2.3)
$$[v]_h(t) := e^{-\frac{t}{h}} v_o + \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} v(s) ds.$$

The basic properties of the mollification $[\cdot]_h$ are provided in the following Lemma, cf. [29, Lemma 2.2], or [11, Appendix B] for the proofs of the particular statements.

LEMMA 2.1. – Suppose X is a separable Banach space and $v_o \in X$. If $v \in L^r(0,T;X)$ for some $r \ge 1$, then the mollification $[v]_h$ defined in (2.3) fulfills $[v]_h \in L^r(0,T;X)$ and for any $t_o \in (0,T]$ there holds

$$\left\| [v]_h \right\|_{L^r(0,t_o;X)} \le \|v\|_{L^r(0,t_o;X)} + \left[\frac{h}{r} \left(1 - e^{-\frac{t_o r}{h}} \right) \right]^{\frac{1}{r}} \|v_o\|_X.$$

In the case $r = \infty$ the bracket $[\ldots]^{\frac{1}{r}}$ in the preceding inequality has to be interpreted as 1. Moreover, in the case $r < \infty$ we have $[v]_h \to v$ in $L^r(0,T;X)$ as $h \downarrow 0$. Finally, if $v \in C^0([0,T];X)$ and $v_0 = v(0)$, then $[v]_h \in C^0([0,T];X)$, $[v]_h(0) = v_o$, and moreover $[v]_h \to v$ in $C^0([0,T];X)$ as $h \downarrow 0$.

For maps $v \in L^r(0,T;X)$ with $\partial_t v \in L^r(0,T;X)$ we have the following assertion.

LEMMA 2.2. – Let X be a separable Banach space and $r \ge 1$. Assume that $v \in L^r(0,T;X)$ with $\partial_t v \in L^r(0,T;X)$. Then, for the mollification in time defined by

$$[v]_h(t) := \mathrm{e}^{-\frac{t}{h}}v(0) + \frac{1}{h}\int_0^t \mathrm{e}^{\frac{s-t}{h}}v(s)\mathrm{d}s$$

the time derivative can be computed by

$$\partial_t [v]_h(t) = \frac{1}{h} \int_0^t e^{\frac{s-t}{h}} \partial_s v(s) ds,$$

and, moreover we have that

$$\left\|\partial_t [v]_h\right\|_{L^r(0,T;X)} \le \|\partial_t v\|_{L^r(0,T;X)}$$

holds true.

The next Lemma ensures the convergence of the total variation $||D[v]_h|| \rightarrow ||Dv||$ in the limit $h \downarrow 0$, provided that $v \in L^1_{w*}(0, T; BV(\Omega))$. The proof can for instance be deduced from [13, Lemma 2.6].

LEMMA 2.3. – Let
$$T > 0$$
. Assume that

$$v \in L^1_{w*}(0,T; \mathrm{BV}(\Omega)), \quad and \quad v_o \in \mathrm{BV}(\Omega).$$

Then, we have

$$[v]_h \in L^1_{w*}(0,T;\mathrm{BV}(\Omega)),$$

with the estimates $||D[v]_h(\cdot,t)||(\Omega) \leq [||Dv(\cdot,t)||(\Omega)]_h$ for any $t \in (0,T)$ and

$$\lim_{h\downarrow 0}\int_0^T \|D[v]_h(t)\|(\Omega)\mathrm{d}t = \int_0^T \|Dv(t)\|(\Omega)\mathrm{d}t.$$

2.4. Lower semicontinuity of the integrated total variation

LEMMA 2.4. – Assume that the sequence $u_i \in L^1_{w*}(0,T; BV(\Omega))$ satisfies $u_i \to u$ weakly in $L^1(\Omega_T)$ as $i \to \infty$, for some $u \in L^1(\Omega_T)$, and

$$\liminf_{i\to\infty}\int_0^T \|Du_i(t)\|(\Omega)\mathrm{d} t<\infty.$$

Then we have $u \in L^1_{w*}(0,T; BV(\Omega))$ and

(2.4)
$$\int_0^T \|Du(t)\|(\Omega)dt \le \liminf_{i\to\infty} \int_0^T \|Du_i(t)\|(\Omega)dt.$$

Proof. – Since $u \in L^1(\Omega_T)$, the map $[0,T] \ni t \mapsto \int_{\Omega} u(\cdot,t) \operatorname{div} \zeta dx$ is measurable for any fixed $\zeta \in C_0^1(\Omega, \mathbb{R}^n)$. Because $C_0^1(\Omega, \mathbb{R}^n)$ is separable, also the supremum of the above integrals over $\zeta \in C_0^1(\Omega, \mathbb{R}^n)$ with $\|\zeta\|_{L^{\infty}} \leq 1$ is measurable, from which we infer that the total variation $t \mapsto \|Du(t)\|(\Omega)$ depends measurably on time.

Next, we consider the time mollifications $[u_i]_h$ from §2.3 with $v_o = 0$, for any $h \in (0, T]$. We observe that for a fixed h, the weak convergence $u_i \rightarrow u$ in $L^1(\Omega_T)$ implies

(2.5)
$$[u_i]_h(t) \rightarrow [u]_h(t)$$
 weakly in $L^1(\Omega)$ for every $t \in (0,T)$, as $i \rightarrow \infty$.

Next, we use the lower semicontinuity of the total variation defined in (2.1) with respect to weak L^1 -convergence. This can be checked by noting that the integrals in (2.1) are continuous with respect to weak L^1 -convergence and consequently, their supremum is lower semicontinuous. In view of (2.5), we thereby obtain

$$\|D[u]_h(t)\|(\Omega) \le \liminf_{i \to \infty} \|D[u_i]_h(t)\|(\Omega) \le \liminf_{i \to \infty} \left\lfloor \|Du_i\|(\Omega)\right\rfloor_h(t)$$

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for any $t \in (0, T)$. The last estimate is a consequence of Lemma 2.3. Fatou's lemma further implies

(2.6)
$$\int_0^T \|D[u]_h(t)\|(\Omega)dt \le \liminf_{i\to\infty} \int_0^T \left[\|Du_i\|(\Omega)\right]_h(t)dt$$
$$\le \liminf_{i\to\infty} \int_0^T \|Du_i(t)\|(\Omega)dt,$$

for every $h \in (0, T]$, where in the last estimate, we used Lemma 2.1 with r = 1. For a sequence $0 < h_{\ell} \downarrow 0$ we have $[u]_{h_{\ell}}(t) \rightarrow u(t)$ in $L^{1}(\Omega)$ for a.e. $t \in (0, T)$ and thereby, for every $\zeta \in C_{0}^{1}(\Omega, \mathbb{R}^{n})$ with $\|\zeta\|_{L^{\infty}(\Omega)} \leq 1$ we get

$$\int_{\Omega} u(t) \operatorname{div} \zeta \mathrm{d}x = \lim_{\ell \to \infty} \int_{\Omega} [u]_{h_{\ell}}(t) \operatorname{div} \zeta \mathrm{d}x \le \lim_{\ell \to \infty} \|D[u]_{h_{\ell}}(t)\|(\Omega)$$

by the definition of the total variation. Next, we take the supremum over ζ on the left-hand side and integrate over (0,T). Then we use first Fatou's lemma and then (2.6) in order to estimate

(2.7)
$$\int_0^T \|Du(t)\|(\Omega)dt \le \int_0^T \lim_{\ell \to \infty} \|D[u]_{h_\ell}(t)\|(\Omega)dt$$
$$\le \liminf_{\ell \to \infty} \int_0^T \|D[u]_{h_\ell}(t)\|(\Omega)dt$$
$$\le \liminf_{i \to \infty} \int_0^T \|Du_i(t)\|(\Omega)dt < \infty.$$

This proves (2.4) and implies in particular $u(t) \in BV(\Omega)$ for a.e. $t \in (0,T)$. It remains to prove the weak*-measurability of $u: [0,T] \to BV(\Omega)$. Because of $u \in L^1(\Omega_T)$ we know that $t \mapsto \int_{\Omega} u(t)\varphi dx$ is measurable for any $\varphi \in C_0^0(\Omega)$. Since $u(t) \in BV(\Omega)$ for a.e. $t \in (0,T)$, an approximation argument implies that $t \mapsto \langle u(t), \varphi \rangle$ is measurable for any φ of the form $\varphi = g - \operatorname{div} G$ with $(g,G) \in C_0^0(\Omega, \mathbb{R}^{n+1})$. Consequently, $u: [0,T] \to BV(\Omega)$ is weakly*-measurable and because of (2.7) we have $u \in L^1_{w*}(0,T; BV(\Omega))$.

2.5. The initial condition

Here we establish that variational solutions to the obstacle problem in the sense of Definition 1.1 fulfill the initial condition $u(0) = u_o$ on Ω^* in the L^2 -sense. This follows from the fact that the difference $||u(t) - u_o||^2_{L^2(\Omega)}$ depends continuously on the time t > 0 for a.e. t, cf. the estimate (2.8) below.

LEMMA 2.5. – Assume that u_o, ψ satisfy (1.3), (1.4), (1.5) and the compatibility condition $\psi(0) \leq u_o$ a.e. in Ω^* . Then, any variational solution u to the obstacle problem for the total variation flow in the sense of Definition 1.1 fulfills the initial condition $u(0) = u_o$ in the L^2 -sense, that is

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^h \|u(t) - u_o\|_{L^2(\Omega^*)}^2 \mathrm{d}t = 0.$$

Proof. – Since u is a variational solution in the sense of Definition 1.1, it satisfies the variational inequality (1.6) for a.e. $\tau \in [0, T]$. Therefore, choosing $v \equiv g$ in (1.6), where g is from (1.5), we obtain for a.e. $\tau \in [0, T]$ that

(2.8)
$$\begin{aligned} \int_{0}^{\tau} \|Du\|(\Omega^{*}) dt + \frac{1}{2}\|(g-u)(\tau)\|_{L^{2}(\Omega^{*})}^{2} \\ &\leq \iint_{\Omega^{*}_{\tau}} \partial_{t}g(g-u) dx dt + \int_{0}^{\tau} \|Dg\|(\Omega^{*}) dt \\ &\leq \iint_{\Omega^{*}_{\tau}} \left(|\partial_{t}g|^{2} + |g|^{2} + |u|^{2}\right) dx dt + \int_{0}^{\tau} \|Dg\|(\Omega^{*}) dt. \end{aligned}$$

Here, we discard the energy term in the left-hand side, which is non-negative. Now, we let $h \in (0,T)$, integrate with respect to τ over (0,h) and divide both sides of the resulting inequality by h to infer that

$$\frac{1}{2h} \int_0^h \|(g-u)(\tau)\|_{L^2(\Omega^*)}^2 \mathrm{d}\tau \le \iint_{\Omega_h^*} \left(|\partial_t g|^2 + |g|^2 + |u|^2\right) \mathrm{d}x \mathrm{d}t + \int_0^h \|Dg\|(\Omega^*) \mathrm{d}t.$$

Since the right-hand side vanishes as $h \downarrow 0$ and $g \in C^0([0,T]; L^2(\Omega^*))$ with $g(0) = u_o$, this proves the claim.

2.6. Energy estimates

LEMMA 2.6. – Assume that $u \in L^{\infty}(0,T; L^{2}(\Omega^{*})) \cap L^{1}_{w*}(0,T; BV_{u_{o}}(\Omega))$ with $u \geq \psi$ a.e. on Ω_{T} is a variational solution of the obstacle problem in the sense of Definition 1.1, and that $v \in L^{1}_{w*}(0,T; BV_{u_{o}}(\Omega))$ is an admissible comparison map, i.e., it holds $\partial_{t}v \in L^{2}(\Omega^{*}_{T})$, $v(0) \in L^{2}(\Omega)$ and $v \geq \psi$ a.e. on Ω_{T} . Then the solution satisfies the energy bound

$$\begin{split} \sup_{t \in [0,T]} & \|u(t)\|_{L^{2}(\Omega^{*})}^{2} + \int_{0}^{T} \|Du\|(\Omega^{*}) \mathrm{d}t \\ & \leq 16 \bigg(\int_{0}^{T} \|\partial_{t}v(\cdot,t)\|_{L^{2}(\Omega)} \, \mathrm{d}t \bigg)^{2} + 16 \int_{0}^{T} \|Dv\|(\Omega^{*}) \mathrm{d}t \\ & + 2 \sup_{t \in [0,T]} \|v(t)\|_{L^{2}(\Omega^{*})}^{2} + 8 \|v(0) - u_{o}\|_{L^{2}(\Omega^{*})}^{2}. \end{split}$$

Proof. – Since v is admissible as comparison function in the variational inequality (1.6), we deduce the estimate

$$\begin{split} &\frac{1}{2} \| (v-u)(\tau) \|_{L^{2}(\Omega^{*})}^{2} + \int_{0}^{\tau} \| Du \| (\Omega^{*}) \, dt \\ &\leq \iint_{\Omega^{*}_{\tau}} \partial_{t} v(v-u) \mathrm{d}x \mathrm{d}t + \int_{0}^{\tau} \| Dv \| (\Omega^{*}) \mathrm{d}t + \frac{1}{2} \| v(0) - u_{o} \|_{L^{2}(\Omega^{*})}^{2} \\ &\leq \frac{1}{4} \sup_{t \in [0,T]} \| (v-u)(t) \|_{L^{2}(\Omega^{*})}^{2} \\ &\quad + \left(\int_{0}^{T} \| \partial_{t} v(\cdot,t) \|_{L^{2}(\Omega)} \, \mathrm{d}t \right)^{2} + \int_{0}^{T} \| Dv \| (\Omega^{*}) \mathrm{d}t + \frac{1}{2} \| v(0) - u_{o} \|_{L^{2}(\Omega^{*})}^{2} \end{split}$$

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for a.e. $\tau \in (0,T]$. Taking the supremum over $\tau \in (0,T]$ on the left-hand side we can reabsorb the first term of the right. We therefore obtain

(2.9)

$$\frac{\frac{1}{8} \sup_{t \in [0,T]} \|u(t)\|_{L^{2}(\Omega^{*})}^{2} + \int_{0}^{T} \|Du\|(\Omega^{*}) dt \\
\leq 2 \left(\int_{0}^{T} \|\partial_{t} v(\cdot,t)\|_{L^{2}(\Omega)} dt \right)^{2} + 2 \int_{0}^{T} \|Dv\|(\Omega^{*}) dt \\
+ \frac{1}{4} \sup_{t \in [0,T]} \|v(t)\|_{L^{2}(\Omega^{*})}^{2} + \|v(0) - u_{o}\|_{L^{2}(\Omega^{*})}^{2}.$$

This implies the claim.

2.7. Strong variational solutions

In this section we consider some sort of strong variational solutions. By this we mean that the regularity requirement $u \in L^{\infty}([0,T]; L^2(\Omega^*))$ from Definition 1.1 is replaced by the stronger assumption $u \in C^0([0,T]; L^2(\Omega^*))$. In this case it suffices to assume that the variational inequality (1.6) holds on the whole time interval [0,T]. In this context we consider obstacle functions $\psi: \Omega_T^* \to \mathbb{R}$ satisfying the stronger assumption

(2.10)
$$\psi \in L^1_{w*}(0,T; \mathrm{BV}_{u_o}(\Omega^*))$$
 and $\partial_t \psi \in L^2(\Omega^*_T)$.

The following definition now describes the concept of strong variational solutions to the obstacle problem for the total variation flow that will be used for obstacles as in (2.10).

DEFINITION 2.7 (Strong variational solutions). – Assume that ψ fulfills (2.10), that u_o is as in (1.3), and moreover, that the compatibility condition $\psi(0) \leq u_o$ a.e. in Ω^* holds true. Furthermore, suppose that (1.5) is in force. In this situation a measurable map $u: \Omega_T^* \to \mathbb{R}$ in the class

 $u \in C^0([0,T]; L^2(\Omega^*)) \cap L^1_{w*}(0,T; \mathrm{BV}_{u_o}(\Omega))$ with $u \ge \psi$ a.e. in Ω_T

is called *strong variational solution* to the obstacle problem for the total variation flow if and only if the variational inequality

(2.11)
$$\int_{0}^{T} \|Du\|(\Omega^{*}) dt \leq \iint_{\Omega_{T}^{*}} \partial_{t} v(v-u) dx dt + \int_{0}^{T} \|Dv\|(\Omega^{*}) dt - \frac{1}{2} \|(v-u)(T)\|_{L^{2}(\Omega^{*})}^{2} + \frac{1}{2} \|v(0) - u_{o}\|_{L^{2}(\Omega^{*})}^{2}$$

holds for any $v \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$ with $\partial_t v \in L^2(\Omega^*_T)$, $v(0) \in L^2(\Omega^*)$, and $v \ge \psi$ a.e. in Ω_T .

2.7.1. Localization. – In this section our aim is to ensure that a strong variational solution u to the obstacle problem in the sense of Definition 2.7 on the cylinder Ω_T is also a variational solution on the smaller cylinder Ω_{τ} for any $\tau \in (0, T)$. This will hold, provided the obstacle's regularity guarantees

(2.12)
$$\lim_{h \downarrow 0} \int_0^T \left\| D\psi - D[\psi]_h \right\| (\Omega^*) \mathrm{d}t = 0.$$

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We let $\tau \in (0,T)$ and start with a testing function $v \in L^1_{w*}(0,\tau; BV_{u_o}(\Omega))$ with $\partial_t v \in L^2(\Omega^*_{\tau})$, $v(0) \in L^2(\Omega^*)$, and $v \ge \psi$ a.e. in Ω_{τ} . For $\theta \in (0,\tau)$ we consider the cut-off function

$$\xi_{\theta}(t) := \chi_{[0,\tau-\theta]}(t) + \frac{\tau-t}{\theta} \chi_{(\tau-\theta,\tau]}(t).$$

As comparison function in (2.11) we choose

$$\tilde{v} := \xi_{\theta} v + (1 - \xi_{\theta}) \left([u - \psi]_h + \psi \right) \in L^1_{w*}(0, T; \mathrm{BV}_{u_o}(\Omega)),$$

where $[u - \psi]_h$ is defined according to (2.3) with $u_o - \psi(0)$ instead of v_o . Choosing this comparison function (whose admissibility can be checked easily; observe that \tilde{v} is a convex combination of two functions, each satisfying the pointwise constraint and the Dirichlet boundary condition on the lateral boundary) we get

(2.13)
$$\int_{0}^{T} \|Du\|(\Omega^{*}) dt \leq \iint_{\Omega_{T}^{*}} \partial_{t} \tilde{v}(\tilde{v}-u) dx dt + \int_{0}^{T} \|D\tilde{v}\|(\Omega^{*}) dt + \frac{1}{2} \|v(0) - u_{o}\|_{L^{2}}^{2} - \frac{1}{2} \|([u-\psi]_{h} - (u-\psi))(T)\|_{L^{2}}^{2}.$$

Our next aim is to pass to the limit $\theta \downarrow 0$. Therefore, we analyze the first two terms of the previous inequality. The first one can be rewritten to

$$\begin{split} \iint_{\Omega_{T}^{*}} \partial_{t} \tilde{v}(\tilde{v}-u) \mathrm{d}x \mathrm{d}t \\ &= \iint_{\Omega^{*} \times (0,\tau-\theta)} \partial_{t} v(v-u) \mathrm{d}x \mathrm{d}t \\ &+ \iint_{\Omega^{*} \times (\tau,T)} \partial_{t} \left([u-\psi]_{h} + \psi \right) \left([u-\psi]_{h} - (u-\psi) \right) \mathrm{d}x \mathrm{d}t \\ &+ \iint_{\Omega^{*} \times (\tau-\theta,\tau)} \xi_{\theta}' \xi_{\theta} | v - \psi - [u-\psi]_{h} |^{2} \mathrm{d}x \mathrm{d}t \\ &+ \iint_{\Omega^{*} \times (\tau-\theta,\tau)} \xi_{\theta}' \left([u-\psi]_{h} - (u-\psi) \right) \left(v - \psi - [u-\psi]_{h} \right) \mathrm{d}x \mathrm{d}t \\ &+ \iint_{\Omega^{*} \times (\tau-\theta,\tau)} \left[\xi_{\theta} \partial_{t} v + (1-\xi_{\theta}) \partial_{t} \left([u-\psi]_{h} + \psi \right) \right] \\ &\cdot \left[\xi_{\theta} (v-u) + (1-\xi_{\theta}) \left([u-\psi]_{h} - (u-\psi) \right) \right] \mathrm{d}x \mathrm{d}t \\ &=: \mathbf{I}_{\theta} + \mathbf{II} + \mathbf{III}_{\theta} + \mathbf{IV}_{\theta} + \mathbf{V}_{\theta}, \end{split}$$

where the meaning of the terms I_{θ} , II and $III_{\theta} - V_{\theta}$ is obvious in this context. Note, that II is independent of θ . Further, using (2.2) we conclude that

$$\partial_t \big([u-\psi]_h + \psi \big) \big([u-\psi]_h - (u-\psi) \big) \\= -\frac{1}{h} \big| [u-\psi]_h - (u-\psi) \big|^2 + \partial_t \psi \big([u-\psi]_h - (u-\psi) \big) \\\leq \partial_t \psi \big([u-\psi]_h - (u-\psi) \big),$$

so that

$$\mathbf{II} \leq \iint_{\Omega^* \times (\tau,T)} \partial_t \psi \big([u - \psi]_h - (u - \psi) \big) \mathrm{d}x \mathrm{d}t.$$

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In the limit $\theta \downarrow 0$ the terms \mathbf{I}_{θ} , $\mathbf{III}_{\theta} - \mathbf{V}_{\theta}$ show the following behavior, for a fixed h > 0:

$$\lim_{\theta \downarrow 0} \mathbf{I}_{\theta} = \iint_{\Omega^* \times (0,\tau)} \partial_t v(v-u) \mathrm{d}x \mathrm{d}t, \quad \lim_{\theta \downarrow 0} \mathbf{III}_{\theta} = -\frac{1}{2} \left\| \left(v - \psi - [u - \psi]_h \right)(\tau) \right\|_{L^2}^2,$$

and

$$\lim_{\theta \downarrow 0} \mathbf{V}_{\theta} = 0, \quad \limsup_{\theta \downarrow 0} \mathbf{IV}_{\theta} \le \left\| \left([u - \psi]_h - (u - \psi) \right) \left(v - \psi - [u - \psi]_h \right) (\tau) \right\|_{L^1}.$$

The second term appearing on the right-hand side of the minimality condition (2.13) can be decomposed as follows:

$$\begin{split} \int_0^T \|D\tilde{v}\|(\Omega^*) \mathrm{d}t &= \int_0^{\tau-\theta} \|Dv\|(\Omega^*) \mathrm{d}t \\ &+ \int_{\tau-\theta}^{\tau} \|D[\xi_\theta v + (1-\xi_\theta) \big([u-\psi]_h + \psi\big)\big] \|(\Omega^*) \mathrm{d}t \\ &+ \int_{\tau}^T \|D[u-\psi]_h + D\psi\|(\Omega^*) \mathrm{d}t. \end{split}$$

Since $\tilde{v} \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$ the second integral on the right-hand side vanishes in the limit $\theta \downarrow 0$, and therefore we conclude that

$$\lim_{\theta \downarrow 0} \int_0^T \|D\tilde{v}\|(\Omega^*) \mathrm{d}t = \int_0^\tau \|Dv\|(\Omega^*) \mathrm{d}t + \int_\tau^T \|D[u-\psi]_h + D\psi\|(\Omega^*) \mathrm{d}t.$$

Summarizing, in the limit $\theta \downarrow 0$ we conclude from the variational inequality (2.13) that

$$\begin{split} \int_{0}^{\tau} \|Du\|(\Omega^{*}) \mathrm{d}t &\leq \iint_{\Omega^{*} \times (0,\tau)} \partial_{t} v(v-u) \mathrm{d}x \mathrm{d}t + \int_{0}^{\tau} \|Dv\|(\Omega^{*}) \mathrm{d}t \\ &\quad - \frac{1}{2} \| \left(v - \psi - [u - \psi]_{h} \right)(\tau) \|_{L^{2}}^{2} + \frac{1}{2} \|v(0) - u_{o}\|_{L^{2}}^{2} \\ &\quad - \frac{1}{2} \| \left([u - \psi]_{h} - (u - \psi) \right)(T) \|_{L^{2}}^{2} \\ &\quad + \| \left([u - \psi]_{h} - (u - \psi) \right) \left(v - \psi - [u - \psi]_{h} \right)(\tau) \|_{L^{1}} \\ &\quad + \iint_{\Omega^{*} \times (\tau,T)} \partial_{t} \psi([u - \psi]_{h} - (u - \psi)) \mathrm{d}x \mathrm{d}t \\ &\quad + \int_{\tau}^{T} \left[\|D[u]_{h}\|(\Omega^{*}) - \|Du\|(\Omega^{*}) + \|D\psi - D[\psi]_{h}\|(\Omega^{*}) \right] \mathrm{d}t \end{split}$$

Taking into account the facts that $[u - \psi]_h \to u - \psi$ in $L^{\infty}(0, T; L^2(\Omega^*))$ by Lemma 2.1, that $\int_{\tau}^{T} \|D[u]_h\|(\Omega^*) dt \to \int_{\tau}^{T} \|Du\|(\Omega^*) dt$ by Lemma 2.3 and the assumption (2.12) we see that the terms in the last four lines vanish in the limit $h \downarrow 0$. Moreover, for the third term we have the convergence $\|(v - \psi - [u - \psi]_h)(\tau)\|_{L^2}^2 \to \|(v - u)(\tau)\|_{L^2}^2$. This ensures that u is a variational solution to the obstacle problem also on the smaller cylinder Ω_{τ} , provided the obstacle ψ fulfills the assumption (2.12).

2.7.2. The initial condition. – Here we establish, assuming again (2.12), that strong variational solutions to the obstacle problem in the sense of Definition 2.7 fulfill the initial condition $u(0) = u_o$ on Ω^* in the $C^0 - L^2$ -sense. Note that this property is stronger than the one from Lemma 2.5 for weak variational solutions. The reason is that here we can show that

 $||u(t) - u_o||^2_{L^2(\Omega)}$ depends continuously on t > 0, whereas in Lemma 2.5 the corresponding estimate holds only for a.e. t > 0.

LEMMA 2.8. – Assume that the obstacle ψ fulfills the hypotheses (2.10) and (2.12). Then, any variational solution u to the obstacle problem for the total variation flow in the sense of Definition 2.7 fulfills the initial condition $u(0) = u_o$ in the $C^0 - L^2$ -sense, that is

$$\lim_{t \downarrow 0} \|u(t) - u_o\|_{L^2(\Omega^*)}^2 = 0.$$

Proof. – From § 2.7.1 we know that u is a variational solution on any sub-cylinder Ω_t^* with $t \in (0,T)$. We fix $t \in (0,T)$ and test the minimality condition (1.6) for Ω_t with $v \equiv g$, where g is from (1.5). As in the proof of Lemma 2.5 (see estimate (2.8)), we can show that

$$\frac{1}{2} \| (g-u)(t) \|_{L^2(\Omega^*)}^2 \le \iint_{\Omega_t^*} \left(|\partial_\tau g|^2 + |g|^2 + |u|^2 \right) \mathrm{d}x \mathrm{d}\tau + \int_0^t \| Dg\|(\Omega^*) \mathrm{d}\tau$$

holds true for any $t \in [0, T]$. Letting $t \downarrow 0$ in the right-hand side and recalling that $g(0) = u_o$, this proves the claim that u satisfies the initial boundary condition $u(0) = u_o$.

2.7.3. *A comparison principle.* – For the proof of the comparison principle we need the following simple property of BV-functions, which can for example be deduced from [24, Theorem 2.8 (iii)].

LEMMA 2.9. – For two functions $v, w \in BV(\Omega)$ we have $\min\{v, w\}, \max\{v, w\} \in BV(\Omega)$ and

$$||D\min\{v,w\}||(\Omega) + ||D\max\{v,w\}||(\Omega) \le ||Dv||(\Omega) + ||Dw||(\Omega).$$

LEMMA 2.10. – Assume that the obstacle ψ fulfills the hypotheses (2.10) and (2.12). Further, let u be a strong variational solution to the obstacle problem for the total variation flow in the sense of Definition 2.7 and let $[u]_h$ respectively $[\psi]_h$ for h > 0 denote the time mollification of u respectively ψ as defined in (2.3) with the initial datum u_o , respectively $\psi(0)$ instead of v_o . Then, there holds

(2.14)
$$\lim_{h \downarrow 0} \frac{1}{h} \iint_{\Omega^* \times (0,T)} |[u - \psi]_h - (u - \psi)|^2 dx dt = 0.$$

Proof. – Since $[u - \psi]_h + \psi$ is an admissible comparison function in (2.11) (this can be seen from the properties of the time mollification and Lemma 2.3), we have

$$\int_{0}^{T} \|Du\|(\Omega^{*}) dt + \frac{1}{2} \| ([u - \psi]_{h} - (u - \psi))(T) \|_{L^{2}(\Omega^{*})}^{2} \\ \leq \iint_{\Omega^{*}_{T}} \partial_{t} ([u - \psi]_{h} + \psi) ([u - \psi]_{h} - (u - \psi)) dx dt \\ + \int_{0}^{T} \|D([u - \psi]_{h} + \psi)\|(\Omega^{*}) dt.$$

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Applying (2.2) and Lemma 2.3 and utilizing the assumption (2.12) we therefore find that

$$\begin{split} \lim_{h \downarrow 0} \frac{1}{h} \iint_{\Omega_T^*} \left| [u - \psi]_h - (u - \psi) \right|^2 \mathrm{d}x \mathrm{d}t \\ &\leq \liminf_{h \downarrow 0} \left[\int_0^T \left[\|D[u]_h\|(\Omega^*) - \|Du\|(\Omega^*) \right] \mathrm{d}t + \int_0^T \left\| D\psi - D[\psi]_h \right\|(\Omega^*) \mathrm{d}t \\ &\quad + \iint_{\Omega_T^*} \partial_t \psi \left([u - \psi]_h - (u - \psi) \right) \mathrm{d}x \mathrm{d}t \right] = 0. \end{split}$$

This proves the claim.

REMARK 2.11. – In (2.14) it is possible to eliminate the terms involving the obstacle. Indeed, using (2.2) and Lemma 2.2 we conclude that

$$\lim_{h\downarrow 0} \frac{1}{h} \iint_{\Omega_T^*} \left| \psi - [\psi]_h \right|^2 \mathrm{d}x \mathrm{d}t = \lim_{h\downarrow 0} h \iint_{\Omega_T^*} \left| \partial_t [\psi]_h \right|^2 \mathrm{d}x \mathrm{d}t \le \lim_{h\downarrow 0} h \|\partial_t \psi\|_{L^2}^2 = 0.$$

Using this in (2.14) we deduce

$$\lim_{h \downarrow 0} \frac{1}{h} \iint_{\Omega_T^*} \left| [u]_h - u \right|^2 \mathrm{d}x \mathrm{d}t = 0.$$

Having arrived at this stage we have all ingredients at hand to prove the comparison principle. The corresponding result for the obstacle-free case can be found in [25, Thm. 3.1].

LEMMA 2.12 (Comparison principle). – Let u_o , \tilde{u}_o fulfill the assumptions (1.3) and (1.5) and suppose that $u_o \leq \tilde{u}_o$ a.e. in Ω^* . Furthermore, let $\psi \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$, $\tilde{\psi} \in L^1_{w*}(0,T; BV_{\tilde{u}_o}(\Omega))$, with $\partial_t \psi, \partial_t \tilde{\psi} \in L^2(\Omega^*_T)$ and $\psi \leq \tilde{\psi}$ a.e. in Ω^*_T and let $\psi, \tilde{\psi}$ satisfy (2.12). Finally, let u, \tilde{u} be the strong variational solutions to the associated obstacle problems in the sense of Definition 2.7 with initial and lateral boundary values u_o, \tilde{u}_o and obstacles $\psi, \tilde{\psi}$ respectively. Then, we have

$$u \leq \tilde{u}$$
 a.e. in Ω_T .

REMARK 2.13. – The above comparison principle implies in particular that strong variational solutions are unique for given Cauchy-Dirichlet data $u_o \in L^2(\Omega^*) \cap BV(\Omega^*)$, provided the obstacle function satisfies (2.10) and (2.12).

Proof. – Let $[u]_h, [\tilde{\psi}]_h, [\tilde{u}]_h, [\tilde{\psi}]_h$ be defined as in (2.3) with $u_o, \psi(0), \tilde{u}_o, \tilde{\psi}(0)$ instead of v_o and abbreviate $u_h := [u - \psi]_h + \psi$ and $\tilde{u}_h := [\tilde{u} - \tilde{\psi}]_h + \tilde{\psi}$. Taking $v_h := \min\{u_h, \tilde{u}_h\}$ as comparison function in the variational inequality for u and $w_h := \max\{u_h, \tilde{u}_h\}$ as comparison function in the variational inequality for \tilde{u} , we obtain, after adding the resulting inequalities, that for any $\tau \in [0, T]$ there holds:

(2.15)

$$\int_{0}^{\tau} \left[\|Du\|(\Omega^{*}) + \|D\tilde{u}\|(\Omega^{*})\right] dt \leq \int_{0}^{\tau} \left[\|Dv_{h}\|(\Omega^{*}) + \|Dw_{h}\|(\Omega^{*})\right] dt \\
+ \iint_{\Omega_{\tau}^{*}} \left[\partial_{t}v_{h}(v_{h}-u) + \partial_{t}w_{h}(w_{h}-\tilde{u})\right] dx dt \\
- \frac{1}{2} \|(v_{h}-u)(\tau)\|_{L^{2}}^{2} - \frac{1}{2} \|(w_{h}-\tilde{u})(\tau)\|_{L^{2}}^{2}.$$

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Here, we also used that $v_h(0) = u_o$ and $w_h(0) = \tilde{u}_o$. The aim of the following is to estimate the first and second term on the right-hand side of (2.15). We start with the term involving the time derivatives. On $\{(x,t) \in \Omega^*_{\tau} : u_h(x,t) \le \tilde{u}_h(x,t)\}$ we have

$$\begin{aligned} \partial_t v_h(v_h - u) &+ \partial_t w_h(w_h - \tilde{u}) \\ &= \partial_t \big([u - \psi]_h + \psi \big) \big([u - \psi]_h - (u - \psi) \big) + \partial_t \big([\tilde{u} - \tilde{\psi}]_h + \tilde{\psi} \big) \big([\tilde{u} - \tilde{\psi}]_h - (\tilde{u} - \tilde{\psi}) \big) \\ &= -\frac{1}{h} \big| [u - \psi]_h - (u - \psi) \big|^2 - \frac{1}{h} \big| [\tilde{u} - \tilde{\psi}]_h - (\tilde{u} - \tilde{\psi}) \big|^2 \\ &+ \partial_t \psi \big([u - \psi]_h - (u - \psi) \big) + \partial_t \tilde{\psi} \big([\tilde{u} - \tilde{\psi}]_h - (\tilde{u} - \tilde{\psi}) \big) \\ &\leq \partial_t \psi (u_h - u) + \partial_t \tilde{\psi} (\tilde{u}_h - \tilde{u}). \end{aligned}$$

For the second conversion we used (2.2). On the other hand, on the complement, i.e., on the set $\{(x,t) \in \Omega_{\tau}^* : u_h(x,t) > \tilde{u}_h(x,t)\}$, we compute, using again (2.2), the facts that $\partial_t \tilde{u}_h = \partial_t \tilde{\psi} - \frac{1}{h}(\tilde{u}_h - \tilde{u})$ and $\partial_t u_h = \partial_t \psi - \frac{1}{h}(u_h - u)$ and moreover Young's inequality, that

$$\begin{aligned} \partial_t v_h(v_h - u) &+ \partial_t w_h(w_h - \tilde{u}) \\ &= \partial_t \tilde{u}_h(\tilde{u}_h - u) + \partial_t u_h(u_h - \tilde{u}) \\ &= \partial_t \tilde{u}_h(\tilde{u}_h - u_h) + \partial_t \tilde{u}_h(u_h - u) + \partial_t u_h(u_h - \tilde{u}_h) + \partial_t u_h(\tilde{u}_h - \tilde{u}) \\ &= \partial_t (u_h - \tilde{u}_h)(u_h - \tilde{u}_h) + \partial_t \tilde{\psi}(u_h - u) + \partial_t \psi(\tilde{u}_h - \tilde{u}) - \frac{2}{h}(\tilde{u}_h - \tilde{u})(u_h - u) \\ &\leq \frac{1}{2} \partial_t |u_h - \tilde{u}_h|^2 + \partial_t \tilde{\psi}(u_h - u) + \partial_t \psi(\tilde{u}_h - \tilde{u}) + \frac{1}{h} |\tilde{u}_h - \tilde{u}|^2 + \frac{1}{h} |u_h - u|^2. \end{aligned}$$

Joining the preceding inequalities, we have shown that the term involving the time derivative is estimated as follows:

$$\iint_{\Omega_{\tau}^{*}} \left[\partial_{t} v_{h}(v_{h}-u) + \partial_{t} w_{h}(w_{h}-\tilde{u})\right] \mathrm{d}x \mathrm{d}t \leq \frac{1}{2} \iint_{\Omega_{\tau}^{*}} \partial_{t} (u_{h}-\tilde{u}_{h})^{2}_{+} \mathrm{d}x \mathrm{d}t + \mathbf{I}_{h} + \mathbf{II}_{h}$$
$$= \frac{1}{2} \int_{\Omega^{*} \times \{\tau\}} (u_{h}-\tilde{u}_{h})^{2}_{+} \mathrm{d}x + \mathbf{I}_{h} + \mathbf{II}_{h}$$

with the abbreviations

$$\mathbf{I}_{h} := \iint_{\Omega_{\tau}^{*}} \left[|\partial_{t}\psi||u_{h} - u| + \left|\partial_{t}\tilde{\psi}\right| \left|\tilde{u}_{h} - \tilde{u}\right| + \left|\partial_{t}\tilde{\psi}\right| \left|u_{h} - u\right| + \left|\partial_{t}\psi\right| \left|\tilde{u}_{h} - \tilde{u}\right| \right] \mathrm{d}x\mathrm{d}t$$

and

$$\mathbf{II}_h := \iint_{\Omega^*_{\tau}} \left[\frac{1}{h} |\tilde{u}_h - \tilde{u}|^2 + \frac{1}{h} |u_h - u|^2 \right] \mathrm{d}x \mathrm{d}t$$

for the remainder terms. We note that $\lim_{h\downarrow 0} I_h = 0$ by Lemma 2.1 and $\lim_{h\downarrow 0} II_h = 0$ by Lemma 2.10. Next, we consider the $L^2(\Omega^*)$ -terms appearing in the third line of (2.15). Restricting the domain of integration we obtain

$$\begin{aligned} -\frac{1}{2} \| (v_h - u)(\tau) \|_{L^2}^2 &\leq -\frac{1}{2} \int_{\Omega^* \times \{\tau\} \cap \{u_h > \tilde{u}_h\}} |\tilde{u}_h - u|^2 \mathrm{d}x \\ &\leq -\frac{1}{2} \int_{\Omega^* \times \{\tau\}} (u_h - \tilde{u}_h)_+^2 \mathrm{d}x + \mathbf{III}_h, \end{aligned}$$

where

$$\mathbf{III}_h := \int_{\Omega^* \times \{\tau\}} \left[\frac{1}{2} |u_h - u|^2 + |\tilde{u}_h - u||u_h - u| \right] \mathrm{d}x$$

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Note that $\lim_{h\downarrow 0} \mathbf{III}_h = 0$ since $u_h - u = [u - \psi]_h - (u - \psi) \to 0$ in $L^{\infty}(0, T; L^2(\Omega^*))$ as $h \downarrow 0$ by Lemma 2.1. Similarly, we can show that

$$-\frac{1}{2} \| (w_h - \tilde{u})(\tau) \|_{L^2(\Omega^*)}^2 \le -\frac{1}{2} \int_{\Omega^* \times \{\tau\}} (u_h - \tilde{u}_h)_+^2 \mathrm{d}x + \mathrm{IV}_h,$$

where we abbreviated this time

$$\mathbf{IV}_h := \int_{\Omega^* \times \{\tau\}} \left[\frac{1}{2} |\tilde{u}_h - \tilde{u}|^2 + |u_h - \tilde{u}| |\tilde{u}_h - \tilde{u}| \right] \mathrm{d}x.$$

As before, we can conclude $\lim_{h\downarrow 0} IV_h = 0$. Joining the preceding estimates with (2.15) and applying Lemma 2.9, we find that

$$\begin{split} \int_0^\tau \left[\|Du\|(\Omega^*) + \|D\tilde{u}\|(\Omega^*)\right] \mathrm{d}t &+ \frac{1}{2} \int_{\Omega^* \times \{\tau\}} (u_h - \tilde{u}_h)_+^2 \mathrm{d}x \\ &\leq \int_0^\tau \left[\|Du_h\|(\Omega^*) + \|D\tilde{u}_h\|(\Omega^*)\right] \mathrm{d}t + \mathbf{I}_h + \mathbf{II}_h + \mathbf{II}_h + \mathbf{IV}_h \\ &\leq \int_0^\tau \left[\|D[u]_h\|(\Omega^*) + \|D[\tilde{u}]_h\|(\Omega^*)\right] \mathrm{d}t \\ &+ \int_0^\tau \left[\|D\psi - D[\psi]_h\|(\Omega^*) + \|D\tilde{\psi} - D[\tilde{\psi}]_h\|(\Omega^*)\right] \mathrm{d}t \\ &+ \mathbf{I}_h + \mathbf{II}_h + \mathbf{III}_h + \mathbf{IV}_h. \end{split}$$

Here, we let $h \downarrow 0$ and use Lemma 2.3, $\tilde{u}_h \rightarrow \tilde{u}$ and $u_h \rightarrow u$ in $L^{\infty}(0,T; L^2(\Omega^*))$ and the assumption (2.12). In this way we end up with

$$\int_{\Omega^* \times \{\tau\}} (u - \tilde{u})_+^2 \, dx \le 0.$$

Since $\tau \in [0, T]$ was arbitrary, this proves the claim that $u \leq \tilde{u}$ a.e. in Ω_T and finishes the proof of the lemma.

3. Proof of the existence results

3.1. Existence of solutions for regularized obstacle problems

Our aim in this section is to prove the following

THEOREM 3.1. – Assume that initial and boundary values $u_o \in L^2(\Omega^*) \cap BV(\Omega^*)$ with $u_o|_{\Omega^*\setminus\overline{\Omega}} \in W^{1,1}(\Omega^*\setminus\overline{\Omega})$ are given and the obstacle $\psi \in W^{1,1}(\Omega^*_T)$ with $\psi = u_o$ a.e. on $(\Omega^*\setminus\Omega) \times (0,T)$ satisfies

(3.1)
$$\partial_t \psi \in L^2(\Omega_T^*) \text{ and } \partial_t D \psi \in L^1(\Omega_T^*, \mathbb{R}^n),$$

and moreover $\psi(0) \in L^2(\Omega^*)$ with $\psi(0) \leq u_o$ a.e. on Ω^* . Then there exists a strong variational solution

$$u \in L^{\infty}_{w*}(0,T; \mathrm{BV}_{u_o}(\Omega))$$
 with $\partial_t u \in L^2(\Omega^*_T)$

to the obstacle problem for the total variation flow subject to the obstacle ψ and with initial boundary data u_o in the sense of Definition 2.7. Moreover, there holds

(3.2)
$$\frac{\frac{1}{2}}{\int_{\Omega_T^*} |\partial_t u|^2 \mathrm{d}x \mathrm{d}t} + \sup_{t \in [0,T]} \|Du(t)\|(\Omega^*) \\ \leq \|Du_o\|(\Omega^*) + \int_{\Omega_T^*} \left[\frac{1}{2} |\partial_t \psi|^2 + |\partial_t D\psi|\right] \mathrm{d}x \mathrm{d}t.$$

REMARK 3.2. – Some remarks concerning the assumptions on the obstacle's regularity are in order. Firstly, $\partial_t D\psi \in L^1(\Omega_T^*, \mathbb{R}^n)$ implies that $D\psi \in C^0([0,T]; L^1(\Omega^*, \mathbb{R}^n))$, so that $\psi \in C^0([0,T]; W^{1,1}(\Omega^*))$. Secondly, since $\partial_t \psi \in L^2(\Omega_T^*)$ and $\psi(0) \in L^2(\Omega^*)$ we have $\psi \in C^{0,\frac{1}{2}}([0,T]; L^2(\Omega^*))$, which in particular yields $\psi \in L^2(\Omega_T^*) \cong L^2(0,T; L^2(\Omega^*))$. As a consequence of the above reasoning we have on the time slices that $\psi(t) \in L^2(\Omega^*) \cap W^{1,1}(\Omega^*)$. Finally, in the setting of Theorem 3.1 in which $\partial_t \psi \in L^2(\Omega_T^*)$, the assumption (1.5) on g is satisfied, since we may choose $g = \max\{\psi, u_o\}$ in this case.

For the proof we shall proceed in several steps. We start with the construction of

3.1.1. A sequence of minimizers to elliptic variational functionals. – We fix a step size h>0 and write $\psi_i := \psi(ih) \in L^2(\Omega^*) \cap W^{1,1}(\Omega^*)$ for each $i \in \mathbb{N}_o$ with $ih \leq T$ for the time-discretized obstacle. Our goal is to inductively construct a sequence $u_i \in L^2(\Omega^*) \cap BV_{u_o}(\Omega)$ of minimizers to certain elliptic variational functionals. The construction is as follows. Suppose that $u_{i-1} \in L^2(\Omega^*) \cap BV_{u_o}(\Omega)$ for some $i \in \mathbb{N}$ has already been defined. If i = 1, then $u_0 = u_o$ is the initial boundary datum. Then, we let u_i be the minimizer of the variational functional

$$\mathbf{F}_{i}[v] := \|Dv\|(\Omega^{*}) + \frac{1}{2h} \int_{\Omega^{*}} |v - u_{i-1}|^{2} \mathrm{d}x$$

in the class of functions $v \in L^2(\Omega^*) \cap BV_{u_o}(\Omega)$ with $v \ge \psi_i$ a.e. in Ω . Note that this class is non-empty since $v = \psi_i$ is admissible. The existence of u_i can be deduced by means of standard compactness arguments. We note that $u_i \ge \psi_i$ a.e. in Ω for any $i \in \mathbb{N}_0$, with $ih \le T$, by construction.

3.1.2. Energy estimates. – We first observe that $u_{i-1} + \psi_i - \psi_{i-1} \ge \psi_i$ is an admissible comparison function in the functional \mathbf{F}_i for any $i \in \mathbb{N}$. Therefore, using the minimality of u_i , we find that

$$\begin{aligned} \mathbf{F}_{i}[u_{i}] &\leq \mathbf{F}_{i}\left[u_{i-1} + \psi_{i} - \psi_{i-1}\right] \\ &= \frac{1}{2h} \int_{\Omega^{*}} \left|\psi_{i} - \psi_{i-1}\right|^{2} \mathrm{d}x + \left\|Du_{i-1} + D\psi_{i} - D\psi_{i-1}\right\|(\Omega^{*}) \\ &\leq \frac{1}{2h} \int_{\Omega^{*}} \left|\psi_{i} - \psi_{i-1}\right|^{2} \mathrm{d}x + \left\|Du_{i-1}\right\|(\Omega^{*}) + \left\|D\psi_{i} - D\psi_{i-1}\right\|(\Omega^{*}). \end{aligned}$$

Taking into account that

$$\frac{1}{2h} \int_{\Omega^*} \left| \psi_i - \psi_{i-1} \right|^2 \mathrm{d}x = \frac{1}{2h} \int_{\Omega^*} \left| \int_{(i-1)h}^{ih} \partial_t \psi \mathrm{d}t \right|^2 \mathrm{d}x \le \frac{1}{2} \iint_{\Omega^* \times [(i-1)h, ih]} \left| \partial_t \psi \right|^2 \mathrm{d}x \mathrm{d}t$$

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and

$$\begin{split} \left\| D\psi_i - D\psi_{i-1} \right\| (\Omega^*) &= \int_{\Omega^*} \left| D\psi_i - D\psi_{i-1} \right| \mathrm{d}x \\ &= \int_{\Omega^*} \left| \int_{(i-1)h}^{ih} \partial_t D\psi \mathrm{d}t \right| \mathrm{d}x \le \iint_{\Omega^* \times [(i-1)h, ih]} \left| \partial_t D\psi \right| \mathrm{d}x \mathrm{d}t, \end{split}$$

we conclude that

$$\frac{1}{2h} \int_{\Omega^*} |u_i - u_{i-1}|^2 dx + ||Du_i||(\Omega^*)$$

$$\leq ||Du_{i-1}||(\Omega^*) + \iint_{\Omega^* \times [(i-1)h, ih]} \left[\frac{1}{2} |\partial_t \psi|^2 + |\partial_t D\psi|\right] dx dt$$

Summing up this inequality from $i = 1, ..., \ell$ for some $\ell \in \mathbb{N}$ with $\ell h \leq T$, we find that

(3.3)
$$\frac{1}{2h} \sum_{i=1}^{\ell} \int_{\Omega^*} |u_i - u_{i-1}|^2 \, dx + \|Du_\ell\|(\Omega^*) \le \Psi(\ell h),$$

where we have abbreviated

$$\Psi(\tau) := \|Du_o\|(\Omega^*) + \iint_{\Omega^*_{\tau}} \left[\frac{1}{2}|\partial_t \psi|^2 + |\partial_t D\psi|\right] \mathrm{d}x \mathrm{d}t$$

for $\tau \in (0, T]$. From (3.3) we also conclude that

(3.4)
$$\int_{\Omega^*} |u_{\ell}|^2 dx \leq \int_{\Omega^*} 2 \left[\sum_{i=1}^{\ell} |u_i - u_{i-1}| \right]^2 dx + 2 \int_{\Omega^*} |u_o|^2 dx \\ \leq 2\ell \sum_{i=1}^{\ell} \int_{\Omega^*} |u_i - u_{i-1}|^2 dx + 2 \int_{\Omega^*} |u_o|^2 dx \\ \leq 4\ell h \Psi(\ell h) + 2 \|u_o\|_{L^2(\Omega^*)}^2,$$

for any $\ell \in \mathbb{N}$ with $\ell h \leq T$. From now on, we consider only such values $h \in (0, 1]$ that satisfy $\ell := \frac{T}{h} \in \mathbb{N}$. Then we define a function $u^{(h)} : \Omega^* \times (-h, T] \to \mathbb{R}$ by

(3.5)
$$u^{(h)}(\cdot,t) := u_i \quad \text{for } t \in ((i-1)h, ih] \text{ with } i \in \{0, \dots, \ell\}.$$

From (3.3) and (3.4) we know that

(3.6)

$$\frac{1}{2h^{2}} \iint_{\Omega_{T}^{*}} |u^{(h)}(t) - u^{(h)}(t-h)|^{2} dx dt
+ \sup_{t \in [0,T]} ||u^{(h)}(t)||^{2}_{L^{2}(\Omega^{*})} + \sup_{t \in [0,T]} ||Du^{(h)}(t)||(\Omega^{*})
\leq 2(1+2T)\Psi(T) + 2||u_{o}||^{2}_{L^{2}(\Omega^{*})}.$$

3.1.3. The limit map. – The energy estimate (3.6) ensures that $u^{(h)}$ is bounded in $L^{\infty}(0,T; L^{2}(\Omega^{*}))$ and in $L^{\infty}_{w*}(0,T; BV_{u_{o}}(\Omega))$ and therefore there exists a subsequence $(h_{k})_{k\in\mathbb{N}}$ with $h_{k} \downarrow 0$ as $k \to \infty$ and a function $u \in L^{\infty}(0,T; L^{2}(\Omega^{*})) \cap L^{\infty}_{w*}(0,T; BV_{u_{o}}(\Omega))$ such that

(3.7)
$$\begin{cases} u^{(h_k)} \stackrel{*}{\rightharpoondown} u & \text{ in } L^{\infty}_{w*}(0,T; \mathrm{BV}_{u_o}(\Omega)), \\ u^{(h_k)} \stackrel{*}{\rightharpoondown} u & \text{ in } L^{\infty}(0,T; L^2(\Omega^*)), \end{cases}$$

in the limit $k \to \infty$. Next, we define a second function $\tilde{u}^{(h)} \colon \Omega^* \times (-h, T] \to \mathbb{R}$ by linearly interpolating u_{i-1} and u_i on the interval ((i-1)h, ih], i.e., by

$$\tilde{u}^{(h)}(\cdot,t) := \left(i - \frac{t}{h}\right)u_{i-1} + \left(1 - i + \frac{t}{h}\right)u_i \quad \text{for } t \in ((i-1)h, ih] \text{ with } i \in \{1, \dots, \ell\},$$

and $\tilde{u}^{(h)}(\cdot,t) := u_o$ for $t \in (-h, 0]$. For $t \in ((i-1)h, ih]$ we compute

$$\partial_t \tilde{u}^{(h)}(\cdot, t) = \frac{1}{h} (u_i - u_{i-1}) \in L^2(\Omega^*)$$

which together with (3.3) yields

(3.8)
$$\frac{1}{2} \iint_{\Omega_T^*} |\partial_t \tilde{u}^{(h)}|^2 \mathrm{d}x \mathrm{d}t + \sup_{t \in [0,T]} \|D\tilde{u}^{(h)}(t)\|(\Omega^*) \le \Psi(T).$$

This implies that the sequence $\tilde{u}^{(h)}$ is bounded in $L^{\infty}_{w*}(0,T; BV_{u_o}(\Omega))$ and that $\partial_t \tilde{u}^{(h)}$ is bounded in $L^2(\Omega^*_T)$. Hence $\tilde{u}^{(h)}$ is bounded in $BV(\Omega^*_T)$. Therefore, there exists a subsequence $(h_k)_{k\in\mathbb{N}}$ with $h_k \downarrow 0$ as $k \to \infty$ and a function $\tilde{u} \in L^{\infty}_{w*}(0,T; BV_{u_o}(\Omega))$ with $\partial_t \tilde{u} \in L^2(\Omega^*_T)$, so that

(3.9)
$$\begin{cases} \tilde{u}^{(h_k)} \to \tilde{u} & \text{strongly in } L^1(\Omega_T^*), \\ \tilde{u}^{(h_k)} \stackrel{*}{\to} \tilde{u} & \text{weak}^* \text{ in } L^{\infty}_{w*}(0, T; BV_{u_o}(\Omega)), \\ \partial_t \tilde{u}^{(h_k)} \to \partial_t \tilde{u} & \text{weakly in } L^2(\Omega_T^*), \end{cases}$$

in the limit $k \to \infty$. Since

$$\left| \left(\tilde{u}^{(h)} - u^{(h)} \right)(t) \right| \le |u_i - u_{i-1}| \text{ for } t \in ((i-1)h, ih],$$

we conclude from (3.3) that

$$\iint_{\Omega_T^*} \left| \tilde{u}^{(h)} - u^{(h)} \right|^2 \mathrm{d}x \mathrm{d}t \le h \sum_{i=1}^{\frac{T}{h}} \int_{\Omega^*} |u_i - u_{i-1}|^2 \mathrm{d}x \le 2h^2 \Psi(T),$$

which by Hölder's inequality implies

$$\iint_{\Omega_T^*} \left| \tilde{u}^{(h)} - u^{(h)} \right| \mathrm{d}x \mathrm{d}t \le h \sqrt{2\Psi(T) |\Omega_T^*|}.$$

Together with $(3.9)_1$ this shows that also $u^{(h_k)} \to \tilde{u}$ strongly in $L^1(\Omega_T^*)$ as $k \to \infty$ and hence $\tilde{u} = u$. By weak lower semi-continuity and (3.8) we infer that

$$\frac{1}{2} \iint_{\Omega_T^*} |\partial_t u|^2 \mathrm{d}x \mathrm{d}t + \sup_{t \in [0,T]} \|Du(t)\|(\Omega^*) \le \Psi(T).$$

Moreover, for a further subsequence we get that $u^{(h_k)} \to u$ a.e. in Ω_T^* in the limit $k \to \infty$. Since $u^{(h)} \ge \psi^{(h)}$, where $\psi^{(h)}$ is defined as in (3.5) with ψ instead of u, and $\psi^{(h)} \to \psi$ a.e. in Ω_T^* (after passing to another not relabelled subsequence) we conclude that $u \ge \psi$ a.e. in Ω_T^* .

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3.1.4. *Minimizing property of the approximations.* – Here, we first observe that $u^{(h)}$ is a minimizer of the functional

$$\mathbf{F}^{(h)}[v] := \frac{1}{2h} \iint_{\Omega_T^*} |v(t) - u^{(h)}(t-h)|^2 \mathrm{d}x \mathrm{d}t + \int_0^T \|Dv(t)\|(\Omega^*) \mathrm{d}t$$

in the class of functions $v \in L^2(\Omega_T^*) \cap L^1_{w*}(0,T; BV_{u_o}(\Omega))$ with $v \ge \psi^{(h)}$ a.e. in Ω_T^* . This can be seen from the following computation.

$$\begin{aligned} \mathbf{F}^{(h)} \left[u^{(h)} \right] &= \sum_{i=1}^{\frac{T}{h}} \int_{(i-1)h}^{ih} \left[\frac{1}{2h} \int_{\Omega^*} |u_i - u_{i-1}|^2 \mathrm{d}x + \|Du_i\|(\Omega^*) \right] \mathrm{d}t \\ &= \sum_{i=1}^{\frac{T}{h}} \int_{(i-1)h}^{ih} \mathbf{F}_i[u_i] \mathrm{d}t \le \sum_{i=1}^{\frac{T}{h}} \int_{(i-1)h}^{ih} \mathbf{F}_i[v(t)] \mathrm{d}t \\ &= \sum_{i=1}^{\frac{T}{h}} \int_{(i-1)h}^{ih} \left[\frac{1}{2h} \int_{\Omega^*} |v(t) - u^{(h)}(t-h)|^2 \mathrm{d}x + \|Dv(t)\|(\Omega^*) \right] \mathrm{d}t \\ &= \mathbf{F}^{(h)}[v], \end{aligned}$$

which holds true for any function v as above. In turn we used the definition of $u^{(h)}$ from (3.5), the minimizing property of u_i , and the very definition of the functional $\mathbf{F}^{(h)}$. The minimality property of $u^{(h)}$ can be re-written in the form

$$\begin{split} &\int_{0}^{T} \left\| Du^{(h)} \right\| (\Omega^{*}) \, dt \\ &\leq \int_{0}^{T} \left\| Dv \right\| (\Omega^{*}) \, dt + \frac{1}{2h} \iint_{\Omega_{T}^{*}} \left[\left| v - u^{(h)}(t-h) \right|^{2} - \left| u^{(h)} - u^{(h)}(t-h) \right|^{2} \right] \mathrm{d}x \mathrm{d}t \\ &= \int_{0}^{T} \left\| Dv \right\| (\Omega^{*}) \, dt + \frac{1}{h} \iint_{\Omega_{T}^{*}} \left[\frac{1}{2} \left| v - u^{(h)} \right|^{2} + \left(v - u^{(h)} \right) \left(u^{(h)} - u^{(h)}(t-h) \right) \right] \mathrm{d}x \mathrm{d}t \end{split}$$

for any $v \in L^2(\Omega_T^*) \cap L^1_{w*}(0,T; BV_{u_o}(\Omega))$ with $v \ge \psi^{(h)}$ a.e. in Ω_T^* . We note that for any $s \in (0,1)$, the function $w^{(h)} := u^{(h)} + s(v - u^{(h)})$ still satisfies $w^{(h)} \in L^2(\Omega_T^*) \cap L^1_{w*}(0,T; BV_{u_o}(\Omega))$ and $w^{(h)} \ge \psi^{(h)}$ a.e. in Ω_T^* . Therefore, we are allowed to replace v by $w^{(h)}$ in the preceding inequality, which yields that

$$\begin{split} \int_{0}^{T} \|Du^{(h)}\|(\Omega^{*}) \mathrm{d}t \\ &\leq \int_{0}^{T} \|Du^{(h)} + s(Dv - Du^{(h)})\|(\Omega^{*}) \, dt \\ &\quad + \frac{1}{h} \iint_{\Omega_{T}^{*}} \left[\frac{s^{2}}{2} |v - u^{(h)}|^{2} + s\left(v - u^{(h)}\right) \left(u^{(h)} - u^{(h)}(t - h)\right) \right] \mathrm{d}x \mathrm{d}t \\ &\leq \int_{0}^{T} \left[(1 - s) \|Du^{(h)}\|(\Omega^{*}) + s\|Dv\|(\Omega^{*}) \right] \mathrm{d}t \\ &\quad + \frac{1}{h} \iint_{\Omega_{T}^{*}} \left[\frac{s^{2}}{2} |v - u^{(h)}|^{2} + s\left(v - u^{(h)}\right) \left(u^{(h)} - u^{(h)}(t - h)\right) \right] \mathrm{d}x \mathrm{d}t, \end{split}$$

or equivalently (after re-absorbing the first integral appearing on the right-hand side on the left and dividing the result by s > 0)

$$\begin{split} &\int_{0}^{T} \|Du^{(h)}\|(\Omega^{*}) \mathrm{d}t \\ &\leq \int_{0}^{T} \|Dv\|(\Omega^{*}) \mathrm{d}t \\ &\quad + \frac{1}{h} \iint_{\Omega_{T}^{*}} \left[\frac{s}{2} |v - u^{(h)}|^{2} + (v - u^{(h)}) (u^{(h)} - u^{(h)}(t - h)) \right] dx dt. \end{split}$$

We extend v to negative times by letting $v(t) := v(0) \in L^2(\Omega^*)$ for t < 0. Letting $s \downarrow 0$, we find that

$$\begin{split} &\int_{0}^{T} \|Du^{(h)}\|(\Omega^{*})dt \\ &\leq \int_{0}^{T} \|Dv\|(\Omega^{*})dt + \frac{1}{h} \iint_{\Omega_{T}^{*}} \left(v - u^{(h)}\right) \left(u^{(h)} - u^{(h)}(t - h)\right) dxdt \\ &= \int_{0}^{T} \|Dv\|(\Omega^{*})dt + \frac{1}{h} \iint_{\Omega_{T}^{*}} \left(v - u^{(h)}\right) \left(v - v(t - h)\right) dxdt \\ &\quad + \frac{1}{2h} \iint_{\Omega_{T}^{*}} \left[|v - u^{(h)}|^{2}(t - h) - |v - u^{(h)}|^{2} \right] dxdt \\ &\quad - \frac{1}{2h} \iint_{\Omega_{T}^{*}} |v - v(t - h) - u^{(h)} + u^{(h)}(t - h)|^{2} dxdt \\ &\leq \int_{0}^{T} \|Dv\|(\Omega^{*})dt + \frac{1}{h} \iint_{\Omega_{T}^{*}} \left(v - u^{(h)}\right) \left(v - v(t - h)\right) dxdt \\ &\qquad - \frac{1}{2h} \iint_{\Omega^{*} \times [T - h, T]} |v - u^{(h)}|^{2} dxdt + \frac{1}{2h} \iint_{\Omega^{*} \times [-h, 0]} |v - u_{o}|^{2} dxdt \,. \end{split}$$

3.1.5. Variational inequality for the limit map. – Now we consider a comparison map $v \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$ with $\partial_t v \in L^2(\Omega^*_T)$, $v(0) \in L^2(\Omega^*)$, and $v \ge \psi$ a.e. in Ω_T . Again, we extend v to negative times by letting $v(t) := v(0) \in L^2(\Omega^*)$ for t < 0. Then we use $v_h := v + \psi^{(h)} - \psi \ge \psi^{(h)}$ as comparison map in (3.10). In the limit $h \downarrow 0$, we have

(3.11)
$$\int_{0}^{T} \|Dv_{h}\|(\Omega^{*})dt \leq \int_{0}^{T} \|Dv\|(\Omega^{*})dt + \iint_{\Omega_{T}} |D\psi^{(h)} - D\psi|dxdt$$
$$\longrightarrow \int_{0}^{T} \|Dv\|(\Omega^{*})dt.$$

For the convergence of the time term, we observe that

$$\frac{1}{h}(v_h - v_h(t-h)) \to \partial_t v + \partial_t \psi - \partial_t \psi = \partial_t v \qquad \text{strongly in } L^2(\Omega_T^*).$$

since $\partial_t v, \partial_t \psi \in L^2(\Omega_T^*)$ by assumption. Together with (3.7) and $v_h \to v$ in $L^2(\Omega_T^*)$, this implies

$$(3.12) \quad \frac{1}{h} \iint_{\Omega^* \times [0,T]} \left(v_h - u^{(h)} \right) \left(v_h - v_h(t-h) \right) \mathrm{d}x \mathrm{d}t \longrightarrow \iint_{\Omega^* \times [0,T]} \partial_t v(v-u) \mathrm{d}x \mathrm{d}t$$

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as $h \downarrow 0$. Next, we turn our attention to the last two integrals of the right-hand side in (3.10). Using the fact $v_h(t) = v(0)$ for $t \in (-h, 0)$ the last integral in (3.10) takes the form

$$\frac{1}{2h} \iint_{\Omega^* \times [-h,0]} |v - u_o|^2 \mathrm{d}x \mathrm{d}t = \frac{1}{2} \int_{\Omega^*} |v(0) - u_o|^2 \mathrm{d}x.$$

For the second last integral we observe that $v \in C^0([0,T]; L^2(\Omega^*))$ implies

$$\lim_{h \downarrow 0} \frac{1}{2h} \iint_{\Omega^* \times [T-h,T]} \left| v - v(T) \right|^2 \mathrm{d}x \mathrm{d}t = 0.$$

This allows us to replace in the second last integral of the right-hand side of (3.10) the function v by its time slice v(T). Since also $u^{(h)}(t)$ is constant on (T - h, T], it remains to consider the integral

$$\int_{\Omega^*} \left| v(T) - u^{(h)}(T) \right|^2 \mathrm{d}x.$$

At this stage we claim that $u^{(h)}(T) \rightarrow u(T)$ weakly in $L^2(\Omega^*)$. Indeed, observing that $u^{(h)}(T) = \tilde{u}^{(h)}(T)$ and $\tilde{u}^{(h)}(0) = u_o = u(0)$ we conclude for any $\eta \in L^2(\Omega^*)$ that

$$\int_{\Omega^*} u(T)\eta dx = \iint_{\Omega^*_T} \partial_t u\eta dx dt + \int_{\Omega^*} u_o \eta dx$$
$$= \lim_{h \downarrow 0} \iint_{\Omega^*_T} \partial_t \tilde{u}^{(h)} \eta dx dt + \int_{\Omega^*} u_o \eta dx$$
$$= \lim_{h \downarrow 0} \int_{\Omega^*} \tilde{u}^{(h)}(T) \eta dx$$

holds true. By lower semicontinuity we therefore have

(3.13)
$$\int_{\Omega^*} |v(T) - u(T)|^2 dx \le \liminf_{h \downarrow 0} \int_{\Omega^*} |v(T) - u^{(h)}(T)|^2 dx$$

Using (3.11), (3.12) and (3.13) in (3.10), we find after *passing to the limit* $h \downarrow 0$ that

$$\int_0^T \|Du\|(\Omega^*) dt \le \int_0^T \|Dv\|(\Omega^*) dt + \int_{\Omega_T^*} \partial_t v(v-u) dx dt$$
$$-\frac{1}{2} \|(v-u)(T)\|_{L^2(\Omega^*)}^2 + \frac{1}{2} \|v(0) - u_o\|_{L^2(\Omega^*)}^2$$

for any $v \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$ with $\partial_t v \in L^2(\Omega_T^*)$, $v(0) \in L^2(\Omega^*)$, and $v \ge \psi$ a.e. in Ω_T . This means that u is a variational solution of the obstacle problem, as claimed. The asserted energy estimate (3.2) follows by letting $h \downarrow 0$ in (3.8) and using lower semi-continuity. This concludes the proof of Theorem 3.1.

3.2. Approximation by regular obstacles: Proof of Theorem 1.2

In this section, we give the proof of the existence result from Theorem 1.2.

3.2.1. *Preparatory results.* – The existence result in Theorem 1.2 relies on the assumption that Ω is a bounded Lipschitz domain. From [2, Thm. 3.87] we know that under this assumption, we have bounded inner and outer trace operators

 $T_{\Omega}: \mathrm{BV}(\Omega) \to L^1(\partial \Omega) \qquad \text{and} \qquad T_{\mathbb{R}^n \setminus \overline{\Omega}}: \mathrm{BV}(\mathbb{R}^n \setminus \overline{\Omega}) \to L^1(\partial \Omega).$

The *inner trace* T_{Ω} is characterized by the condition

(3.14)
$$\lim_{\varrho \downarrow 0} \varrho^{-n} \int_{\Omega \cap B_{\varrho}(x)} |u(y) - T_{\Omega}u(x)| \, dy = 0$$

for \mathcal{H}^{n-1} -a.e. $x \in \partial \Omega$, and analogously for the outer trace. For the extension of a *BV*-function outside of Ω , we have

LEMMA 3.3 ([2, Cor. 3.89]). – Assume that $\Omega \subset \mathbb{R}^n$ is a domain with bounded Lipschitz boundary, $u \in BV(\Omega)$ and $v \in BV(\mathbb{R}^n \setminus \overline{\Omega})$. Then the function

$$w(x) = \begin{cases} u(x), & \text{for } x \in \Omega, \\ v(x), & \text{for } x \in \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

belongs to $BV(\mathbb{R}^n)$, and its derivative is given by the measure

$$Dw = Du + Dv + \left(T_{\Omega}u - T_{\mathbb{R}^n \setminus \overline{\Omega}}v\right)\nu_{\Omega}\mathcal{H}^{n-1} \sqcup \partial\Omega,$$

where ν_{Ω} denotes the generalized inner unit normal to Ω . In the above formula, we interpret Du and Dv as vector-valued measures on the whole \mathbb{R}^n that are concentrated in Ω , respectively in $\mathbb{R}^n \setminus \overline{\Omega}$.

The following lemma can be retrieved from [2, Prop. 3.21].

LEMMA 3.4. – Assume that $\Omega \subset \mathbb{R}^n$ is a domain with bounded Lipschitz boundary. Then there is a bounded linear extension operator $E : BV(\Omega) \to BV(\mathbb{R}^n)$ with the properties

(i) $T_{\mathbb{R}^n \setminus \overline{\Omega}}(Eu) = T_{\Omega}u$ holds \mathcal{H}^{n-1} -a.e. on $\partial\Omega$ for every $u \in BV(\Omega)$; (ii) if $u \in L^2(\Omega) \cap BV(\Omega)$ then $Eu \in L^2(\mathbb{R}^n) \cap BV(\mathbb{R}^n)$.

Proof. – The extension operator constructed in [2, Prop. 3.21] satisfies $||D(Eu)||(\partial \Omega) = 0$. In view of Lemma 3.3, this is equivalent to (i). The operator is constructed by locally flattening the boundary and then reflecting the function across the flat boundary. Hence, it can be easily seen from the construction that $Eu \in L^2(\mathbb{R}^n)$ holds for any $u \in L^2(\Omega)$. This yields (ii).

For a cut-off procedure, we will need the following parabolic variant of [17, Lemma 7.2].

LEMMA 3.5. – Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open subset with Lipschitz boundary and let $\Omega_{\varepsilon} := \{x \in \Omega : \operatorname{dist}(x, \partial \Omega) > \varepsilon\}$ for any $\varepsilon > 0$. Then for any $u \in L^1_{w*}(0, T; \operatorname{BV}(\Omega))$ we have

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \iint_{(\Omega \setminus \Omega_{\varepsilon}) \times (0,T)} |u| \mathrm{d}x \mathrm{d}t \leq \iint_{\partial \Omega \times (0,T)} |T_{\Omega}u| \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}t.$$

The proof follows by applying the arguments from [17, Lemma 7.2] separately on each time slice and using the fact $\int_0^T \|Du(t)\|(\Omega \setminus \Omega_{\varepsilon}) dt \to 0$ in the limit $\varepsilon \downarrow 0$.

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3.2.2. Regularization in space. – In this section we will give the proof of Theorem 1.2 under the additional assumption $\partial_t \psi \in L^2(\Omega^*_T)$. Without loss of generality, we may restrict ourselves to the case $\Omega^* = \mathbb{R}^n$, since otherwise, we can replace $u_o \in L^2(\Omega^*) \cap BV(\Omega^*)$ by a map in $L^2(\mathbb{R}^n) \cap BV(\mathbb{R}^n)$ that agrees with u_o in a neighborhood of $\overline{\Omega}$.

We begin by introducing the mollification procedure with respect to the spatial variable. We apply Lemma 3.4 to $\mathbb{R}^n \setminus \overline{\Omega}$ instead of Ω in order to find an extension $w_o \in L^2(\mathbb{R}^n) \cap BV(\mathbb{R}^n)$ of $u_o|_{\mathbb{R}^n\setminus\overline{\Omega}} \in L^2(\mathbb{R}^n\setminus\overline{\Omega}) \cap \mathrm{BV}(\mathbb{R}^n\setminus\overline{\Omega})$ with

(3.15)
$$T_{\Omega}w_o = T_{\mathbb{R}^n \setminus \overline{\Omega}} u_o \qquad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega$$

Then we define $\hat{u}_o \in L^2(\mathbb{R}^n \times (0,T)) \cap L^1_{w*}(0,T; BV_{u_o}(\Omega))$ by

(3.16)
$$\hat{u}_o := \max\{w_o, \psi\} \quad \text{in } \mathbb{R}^n \times (0, T).$$

As in Lemma 3.5 we write $\Omega_{\varepsilon} := \{x \in \Omega : \operatorname{dist}(x, \partial \Omega) > \varepsilon\}$ for $\varepsilon > 0$. We define a cut-off function $\eta_{\varepsilon} \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}_{>0})$ by $\eta_{\varepsilon} \equiv 0$ on $\mathbb{R}^n \setminus \Omega_{\varepsilon}, \eta_{\varepsilon} \equiv 1$ on $\Omega_{\varepsilon + \sqrt{\varepsilon}}$ and

$$\eta_{\varepsilon}(x) := \frac{\operatorname{dist}(x, \partial \Omega) - \varepsilon}{\sqrt{\varepsilon}} \qquad \text{for } \varepsilon < \operatorname{dist}(x, \partial \Omega) \le \varepsilon + \sqrt{\varepsilon}.$$

We fix a standard mollifier $\phi \in C_0^{\infty}(B_1, \mathbb{R}_{\geq 0})$ and abbreviate $\phi_{\varepsilon}(x) := \varepsilon^{-n} \phi(\frac{x}{\varepsilon})$. Then we define a mollification with respect to the spatial variables by

(3.17)
$$\mathbf{M}_{\varepsilon}[v] := \left[\eta_{\varepsilon} v + (1 - \eta_{\varepsilon}) \hat{u}_{o} \right] * \phi_{\varepsilon}$$

for any $v \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$. We list the main properties of this mollification in the following

LEMMA 3.6. – Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Then for any $v \in L^2(\mathbb{R}^n \times (0,T)) \cap L^1_{w*}(0,T; BV_{u_o}(\Omega))$ we have:

- (i) $M_{\varepsilon}[v] = M_{\varepsilon}[u_{o}]$ on $(\mathbb{R}^{n} \setminus \Omega) \times (0, T)$ for every $\varepsilon > 0$;
- (ii) if $v \ge \psi$ a.e. on Ω_T , then $M_{\varepsilon}[v] \ge M_{\varepsilon}[\psi]$ a.e. on Ω_T ;

(iii)
$$M_{\varepsilon}[v] \to v \text{ in } L^2(\mathbb{R}^n \times (0,T)) \text{ as } \varepsilon \downarrow 0;$$

(iv)
$$\int_0^1 \|D(\mathbf{M}_{\varepsilon}[v])\|(\mathbb{R}^n) dt \to \int_0^1 \|Dv\|(\mathbb{R}^n) dt \text{ in the limit } \varepsilon \downarrow 0;$$

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(v) if
$$v, \psi \in L^{\infty}(0,T; L^2(\mathbb{R}^n))$$
 then for every $\varepsilon > 0$ we have

$$\sup_{t \in [0,T]} \left\| \mathbf{M}_{\varepsilon}[v(t)] \right\|_{L^{2}(\mathbb{R}^{n})} \leq \sup_{t \in [0,T]} \left(\|v(t)\|_{L^{2}(\mathbb{R}^{n})} + \|\psi(t)\|_{L^{2}(\mathbb{R}^{n})} \right) + \|w_{o}\|_{L^{2}(\mathbb{R}^{n})};$$

(vi) if
$$\partial_t v \in L^2(\Omega_T)$$
, then $\partial_t(\mathcal{M}_{\varepsilon}[v]) \to \partial_t v$ in $L^2(\Omega_T)$ as $\varepsilon \downarrow 0$.

Proof. – The properties (i)–(iii) are straightforward consequences of the definition of M_{ε} and standard properties of mollifications. For the proof of (iv), we first note that by the lower semicontinuity of the total variation with respect to L^1 -convergence (cf. Lemma 2.4), we have

(3.18)
$$\int_0^T \|Dv\|(\mathbb{R}^n) \mathrm{d}t \le \liminf_{\varepsilon \downarrow 0} \int_0^T \|D(\mathrm{M}_\varepsilon[v])\|(\mathbb{R}^n) \mathrm{d}t$$

For the reverse inequality, we observe first that

(3.19)
$$\|D(\mathbf{M}_{\varepsilon}[v](t))\|(\mathbb{R}^n) \leq \|D[\eta_{\varepsilon}v(t) + (1-\eta_{\varepsilon})\hat{u}_o(t)]\|(\mathbb{R}^n)$$

holds for a.e. $t \in [0, T]$. In order to bound the right-hand side further, we estimate for any $\zeta \in C_0^1(\mathbb{R}^n, \mathbb{R}^n)$ with $\|\zeta\|_{L^{\infty}} \leq 1$

$$\begin{split} &\int_{\mathbb{R}^n \times \{t\}} [\eta_{\varepsilon} v + (1 - \eta_{\varepsilon}) \hat{u}_o] \operatorname{div} \zeta \mathrm{d}x \\ &= \int_{\mathbb{R}^n \times \{t\}} v \operatorname{div}(\eta_{\varepsilon} \zeta) \mathrm{d}x + \int_{\mathbb{R}^n \times \{t\}} \hat{u}_o \operatorname{div}((1 - \eta_{\varepsilon}) \zeta) \mathrm{d}x - \int_{\mathbb{R}^n \times \{t\}} (v - \hat{u}_o) \nabla \eta_{\varepsilon} \cdot \zeta \mathrm{d}x \\ &\leq \| Dv(t) \|(\Omega) + \| D\hat{u}_o(t) \| (\mathbb{R}^n \setminus \Omega_{\varepsilon + \sqrt{\varepsilon}}) + \frac{1}{\sqrt{\varepsilon}} \int_{\Omega \setminus \Omega_{\varepsilon + \sqrt{\varepsilon}}} |v - \hat{u}_o| \mathrm{d}x. \end{split}$$

Taking the supremum over all functions ζ as above and integrating with respect to time, we deduce

(3.20)

$$\int_{0}^{T} \left\| D \left[\eta_{\varepsilon} v + (1 - \eta_{\varepsilon}) \hat{u}_{o} \right] \right\| (\mathbb{R}^{n}) dt$$

$$\leq \int_{0}^{T} \| D v \| (\Omega) dt + \int_{0}^{T} \| D \hat{u}_{o} \| (\mathbb{R}^{n} \setminus \Omega_{\varepsilon + \sqrt{\varepsilon}}) dt$$

$$+ \frac{1}{\sqrt{\varepsilon}} \iint_{(\Omega \setminus \Omega_{\varepsilon + \sqrt{\varepsilon}}) \times (0, T)} |v - \hat{u}_{o}| dx dt.$$

For the last integral, we infer from Lemma 3.5 that

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon + \sqrt{\varepsilon}} \iint_{(\Omega \setminus \Omega_{\varepsilon + \sqrt{\varepsilon}}) \times (0,T)} |v - \hat{u}_o| \mathrm{d}x \mathrm{d}t \le \iint_{\partial \Omega \times (0,T)} |T_\Omega v - T_\Omega \hat{u}_o| \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}t.$$

According to Lemma 3.3 and the definition of \hat{u}_o , we have

$$\begin{split} \lim_{\varepsilon \downarrow 0} \int_0^T \|D\hat{u}_o\|(\mathbb{R}^n \setminus \Omega_{\varepsilon + \sqrt{\varepsilon}}) \mathrm{d}t \\ &= \iint_{\partial\Omega \times (0,T)} |T_\Omega \hat{u}_o - T_{\mathbb{R}^n \setminus \overline{\Omega}} u_o| \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}t + \|Du_o\|(\mathbb{R}^n \setminus \overline{\Omega}). \end{split}$$

Finally, the left-hand side of (3.20) can be estimated from below using (3.19). Consequently, from (3.20) we infer

(3.21)
$$\limsup_{\varepsilon \downarrow 0} \int_{0}^{T} \|D(\mathbf{M}_{\varepsilon}[v])\|(\mathbb{R}^{n}) dt$$
$$\leq \int_{0}^{T} \|Dv\|(\Omega) dt + \int_{0}^{T} \|Du_{o}\|(\mathbb{R}^{n} \setminus \overline{\Omega}) dt$$
$$+ \iint_{\partial\Omega \times (0,T)} \left[|T_{\Omega}v - T_{\Omega}\hat{u}_{o}| + |T_{\Omega}\hat{u}_{o} - T_{\mathbb{R}^{n} \setminus \overline{\Omega}} u_{o}| \right] d\mathcal{H}^{n-1} dt$$

Since the function $\max\{\cdot, \cdot\}$ is Lipschitz continuous on \mathbb{R}^2 , we deduce from the characterization (3.14) of the trace and (3.15) that

$$T_{\Omega}\hat{u}_o = T_{\Omega}\max\{w_o,\psi\} = \max\{T_{\Omega}w_o,T_{\Omega}\psi\} = \max\{T_{\mathbb{R}^n\setminus\overline{\Omega}}u_o,T_{\Omega}\psi\}$$

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holds \mathcal{H}^{n-1} -a.e. on $\partial\Omega$ and at a.e. time $t \in [0, T]$. Distinguishing whether $T_{\Omega}\psi$ is larger or smaller than $T_{\mathbb{R}^n\setminus\overline{\Omega}}u_o$ and using $T_{\Omega}v \ge T_{\Omega}\psi$, we thus deduce from (3.21) that

$$\begin{split} &\limsup_{\varepsilon \downarrow 0} \int_0^T \left\| D(\mathbf{M}_{\varepsilon}[v]) \right\| (\mathbb{R}^n) \mathrm{d}t \\ &\leq \int_0^T \| Dv \| (\Omega) \mathrm{d}t + \int_0^T \| Du_o \| (\mathbb{R}^n \setminus \overline{\Omega}) \mathrm{d}t + \iint_{\partial \Omega \times (0,T)} \left| T_{\Omega} v - T_{\mathbb{R}^n \setminus \overline{\Omega}} u_o \right| \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}t \\ &= \int_0^T \| Dv \| (\mathbb{R}^n) \mathrm{d}t, \end{split}$$

according to Lemma 3.3. Together with (3.18), this completes the proof of (iv).

The estimate (v) follows from

$$\begin{split} \|\mathbf{M}_{\varepsilon}[v(t)]\|_{L^{2}(\mathbb{R}^{n})} &\leq \|\eta_{\varepsilon}v(t) + (1 - \eta_{\varepsilon})\hat{u}_{o}(t)\|_{L^{2}(\mathbb{R}^{n})} \\ &\leq \|v(t)\|_{L^{2}(\mathbb{R}^{n})} + \|\psi(t)\|_{L^{2}(\mathbb{R}^{n})} + \|w_{o}\|_{L^{2}(\mathbb{R}^{n})} \end{split}$$

...

for a.e. $t \in [0, T]$.

For the remaining part (vi), we recall that η_{ε} and ϕ_{ε} are independent of time and therefore $\partial_t M_{\varepsilon}[v] = [\eta_{\varepsilon} \partial_t v + (1 - \eta_{\varepsilon}) \partial_t \hat{u}_o] * \phi_{\varepsilon}$. This implies

$$\begin{split} &\iint_{\Omega_{T}} |\partial_{t} \mathbf{M}_{\varepsilon}[v] - \partial_{t} v|^{2} \mathrm{d}x \mathrm{d}t \\ &\leq 3 \iint_{\Omega_{T}} \left| \left[\eta_{\varepsilon} \partial_{t} v - \partial_{t} v \right] * \phi_{\varepsilon} \right|^{2} + \left| \partial_{t} v * \phi_{\varepsilon} - \partial_{t} v \right|^{2} + \left| \left[(1 - \eta_{\varepsilon}) \partial_{t} \hat{u}_{o} \right] * \phi_{\varepsilon} \right|^{2} \mathrm{d}x \mathrm{d}t \\ &\leq 3 \iint_{\Omega_{T}} \left| (\eta_{\varepsilon} - 1) \partial_{t} v \right|^{2} + \left| \partial_{t} v * \phi_{\varepsilon} - \partial_{t} v \right|^{2} + \left| (1 - \eta_{\varepsilon}) \partial_{t} \hat{u}_{o} \right|^{2} \mathrm{d}x \mathrm{d}t \to 0 \end{split}$$

in the limit $\varepsilon \downarrow 0$, which completes the proof of the lemma.

3.2.3. Proof of Theorem 1.2 in the case $\partial_t \psi \in L^2(\Omega_T)$. – The proof is divided into several steps.

Step 1: Regularization. – First observe that $\psi \equiv u_o$ on $(\mathbb{R}^n \setminus \overline{\Omega}) \times (0,T)$, so that $\partial_t \psi \in L^2(\mathbb{R}^n \times (0,T))$, and also $\partial_t \psi \in L^1(\mathbb{R}^n \times (0,T))$. The same holds true for $\hat{u}_o = \max\{w_o, \psi\}$, i.e., $\hat{u}_o \in L^1(\mathbb{R}^n \times (0,T)) \cap L^2(\mathbb{R}^n \times (0,T))$. Now, for some sequence $0 < \varepsilon_i \downarrow 0$ as $i \to \infty$, we define mollifications

$$u_{o,i} := M_{\varepsilon_i}[u_o]$$
 and $\psi_i := M_{\varepsilon_i}[\psi]$

of the initial values and the obstacle, where the operator M_{ε_i} from (3.17) is applied only on the slice $\mathbb{R}^n \times \{0\}$ in the first case, and slice-wise on the time slices $\mathbb{R}^n \times \{t\}$ (exactly as in (3.17)) in the latter case. By Lemma 3.6 (iii) we have the convergences

(3.22)
$$\begin{cases} \psi_i \to \psi & \text{in } L^2(\mathbb{R}^n \times (0,T)) \text{ in the limit } i \to \infty, \\ u_{o,i} \to u_o & \text{in } L^2(\mathbb{R}^n) \text{ in the limit } i \to \infty. \end{cases}$$

Moreover, as stated in Lemma 3.6 (i), we have $\psi_i = M_{\varepsilon_i}[u_o] = u_{o,i}$ on $(\mathbb{R}^n \setminus \Omega) \times (0,T)$ and $\psi_i(0) \leq u_{o,i}$ holds a.e. on Ω . Since $\partial_t \psi_i = [\eta_{\varepsilon_i} \partial_t \psi + (1 - \eta_{\varepsilon_i}) \partial_t \hat{u}_o] * \phi_{\varepsilon_i}$ and $\partial_t \psi \in L^2(\Omega_T)$, we have $\psi_i \in W^{1,1}(\mathbb{R}^n \times (0,T))$ and

$$\partial_t \psi_i \in L^1(\mathbb{R}^n \times (0,T)) \cap L^2(\mathbb{R}^n \times (0,T)) \text{ and } \partial_t D\psi_i \in L^1(\mathbb{R}^n \times (0,T),\mathbb{R}^n)$$

for each $i \in \mathbb{N}$. Note also that $u_{o,i} \in L^2(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n)$, so that the hypotheses of Theorem 3.1 (concerning the initial and boundary values and the obstacle function) are fulfilled for $u_{o,i}$ and ψ_i . We may therefore apply Theorem 3.1 to the obstacles ψ_i and initial and boundary data $u_{o,i}$ in order to obtain solutions $u_i \in C^0([0,T]; L^2(\mathbb{R}^n)) \cap L^{\infty}_{w*}(0,T; BV_{u_{o,i}}(\Omega))$ with $\partial_t u_i \in L^2(\mathbb{R}^n \times (0,T))$ and $u_i \geq \psi_i$ a.e. on Ω_T of the variational inequalities

(3.23)
$$\int_{0}^{T} \|Du_{i}\|(\mathbb{R}^{n}) dt \leq \iint_{\mathbb{R}^{n} \times (0,T)} \partial_{t} v(v-u_{i}) dx dt + \int_{0}^{T} \|Dv\|(\mathbb{R}^{n}) dt - \frac{1}{2} \|(v-u_{i})(T)\|_{L^{2}(\mathbb{R}^{n})}^{2} + \frac{1}{2} \|v(0) - u_{o,i}\|_{L^{2}(\mathbb{R}^{n})}^{2}$$

for any comparison function $v \in C^0([0,T]; L^2(\mathbb{R}^n)) \cap L^1_{w*}(0,T; BV_{u_{o,i}}(\Omega))$ with $\partial_t v \in L^2(\mathbb{R}^n \times (0,T))$ and $v \ge \psi_i$ a.e. in Ω_T .

Step 2: Weak convergence. – The obstacle functions ψ_i fulfill the Hypothesis (2.12) by Lemma 3.6 (iii), since $\psi_i \in L^1(0,T; W^{1,1}(\mathbb{R}^n))$. Hence, we know from §2.7.1 that the map u_i is also a variational solution on any smaller cylinder Ω_{τ} for any $\tau \in (0,T]$. Therefore, applying Lemma 2.6 to the variational solution u_i and the comparison function $v = \psi_i$ we deduce the energy estimate

(3.24)
$$\sup_{t \in [0,T]} \|u_i(t)\|_{L^2(\Omega^*)}^2 + \int_0^T \|Du_i\|(\Omega^*) dt$$
$$\leq 16 T \iint_{\Omega_T} |\partial_t \psi_i|^2 dx dt + 16 \int_0^T \|D\psi_i\|(\Omega^*) dt$$
$$+ 2 \sup_{t \in [0,T]} \|\psi_i(t)\|_{L^2(\Omega^*)}^2 + 8 \|\psi_i(0) - u_{o,i}\|_{L^2(\Omega^*)}^2,$$

for any $i \in \mathbb{N}$. According to Lemma 3.6 (v), we have

$$\sup_{i\in\mathbb{N}}\sup_{t\in[0,T]}\|\psi_i(t)\|_{L^2(\mathbb{R}^n)}^2<\infty,$$

and the remaining terms on the right-hand side of (3.24) converge to the corresponding terms with ψ , u_o instead of ψ_i , $u_{o,i}$ in the limit as $i \to \infty$. Consequently, the sequence u_i is bounded in the spaces $L^{\infty}(0,T; L^2(\mathbb{R}^n))$ and $L^1_{w*}(0,T; BV(\mathbb{R}^n))$. By passing to a subsequence, we can thus ensure convergence $u_i \xrightarrow{*} u$ in $L^{\infty}(0,T; L^2(\mathbb{R}^n))$ as $i \to \infty$ for some limit map $u \in L^{\infty}(0,T; L^2(\mathbb{R}^n))$. From Lemma 2.4 we infer that $u \in L^1_{w*}(0,T; BV(\mathbb{R}^n))$ with

(3.25)
$$\int_0^{t_o} \|Du\|(\mathbb{R}^n) \mathrm{d}t \le \liminf_{i \to \infty} \int_0^{t_o} \|Du_i\|(\mathbb{R}^n) \mathrm{d}t$$

for every $t_o \in [0,T]$. Moreover, from the convergences $u_i \stackrel{*}{\to} u$ in $L^{\infty}(0,T;L^2(\mathbb{R}^n))$ and $u_{o,i} \to u_o$ in $L^2(\mathbb{R}^n)$ as $i \to \infty$ and the fact that $u_i \in L^{\infty}_{w*}(0,T;BV_{u_{o,i}}(\Omega))$ for any $i \in \mathbb{N}$ we conclude that $u \in L^1_{w*}(0,T;BV_{u_o}(\Omega))$. Combining the weak* convergence $u_i \stackrel{*}{\to} u$ in $L^{\infty}(0,T;L^2(\mathbb{R}^n))$ with (3.22) we infer

(3.26)
$$\iint_{\Omega_T} (\psi - u)_+^2 dx dt = \int_{\Omega_T} (\psi - u)(\psi - u)_+ dx dt$$
$$= \lim_{i \to \infty} \int_{\Omega_T} (\psi_i - u_i)(\psi - u)_+ dx dt \le 0,$$

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since the maps u_i satisfy the obstacle constraint with the obstacles ψ_i . We deduce that the obstacle constraint $u \ge \psi$ a.e. on Ω_T also holds for the limit map.

Step 3: Variational inequality for the limit map. – Finally, we prove that the limit map u is a variational solution. To this end, we consider an arbitrary comparison function $v \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$ with $v \ge \psi$ a.e. on Ω_T , $\partial_t v \in L^2(\Omega^*_T)$, and $v(0) \in L^2(\Omega^*)$. The mollifications $v_i := M_{\varepsilon_i}[v]$ for $i \in \mathbb{N}$ satisfy $v_i \ge \psi_i$ a.e. on Ω_T and $\partial_t v_i \in L^2(\mathbb{R}^n \times (0,T))$ by Lemma 3.6 (vi). Moreover, we have $v_i = M_{\varepsilon_i}[u_o] = u_{o,i}$ a.e. $(\mathbb{R}^n \setminus \Omega) \times (0,T)$, which implies $v_i \in L^1_{w*}(0,T; BV_{u_{o,i}}(\Omega))$. Hence, the map v_i is admissible in the variational inequality (3.23) for u_i , from which we infer

$$\int_0^\tau \|Du_i\|(\mathbb{R}^n) \, dt \le \iint_{\mathbb{R}^n \times (0,\tau)} \partial_t v_i(v_i - u_i) \, \mathrm{d}x \, \mathrm{d}t + \int_0^\tau \|Dv_i\|(\mathbb{R}^n) \, \mathrm{d}t \\ - \frac{1}{2} \|(v_i - u_i)(\tau)\|_{L^2(\mathbb{R}^n)}^2 + \frac{1}{2} \|v_i(0) - u_{o,i}\|_{L^2(\mathbb{R}^n)}^2$$

for any $\tau \in (0, T]$. Before passing to the limit $i \to \infty$, we integrate both sides of the inequality over $\tau \in (t_o, t_o + \delta) \subset (0, T)$ and take means. This provides us with the estimate

$$\int_{0}^{t_{o}} \|Du_{i}\|(\mathbb{R}^{n}) \mathrm{d}t \leq \int_{t_{o}}^{t_{o}+\delta} \left[\iint_{\mathbb{R}^{n}\times(0,\tau)} \partial_{t}v_{i}(v_{i}-u_{i})\mathrm{d}x\mathrm{d}t + \int_{0}^{\tau} \|Dv_{i}\|(\mathbb{R}^{n})\mathrm{d}t \right] \mathrm{d}\tau$$
(3.27)
$$- \frac{1}{2} \int_{t_{o}}^{t_{o}+\delta} \|(v_{i}-u_{i})(\tau)\|_{L^{2}(\mathbb{R}^{n})}^{2} \mathrm{d}\tau + \frac{1}{2} \|v_{i}(0) - u_{o,i}\|_{L^{2}(\mathbb{R}^{n})}^{2}.$$

According to (3.25) the left-hand side is lower semicontinuous. From Lemma 3.6 (iii) we obtain $v_i \rightarrow v$ in $L^2(\mathbb{R}^n \times (0,T))$, and moreover, we have weak* convergence $u_i \stackrel{*}{\rightarrow} u$ in $L^{\infty}(0,T; L^2(\mathbb{R}^n))$. This implies the weak convergence $v_i - u_i \rightarrow v - u$ in $L^2(\mathbb{R}^n \times (0,T))$, from which we infer that the second last integral on the right-hand side of (3.27) is also lower semicontinuous. Finally, the remaining terms on the right-hand side converge by Lemma 3.6 (iii), (iv) and (vi). Letting $i \rightarrow \infty$, we therefore arrive at

$$\begin{split} &\frac{1}{2} \int_{t_o}^{t_o+\delta} \|(v-u)(\tau)\|_{L^2(\mathbb{R}^n)}^2 \mathrm{d}\tau + \int_0^{t_o} \|Du\|(\mathbb{R}^n) \, dt \\ & \qquad \leq \int_{t_o}^{t_o+\delta} \left[\iint_{\mathbb{R}^n \times (0,\tau)} \partial_t v(v-u) \mathrm{d}x \mathrm{d}t + \int_0^\tau \|Dv\|(\mathbb{R}^n) \mathrm{d}t \right] \mathrm{d}\tau + \frac{1}{2} \|v(0) - u_o\|_{L^2(\mathbb{R}^n)}^2 \end{split}$$

for every $t_o \in (0,T)$ and $\delta \in (0,T-t_o]$. Letting $\delta \downarrow 0$, we deduce

(3.28)
$$\frac{1}{2} \| (v-u)(t_o) \|_{L^2(\mathbb{R}^n)}^2 + \int_0^{t_o} \| Du \| (\mathbb{R}^n) dt$$
$$\leq \iint_{\mathbb{R}^n \times (0,t_o)} \partial_t v(v-u) dx dt + \int_0^{t_o} \| Dv \| (\mathbb{R}^n) dt + \frac{1}{2} \| v(0) - u_o \|_{L^2(\mathbb{R}^n)}^2$$

for a.e. $t_o \in [0, T]$, which is the desired variational inequality. This completes the existence proof under the additional assumption $\partial_t \psi \in L^2(\Omega_T^*)$.

3.2.4. *Regularization in time: Proof of Theorem 1.2 in the general case.* – As above, we divide the proof in several steps.

Step 1: Regularization. – We consider a sequence $0 < h_i \downarrow 0$ and let $\varepsilon_i := \sqrt{h_i}$. For a standard mollifier $\phi_{\varepsilon}(x) := \varepsilon^{-n} \phi(\frac{x}{\varepsilon})$ we define mollifications of the initial values by

$$u_{o,i} := u_o * \phi_{\varepsilon_i}, \quad \psi_{o,i} := \psi(0) * \phi_{\varepsilon_i} \quad ext{and} \quad g_{o,i} := g(0) * \phi_{\varepsilon_i} \quad ext{in } \Omega_{\varepsilon_i}$$

and extend all three functions by u_o outside of Ω , so that $u_{o,i}, \psi_{o,i}, g_{o,i} \in BV_{u_o}(\Omega)$. Then we let

$$\psi_i := [\psi]_{h_i}$$
 and $g_i := [g]_{h_i}$ for any $i \in \mathbb{N}$,

where the time mollifications $[\cdot]_{h_i}$ are defined according to (2.3) with initial values $\psi_{o,i}$, respectively $g_{o,i}$. Lemma 2.1 implies

$$\psi_i \to \psi$$
 in $L^2(\mathbb{R}^n \times (0,T))$ when $i \to \infty$.

From the construction, it is clear that $g_i \ge \psi_i$ holds a.e. on Ω_T for every $i \in \mathbb{N}$. The mollified obstacles satisfy

$$\psi_i(0) = \psi_{o,i} \ge u_{o,i}$$

and $\psi_i \in L^2(\Omega_T^*) \cap L^1_{w*}(0,T; BV_{u_a}(\Omega))$, as well as

$$\partial_t \psi_i = \frac{1}{h_i} (\psi - \psi_i) \in L^2(\Omega_T^*).$$

Step 2: Solutions of the regularized problem. – We are therefore in the situation covered by the preceding \S 3.2.2, in which we already proved the existence result. We therefore obtain solutions

$$u_i \in L^{\infty}(0,T; L^2(\Omega^*)) \cap L^1_{w*}(0,T; BV_{u_0}(\Omega))$$

of the variational inequalities

(3.29)
$$\int_{0}^{\tau} \|Du_{i}\|(\Omega^{*}) dt \leq \iint_{\Omega^{*}_{\tau}} \partial_{t} v(v-u_{i}) dx dt + \int_{0}^{\tau} \|Dv\|(\Omega^{*}) dt - \frac{1}{2} \|(v-u_{i})(\tau)\|_{L^{2}(\Omega^{*})}^{2} + \frac{1}{2} \|v(0) - u_{o,i}\|_{L^{2}(\Omega^{*})}^{2}$$

for a.e. $\tau \in [0, T]$ and every $v \in L^1_{w*}(0, T; BV_{u_o}(\Omega))$ with $v \ge \psi_i$ a.e. on Ω_T , $\partial_t v \in L^2(\Omega_T^*)$, and $v(0) \in L^2(\Omega^*)$. In particular, we can apply Lemma 2.6 to the variational solutions u_i and with the comparison functions $v = g_i$ in order to deduce the estimate

(3.30)

$$\sup_{t \in [0,T]} \|u_i(t)\|_{L^2(\Omega^*)}^2 + \int_0^T \|Du_i\|(\Omega^*) dt$$

$$\leq 16 \left(\int_0^T \|\partial_t g_i(\cdot,t)\|_{L^2(\Omega)} dt \right)^2 + 16 \int_0^T \|Dg_i\|(\Omega^*) dt$$

$$+ 2 \sup_{t \in [0,T]} \|g_i(t)\|_{L^2(\Omega^*)}^2 + 8 \|g_i(0) - u_{o,i}\|_{L^2(\Omega^*)}^2.$$

From the definition of g_i , Lemma 2.3 and Lemma 2.1 with r = 1 we infer

(3.31)
$$\int_{0}^{T} \|Dg_{i}\|(\Omega^{*})dt \leq \int_{0}^{T} \left[\|Dg\|(\Omega^{*})\right]_{h_{i}}dt$$
$$\leq \int_{0}^{T} \|Dg\|(\Omega^{*})dt + h_{i}\|Dg_{o,i}\|(\Omega^{*}).$$

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For the estimate of the last term in the preceding inequality, we use Lemma 3.3 and the boundedness of the trace operator to infer that

$$\begin{split} h_i \|Dg_{o,i}\|(\Omega^*) &\leq h_i \left[\|Du_o\|(\Omega^* \setminus \overline{\Omega}) + \|D[g(0) * \phi_{\varepsilon_i}]\|(\Omega) \\ &+ \int_{\partial\Omega} |T_{\mathbb{R}^n \setminus \overline{\Omega}} u_o| + |T_{\Omega}(g(0) * \phi_{\varepsilon_i})| \mathrm{d}\mathcal{H}^{n-1} \right] \\ &\leq Ch_i \Big[\|Du_o\|(\Omega^* \setminus \overline{\Omega}) + \|D[g(0) * \phi_{\varepsilon_i}]\|(\Omega) \Big] \\ &\leq Ch_i \|Du_o\|(\Omega^* \setminus \overline{\Omega}) + \frac{Ch_i}{\varepsilon_i} \int_{\Omega^*} |g(0)| \mathrm{d}x \\ &= Ch_i \|Du_o\|(\Omega^* \setminus \overline{\Omega}) + C\sqrt{h_i} \int_{\Omega^*} |g(0)| \mathrm{d}x \longrightarrow 0 \end{split}$$

in the limit $i \to \infty$, by our choice of $\varepsilon_i = \sqrt{h_i}$. Combining this with (3.31), we deduce

(3.32)
$$\limsup_{i \to \infty} \int_0^T \|Dg_i\|(\Omega^*) \mathrm{d}t \le \int_0^T \|Dg\|(\Omega^*) \mathrm{d}t < \infty.$$

Furthermore, by the properties of the time mollification from Lemmas 2.1 and 2.2, we know that also the other terms on the right-hand side of (3.30) are bounded independently of $i \in \mathbb{N}$. We deduce that the sequence u_i is bounded in the spaces $L^{\infty}(0,T; L^2(\Omega^*))$ and $L^1_{w*}(0,T; BV(\Omega^*))$.

Step 3: Passage to the limit. - By passing to a subsequence, we can achieve convergence

(3.33)
$$u_i \stackrel{*}{\rightharpoondown} u \quad \text{in } L^{\infty}(0,T;L^2(\Omega^*)), \text{ as } i \to \infty.$$

Keeping in mind the bound (3.30), i.e., the part concerning the integrated slice-wise total variation of u_i , we deduce from Lemma 2.4 that $u \in L^1_{w*}(0,T; BV(\Omega))$ with the estimate

$$\int_0^T \|Du\|(\Omega^*) \, dt \le \liminf_{i \to \infty} \int_0^T \|Du_i\|(\Omega^*) \, dt < \infty.$$

As in (3.26) we infer that the obstacle condition is preserved under weak^{*} convergence and therefore, $u \ge \psi$ holds a.e. on Ω_T . Finally, we have to check that u is a variational solution of the obstacle problem. To this end, we consider an arbitrary comparison map $v \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$ with $\partial_t v \in L^2(\Omega_T^*)$, $v(0) \in L^2(\Omega^*)$, and $v \ge \psi$ a.e. in Ω_T . For any $i \in \mathbb{N}$, we define time mollifications $v_i := [v]_{h_i}$ according to (2.3) with initial values

$$v_{o,i} := \begin{cases} v(0) * \phi_{\varepsilon_i} \text{ in } \Omega, \\ u_o & \text{ in } \Omega^* \setminus \Omega \end{cases}$$

Since the mollifications ψ_i of ψ had been defined in exactly the same way, we know that $v_i \ge \psi_i$ holds a.e. in Ω_T for every $i \in \mathbb{N}$ and therefore, v_i is admissible as comparison map in the variational inequality (3.29) for u_i . This means that for a.e. $\tau \in [0, T]$ we have

(3.34)
$$\frac{1}{2} \| (v_i - u_i)(\tau) \|_{L^2(\Omega^*)}^2 + \int_0^\tau \| Du_i \| (\Omega^*) dt \\ \leq \iint_{\Omega^*_\tau} \partial_t v_i (v_i - u_i) dx dt + \int_0^\tau \| Dv_i \| (\Omega^*) dt + \frac{1}{2} \| v_i(0) - u_{o,i} \|_{L^2(\Omega^*)}^2.$$

Next, we note that according to Lemma 2.1 and Lemma 2.2 we have $v_i \rightarrow v$ as $i \rightarrow \infty$ in the sense that

(3.35)
$$\begin{cases} v_i \to v & \text{in } L^{\infty}(0,T;L^2(\Omega^*)), \\ \partial_t v_i \to \partial_t v & \text{in } L^1(0,T;L^2(\Omega^*)). \end{cases}$$

For the last convergence, we used the fact

$$\int_0^T \|\partial_t (e^{-\frac{t}{h_i}} (v_{o,i} - v(0)))\|_{L^2(\Omega)} \, \mathrm{d}t = \int_0^T \frac{1}{h_i} e^{-\frac{t}{h_i}} \, \mathrm{d}t \, \|v_{o,i} - v(0)\|_{L^2(\Omega)} \to 0$$

as $i \to \infty$.

Analogously to (3.32) we deduce

$$\limsup_{i\to\infty}\int_0^T\|Dv_i\|(\Omega^*)\mathrm{d} t\leq\int_0^T\|Dv\|(\Omega^*)\mathrm{d} t,$$

and since the lower semicontinuity of the total variation provides us with the reverse estimate, we infer even that

(3.36)
$$\lim_{i \to \infty} \int_0^T \|Dv_i\|(\Omega^*) \mathrm{d}t = \int_0^T \|Dv\|(\Omega^*) \mathrm{d}t$$

holds true. Now we proceed analogously as in (3.27). We take the means of both sides of the inequality (3.34) over $\tau \in (t_o, t_o + \delta)$ and let $i \to \infty$. Using the lower semicontinuity of the left-hand side and (3.33), (3.35), (3.36) and the fact $u_{o,i} \to u_o$ in $L^2(\Omega^*)$ on the right-hand side, we deduce

$$\begin{split} &\frac{1}{2} \int_{t_o}^{t_o+\delta} \|(v-u)(\tau)\|_{L^2(\Omega^*)}^2 \mathrm{d}\tau + \int_0^{t_o} \|Du\|(\Omega^*) \, dt \\ & \qquad \leq \int_{t_o}^{t_o+\delta} \left[\iint_{\Omega^*_{\tau}} \partial_t v(v-u) \mathrm{d}x \mathrm{d}t + \int_0^{\tau} \|Dv\|(\Omega^*) \mathrm{d}t \right] \mathrm{d}\tau + \frac{1}{2} \|v(0) - u_o\|_{L^2(\Omega^*)}^2 \end{split}$$

for any $t_o \in (0,T)$ and $\delta \in (0,T-t_o]$. Letting $\delta \downarrow 0$, we infer

$$\begin{split} &\frac{1}{2} \| (v-u)(t_o) \|_{L^2(\Omega^*)}^2 + \int_0^{t_o} \| Du \| (\Omega^*) \mathrm{d}t \\ & \leq \iint_{\Omega^*_{t_o}} \partial_t v(v-u) \mathrm{d}x \mathrm{d}t + \int_0^{t_o} \| Dv \| (\Omega^*) \mathrm{d}t + \frac{1}{2} \| v(0) - u_o \|_{L^2(\Omega^*)}^2 \end{split}$$

for a.e. $t_o \in [0,T]$. This means that u is the desired variational solution to the obstacle problem. This concludes the proof of the existence result from Theorem 1.2 in the general case.

3.3. Lower semicontinuous obstacles: Proof of Theorem 1.3

A further application of Theorem 3.1 leads to an existence result for the obstacle problem for the total variation flow in cases where the obstacle is given by a lower semicontinuous function.

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Proof of Theorem 1.3. – Since $\psi - u_o \colon \Omega_T \to \mathbb{R}$ is lower semicontinuous, there exists a sequence of smooth functions $\psi_i^* \in C^{\infty}(\Omega_T, \mathbb{R})$ with

 $\psi_i^*(z) \uparrow (\psi - u_o)(z)$ monotonically increasing for every $z \in \Omega_T$, as $i \to \infty$.

Since $\operatorname{spt}(\psi - u_o) \Subset \Omega_T$ by assumption, we can achieve $\operatorname{spt}(\psi_i^*) \Subset \Omega_T$ by a standard cut-off procedure. Then the regularized obstacles $\psi_i := \psi_i^* + u_o \in W^{1,1}(\Omega_T^*)$ satisfy

$$\partial_t \psi_i = \partial_t \psi_i^* \in L^2(\Omega_T^*)$$
 and $\partial_t D \psi_i = \partial_t D \psi_i^* \in L^1(\Omega_T^*)$

and

 $\psi_i \leq \psi$ on Ω_T and $\psi_i = u_o$ in a neighborhood of $\partial_{\mathscr{P}}\Omega_T$.

Consequently, Theorem 3.1 is applicable to the obstacles ψ_i and the initial and boundary values u_o , from which we obtain solutions $u_i \in C^0([0,T]; L^2(\Omega^*)) \cap L^1_{w*}(0,T; BV_{u_o}(\Omega))$ with $u_i \geq \psi_i$ a.e. on Ω_T of the variational inequalities

(3.37)
$$\int_{0}^{T} \|Du_{i}\|(\Omega^{*}) dt \leq \iint_{\Omega_{T}^{*}} \partial_{t} v(v-u_{i}) dx dt + \int_{0}^{T} \|Dv\|(\Omega^{*}) dt - \frac{1}{2} \|(v-u_{i})(T)\|_{L^{2}(\Omega^{*})}^{2} + \frac{1}{2} \|v(0) - u_{o}\|_{L^{2}(\Omega^{*})}^{2}$$

for any comparison map $v \in C^0([0,T]; L^2(\Omega^*)) \cap L^1_{w*}(0,T; BV_{u_o}(\Omega))$ with $v \ge \psi_i$ a.e. in Ω_T and $\partial_t v \in L^2(\Omega_T^*)$. From §2.7.1 (note that ψ_i fulfills the Hypothesis (2.12)) we know that this inequality is also satisfied on every smaller domain Ω_τ , for any $\tau \in (0,T)$. According to (1.5), we have $g \ge \psi \ge \psi_i$, and hence, v = g is an admissible comparison map in (3.37) for every $i \in \mathbb{N}$. From Lemma 2.6 we thus infer the energy estimate

$$\begin{split} \sup_{t \in [0,T]} & \|u_i(t)\|_{L^2(\Omega^*)}^2 + \int_0^T \|Du_i\|(\Omega^*) \mathrm{d}t \\ & \leq 16 T \iint_{\Omega_T} |\partial_t g|^2 \mathrm{d}x \mathrm{d}t + 16 \int_0^T \|Dg\|(\Omega^*) \mathrm{d}t \\ & + 2 \sup_{t \in [0,T]} \|g(t)\|_{L^2(\Omega^*)}^2 + 8 \|g(0) - u_o\|_{L^2(\Omega^*)}^2. \end{split}$$

After extraction of a subsequence we thereby get convergence

(3.38)
$$u_i \stackrel{*}{\rightharpoondown} u$$
 weakly* in $L^{\infty}(0,T;L^2(\Omega^*))$

in the limit $i \to \infty$ for a limit map $u \in L^{\infty}(0,T; L^2(\Omega^*))$ which in addition satisfies by Lemma 2.4 that $u \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$ with the estimate

$$\int_0^T \|Du\|(\Omega) dt \le \liminf_{i \to \infty} \int_0^T \|Du_i\|(\Omega) dt.$$

Using (3.38) and the monotone convergence $\psi_i \uparrow \psi$, we deduce

$$\int_{\Omega_T} (\psi - u)_+^2 \mathrm{d}z = \int_{\Omega_T} (\psi - u)(\psi - u)_+ \mathrm{d}z = \lim_{i \to \infty} \int_{\Omega_T} (\psi_i - u_i)(\psi - u)_+ \mathrm{d}z \le 0,$$

which implies that $u \ge \psi$ a.e. on Ω_T . In order to verify that u is the desired variational solution, we consider an arbitrary $v \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$ with $v \ge \psi$ a.e. on Ω_T , $\partial_t v \in L^2(\Omega_T^*)$, and $v(0) \in L^2(\Omega^*)$. Because of $v \ge \psi \ge \psi_i$ a.e. on Ω_T for every $i \in \mathbb{N}$, this map is also

admissible in the variational inequality (3.37) for u_i , on any sub-cylinder $\Omega_{\tau} \subset \Omega_T$. We therefore have

$$\begin{split} &\frac{1}{2} \| (v - u_i)(\tau) \|_{L^2(\Omega^*)}^2 + \int_0^\tau \| D u_i \| (\Omega^*) \mathrm{d}t \\ & \leq \iint_{\Omega^*_\tau} \partial_t v(v - u_i) \mathrm{d}x \mathrm{d}t + \int_0^\tau \| D v \| (\Omega^*) \mathrm{d}t + \frac{1}{2} \| v(0) - u_o \|_{L^2(\Omega^*)}^2 \end{split}$$

for every $\tau \in (0, T)$ and any $i \in \mathbb{N}$. For the passage to the limit $i \to \infty$, we take mean integrals of both sides with respect to $\tau \in (t_o, t_o + \delta)$ for some $t_o \in (0, T)$ and $\delta \in (0, T - t_o]$. Letting first $i \to \infty$ and then $\delta \downarrow 0$, we argue similarly as for the derivation of (3.28) and arrive at

$$\begin{split} &\frac{1}{2} \| (v-u)(t_o) \|_{L^2(\Omega^*)}^2 + \int_0^{t_o} \| Du \| (\Omega^*) \mathrm{d}t \\ & \leq \iint_{\Omega^*_{t_o}} \partial_t v(v-u) \mathrm{d}x \mathrm{d}t + \int_0^{t_o} \| Dv \| (\Omega^*) \mathrm{d}t + \frac{1}{2} \| v(0) - u_o \|_{L^2(\Omega^*)}^2 \end{split}$$

for a.e. $t_o \in [0, T]$. This completes the proof of the theorem.

3.4. Upper semicontinuous obstacles: Proof of Theorem 1.4

In this section, we consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ and an obstacle function $\psi \colon \Omega_T \to \mathbb{R}$ such that $\psi - u_o$ is upper semicontinuous. According to [18], the *De Giorgi* measure can be defined as follows.

DEFINITION 3.7. – For any $\varepsilon > 0$, we define a measure σ_{ε} on \mathbb{R}^n by letting

$$\sigma_{\varepsilon}(E) := \inf \left\{ P(B) + \frac{1}{\varepsilon} \mathcal{L}^n(B) : B \subset \mathbb{R}^n \text{ is open with } B \supset E \right\}$$

for every $E \subset \mathbb{R}^n$ and then

(3.39)
$$\sigma(E) = \lim_{\varepsilon \downarrow 0} \sigma_{\varepsilon}(E) = \sup_{\varepsilon > 0} \sigma_{\varepsilon}(E).$$

From the analysis in [20] it follows that σ is a Borel regular measure which satisfies

$$\sigma(E) = 2\mathcal{H}^{n-1}(E)$$

for every Borel set E that is contained in a countable union of (n - 1)-dimensional regular surfaces. However, a counterexample in [26] shows that this is not true for arbitrary Borel sets. Nevertheless, the De Giorgi measure is comparable to the Hausdorff measure in the sense that

$$C_1 \mathcal{H}^{n-1}(E) \le \sigma(E) \le C_2 \mathcal{H}^{n-1}(E)$$

holds for every subset $E \subset \mathbb{R}^n$ and dimensional constants $C_1(n), C_2(n) > 0$, cf. [20].

For a function $u \in BV(\Omega^*)$ we define the *upper approximate limit* by

$$u^{+}(x) := \inf \left\{ \lambda \in \mathbb{R} : \limsup_{r \downarrow 0} \frac{\mathcal{L}^{n}(\{y \in B_{r}(x) : u(y) > \lambda\})}{r^{n}} = 0 \right\}$$

This defines a Borel function $u^+: \Omega^* \to \mathbb{R}$, which equals the Lebesgue values of u in the approximate continuity points and the larger of the two jump values in the approximate jump points of u. Motivated by [17, §5], we define the functional

$$\mathbf{TV}_{\psi}(u) := \|Du\|(\Omega^*) + \int_{\Omega} (\psi - u^+)_+ \mathrm{d}\sigma$$

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for any $u \in BV_{u_o}(\Omega)$, where we used the abbreviation $(\psi - u^+)_+ := \max\{\psi - u^+, 0\}$. This choice of the functional is justified by [17, Thm. 7.1], which implies that the functional \mathbf{TV}_{ψ} is the relaxation of the functional $TV_{\psi} : BV_{u_o}(\Omega) \to [0, \infty]$ defined by

$$\mathrm{TV}_{\psi}(u) := \begin{cases} \int_{\Omega^*} |Du| \mathrm{d}x \text{ if } u \in W^{1,1}(\Omega), \, u = u_o \text{ on } \Omega^* \backslash \Omega \text{ and } u \ge \psi \, \mathcal{H}^{n-1} \text{-a.e.}, \\ \infty \quad \text{otherwise}, \end{cases}$$

provided this functional is not constantly infinite. To be more precise, the functional \mathbf{TV}_{ψ} is the greatest functional smaller than TV_{ψ} that is lower semicontinuous with respect to L^1 -convergence. Therefore, one of the consequences of [17, Thm. 7.1] is the fact that

(3.40)
$$\mathbf{TV}_{\psi}(u) \le \liminf_{i \to \infty} \mathbf{TV}_{\psi}(u_i)$$

holds whenever $u_i \in BV_{u_o}(\Omega)$ is a sequence with $u_i \to u \in BV_{u_o}(\Omega)$ in L^1 -norm in the limit $i \to \infty$.

REMARK 3.8. - We note that the penalization term in (1.9) is well-defined, since

$$t \mapsto \int_{\Omega \times \{t\}} (\psi - u^+)_+ \mathrm{d}\sigma$$

is a \mathcal{L}^1 -measurable function on [0, T]. This can be checked by observing that the definition (3.39) implies

$$\int_{\Omega \times \{t\}} (\psi - u^+)_+ \mathrm{d}\sigma = \lim_{\varepsilon \downarrow 0} \int_{\Omega \times \{t\}} (\psi - u^+)_+ \mathrm{d}\sigma_\varepsilon,$$

and the right-hand side depends measurably on t by Fubini's theorem, because the measures σ_{ε} are σ -finite on \mathbb{R}^n .

However, we note that Fubini's theorem is not applicable to the double integral

(3.41)
$$\int_0^\tau \left[\int_\Omega (\psi - u^+)_+ \mathrm{d}\sigma \right] \mathrm{d}t$$

that appears in the formulation (1.9) of the variational inequality in the case of thin obstacles. This becomes evident from the example of the characteristic function of the cone $C := \{(x,t) \in B_1^n \times (0,1) : |x| = t\}$, for which we have

$$\int_{B_1^n} \left[\int_0^1 \chi_C(x,t) \mathrm{d}t \right] \mathrm{d}\sigma = 0, \quad \text{but} \quad \int_0^1 \left[\int_{B_1^n} \chi_C(x,t) \mathrm{d}\sigma \right] \mathrm{d}t = \frac{2}{n} \mathcal{H}^{n-1}(\partial B_1^n).$$

Consequently, it is crucial to evaluate the integrals in (3.41) only in the specified order.

REMARK 3.9. – The solution u constructed in Theorem 1.4 may violate the obstacle constraint $u \ge \psi$. However, as a consequence of the inequality (1.9), the exceptional set

$$E := \{ (x,t) \in \Omega_T : u^+(x,t) < \psi(x,t) \} = \{ (x,t) \in \Omega_T : (\psi - u^+)_+ \neq 0 \}$$

is small in the sense that

(3.42)
$$\mathscr{H}\operatorname{-dim}(E_t) \leq n-1 \quad \text{for a.e. } t \in [0,T],$$

where \mathcal{H} -dim denotes the Hausdorff-dimension and $E_t := E \cap (\Omega \times \{t\})$. In fact, if (3.42) did not hold, the sets E_t would not be σ -finite with respect to \mathcal{H}^{n-1} and therefore

$$\int_{\Omega \times \{t\}} (\psi - u^+)_+ \mathrm{d}\sigma \ge C_1 \int_{\Omega \times \{t\}} (\psi - u^+)_+ \mathrm{d}\mathcal{H}^{n-1} = \infty$$

on a set of times with positive \mathcal{L}^1 -measure, in contradiction to the variational inequality (1.9) which guarantees the finiteness of the penalization term.

Proof of Theorem 1.4. – Step 1: Yosida regularization. – We consider the Yosida regularizations of $\varphi := \psi - u_o$, which are defined by

$$\widehat{\varphi}_k(z) := \sup_{w \in \Omega_T} \left[\varphi(w) - k |z - w| \right]$$

for each $z \in \Omega_T$ and $k \in \mathbb{N}$. Since φ is upper semicontinuous with compact support, the sequence $\widehat{\varphi}_k$ is nonincreasing with $\widehat{\varphi}_k(z) \downarrow \varphi(z)$ for every $z \in \Omega_T$ (cf. [23, Thm. 9.85]). Moreover, it is well-known that the Yosida regularization is Lipschitz continuous with $\operatorname{Lip}(\widehat{\varphi}_k) \leq k$. Finally, we observe that for each $z \in \Omega_T$ with $\operatorname{dist}(z, \operatorname{spt} \varphi) \geq \frac{1}{k} \sup \varphi$, we have $\widehat{\varphi}_k(z) = 0$, which implies

spt
$$(\widehat{\varphi}_k) \Subset \Omega_T$$
 for any $k \ge k_o$,

if we choose $k_o \in \mathbb{N}$ sufficiently large. An additional smoothing procedure of the strictly decreasing sequence $\widehat{\varphi}_k + \frac{1}{k}$ yields a sequence of smooth functions $\varphi_k \in C_0^{\infty}(\Omega_T)$ with $\varphi_k \to \varphi$ pointwise in Ω_T as $k \to \infty$, and

(3.43)
$$\varphi_{k+1} \le \varphi_k \le \widehat{\varphi}_k + \frac{2}{k} \quad \text{for any } k \in \mathbb{N}.$$

Now we apply Theorem 3.1 to the obstacles $\psi_k := \varphi_k + u_o$ for $k \ge k_o$ (observe that the hypotheses in (3.1) are satisfied by the obstacles ψ_k ; in particular $\psi_k \in W^{1,1}(\Omega_T^*)$) and infer the existence of solutions $u_k \in L^{\infty}_{w*}(0,T; BV_{u_o}(\Omega))$ with $\partial_t u_k \in L^2(\Omega_T^*)$ of the variational inequality

(3.44)
$$\int_{0}^{\tau} \|Du_{k}\|(\Omega^{*}) dt \leq \iint_{\Omega_{\tau}^{*}} \partial_{t} v(v-u_{k}) dx dt + \int_{0}^{\tau} \|Dv\|(\Omega^{*}) dt - \frac{1}{2} \|(v-u_{k})(\tau)\|_{L^{2}(\Omega^{*})}^{2} + \frac{1}{2} \|v(0) - u_{o}\|_{L^{2}(\Omega^{*})}^{2}$$

for every $\tau \in [0, T]$ and any $v \in L^1_{w*}(0, T; BV_{u_o}(\Omega))$ such that $\partial_t v \in L^2(\Omega^*_T)$, $v(0) \in L^2(\Omega^*)$, and $v \ge \psi_k$ a.e. in Ω_T . The variational inequality holds for every $\tau \in [0, T]$ as a consequence of §2.7.1, since assumption (2.12) is fulfilled by the obstacle functions $\psi_k \in W^{1,1}(\Omega^*_T)$. Since $\psi_{k+1} \le \psi_k$ for every $k \ge k_o$, the comparison principle from Lemma 2.12 implies

(3.45)
$$u_{k+1} \le u_k$$
 a.e. on Ω_T for any $k \ge k_o$.

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3.4.1. Step 2: Weak convergence. – Since ψ_{k_o} is an admissible comparison function for each u_k with $k \ge k_o$, Lemma 2.6 provides us with the energy bound

$$\begin{split} \sup_{t \in [0,T]} & \|u_k(t)\|_{L^2(\Omega^*)}^2 + \int_0^T \|Du_k\|(\Omega^*) \mathrm{d}t \\ & \leq 16 \iint_{\Omega_T} \left[T |\partial_t \psi_{k_o}|^2 + |D\psi_{k_o}| \right] \mathrm{d}x \mathrm{d}t \\ & + 2 \sup_{t \in [0,T]} \|\psi_{k_o}(t)\|_{L^2(\Omega^*)}^2 + 8 \|\psi_{k_o}(0) - u_o\|_{L^2(\Omega^*)}^2. \end{split}$$

We infer that the sequence $(u_k)_{k\geq k_o}$ is bounded in the spaces $L^{\infty}(0,T;L^2(\Omega^*))$ and $L^1_{w*}(0,T;BV_{u_o}(\Omega))$. Therefore and because of the monotonicity (3.45) we can find a subsequence $\{k_i\}$ and a limit map $u \in L^{\infty}(0,T;L^2(\Omega^*))$ with

$$\begin{cases} u_{k_i} \stackrel{*}{\rightarrow} u \text{ in } L^{\infty}(0,T;L^2(\Omega^*)), \\ \\ u_{k_i} \rightarrow u \text{ in } L^1(\Omega_T) \text{ and a.e.}, \end{cases}$$

in the limit $i \to \infty$. Lemma 2.4 implies $u \in L^1_{w*}(0, T; BV_{u_o}(\Omega))$. Since for a further subsequence, we have convergence $u_{k_i}(t) \to u(t)$ in $L^1(\Omega)$ in the limit $i \to \infty$ for a.e. $t \in [0, T]$, the lower semicontinuity (3.40) implies

$$\|Du(t)\|(\Omega^*) + \int_{\Omega \times \{t\}} (\psi - u^+)_+ \mathrm{d}\sigma$$

$$\leq \liminf_{i \to \infty} \left[\|Du_{k_i}(t)\|(\Omega^*) + \int_{\Omega \times \{t\}} (\psi - u^+_{k_i})_+ \mathrm{d}\sigma \right]$$

for a.e. $t \in [0, T]$. Next, we note that the obstacle condition $u_{k_i} \ge \psi_{k_i}$ a.e. on Ω_T implies by the definition of the approximate upper limit that $u_{k_i}^+ \ge \psi_{k_i}^+ = \psi_{k_i}$ holds everywhere on Ω for a.e. $t \in [0, T]$. For the last identity we used that ψ_{k_i} is smooth. In particular, we have that $u_{k_i}^+ \ge \psi_{k_i} \ge \psi$ holds true σ -a.e. on $\Omega \times \{t\}$, for a.e. $t \in [0, T]$. Consequently, the last integral in the preceding formula vanishes, and we deduce

$$\|Du(t)\|(\Omega^*) + \int_{\Omega \times \{t\}} (\psi - u^+)_+ \mathrm{d}\sigma \le \liminf_{i \to \infty} \|Du_{k_i}(t)\|(\Omega^*)\|$$

for a.e. $t \in [0, T]$. Integrating the preceding estimate and applying Fatou's lemma, we deduce

(3.46)
$$\int_{0}^{\tau} \|Du\|(\Omega^{*})dt + \int_{0}^{\tau} \left[\int_{\Omega} (\psi - u^{+})_{+} d\sigma\right] dt$$
$$\leq \int_{0}^{\tau} \liminf_{i \to \infty} \|Du_{k_{i}}\|(\Omega^{*})dt \leq \liminf_{i \to \infty} \int_{0}^{\tau} \|Du_{k_{i}}\|(\Omega^{*})dt$$

for every $\tau \in [0, T]$.

Step 3: Variational inequality for Lipschitz comparison maps. – For the proof of the variational inequality (1.9), we begin with a comparison map v satisfying

(3.47) $v \ge \psi$ everywhere on Ω_T and $v - u_o$ is Lipschitz continuous.

For any $k \in \mathbb{N}$ with $k \geq \operatorname{Lip}(v - u_o)$ and any $w, z \in \Omega_T$ we observe that

$$\begin{split} \varphi(w) - k|z - w| &= \psi(w) - u_o(w) - k|z - w| \le v(w) - u_o(w) - k|z - w| \\ &\le v(z) - u_o(z) + \left[\text{Lip}(v - u_o) - k \right] |z - w| \\ &\le v(z) - u_o(z) \end{split}$$

holds true. Taking the supremum over $w \in \Omega_T$ and keeping in mind (3.43), we deduce

$$v(z) + \frac{2}{k} \ge \widehat{\varphi}_k(z) + u_o(z) + \frac{2}{k} \ge \varphi_k(z) + u_o(z) = \psi_k(z)$$

for every $z \in \Omega_T$. We therefore may choose $v + \frac{2}{k_i}$ (extended by u_o on $(\Omega^* \setminus \Omega) \times (0, T)$) as comparison map in the variational inequality (3.44) for u_{k_i} , which means

$$\begin{split} \frac{1}{2} \| (v + \frac{2}{k_i} - u_{k_i})(\tau) \|_{L^2(\Omega^*)}^2 + \int_0^\tau \| Du_{k_i} \| (\Omega^*) \, dt \\ & \leq \iint_{\Omega^*_\tau} \partial_t v(v + \frac{2}{k_i} - u_{k_i}) \mathrm{d}x \mathrm{d}t + \int_0^\tau \| Dv \| (\Omega^*) \mathrm{d}t + \frac{2\tau}{k_i} \mathcal{H}^{n-1}(\partial \Omega) \\ & \quad + \frac{1}{2} \| v(0) + \frac{2}{k_i} - u_o \|_{L^2(\Omega^*)}^2 \\ & \quad \to \iint_{\Omega^*_\tau} \partial_t v(v - u) \mathrm{d}x \mathrm{d}t + \int_0^\tau \| Dv \| (\Omega^*) \mathrm{d}t + \frac{1}{2} \| v(0) - u_o \|_{L^2(\Omega^*)}^2 \end{split}$$

holds true in the limit $i \to \infty$ for every $\tau \in [0, T]$. On the left-hand side, we use the convergence $u_{k_i} \to u$ a.e. on Ω_T and Fatou's lemma in order to pass to the limit in the first term. For the second term on the left-hand side, we apply (3.46), with the result that

(3.48)
$$\frac{1}{2} \| (v-u)(\tau) \|_{L^{2}(\Omega^{*})}^{2} + \int_{0}^{\tau} \| Du \| (\Omega^{*}) dt + \int_{0}^{\tau} \left[\int_{\Omega} \left(\psi - u^{+} \right)_{+} d\sigma \right] dt \\ \leq \iint_{\Omega_{\tau}} \partial_{t} v(v-u) dx dt + \int_{0}^{\tau} \| Dv \| (\Omega^{*}) dt + \frac{1}{2} \| v(0) - u_{o} \|_{L^{2}(\Omega^{*})}^{2}$$

holds true for a.e. $\tau \in [0, T]$. This establishes the claimed inequality under the additional assumption (3.47), i.e., $v - u_o \in C^{0,1}(\Omega_T)$ and $v \ge \psi$ everywhere on Ω_T .

Step 4: Variational inequality for more general comparison maps. – Finally, we consider (as claimed in the statement of Theorem 1.4) a general comparison function $v \in L^1_{w*}(0,T; BV_{u_o}(\Omega))$ with $\partial_t v \in L^2(\Omega_T^*)$ and $v(0) \in L^2(\Omega^*)$, for which $v - u_o$ is lower semicontinuous in $\overline{\Omega}_T$ and $v \geq \psi$ holds everywhere in Ω_T . In order to approximate it with more regular comparison maps, we recall the mollification defined in (3.17). Since $u_o \in W^{1,1}(\Omega^*)$ and $\operatorname{spt}(\psi - u_o) \subseteq \Omega_T$, we can discard the complicated construction of \hat{u}_o in the definition of M_{ε} and use u_o instead. Then the definition reads

$$\mathbf{M}_{\varepsilon}[v] := \left[\eta_{\varepsilon}v + (1 - \eta_{\varepsilon})u_{o}\right] * \phi_{\varepsilon}$$

for $\varepsilon > 0$ sufficiently small, where $\phi_{\varepsilon}(x) \in C_0^{\infty}(B_{\varepsilon}^n, \mathbb{R}_{\geq 0})$ denotes a standard mollifier in space. From Lemma 3.6 we know that $M_{\varepsilon}[v] = M_{\varepsilon}[u_o] = u_o * \phi_{\varepsilon}$ holds outside of Ω_T , as well as $\partial_t(M_{\varepsilon}[v]) \to \partial_t v$ in $L^2(\Omega_T)$ as $\varepsilon \downarrow 0$, and

(3.49)
$$\lim_{\varepsilon \downarrow 0} \int_0^T \left\| D(\mathcal{M}_{\varepsilon}[v]) \right\| (\Omega^*) dt = \int_0^T \| Dv \| (\Omega^*) dt.$$

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For the mollification procedure in time, we choose a mollifying kernel $\zeta_{\varepsilon}(t) := \frac{1}{\varepsilon}\zeta(\frac{t}{\varepsilon})$ with $\zeta_{\varepsilon} \in C_0^{\infty}((-\varepsilon,\varepsilon), \mathbb{R}_{\geq 0})$ and extend v to $\Omega^* \times [-T, 2T]$ by letting v(-t) := v(t) and v(T+t) := v(T-t) for every $t \in [0,T]$. For a sequence $\varepsilon_i \downarrow 0$ we then define

$$v_i := \left[\mathbf{M}_{\varepsilon_i}[v] - \mathbf{M}_{\varepsilon_i}[u_o] + u_o \right] * \zeta_{\varepsilon_i} = \mathbf{M}_{\varepsilon_i}[v] * \zeta_{\varepsilon_i} - u_o * \phi_{\varepsilon_i} + u_o$$

where the last equality holds because u_o is independent from time. Then we have $v_i = u_o$ on $(\Omega^* \setminus \Omega)_T$, the map $v_i - u_o$ is Lipschitz continuous and we have convergence

(3.50)
$$\begin{cases} v_i \to v \quad \text{in } L^2(\Omega_T) \text{ and a.e. on } \Omega_T, \\ v_i(0) \to v(0) \text{ in } L^2(\Omega), \end{cases}$$

as well as

(3.51)
$$\partial_t v_i = \left[\partial_t \mathcal{M}_{\varepsilon_i}[v]\right] * \zeta_{\varepsilon_i} \to \partial_t v \qquad \text{in } L^2(\Omega_T)$$

in the limit $i \to \infty$. Moreover, we use standard properties of mollifications and (3.49) to deduce

(3.52)
$$\int_{0}^{T} \|Dv_{i}\|(\Omega^{*})dt \leq \int_{-\varepsilon_{i}}^{T+\varepsilon_{i}} \|DM_{\varepsilon_{i}}[v]\|(\Omega^{*})dt + T\int_{\Omega^{*}} |D(u_{o}*\phi_{\varepsilon_{i}}-u_{o})|dx - \int_{0}^{T} \|Dv\|(\Omega^{*})dt$$

in the limit $i \to \infty$. Next, we claim that for each $\kappa > 0$ there exists $i_o(\kappa) \in \mathbb{N}$ such that

(3.53)
$$\psi \le v_i + \kappa \quad \text{in } \Omega_T \text{ for any } i \ge i_o(\kappa)$$

In fact, if this did not hold, after passing to a subsequence we could find $\kappa_o > 0$ and a sequence $z_i \in \Omega_T$ so that

$$(3.54) \qquad (\psi - u_o)(z_i) > (v_i - u_o)(z_i) + \kappa_o$$
$$= \left(\left[\eta_{\varepsilon_i}(v - u_o) \right] * \phi_{\varepsilon_i} \right) * \zeta_{\varepsilon_i}(z_i) + \kappa_o$$
$$\geq \inf_{Q_{\varepsilon_i}(z_i)} \left[\eta_{\varepsilon_i}(v - u_o) \right] + \kappa_o$$

by the properties of the mollifying kernels ϕ and ζ , where we abbreviated $Q_{\varepsilon_i}(z_i) := B_{\varepsilon_i}(x_i) \times (t_i - \varepsilon_i, t_i + \varepsilon_i)$ for $z_i = (x_i, t_i)$. By passing to another subsequence we can achieve $z_i \to z \in \overline{\Omega}_T$ as $i \to \infty$. In the case $z \notin \operatorname{spt}(\psi - u_o)$, the preceding inequality implies the contradiction

$$0>\inf_{Q_{\varepsilon_i}(z_i)} \left[\eta_{\varepsilon_i}(v-u_o)\right]+\kappa_o\geq \inf_{Q_{\varepsilon_i}(z_i)} \left[\eta_{\varepsilon_i}(\psi-u_o)\right]+\kappa_o=\kappa_o>0$$

for sufficiently large $i \in \mathbb{N}$. On the other hand, in the case $z \in \operatorname{spt}(\psi - u_o) \Subset \Omega_T$, we have $\eta_{\varepsilon_i} \equiv 1$ on $Q_{\varepsilon_i}(z_i)$ for large i, so that (3.54) implies

$$\begin{split} (\psi - u_o)(z) &\geq \limsup_{i \to \infty} (\psi - u_o)(z_i) \\ &\geq \liminf_{i \to \infty} \inf_{Q_{\varepsilon_i}(z_i)} (v - u_o) + \kappa_o \\ &\geq (v - u_o)(z) + \kappa_o. \end{split}$$

Here we used the upper semicontinuity of $\psi - u_o$ and the lower semicontinuity of $v - u_o$. However, this contradicts the obstacle condition $v \ge \psi$ on Ω_T , so that we reach a contradiction in either case. This establishes the claim (3.53). Hence, we know that $v_i + \kappa$ —extended by u_o on $\Omega^* \setminus \Omega$ —is admissible as comparison map in (3.48) for every $\kappa > 0$ and $i \ge i_o(\kappa)$, from which we infer

$$\begin{split} &\frac{1}{2} \left\| (v_i + \kappa - u)(\tau) \right\|_{L^2(\Omega^*)}^2 + \int_0^\tau \|Du\|(\Omega^*) \mathrm{d}t + \int_0^\tau \left[\int_\Omega \left(\psi - u^+ \right)_+ \mathrm{d}\sigma \right] \mathrm{d}t \\ & \leq \iint_{\Omega_\tau} \partial_t v_i (v_i + \kappa - u) \mathrm{d}x \mathrm{d}t + \int_0^\tau \|Dv_i\|(\Omega^*) \mathrm{d}t + \kappa \tau \mathcal{H}^{n-1}(\partial\Omega) \\ & \quad + \frac{1}{2} \left\| v_i(0) + \kappa - u_o \right\|_{L^2(\Omega^*)}^2 \end{split}$$

for a.e. $\tau \in [0,T]$ and $i \ge i_o(\kappa)$. Letting $i \to \infty$ and using (3.50), (3.51), and (3.52), we deduce

$$\begin{split} &\frac{1}{2} \left\| (v+\kappa-u)(\tau) \right\|_{L^2(\Omega^*)}^2 + \int_0^\tau \|Du\|(\Omega^*) \mathrm{d}t + \int_0^\tau \left[\int_\Omega \left(\psi - u^+ \right)_+ \mathrm{d}\sigma \right] \mathrm{d}t \\ &\leq \iint_{\Omega_\tau} \partial_t v(v+\kappa-u) \mathrm{d}x \mathrm{d}t + \int_0^\tau \|Dv\|(\Omega^*) \mathrm{d}t + \kappa \tau \mathcal{H}^{n-1}(\partial\Omega) \\ &\quad + \frac{1}{2} \|v(0) + \kappa - u_o\|_{L^2(\Omega^*)}^2 \end{split}$$

for a.e. $\tau \in [0, T]$ and every $\kappa > 0$. Letting $\kappa \downarrow 0$, this yields the claimed inequality (1.9) and finishes the proof of the theorem.

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