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SHARP STRICHARTZ ESTIMATES FOR THE WAVE EQUATION ON A ROUGH BACKGROUND

BY JÉRÉMIE SZEFTEL

ABSTRACT. – In this paper, we obtain sharp Strichartz estimates for solutions of the wave equation $\square_{\mathbf{g}}\phi = 0$ where $\mathbf{g}$ is a rough Lorentzian metric on a 4 dimensional space-time $\mathcal{M}$. This is the last step of the proof of the bounded L^2 curvature conjecture proposed in [3], and solved by S. Klainerman, I. Rodnianski and the author in [7], which also relies on the sequence of papers [15] [16] [17] [18]. Obtaining such estimates is at the core of the low regularity well-posedness theory for quasilinear wave equations. The difficulty is intimately connected to the regularity of the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ for a rough metric $\mathbf{g}$. In order to be consistent with the final goal of proving the bounded L^2 curvature conjecture, we prove Strichartz estimates for all admissible Strichartz pairs under minimal regularity assumptions on the solutions of the eikonal equation.

RÉSUMÉ. – Dans cet article, nous obtenons des estimations de Strichartz optimales pour les solutions de l'équation des ondes $\square_{\mathbf{g}}\phi = 0$ où $\mathbf{g}$ est une métrique lorentzienne peu régulière sur un espace-temps $\mathcal{M}$ de dimension 4. Il s'agit de la dernière étape de la preuve de la conjecture de courbure L^2 proposée dans [3], et résolue par S. Klainerman, I. Rodnianski et l'auteur dans [7], qui repose également sur la série d'articles [15] [16] [17] [18]. De telles estimations sont au cœur de la théorie de l'existence locale pour les équations d'ondes non linéaires en faible régularité. La difficulté est intimement liée à la régularité de l'équation eikonale $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ pour une métrique peu régulière $\mathbf{g}$. Avec pour but final la preuve de la conjecture de courbure L^2, nous prouvons des estimations de Strichartz pour toutes les paires admissibles sous des hypothèses minimales de régularité pour l'équation eikonale.

1. Introduction

In this paper, we obtain sharp Strichartz estimates for solutions of the wave equation $\square_{\mathbf{g}}\phi = 0$ where $\mathbf{g}$ is a rough Lorentzian metric on a 4 dimensional space-time $\mathcal{M}$. This is the last step of the proof of the bounded L^2 curvature conjecture proposed in [3], and solved by S. Klainerman, I. Rodnianski and the author in [7], which also relies on the sequence of papers [15] [16] [17] [18]. Obtaining such estimates is at the core of the low regularity well-posedness theory for quasilinear wave equations. The difficulty is intimately connected to the regularity of the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ for a rough metric $\mathbf{g}$. In order to be consistent with the final goal of proving the bounded L^2 curvature conjecture, we prove Strichartz estimates for all admissible Strichartz pairs under minimal regularity assumptions on the solutions of the eikonal equation.

Since we are ultimately interested in local well-posedness, it is enough to prove local in time Strichartz estimates. Also, it is natural to prove Strichartz estimates which are localized in frequency (1). Finally, an $L^\infty_t L^2_x$ type bound in the context of the bounded L^2 curvature conjecture follows from the analysis in [16] [18], so we will assume that such a bound holds in this paper (2). Thus, we focus in this paper on the issue of proving local in time Strichartz estimates which are localized in frequency assuming an a priori $L^\infty_t L^2_x$ bound. In particular, this turns out to be sufficient for the proof of the bounded L^2 curvature conjecture.

We start by recalling the sharp Strichartz estimates for the standard wave equation on $(\mathbb{R}^{1+3}, \mathbf{m})$ where $\mathbf{m}$ is the Minkowski metric. We consider ϕ solution of

$$\begin{cases} \Box\phi = 0,\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\ \phi(0,.) = \phi_0,\ \partial_t\phi(0,.) = \phi_1, \end{cases} \tag{1.1}$$

where

$$\Box = \Box_{\mathbf{m}} = -\partial_t^2 + \Delta_x.$$

Let (p,q) such that $p,q \geq 2$, $q < +\infty$, and

$$\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}.$$

Let r defined by

$$r = \frac{3}{2} - \frac{1}{p} - \frac{3}{q}.$$

We call (p,q,r) an admissible pair. Then, the solution ϕ of (1.1) satisfies the following estimates, called Strichartz estimates [13] [14]

$$\|\phi\|_{L^p(\mathbb{R}^+, L^q(\mathbb{R}^3))} \lesssim \|\phi_0\|_{\dot{H}^r(\mathbb{R}^3)} + \|\phi_1\|_{\dot{H}^{r-1}(\mathbb{R}^3)}. \tag{1.2}$$

Strichartz estimates allow to obtain well-posedness results for nonlinear wave equations with less regularity for the Cauchy data (ϕ_0, ϕ_1) than what is typically possible by relying only on energy methods (see for example [8] in the context of semilinear wave equations). Therefore, as far as low regularity well-posedness theory for quasilinear wave equations is concerned, a considerable effort was put in trying to derive Strichartz estimates for the wave equation

$$\Box_{\mathbf{g}}\phi = 0 \tag{1.3}$$

(1) The standard proof of Strichartz estimates in the flat case proceeds in two steps (see for example [11]). First, one localizes in frequency using Littlewood-Paley theory. Then, one proves the corresponding Strichartz estimates localized in frequency.

(2) The standard proof of Strichartz estimates in the flat case relies in particular on an interpolation argument between the $L^\infty_t L^2_x$ bound and a dispersive bound. The $L^\infty_t L^2_x$ bound is usually obtained by other methods - for the wave equation in Minkowski, it follows from the conservation of energy - so we will focus in this paper on the derivation of the dispersive estimate.

on a space-time $(\mathcal{M}, \mathbf{g})$ where $\mathbf{g}$ has limited regularity, see [9], [2], [1], [19], [20], [4], [5], [10]. All these methods have in common a crucial and delicate analysis of the regularity of solutions u to the eikonal equation

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0.$$

To illustrate the role played by the eikonal equation, let us first recall the plane wave representation of the standard wave equation. The solution ϕ of (1.1) is given by:

$$(1.4)\quad \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i(-t+x\cdot\omega)\lambda}\frac{1}{2}\left(\mathcal{F}\phi_0(\lambda\omega) + i\frac{\mathcal{F}\phi_1(\lambda\omega)}{\lambda}\right)\lambda^2 d\lambda d\omega + \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i(t+x\cdot\omega)\lambda}\frac{1}{2}\left(\mathcal{F}\phi_0(\lambda\omega) - i\frac{\mathcal{F}\phi_1(\lambda\omega)}{\lambda}\right)\lambda^2 d\lambda d\omega,$$

where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^3$. The plane wave representation (1.4) is the sum of two half waves, and Strichartz estimates are derived for each half-wave separately with an identical proof so we may focus on the first half-wave which we rewrite under the form

$$(1.5)\quad \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i(-t+x\cdot\omega)\lambda} f(\lambda\omega)\lambda^2 d\lambda d\omega$$

where the function f on $\mathbb{R}^3$ is explicitly given in term of the Fourier transform of the initial data. Note that $-t + x\cdot\omega$ is a family of solutions to the eikonal equation in the Minkowski space-time depending on the extra parameter $\omega \in \mathbb{S}^2$. The natural generalization of (1.5) to the curved case is the following representation formula - also called parametrix

$$(1.6)\quad \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(t,x,\omega)} f(\lambda\omega)\lambda^2 d\lambda d\omega$$

where u is a family of solutions to the eikonal equation in the curved space-time $(\mathcal{M}, \mathbf{g})$ depending on the extra parameter $\omega \in \mathbb{S}^2$. Thus, our parametrix is a Fourier integral operator with a phase u satisfying the eikonal equation [(3)].

Assume now that the space-time $\mathcal{M}$ is foliated by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. The estimate for the parametrix (1.6) corresponding to the Strichartz estimates of the flat case (1.2) is

$$(1.7)\quad \left\|\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(t,x,\omega)} f(\lambda\omega)\lambda^2 d\lambda d\omega\right\|_{L^p(\mathbb{R}^+, L^q(\Sigma_t))} \lesssim \|\lambda^r f\|_{L^2(\mathbb{R}^3)}.$$

Since we are ultimately interested in local well-posedness, it is enough to restrict the time interval to $[0,1]$, which corresponds to local in time Strichartz estimates. Also, it is natural to prove Strichartz estimates which are localized in frequency (see footnote 1). Finally, an $L^\infty_t L^2_x$ type bound in the context of the bounded L^2 curvature conjecture follows from the analysis in [16] [18], so we will assume that such a bound holds in this paper. Thus we focus on proving Strichartz estimates on the time interval $[0,1]$ for a parametrix localized in a dyadic shell for which an a priori $L^\infty_t L^2_x$ bound is assumed. Let $j \geq 0$, and let ψ a smooth function on $\mathbb{R}$ supported in

$$\frac{1}{2} \leq \lambda \leq 2.$$

(3) We refer to [16] [18] for a precise construction of a parametrix of the form (1.6) which generates any initial data of (1.3) and for its control in the context of the bounded L^2 curvature theorem of [7]

Let φ_j the scalar function on $\mathcal{M}$ defined by the following oscillatory integral:

$$\varphi_j(t,x) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u(t,x,\omega)}\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{1.8}$$

We will prove the following version of (1.7), both localized in time and frequency

$$\|\varphi_j\|_{L^p_{[0,1]}L^q(\Sigma_t)} \lesssim 2^{jr}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)} \tag{1.9}$$

assuming that we already know that (1.9) holds in the case $(p,q,r) = (+\infty, 2, 0)$.

The Strichartz estimates (1.9) are a consequence of the oscillations of the phase u of the Fourier integral operator φ_j. Thus, one should expect to have to perform integrations by parts to obtain (1.9). In turn, this requires u to have enough regularity to be able to perform these integrations by parts. But of course, the rougher the space-time $(\mathcal{M}, \mathbf{g})$ is, the less regularity one can extract from the solution u to the eikonal equation. Our goal is to prove (1.9) in the context of the bounded L^2 curvature theorem obtained in [7]. This forces us to make assumptions on u which are compatible with the one derived in the companion papers [15] [17]. In particular, we may assume the following regularity for u

$$\partial_{t,x}u \in L^\infty,\ \partial_{t,x}\partial_\omega u \in L^\infty. \tag{1.10}$$

Now, the standard procedure for proving (1.9)—which we shall follow here—is to use the TT^* argument to reduce (1.9) to an L^1-L^∞ estimate by interpolation (4), and finally to a L^∞ estimate for an oscillatory integral with a phase involving u. One then typically uses the stationary phase to conclude the proof. This would require at the least (5)

$$\partial_{t,x}u \in L^\infty,\ \partial_{t,x}\partial_\omega^2 u \in L^\infty. \tag{1.11}$$

(1.11) involves unfortunately one more derivative than our assumptions (1.10) and we thus are forced to follow an alternative approach (6) to the stationary phase method inspired by [9] and [10] in order to prove (1.9) under the regularity assumption (1.10) for u.

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2. Assumptions on the phase $u(t,x,\omega)$ and main results

2.1. Time foliation on $\mathcal{M}$

We foliate the space-time $\mathcal{M}$ by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. We consider local in time Strichartz estimates. Thus we may assume

(4) Assuming that the $L^\infty_t L^2_x$ bound, i.e., the case $(p,q,r) = (+\infty,2,0)$ in (1.9), is already known to hold.
(5) The regularity (1.11) is necessary to make sense of the change of variables involved in the stationary phase method (see Remark 4.1).
(6) We refer to the approach based on the overlap estimates for wave packets derived in [9] and [10] in the context of Strichartz estimates respectively for $C^{1,1}$ and $H^{2+\varepsilon}$ metrics. Note however that our approach does not require a wave packet decomposition.

$0 \leq t \leq 1$ so that

$$\mathcal{M} = \bigcup_{0\leq t\leq 1} \Sigma_t. \tag{2.1}$$

We denote by T the unit, future oriented, normal to Σ_t. We also define the lapse n as

$$n^{-1} = T(t). \tag{2.2}$$

Note that we have the following identity between the volume element of $\mathcal{M}$ and the volume element corresponding to the induced metric on Σ_t

$$d\mathcal{M} = n\, d\Sigma_t\, dt. \tag{2.3}$$

We will assume the following assumption on n

$$\frac{1}{2} \leq n \leq 2 \tag{2.4}$$

which together with (2.3) yields

$$d\mathcal{M} \simeq d\Sigma_t\, dt. \tag{2.5}$$

REMARK 2.1. – *The assumption* (2.4) *is very mild. Indeed, even for the very rough space-time* $(\mathcal{M}, \mathbf{g})$ *constructed in* [7], (2.4) *is satisfied, and one has the additional regularity* $\nabla n \in L^\infty$, *where* ∇ *denotes the induced covariant derivative on* Σ_t.

REMARK 2.2. – *In the flat case, we have* $\mathcal{M} = (\mathbb{R}^{1+3}, \mathbf{m})$, *where* $\mathbf{m}$ *is the Minkowski metric, and we can take for example* $\Sigma_t = \{t\} \times \mathbb{R}^3$. *This choice yields* $n = 1$ *so that* n *satisfies* (2.4) *in this case.*

2.2. Geometry of the foliation generated by u on $\mathcal{M}$

Remember that u is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$ depending on a extra parameter $\omega \in \mathbb{S}^2$. The level hypersurfaces $u(t, x, \omega) = u$ of the optical function u are denoted by $\mathcal{H}_u$. Let L' denote the space-time gradient of u, i.e.,:

$$L' = \mathbf{g}^{\alpha\beta}\partial_\beta u \partial_\alpha. \tag{2.6}$$

Using the fact that u satisfies the eikonal equation, we obtain:

$$\mathbf{D}_{L'}L' = 0, \tag{2.7}$$

which implies that L' is the geodesic null generator of $\mathcal{H}_u$.

We have:

$$T(u) = \pm|\nabla u|$$

where $|\nabla u|^2 = \sum_{i=1}^3 |e_i(u)|^2$ relative to an orthonormal frame e_i on Σ_t. Since the sign of $T(u)$ is irrelevant, we choose by convention:

$$T(u) = -|\nabla u| \tag{2.8}$$

so that u corresponds to $-t + x \cdot \omega$ in the flat case.

Let

$$L = bL' = T + N, \tag{2.9}$$

where L' is the space-time gradient of u (2.6), b is the *lapse of the null foliation* (or shortly null lapse)

$$b^{-1} = - < L', T >= -T(u), \tag{2.10}$$

and N is a unit vector field given by

$$N = \frac{\nabla u}{|\nabla u|}. \tag{2.11}$$

Note that we have the following identities:

Lemma 2.3. – *We have*

$$L(u) = 0,\ L(\partial_\omega u) = 0 \tag{2.12}$$

and

$$\mathbf{g}(N, \partial_\omega N) = 0. \tag{2.13}$$

Proof. – Using the Definition (2.6) of L' and the fact that u satisfies the eikonal equation, we have

$$L'(u) = \mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0.$$

In view of the Definition (2.9) of L, we deduce

$$L(u) = 0. \tag{2.14}$$

Also, differentiating the eikonal equation with respect to ω yields

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta \partial_\omega u = 0$$

which yields

$$L'(\partial_\omega u) = 0$$

and thus

$$L(\partial_\omega u) = 0.$$

Together with (2.14), this implies (2.12).

Also, we have in view of the Definition (2.11) of N

$$\mathbf{g}(N, N) = 1.$$

Differentiating in ω, we obtain

$$\mathbf{g}(N, \partial_\omega N) = 0,$$

which is (2.13). This concludes the proof of the lemma. □

2.3. Regularity assumptions for $u(t,x,\omega)$

We now state our assumptions for the phase $u(t,x,\omega)$. These assumptions are compatible with the regularity obtained for the function $u(t,x,\omega)$ constructed in [17]. Let $0<\varepsilon<1$ a small enough universal constant [(7)]. b and N satisfy

$$\|b-1\|_{L^\infty}+\|\partial_\omega b\|_{L^\infty}\lesssim\varepsilon. \tag{2.15}$$

$$\|\mathbf{g}(\partial_\omega N,\partial_\omega N)-I_2\|_{L^\infty}\lesssim\varepsilon. \tag{2.16}$$

$$|N(.,\omega)-N(.,\omega')|=|\omega-\omega'|(1+O(\varepsilon)). \tag{2.17}$$

REMARK 2.4. – *In the flat case, we have $\mathcal{M}=(\mathbb{R}^{1+3},\mathbf{m})$, where $\mathbf{m}$ is the Minkowski metric, $u(t,x,\omega)=-t+x\cdot\omega$, $b=1$, $N=\omega$ and $L=\partial_t+\omega\cdot\partial_x$. Thus, the assumptions* (2.15) (2.16) (2.17) *are clearly satisfied with $\varepsilon=0$.*

REMARK 2.5. – *Note that there is a slight abuse of notations in the assumption* (2.16). *Indeed, considering the standard spherical coordinates system (θ,φ) on $\mathbb{S}^2$, we have in the flat case $N=\omega$, and hence, denoting $\partial_\omega N=(\partial_\theta N,\partial_\varphi N)$, we obtain*

$$\mathbf{g}(\partial_\omega N,\partial_\omega N)=\begin{pmatrix}1&0\\0&(\sin\theta)^2\end{pmatrix}$$

so that $\mathbf{g}(\partial_\omega N,\partial_\omega N)=I_2$ only at $\theta=\pi/2$. However, note that up to suitably choosing the axis of the spherical coordinates, one can always ensure that $\mathbf{g}(\partial_\omega N,\partial_\omega N)=I_2$ in the flat case at a given point of $\mathbb{S}^2$. Thus, the L^∞ norm in (2.16) *is taken in (t,x) and holds for an arbitrarily chosen $\omega\in\mathbb{S}^2$.* (2.16) *will be used in* (5.4) *and* (5.13)*, and in both cases, the L^∞ norm in* (2.16) *is indeed taken in (t,x) at a fixed ω.*

REMARK 2.6. – *In terms of the regularity of $u(t,x,\omega)$, the assumptions* (2.15) (2.16) *correspond to*

$$\nabla u\in L^\infty \text{ and } \nabla\partial_\omega u\in L^\infty$$

which is very weak. In particular, the classical proof for obtaining Strichartz estimates for the wave equation relies on the stationary phase for an oscillatory integral involving u as a phase, and typically requires at the least one more derivative for u (see Remark 4.1).

2.4. A global coordinate system on Σ_t

For all $0\leq t\leq 1$, and for all $\omega\in\mathbb{S}^2$, $(u(t,x,\omega),\partial_\omega u(t,x,\omega))$ is a global coordinate system on Σ_t. Furthermore, the volume element is under control in the sense that in this coordinate system, we have

$$\frac{1}{2}\leq\sqrt{\det g}\leq 2 \tag{2.18}$$

where g is the induced metric on Σ_t, and where $\det g$ denotes the determinant of the matrix of the coefficients of g.

(7) The fact that we may take ε small enough is consistent with the construction in [17] and results from a standard reduction to small data for proving well-posedness results for nonlinear wave equations (see [7] for details on this procedure in the context of the bounded L^2 curvature theorem)

REMARK 2.7. – *In the flat case, we have* $u(t,x,\omega) = -t + x\cdot\omega$ *and we can take* $\Sigma_t = \{t\}\times\mathbb{R}^3$ *so that* $(u(t,x,\omega), \partial_\omega u(t,x,\omega))$ *is clearly a global coordinate system on* Σ_t *and* $\det g = 1$ *in this case. These assumptions are also satisfied by the function* $u(t,x,\omega)$ *constructed in* [17].

2.5. Main results

We next state our main result concerning general Strichartz inequalities in mixed space-time norms of the form $L^p_{[0,1]}L^q(\Sigma_t)$ defined as follows,

$$\|F\|_{L^p_{[0,1]}L^q(\Sigma_t)} = \left(\int_0^1 \|F(t,\cdot)\|^p_{L^p(\Sigma_t)}dt\right)^{\frac{1}{p}}.$$

THEOREM 2.8. – *Let* (p,q) *such that* $p,q\geq 2$, $q<+\infty$, *and*

$$\frac{1}{p}+\frac{1}{q}\leq\frac{1}{2}.$$

Let r *defined by*

$$r = \frac{3}{2}-\frac{1}{p}-\frac{3}{q}.$$

Assume that the parametrix localized at frequency j *defined in* (1.8) *satisfies the following* $L^\infty_t L^2_x$ *bound*

$$\|\varphi_j\|_{L^\infty_{[0,1]}L^2(\Sigma_t)} \lesssim \|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}. \tag{2.19}$$

Then, φ_j *satisfies under the assumptions* (2.4), (2.15), (2.16), (2.17) *and the assumptions in Section 2.4 the following Strichartz inequality*

$$\|\varphi_j\|_{L^p_{[0,1]}L^q(\Sigma_t)} \lesssim 2^{jr}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}. \tag{2.20}$$

We also obtain the following corollary which is needed in the proof of the bounded L^2 curvature conjecture [7].

COROLLARY 2.9. – *The parametrix localized at frequency* j *defined in* (1.8) *satisfies under the assumptions* (2.4), (2.15), (2.16), (2.17), *the assumptions in Section 2.4, and* (2.19), *the following* $L^4(\mathcal{M})$ *Strichartz inequalities*

$$\|\varphi_j\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{j}{2}}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}, \tag{2.21}$$

and

$$\|\nabla\varphi_j\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{3j}{2}}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}. \tag{2.22}$$

Furthermore, assume that u *satisfies the following additional assumption*

$$\sup_{\omega\in\mathbb{S}^2, u_0\in\mathbb{R}} \|\nabla^2 u\|_{L^4({}^{(\omega)}\mathcal{H}_{u_0})} \lesssim 1, \tag{2.23}$$

where for $\omega\in\mathbb{S}^2$ *and* $u_0\in\mathbb{R}$, ${}^{(\omega)}\mathcal{H}_{u_0}$ *denotes the level hypersurface of* $u(.,\omega)$

$${}^{(\omega)}\mathcal{H}_{u_0} = \{(t,x)\,/\,u(t,x,\omega)=u_0\}.$$

Then (1.8) *satisfies the following* $L^4(\mathcal{M})$ *Strichartz inequality*

$$\|\nabla^2\varphi_j\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{5j}{2}}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}. \tag{2.24}$$

REMARK 2.10. – *Recall that the* $L_t^\infty L_x^2$ *bound* (2.19) *in the context of the bounded* L^2 *curvature conjecture follows from the analysis in* [16] [18].

REMARK 2.11. – *The additional regularity assumption* (2.23) *is compatible with the regularity obtained for the function* $u(t,x,\omega)$ *constructed in* [17]. *Note that it also holds in the flat case since we have* $u=-t+x\cdot\omega$ *and hence* $\nabla^2 u = 0$.

The rest of the paper is organized as follows. In Section 3, we use the standard TT^* argument to reduce the proof of Theorem 2.8 and Corollary 2.9 to an upper bound on the kernel K of a certain operator. This kernel is an oscillatory integral with a phase ϕ. In Section 4, we prove the upper bound on the kernel K provided we have a suitable lower bound on ϕ. Finally, in Section 5, we prove the lower bound for ϕ used in Section 4.

3. Proof of Theorem 2.8 and Corollary 2.9

3.1. Proof of Theorem 2.8

Let $a(t,x,\omega)$ a scalar function on $\mathcal{M}\times\mathbb{S}^2$. Let T_j be the operator, applied to functions $f\in L^2(\mathbb{R}^3)$,

$$T_j f(t,x) = \int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u(t,x,\omega)} a(t,x,\omega)\psi(2^{-j}\lambda) f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{3.1}$$

T_j satisfies the following estimate.

PROPOSITION 3.1. – *Let* (p,q) *such that* $p,q\geq 2$, $q<+\infty$, *and*

$$\frac{1}{p}+\frac{1}{q}\leq\frac{1}{2}.$$

Let r *defined by*

$$r=\frac{3}{2}-\frac{1}{p}-\frac{3}{q}.$$

Assume that the scalar function a *satisfies*

$$\|a\|_{L^\infty}\lesssim 1 \tag{3.2}$$

and

$$\|T_j f\|_{L^\infty_{[0,1]}L^2(\Sigma_t)}\lesssim\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}. \tag{3.3}$$

Then, the operator T_j *defined in* (3.1) *satisfies the following Strichartz inequality*

$$\|T_j f\|_{L^p_{[0,1]}L^q(\Sigma_t)}\lesssim 2^{jr}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}. \tag{3.4}$$

The proof of Proposition 3.1 is postponed to Section 3.3. Let us now conclude the proof of Theorem 2.8. Note that T_j satisfies

$$T_j f=\varphi_j \text{ if } a(t,x,\omega)=1 \text{ for all } (t,x,\omega)\in\mathcal{M}\times\mathbb{S}^2$$

where φ_j is the parametrix localized at frequency 2^j defined in (1.8). Thus, the estimate (2.20) follows immediately from (3.4). This concludes the proof of Theorem 2.8.

3.2. Proof of Corollary 2.9

Note first that (2.21) follows immediately from Theorem 2.8 by choosing $p = q = 4$ in (2.20), and noticing in view of (2.1) and (2.5) that

$$L^4_{[0,1]}L^4(\Sigma_t) = L^4(\mathcal{M}). \tag{3.5}$$

Next, we turn to the proof of the estimates (2.22) and (2.24) starting with the first one. In view of the Definition (1.8) of φ_j, we have

$$\nabla\varphi_j(t,x) = i2^j\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u(t,x,\omega)}\nabla u(t,x,\omega)(2^{-j}\lambda)\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{3.6}$$

Note that

$$\nabla\varphi_j = i2^jT_jf,$$

with $\psi(\lambda)$ replaced by $\lambda\psi(\lambda)$, and with the choice

$$a(t,x,\omega) = \nabla u(t,x,\omega).$$

Since we have $\nabla u = b^{-1}N$ in view of (2.8), (2.10) and (2.11), we deduce from the assumption (2.15) that

$$\|a\|_{L^\infty} \lesssim \|\nabla u\|_{L^\infty} \lesssim \|b^{-1}\|_{L^\infty} \lesssim 1$$

so that a satisfies the assumption (3.2). Thus, (3.4) with the choice $p = q = 4$ yields in view of (3.5)

$$\left\|\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u(t,x,\omega)}\nabla u(t,x,\omega)(2^{-j}\lambda)\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega\right\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{j}{2}}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}.$$

Together with (3.6), we obtain

$$\|\nabla\varphi_j\|_{L^4(\mathcal{M})} \lesssim 2^{\frac{3j}{2}}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}$$

which is the desired estimate (2.22).

Finally, we turn to the proof of the estimate (2.24). Differentiating (3.6), we obtain

$$\begin{aligned}\nabla_l\nabla_m\varphi_j(t,x) = \\ &-2^{2j}\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u(t,x,\omega)}\nabla_l u(t,x,\omega)\nabla_m u(t,x,\omega)(2^{-j}\lambda)^2\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega\\ &+i2^j\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u(t,x,\omega)}\nabla_l\nabla_m u(t,x,\omega)(2^{-j}\lambda)\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega.\end{aligned} \tag{3.7}$$

Next, we estimate the two terms in the right-hand side of (3.7) starting with the first one. Note that

$$\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u(t,x,\omega)}\nabla_l u(t,x,\omega)\nabla_m u(t,x,\omega)(2^{-j}\lambda)^2\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega = T_jf,$$

with $\psi(\lambda)$ replaced by $\lambda^2\psi(\lambda)$, and with the choice

$$a(t,x,\omega) = \nabla_l u(t,x,\omega)\nabla_m u(t,x,\omega).$$

Since we have $\nabla u = b^{-1}N$, we deduce from the assumption (2.15) that

$$\|a\|_{L^\infty} \lesssim \|\nabla u\|^2_{L^\infty} \lesssim \|b^{-2}\|_{L^\infty} \lesssim 1$$

so that a satisfies the assumption (3.2). Thus, (3.4) with the choice $p = q = 4$ yields in view of (3.5)

$$\left\|\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u(t,x,\omega)}\nabla_l u(t,x,\omega)\nabla_m u(t,x,\omega)(2^{-j}\lambda)^2\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega\right\|_{L^4(\mathcal{M})}$$
$$\lesssim 2^{\frac{j}{2}}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}. \tag{3.8}$$

Next, we estimate the second term in the right-hand side of (3.7). We have

$$\begin{aligned}
&\left\|\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u(t,x,\omega)}\nabla_l\nabla_m u(t,x,\omega)(2^{-j}\lambda)\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega\right\|_{L^4(\mathcal{M})} \qquad (3.9)\\
&\lesssim \int_{\mathbb{S}^2}\left\|\left(\int_0^\infty e^{i\lambda u(t,x,\omega)}(2^{-j}\lambda)\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda\right)\nabla_l\nabla_m u(t,x,\omega)\right\|_{L^4(\mathcal{M})} d\omega\\
&\lesssim \int_{\mathbb{S}^2}\left\|\int_0^\infty e^{i\lambda u}(2^{-j}\lambda)\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda\right\|_{L^4_u}\|\nabla^2 u(.,\omega)\|_{L^4(\mathcal{H}_u)}d\omega\\
&\lesssim \int_{\mathbb{S}^2}\left\|\int_0^\infty e^{i\lambda u}(2^{-j}\lambda)\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda\right\|_{L^4_u} d\omega,
\end{aligned}$$

where we used in the last inequality the assumption (2.23) on $\nabla^2 u$. Now, we have

$$\begin{aligned}
&\left\|\int_0^\infty e^{i\lambda u}(2^{-j}\lambda)\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda\right\|_{L^4_u}\\
&\lesssim \left\|\int_0^\infty e^{i\lambda u}(2^{-j}\lambda)\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda\right\|^{\frac{1}{2}}_{L^\infty_u}\\
&\quad\times\left\|\int_0^\infty e^{i\lambda u}(2^{-j}\lambda)\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda\right\|^{\frac{1}{2}}_{L^2_u}\\
&\lesssim \left(2^{\frac{j}{2}}\|\psi(2^{-j}\lambda)\lambda^2 f\|_{L^2_\lambda}\right)^{\frac{1}{2}}\|\psi(2^{-j}\lambda)\lambda^2 f\|^{\frac{1}{2}}_{L^2_\lambda}\\
&\lesssim 2^{\frac{j}{4}}\|\psi(2^{-j}\lambda)\lambda^2 f\|_{L^2_\lambda}
\end{aligned}$$

where we used Cauchy-Schwartz in λ to evaluate the L^∞_u norm and Plancherel to evaluate the L^2_u norm. In view of (3.9), this yields [(8)]

$$\begin{aligned}
&\left\|\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u(t,x,\omega)}\nabla_l\nabla_m u(t,x,\omega)(2^{-j}\lambda)\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega\right\|_{L^4(\mathcal{M})}\\
&\lesssim 2^{\frac{j}{4}}\int_{\mathbb{S}^2}\|\psi(2^{-j}\lambda)\lambda^2 f\|_{L^2_\lambda}d\omega\\
&\lesssim 2^{\frac{5j}{4}}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)} \qquad (3.10)
\end{aligned}$$

where we used Cauchy-Schwarz in ω in the last inequality. Together with (3.7) and (3.8), we finally obtain

$$\|\nabla^2\varphi_j\|_{L^4(\mathcal{M})}\lesssim 2^{\frac{5j}{2}}\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}$$

(8) In the proof of (2.24), note that (3.10) leads to a $2^{\frac{j}{4}}$ room while (3.8) is sharp. One should compare with the corresponding estimates in the flat case where (3.8) is sharp while the analog of (3.10) would display a 2^j room. Some of this room - i.e., $2^{\frac{3j}{4}}$ of the 2^j room - is exploited to obtain (3.10).

which is the desired estimate (2.24). This concludes the proof of Corollary 2.9.

3.3. Proof of Proposition 3.1 (the TT^* argument)

We start with the following remark.

REMARK 3.2. – *Fixing a global system of coordinates $x = (x^1, x^2, x^3)$ in Σ_t, such as the one described in Section 2.4, we note in view of* (2.18) *that* (3.4) *is equivalent with the same inequality where the norm $L^q(\Sigma_t)$ on the left-hand side is replaced by the corresponding euclidean norm in the given coordinates. More precisely we can assume from now on that*

$$\|F\|_{L^p_{[0,1]}L^q(\Sigma_t)} = \left(\int_0^1 \left(\int_{\mathbb{R}^3} |F(t,x)|^q dx\right)^{\frac{p}{q}} dt\right)^{\frac{1}{q}}$$

which we will denote by a slight abuse of notation by

$$\|F\|_{L^p_{[0,1]}L^q(\mathbb{R}^3)}.$$

Note also that in the (t,x) coordinates $\mathcal{M} = [0,1] \times \mathbb{R}^3$.

To prove Proposition 3.1, we rely on the standard TT^* argument for the Fourier integral operator (3.1). Note that the operator T_j^* takes real valued functions h on $\mathcal{M}$ to complex valued functions on $\mathbb{R}^3$

$$T_j^* h(\lambda\omega) = \psi(2^{-j}\lambda) \int_{\mathcal{M}} a(s,y,\omega) e^{-i\lambda u(s,y,\omega)} h(s,y) ds dy.$$

Therefore, the operator $U_j := T_j T_j^*$ is given by the formula,

$$U_j h(t,x) = \int_{\mathbb{S}^2}\int_0^\infty \int_{\mathcal{M}} e^{i\lambda u(t,x,\omega) - i\lambda u(s,y,\omega)} a(t,x,\omega) a(s,y,\omega) \psi(2^{-j}\lambda)^2 h(s,y) \lambda^2 d\lambda d\omega ds dy.$$

Note, in view of Remark 3.2, that (3.4) is equivalent to the following estimate

$$\|U_j h\|_{L^p_{[0,1]}L^q(\mathbb{R}^3)} \lesssim 2^{2jr} \|h\|_{L^{p'}_{[0,1]}L^{q'}(\mathbb{R}^3)}, \tag{3.11}$$

where p' (resp. q') is the conjugate exponent to p (resp. q). Observe that,

$$\begin{aligned} U_j h\left(\frac{t}{2^j}, \frac{x}{2^j}\right) = 2^{-j} \int_{\mathbb{S}^2}\int_0^\infty \int_{2^j\mathcal{M}} & e^{i\lambda 2^j u\left(\frac{t}{2^j}, \frac{x}{2^j}, \omega\right) - i\lambda 2^j u\left(\frac{s}{2^j}, \frac{y}{2^j}, \omega\right)} a\left(\frac{t}{2^j}, \frac{x}{2^j}, \omega\right) a\left(\frac{s}{2^j}, \frac{y}{2^j}, \omega\right) \\ & \times \psi(\lambda)^2 h\left(\frac{s}{2^j}, \frac{y}{2^j}\right) \lambda^2 d\lambda d\omega ds dy \end{aligned}$$

with $2^j\mathcal{M} = [0, 2^j] \times \mathbb{R}^3$ relative to the rescaled variables (s,y). Thus, setting,

$$\begin{aligned} Ah(t,x) := \int_{\mathbb{S}^2}\int_0^\infty \int_{2^j\mathcal{M}} & e^{i\lambda 2^j u\left(\frac{t}{2^j}, \frac{x}{2^j}, \omega\right) - i\lambda 2^j u\left(\frac{s}{2^j}, \frac{y}{2^j}, \omega\right)} a\left(\frac{t}{2^j}, \frac{x}{2^j}, \omega\right) a\left(\frac{s}{2^j}, \frac{y}{2^j}, \omega\right) \\ & \times \psi(\lambda)^2 h(s,y) \lambda^2 d\lambda d\omega ds dy \end{aligned}$$

we have

$$U_j h\left(\frac{t}{2^j}, \frac{x}{2^j}\right) = 2^{-j} A h_j(t,x), \;\; h_j(s,y) = h\left(\frac{s}{2^j}, \frac{y}{2^j}\right).$$

We easily infer that (3.11) is equivalent to the estimate,

$$\|Ah\|_{L^p_{[0,2^j]}L^q(\mathbb{R}^3)} \lesssim \|h\|_{L^{p'}_{[0,2^j]}L^{q'}(\mathbb{R}^3)}. \tag{3.12}$$

REMARK 3.3. – *Note in view of the assumption* (3.3) *that* (3.12) *holds in the particular case* $(p,q)=(+\infty,2)$

$$\|Ah\|_{L^\infty_{[0,2^j]}L^2(\mathbb{R}^3)}\lesssim\|h\|_{L^1_{[0,2^j]}L^2(\mathbb{R}^3)}. \tag{3.13}$$

We introduce the kernel K of A

$$K(t,x,s,y)=\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda 2^j u\left(\frac{t}{2^j},\frac{x}{2^j},\omega\right)-i\lambda 2^j u\left(\frac{s}{2^j},\frac{y}{2^j},\omega\right)}a\left(\frac{t}{2^j},\frac{x}{2^j},\omega\right)a\left(\frac{s}{2^j},\frac{y}{2^j},\omega\right)\times\psi(\lambda)^2\lambda^2 d\lambda d\omega. \tag{3.14}$$

REMARK 3.4. – *In the flat case, we have* $u(t,x,\omega)=-t+x\cdot\omega$ *so that*

$$2^j u\left(\frac{t}{2^j},\frac{x}{2^j},\omega\right)=u(t,x,\omega).$$

In particular, in the case $a=1$, K *is independent of* j

$$K(t,x,s,y)=\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda u(t,x,\omega)-i\lambda u(s,y,\omega)}\psi(\lambda)^2\lambda^2 d\lambda d\omega.$$

We have the following proposition.

PROPOSITION 3.5. – *The kernel* K *of the operator* A *satisfies the dispersive estimates,*

$$|K(t,x,s,y)|\lesssim\frac{1}{|t-s|},\quad \forall(t,x)\in 2^j\mathcal{M},\ \ \forall(s,y)\in 2^j\mathcal{M}. \tag{3.15}$$

The proof of Proposition 3.5 is postponed to Section 4. We now conclude the proof of Proposition 3.1. (3.12) follows from (3.15) and (3.13) using interpolation and the Hardy-Littlewood inequality according to the standard procedure, see for example [11] and [12]. Finally, in view of the discussion above, (3.12) yields (3.11) which in turn implies (3.4). This concludes the proof of Proposition 3.1.

4. Proof of Proposition 3.5 (bound on the kernel K)

Let ϕ the scalar function on $\mathcal{M}\times\mathcal{M}\times\mathbb{S}^2$ defined as

$$\phi(t,x,s,y,\omega)=u(t,x,\omega)-u(s,y,\omega). \tag{4.1}$$

In view of (3.14), we may rewrite K as

$$K(t,x,s,y)=\int_{\mathbb{S}^2}\int_0^\infty e^{i\lambda 2^j\phi\left(\frac{t}{2^j},\frac{x}{2^j},\frac{s}{2^j},\frac{y}{2^j},\omega\right)}a\left(\frac{t}{2^j},\frac{x}{2^j},\omega\right)a\left(\frac{s}{2^j},\frac{y}{2^j},\omega\right)\psi(\lambda)^2\lambda^2 d\lambda d\omega.$$

After integrating by parts twice in λ, and using the assumption (3.2) on a and the size of the support of ψ, this yields

$$|K(t,x,s,y)|\lesssim\int_{\mathbb{S}^2}\frac{1}{1+2^{2j}\phi\left(\frac{t}{2^j},\frac{x}{2^j},\frac{s}{2^j},\frac{y}{2^j},\omega\right)^2}d\omega. \tag{4.2}$$

The next section is dedicated to the obtention of a lower bound on $|\phi|$ which will allow us to deduce (3.15) from (4.2).

REMARK 4.1. – *It is at this stage that we depart from the standard strategy for proving Strichartz estimates. Indeed, the usual method consists in using the stationary phase method to derive* (3.15)*. To this end, one considers the neighborhood in* $\mathbb{S}^2$ *of stationary points* ω_0*, i.e., such that* $\partial_\omega \phi_{|\omega=\omega_0} = 0$*. One then needs an identity of the type*

$$\phi = (s-t)A(\omega-\omega_0)\cdot(\omega-\omega_0) + o\left((s-t)(\omega-\omega_0)^2\right) \tag{4.3}$$

for ω *in the neighborhood of* ω_0 *and for some* 3×3 *invertible matrix* A*.* (4.3) *then allows to perform a change of variables in* ω *which ultimately leads to* (3.15)*. In particular, the standard method requires at the least* [9] $\partial_{t,x}\partial_\omega^2 u \in L^\infty$ *just to derive* (4.3)*.*

Our assumptions correspond only to $\partial_{t,x}\partial_\omega u \in L^\infty$*. Thus, in order to obtain* (3.15)*, we instead integrate by parts in* λ *to obtain* (4.2)*, and then look for a suitable lower bound on* $|\phi|$*. In particular, we obtain lower bounds of the following type (see details in Lemma 4.9)*

$$|\phi| \gtrsim |s-t||\omega-\omega_0|^2 \tag{4.4}$$

for ω *in the neighborhood of some* $\omega_0 \in \mathbb{S}^2$*. The fundamental observation is that, as it turns out, the inequality* (4.4) *requires less regularity than the equality* (4.3)*.*

4.1. The key lemma

Let (t,x) and (s,y) in $\mathcal{M}$, and let $\omega \in \mathbb{S}^2$. In this section, we obtain a lower bound on $\phi(t,x,s,y,\omega)$. We may assume

$$0 \leq t < s \leq 1.$$

DEFINITION 4.2. – *For any* $\omega \in \mathbb{S}^2$ *and* $\sigma \in \mathbb{R}$*, let* $\gamma_\omega(\sigma)$ *denote the null geodesic parametrized by proper time and with initial data*

$$\gamma_\omega(0) = (t,x),\ \gamma'_\omega(0) = b^{-1}(t,x,\omega)L(t,x,\omega).$$

Recall from (2.7) and (2.9) that $b^{-1}L$ is geodesic. Thus, for any $\omega \in \mathbb{S}^2$ and any $\sigma \in \mathbb{R}$, we have

$$u(\gamma_\omega(\sigma),\omega) = u(t,x,\omega),\ \gamma'_\omega(\sigma) = b^{-1}(\gamma_\omega(\sigma),\omega)L(\gamma_\omega(\sigma),\omega). \tag{4.5}$$

DEFINITION 4.3. – *For any* (t,x)*, let us define the subset* $S_{t,x,s}$ *of* Σ_s *as*

$$S_{t,x,s} = \bigcup_{\omega\in\mathbb{S}^2} \{\gamma_\omega(s-t)\}. \tag{4.6}$$

Note that $S_{t,x,s}$ *depends on* (t,x) *since all geodesics* γ_ω *originate from* (t,x)*. To ease the notations, we drop the indices* t,x,s *in the rest of the paper and simply refer to this set as* S*.*

We also define for all $(s,z) \in \Sigma_s$

$$m(s,z) = \max_{\omega\in\mathbb{S}^2}(u(s,z,\omega) - u(t,x,\omega)). \tag{4.7}$$

We have the following lemma characterizing the zeros of m.

LEMMA 4.4. – *We have*

$$S = \{p \in \Sigma_s,\ /\ m(p) = 0\}.$$

[9] One also needs to take care of the contribution to K of the angles $\omega \in \mathbb{S}^2$ corresponding to the exterior of the neighborhood of stationary points which may increase the needed regularity.

The proof of Lemma 4.4 is postponed to Appendix A. Next, we define the following two subsets of Σ_s

$$A_{\text{int}} = \{p \in \Sigma_s \,/\, m(p) < 0\},\ A_{\text{ext}} = \{p \in \Sigma_s \,/\, m(p) > 0\}. \tag{4.8}$$

Note in view of Lemma 4.4 that

$$\Sigma_s = S \sqcup A_{\text{int}} \sqcup A_{\text{ext}}. \tag{4.9}$$

REMARK 4.5. – *In the flat case, the picture is the following:*

1. *The null geodesics*[10] γ_ω *span the light cone from* (t,x)*. In particular, the null geodesics* γ_ω *do not intersect except at* (t,x)*.*
2. S *is the intersection*[11] *of the forward light cone from* (t,x) *with* $\{s\}\times\mathbb{R}^3$*.*
3. A_{int} *and* A_{ext} *correspond respectively to the interior and the exterior of* S*.*

Note that we do not need to prove these statements in our case. This is fortunate since these statements—while probably true in our general setting—would be delicate to establish (see for instance [6] *for a proof of (1) on a space-time* $(\mathcal{M},\mathbf{g})$ *with limited regularity).*

Next, we introduce some further notations. First, we denote by m_0 the value of m at (s,y), i.e.,

$$m_0 = \max_{\omega\in\mathbb{S}^2}(u(s,y,\omega) - u(t,x,\omega)). \tag{4.10}$$

We also denote by ω_0 an angle in $\mathbb{S}^2$ where the maximum in (4.10) is achieved, i.e.,

$$m_0 = u(s,y,\omega_0) - u(t,x,\omega_0). \tag{4.11}$$

REMARK 4.6. – *In the flat case,* ω_0 *is unique and corresponds to the angle of the projection of* (s,y) *on* S*. Again, while this may be also true in our general setting, we do not need to prove this statement in our case.*

Note that if $(s,y)\in A_{\text{ext}}$, the function $u(s,y,\omega) - u(t,x,\omega)$ may change sign as ω varies on $\mathbb{S}^2$. We define

$$D = \{\omega \in \mathbb{S}^2 \,/\, u(t,x,\omega) = u(s,y,\omega)\}. \tag{4.12}$$

The following lemma gives a precise description of D.

LEMMA 4.7. – *Let* $(s,y)\in A_{\text{ext}}$*. Let* D *defined as in* (4.12)*. Let* (θ,φ) *denote the spherical coordinates with axis* ω_0*. Then, there exists a* C^1 2π*-periodic function*

$$\theta_1 : [0,2\pi) \to (0,\pi)$$

such that in the coordinate system (θ,φ)*,* D *is parametrized by*

$$D = \{\theta = \theta_1(\varphi),\ 0 \le \varphi < 2\pi\}.$$

The proof of Lemma 4.7 is postponed to Appendix B.

REMARK 4.8. – *In the flat case, recall that* $u(t,x,\omega) = -t + x\cdot\omega$*. In this case, one easily checks that* D *is a circle of axis* ω_0 *on the sphere* S *which is generated by the tangents to* S *through* y *(see Figure 1).*

[10] Which are straight lines in this case
[11] S is a sphere in this case

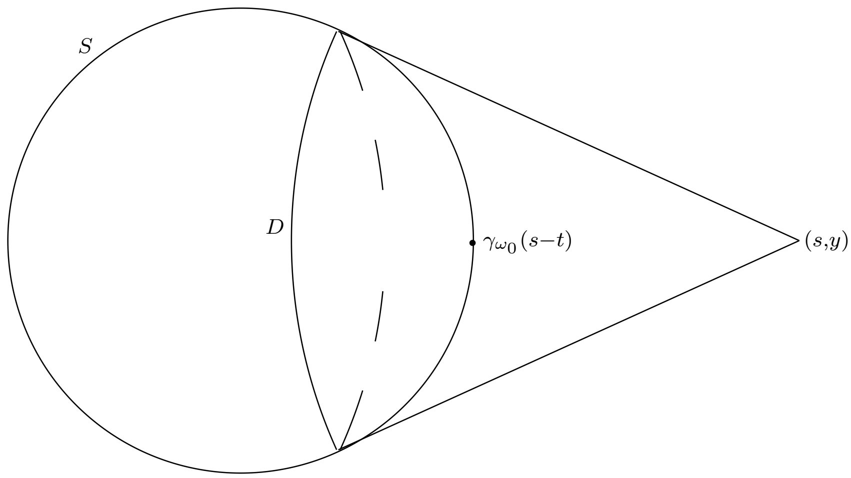

FIGURE 1. Representation of D in the flat case

Let $\omega \in \mathbb{S}^2$. According to Lemma 4.7, the great half circle on $\mathbb{S}^2$ originating at ω_0 and containing ω intersects D at a fixed point ω_1. Let θ and θ_1 respectively denote the positive angles between ω_0 and ω (resp. ω_0 and ω_1).

In order to obtain a lower bound for $|\phi|$, we will argue differently according to whether (s, y) belongs to the region S, $A_{\rm int}$ or $A_{\rm ext}$.

LEMMA 4.9 (Key lemma). – *$|\phi|$ satisfies the following lower bounds*

1. *If $(s, y) \in S$, we have*

$$|\phi(t, x, s, y, \omega)| \geq \frac{1}{4}|t - s||\omega - \omega_0|^2. \tag{4.13}$$

2. *If $(s, y) \in A_{\rm int}$, we have*

$$|\phi(t, x, s, y, \omega)| \geq \frac{1}{8}|t - s||\omega - \omega_0|^2. \tag{4.14}$$

3. *If $(s, y) \in A_{\rm ext}$ and $\theta_1 \leq \theta \leq \pi$, we have*

$$|\phi(t, x, s, y, \omega)| \geq \frac{1}{4}|t - s||\omega - \omega_1|^2. \tag{4.15}$$

4. *If $(s, y) \in A_{\rm ext}$ and $0 \leq \theta \leq \theta_1$, we have*

$$|\phi(t, x, s, y, \omega)| \gtrsim \sqrt{\frac{1 - \cos(\theta - \theta_1)}{1 - \cos(\theta_1)}}m_0. \tag{4.16}$$

The proof of Lemma 4.9 is postponed to Section 5.

REMARK 4.10. – *The proof of Lemma 4.9 is inspired by the overlap estimates for wave packets derived in* [9] *and* [10] *in the context of Strichartz estimates respectively for $C^{1,1}$ and $H^{2+\varepsilon}$ metrics. Note however that the estimates in these papers rely heavily on a direct comparison of various quantities with the corresponding ones in the flat case. Such direct comparisons do not hold in our framework. Here, the closeness to the flat case manifests itself* *in the small constant ε in the right-hand side of* (2.15)*,* (2.16) *and* (2.17)*, and in the existence of the global coordinates systems of Section 2.4.*

4.2. Proof of Proposition 3.5

Recall that we need to show that the kernel K defined in (3.14) satisfies the upper bound (3.15). To this end, we will use the estimate (4.2) for K together with the estimates provided by Lemma 4.9. We argue differently according to whether (s,y) belongs to S, $A_{\rm int}$ or $A_{\rm ext}$.

4.2.1. *The case* $(s,y)\in S$. – If (s,y) belongs to S, we have the lower bound (4.13) for $|\phi|$

$$|\phi(t,x,s,y,\omega)|\geq\frac{1}{4}|t-s||\omega-\omega_0|^2,$$

where $\omega_0\in\mathbb{S}^2$ is an angle satisfying (4.11). Then, we deduce

$$2^j\left|\phi\left(\frac{t}{2^j},\frac{x}{2^j},\frac{s}{2^j},\frac{y}{2^j},\omega\right)\right|\geq\frac{1}{4}|t-s||\omega-\omega_0|^2 \text{ for all } \left(\frac{s}{2^j},\frac{y}{2^j}\right)\in S.$$

Together with (4.2), this yields

$$|K(t,x,s,y)|\lesssim\int_{\mathbb{S}^2}\frac{d\omega}{1+|t-s|^2|\omega-\omega_0|^4}.$$

Using the spherical coordinates (θ,φ) with axis ω_0, we obtain

$$|K(t,x,s,y)|\lesssim\int_0^\pi\frac{\sin(\theta)d\theta}{1+|t-s|^2(1-\cos(\theta))^2}.$$

Performing the change of variables

$$z=|t-s|(1-\cos(\theta))$$

we obtain

$$|K(t,x,s,y)|\lesssim\frac{1}{|t-s|}\int_0^{+\infty}\frac{dz}{1+z^2}.$$

This implies

$$|K(t,x,s,y)|\lesssim\frac{1}{|t-s|},\ \ \forall(t,x)\in 2^j\mathcal{M},\ \ \forall\left(\frac{s}{2^j},\frac{y}{2^j}\right)\in S \tag{4.17}$$

which is the desired estimate.

4.2.2. *The case* $(s,y)\in A_{\rm int}$. – If (s,y) belongs to $A_{\rm int}$, we have the lower bound (4.14) for $|\phi|$

$$|\phi(t,x,s,y,\omega)|\geq\frac{1}{8}|t-s||\omega-\omega_0|^2,$$

where $\omega_0\in\mathbb{S}^2$ is an angle satisfying (4.11). Arguing as in the previous case, we obtain

$$|K(t,x,s,y)|\lesssim\frac{1}{|t-s|},\ \ \forall(t,x)\in 2^j\mathcal{M},\ \ \forall\left(\frac{s}{2^j},\frac{y}{2^j}\right)\in A_{\rm int}. \tag{4.18}$$

4.2.3. *The case* $(s,y)\in A_{\mathrm{ext}}$. – If (s,y) belongs to A_{ext}, recall that ω_1 is in D such that ω, ω_1 and ω_0 are on the same half great circle of $\mathbb{S}^2$, and that θ and θ_1 denote respectively the positive angles between ω_0 and ω (resp. ω_0 and ω_1).

The case $\theta_1\leq\theta\leq\pi$. If $\theta_1\leq\theta\leq\pi$, we have the lower bound (4.15) for $|\phi|$

$$|\phi(t,x,s,y,\omega)|\geq\frac{1}{2}|t-s||\omega-\omega_1|^2.$$

Then, we deduce

$$2^j\left|\phi\left(\frac{t}{2^j},\frac{x}{2^j},\frac{s}{2^j},\frac{y}{2^j},\omega\right)\right|\geq\frac{1}{2}|t-s||\omega-\omega_1|^2.$$

Together with (4.2), this yields

$$|K(t,x,s,y)|\lesssim\int_{\mathbb{S}^2}\frac{d\omega}{1+|t-s|^2|\omega-\omega_1|^4}.$$

Using the spherical coordinates (θ,φ) with axis ω_0, we parametrize $\mathbb{S}^2$ by $(\theta-\theta_1(\varphi),\varphi)$ where $\varphi\to\theta_1(\varphi)$ is defined in Lemma 4.7. We obtain

$$|K(t,x,s,y)|\lesssim\int_0^{2\pi}\int_{\theta_1(\varphi)}^{\pi}\frac{\sin(\theta-\theta_1(\varphi))}{1+|t-s|^2(1-\cos(\theta-\theta_1(\varphi)))^2}d\theta d\varphi$$

and thus

$$|K(t,x,s,y)|\lesssim\int_0^{\pi}\frac{\sin(\theta)}{1+|t-s|^2(1-\cos(\theta))^2}d\theta.$$

Performing the change of variable

$$z=|t-s|(1-\cos(\theta))$$

we obtain

$$|K(t,x,s,y)|\lesssim\frac{1}{|t-s|}\int_0^{+\infty}\frac{dz}{1+z^2}.$$

This implies

$$|K(t,x,s,y)|\lesssim\frac{1}{|t-s|},\ \ \forall(t,x)\in 2^j\mathcal{M},\ \ \forall\left(\frac{s}{2^j},\frac{y}{2^j}\right)\in A_{\mathrm{ext}}\text{ with }\theta_1\leq\theta\leq\pi\tag{4.19}$$

which is the desired estimate.

The case $0\leq\theta\leq\theta_1$. Finally, if $0\leq\theta\leq\theta_1$, we have the lower bound (4.16) for $|\phi|$

$$|\phi(t,x,s,y,\omega)|\gtrsim\sqrt{\frac{1-\cos(\theta-\theta_1)}{1-\cos(\theta_1)}}m_0.$$

We then deduce

$$2^j\left|\phi\left(\frac{t}{2^j},\frac{x}{2^j},\frac{s}{2^j},\frac{y}{2^j},\omega\right)\right|\gtrsim 2^j\sqrt{\frac{1-\cos(\theta-\theta_1)}{1-\cos(\theta_1)}}m_j$$

where m_j is defined as

$$m_j=\max_{\omega\in\mathbb{S}^2}\left(u\left(\frac{s}{2^j},\frac{y}{2^j},\omega\right)-u\left(\frac{t}{2^j},\frac{x}{2^j},\omega\right)\right).$$

Together with (4.2), this yields

$$|K(t,x,s,y)| \lesssim \int_0^{\theta_1} \frac{\sin(\theta)}{1+2^{2j}m_j^2\frac{1-\cos(\theta-\theta_1)}{1-\cos(\theta_1)}}d\theta.$$

Performing the change of variable

$$z = 2^j m_j\sqrt{\frac{1-\cos(\theta-\theta_1)}{1-\cos(\theta_1)}},$$

noticing that

$$\sin(\theta)d\theta = \frac{2\sin(\theta_1)\sqrt{1-\cos(\theta_1)}}{2^j m_j}\frac{\sin(\theta)}{\sin(\theta_1)}\frac{\sqrt{1-\cos(\theta-\theta_1)}}{\sin(\theta-\theta_1)}dz$$

and using (5.20) and the fact that

$$\frac{\sqrt{1-\cos(\theta_1-\theta)}}{\sin(\theta_1-\theta)} \lesssim 1 \text{ and } \frac{\sin(\theta)}{\sin(\theta_1)} \lesssim 1 \text{ on } 0\leq\theta\leq\theta_1\leq\frac{\pi}{2}+O(\varepsilon),$$

we obtain

$$|K(t,x,s,y)| \lesssim \frac{\sin(\theta_1)\sqrt{1-\cos(\theta_1)}}{2^j m_j}\int_0^{+\infty}\frac{dz}{1+z^2} \lesssim \frac{\sin(\theta_1)\sqrt{1-\cos(\theta_1)}}{2^j m_j}. \tag{4.20}$$

Now, in view of (5.19), we have

$$\sin(\theta_1)\sqrt{1-\cos(\theta_1)} \lesssim \frac{1}{1+\frac{|t-s|}{2^j m_j}}.$$

Together with (4.20), we obtain

$$|K(t,x,s,y)| \lesssim \frac{1}{2^j m_j + |t-s|}$$

which implies

$$|K(t,x,s,y)| \lesssim \frac{1}{|t-s|},\ \ \forall (t,x)\in 2^j\mathcal{M},\ \ \forall\left(\frac{s}{2^j},\frac{y}{2^j}\right)\in A_{\text{ext}} \text{ with } 0\leq\theta\leq\theta_1. \tag{4.21}$$

Finally, (4.9), (4.17), (4.18), (4.19) and (4.21) yield (3.15) which concludes the proof of Proposition 3.5.

5. Proof of Lemma 4.9 (Lower bound for $|\phi|$)

5.1. A lower bound for $|\phi|$ when $(s,y)\in S$ (proof of (4.13))

In view of the definition of S, there is $\omega_0\in\mathbb{S}^2$ such that

$$(s,y) = \gamma_{\omega_0}(s-t).$$

In view of (4.5), this yields

$$\begin{aligned}(5.1)\quad u(s,y,\omega) - u(t,x,\omega) &= u(\gamma_{\omega_0}(s-t),\omega) - u(t,x,\omega)\\ &= \int_0^{s-t} \mathbf{g}(\mathbf{D}u, \gamma'_{\omega_0}(\sigma))d\sigma\\ &= \int_0^{s-t} b^{-1}(\gamma_{\omega_0}(\sigma),\omega)b^{-1}(\gamma_{\omega_0}(\sigma),\omega_0)\mathbf{g}(L(\gamma_{\omega_0}(\sigma),\omega), L(\gamma_{\omega_0}(\sigma),\omega_0))d\sigma\\ &= -\frac{1}{2}\int_0^{s-t} b^{-1}(\gamma_{\omega_0}(\sigma),\omega)b^{-1}(\gamma_{\omega_0}(\sigma),\omega_0)|N(\gamma_{\omega_0}(\sigma),\omega) - N(\gamma_{\omega_0}(\sigma),\omega_0)|^2 d\sigma\\ &\le -\frac{1}{4}|t-s||\omega-\omega_0|^2,\end{aligned}$$

where we used in the last inequality the estimates (2.15) and (2.17) with $\varepsilon > 0$ small enough. (5.1) implies for all $(s,y) \in S$ and all $\omega \in \mathbb{S}^2$

$$(5.2)\qquad |\phi(t,x,s,y,\omega)| \ge \frac{1}{4}|t-s||\omega-\omega_0|^2,$$

which is the desired estimate (4.13).

5.2. A lower bound for $|\phi|$ when $(s,y) \in A_{\mathrm{int}}$ (proof of (4.14))

Recall that $m_0 < 0$ since $(s,y) \in A_{\mathrm{int}}$. Let $\omega_0 \in \mathbb{S}^2$ an angle satisfying (4.11). Then, we have in particular

$$\partial_\omega u(s,y,\omega_0) = \partial_\omega u(t,x,\omega_0).$$

Together with (2.12), this yields

$$(5.3)\qquad \partial_\omega u(s,y,\omega_0) = \partial_\omega u(\gamma_{\omega_0}(s-t),\omega_0).$$

In view of the assumption (2.16), the 2×2 matrix

$$\mathbf{g}(\partial_\omega N, \partial_\omega N)$$

is invertible. We define the map a_0 from Σ_s to $T\mathbb{S}^2$ as

$$(5.4)\qquad a_0 = \mathbf{g}(\partial_\omega N(\omega_0,.), \partial_\omega N(\omega_0,.))^{-1}\partial_\omega b(\omega_0,.).$$

Note in view of (2.15) and (2.16) that a_0 satisfies the estimate

$$(5.5)\qquad \|a_0\|_{L^\infty} \lesssim \varepsilon.$$

For $\sigma \in \mathbb{R}$, let us consider the curve $\mu(\sigma)$ defined by

$$(5.6)\qquad \begin{cases}\mu'(\sigma) = b(\mu(\sigma),\omega_0)N(\mu(\sigma),\omega_0) + a_0(\mu(\sigma))\cdot\partial_\omega N(\mu(\sigma),\omega_0),\\ \mu(0) \;= \gamma_{\omega_0}(s-t).\end{cases}$$

REMARK 5.1. – *In the flat case, the curve μ is simply the segment of straight line between $\gamma_{\omega_0}(s-t)$ and (s,y) (see Figure 2).*

LEMMA 5.2. – *Let μ the curve defined in* (5.6). *Then, we have*

$$(5.7)\qquad (s,y) = \mu(m_0).$$

The proof of Lemma 5.2 is postponed to Appendix C. (5.7) yields

$$\begin{aligned}u(s,y,\omega)-u(t,x,\omega) &= u(\mu(m_0),\omega)-u(\mu(0),\omega)+u(\gamma_{\omega_0}(s-t),\omega)-u(t,x,\omega)\\ &= \int_0^{m_0}\mathbf{g}(\nabla u(\mu(\sigma),\omega),\mu'(\sigma))d\sigma+\int_t^s \mathbf{g}(\mathbf{D}u(\gamma_{\omega_0}(\sigma),\omega),\gamma'_{\omega_o}(\sigma))d\sigma\\ &= \int_0^{m_0} b^{-1}(\mu(\sigma),\omega)\Big(b(\mu(\sigma),\omega_0)\mathbf{g}(N(\mu(\sigma),\omega),N(\mu(\sigma),\omega_0))\\ &\quad + a_0(\mu(\sigma))\cdot\mathbf{g}(\partial_\omega N(\mu(\sigma),\omega_0),N(\mu(\sigma),\omega))\Big)\\ &\quad + \int_t^s b^{-1}(\gamma_{\omega_0}(\sigma),\omega)b^{-1}(\gamma_{\omega_0}(\sigma),\omega_0)\mathbf{g}(L(\gamma_{\omega_0}(\sigma),\omega),L(\gamma_{\omega_0}(\sigma),\omega_0)).\end{aligned}$$

We obtain

$$\begin{aligned}u(s,y,\omega)-u(t,x,\omega) = &-\int_0^{|m_0|}\Big(\mathbf{g}(N(\mu(\sigma),\omega),N(\mu(\sigma),\omega_0))+O(\varepsilon)\Big)d\sigma\\ &-\frac{1}{2}\int_t^s |N(\gamma_{\omega_0}(\sigma),\omega)-N(\gamma_{\omega_0}(\sigma),\omega_0)|^2(1+O(\varepsilon))d\sigma\end{aligned}$$

where we used the estimates (2.15) and (5.5), the fact that $s>t$, and the fact that $m_0<0$ since $(s,y)\in A_{\text{int}}$. Together with (2.17), this yields

$$u(s,y,\omega)-u(t,x,\omega) = -\frac{1}{2}|t-s||\omega-\omega_0|^2(1+O(\varepsilon))-|m_0|(\omega\cdot\omega_0+O(\varepsilon)). \tag{5.8}$$

In particular we deduce for $\varepsilon>0$ small enough

$$u(s,y,\omega)-u(t,x,\omega)\leq -\frac{1}{4}|t-s||\omega-\omega_0|^2 \text{ for all } \omega \text{ such that } \omega\cdot\omega_0\geq\frac{1}{4}. \tag{5.9}$$

We now consider the case $\omega\cdot\omega_0\leq 1/4$. Since ω_0 is an angle where the maximum in the Definition (4.10) of m_0 is attained, we have for all $\omega\in\mathbb{S}^2$, in view of (4.11) and (5.8), and

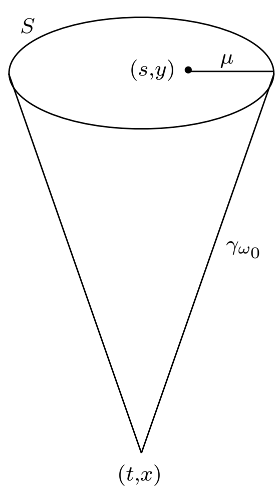

FIGURE 2. The case $(s,y)\in A_{\text{int}}$ in the flat case

the fact that $m_0 < 0$

$$-|m_0| \geq -\frac{1}{2}|t-s||\omega-\omega_0|^2(1+O(\varepsilon)) - |m_0|(\omega\cdot\omega_0 + O(\varepsilon)).$$

This yields

$$|m_0| \leq \frac{1}{2}|t-s||\omega-\omega_0|^2\frac{1+O(\varepsilon)}{1-\omega\cdot\omega_0+O(\varepsilon)}.$$

Injecting back in (5.8), we obtain for $\varepsilon>0$ small enough

$$u(s,y,\omega) - u(t,x,\omega) \leq -\frac{1}{8}|t-s||\omega-\omega_0|^2 \text{ for all } \omega \text{ such that } \omega\cdot\omega_0 \leq \frac{1}{4}.$$

Together with (5.9), we finally obtain for all $\omega\in\mathbb{S}^2$

$$|\phi(t,x,s,y,\omega)| \geq \frac{1}{8}|t-s||\omega-\omega_0|^2 \tag{5.10}$$

which is the desired estimate (4.14).

5.3. A lower bound for $|\phi|$ when $(s,y)\in A_{\text{ext}}$ (proof of (4.15) (4.16))

Let m_0 defined in (4.10). Let $\omega_0\in\mathbb{S}^2$ an angle satisfying (4.11). Note that $m_0>0$ since $(s,y)\in A_{\text{ext}}$. In particular, proceeding as in Section 5.2, we obtain the following analog of (5.8)

$$u(s,y,\omega) - u(t,x,\omega) = -\frac{1}{2}|t-s||\omega-\omega_0|^2(1+O(\varepsilon)) + m_0(\omega\cdot\omega_0+O(\varepsilon)). \tag{5.11}$$

Recall the Definition (4.12) of the set D

$$D = \{\omega\in\mathbb{S}^2 \,/\, u(t,x,\omega) = u(s,y,\omega)\}.$$

In view of (5.11), if $\omega_1\in D$, then

$$1-\omega_1\cdot\omega_0 = \frac{1+O(\varepsilon)}{1+\frac{|t-s|}{m_0}(1+O(\varepsilon))}. \tag{5.12}$$

Next, we consider $\omega_1\in D$. In view of the assumption (2.16), the 2×2 matrix

$$\mathbf{g}(\partial_\omega N, \partial_\omega N)$$

is invertible. We define the map a_1 from Σ_s to $T_{\omega_1}\mathbb{S}^2$ as

$$a_1 = \mathbf{g}(\partial_\omega N(\omega_1,.), \partial_\omega N(\omega_1,.))^{-1}\left(\partial_\omega u(s,y,\omega_1) - \partial_\omega u(\gamma_{\omega_1}(s-t),\omega_1)\right). \tag{5.13}$$

Let us consider the curve $\eta(\sigma)$ defined by

$$\begin{cases} \eta'(\sigma) = b(\eta(\sigma),\omega_1)a_1(\mu(\sigma))\cdot\partial_\omega N(\eta(\sigma),\omega_1), \\ \eta(0) \;\; = \gamma_{\omega_1}(s-t). \end{cases} \tag{5.14}$$

REMARK 5.3. – *In the flat case, the curve η is simply the segment of straight line between $\gamma_{\omega_1}(s-t)$ and (s,y) (see Figure 3).*

LEMMA 5.4. – *Let η the curve defined in* (5.14). *Then, we have*

$$(s,y) = \eta(1). \tag{5.15}$$

The proof of Lemma 5.4 is postponed to Appendix D. (5.15) yields

$$\begin{aligned}
u(s,y,\omega)-u(t,x,\omega) &= u(\eta(1),\omega)-u(\eta(0),\omega)+u(\gamma_{\omega_1}(s-t),\omega)-u(t,x,\omega)\\
&=\int_0^1 \mathbf{g}(\nabla u(\eta(\sigma),\omega),\eta'(\sigma))d\sigma+\int_t^s \mathbf{g}(\mathbf{D}u(\gamma_{\omega_1}(\sigma),\omega),\gamma'_{\omega_1}(\sigma))d\sigma\\
&=\int_0^1 b^{-1}(\eta(\sigma),\omega)b(\eta(\sigma),\omega_1)\mathbf{g}(N(\eta(\sigma),\omega),a_1\cdot\partial_\omega N(\eta(\sigma),\omega_1))\\
&\quad+\int_t^s b^{-1}(\gamma_{\omega_1}(\sigma),\omega)b^{-1}(\gamma_{\omega_1}(\sigma),\omega_1)\mathbf{g}(L(\gamma_{\omega_1}(\sigma),\omega),L(\gamma_{\omega_1}(\sigma),\omega_1)).
\end{aligned}$$

We obtain

$$\begin{aligned}
u(s,y,\omega)-u(t,x,\omega) = &\int_0^1 \mathbf{g}(N(\eta(\sigma),\omega)-N(\eta(\sigma),\omega_1),a_1\cdot\partial_\omega N(\eta(\sigma),\omega_1))(1+O(\varepsilon))\\
&-\frac{1}{2}\int_t^s |N(\gamma_{\omega_1}(\sigma),\omega)-N(\gamma_{\omega_1}(\sigma),\omega_1)|^2(1+O(\varepsilon))d\sigma
\end{aligned}$$

where we used the estimates (2.15) and the identity (2.13). Together with the assumptions (2.16) and (2.17), this yields

(5.16)

$$\begin{aligned}
u(s,y,\omega)-u(t,x,\omega) = &-\frac{1}{2}|t-s||\omega-\omega_1|^2(1+O(\varepsilon))\\
&+(\omega-\omega_1)\cdot(\partial_\omega u(s,y,\omega_1)-\partial_\omega u(\gamma_{\omega_1}(s-t),\omega_1))(1+O(\varepsilon)).
\end{aligned}$$

We introduce the notation v_1 for the following vector in $\mathbb{R}^3$.

$$v_1=\partial_\omega u(s,y,\omega_1)-\partial_\omega u(\gamma_{\omega_1}(s-t),\omega_1). \tag{5.17}$$

Recall that $\omega_0\in\mathbb{S}^2$ is an angle satisfying (4.11). In view of (5.16), this yields

$$m_0=-\frac{1}{2}|t-s||\omega_0-\omega_1|^2(1+O(\varepsilon))+(\omega_0-\omega_1)\cdot v_1(1+O(\varepsilon)).$$

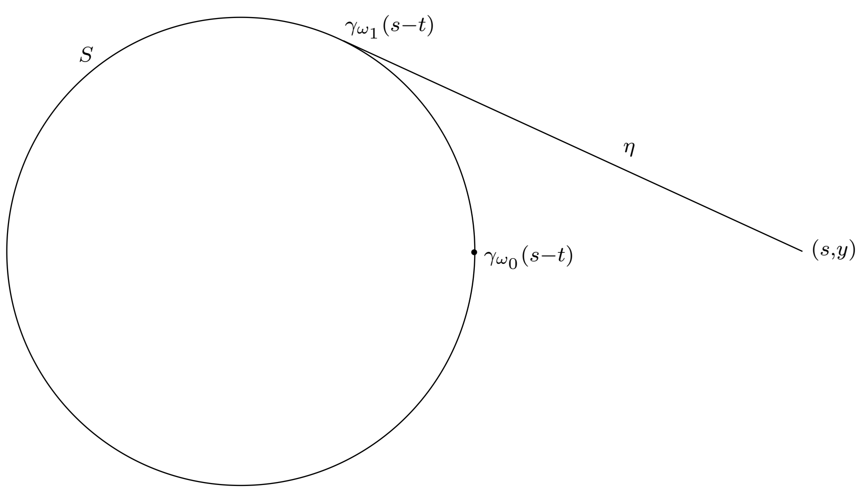

FIGURE 3. The case $(s,y)\in A_{\text{ext}}$ in the flat case

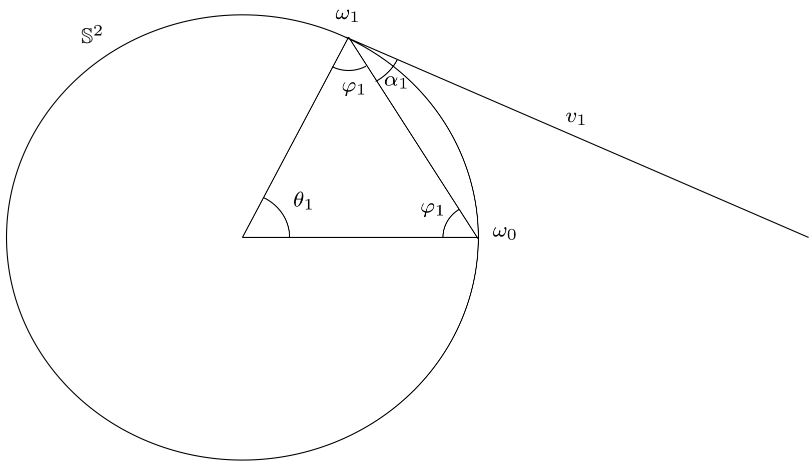

FIGURE 4. Definition of the angles θ_1 and α_1

We deduce

$$|v_1| = \frac{1}{|\omega_0-\omega_1|\cos(\alpha_1)}\left(m_0+\frac{1}{2}|t-s||\omega_0-\omega_1|^2(1+O(\varepsilon))\right)(1+O(\varepsilon)), \tag{5.18}$$

where α_1 denotes the angle between v_1 and $\omega_1-\omega_0$. Let us denote by θ_1 the angle between ω_0 and ω_1. In view of (5.12), we have

$$1-\cos(\theta_1) = \frac{1+O(\varepsilon)}{1+\frac{|t-s|}{m_0}(1+O(\varepsilon))} \tag{5.19}$$

and we deduce in particular

$$0<\theta_1\leq \frac{\pi}{2}+O(\varepsilon). \tag{5.20}$$

Note also in view of the Definition (5.17) that v_1 belongs to $T_{\omega_1}\mathbb{S}^2$ so that

$$v_1\cdot\omega_1 = 0. \tag{5.21}$$

Simple considerations on angles imply [(12)] (see Figure 4)

$$\alpha_1 = \frac{\theta_1}{2}.$$

Together with (5.18), this yields

$$|v_1| = \frac{1}{|\omega_0-\omega_1|\cos\left(\frac{\theta_1}{2}\right)}\left(m_0+\frac{1}{2}|t-s||\omega_0-\omega_1|^2(1+O(\varepsilon))\right)(1+O(\varepsilon)). \tag{5.22}$$

Let $\omega\in\mathbb{S}^2$. According to Lemma 4.7, the half great circle on $\mathbb{S}^2$ originating at ω_0 and containing ω intersects D at a unique point ω_1. Let θ denote the positive angle between

(12) Let φ_1 the angle defined on Figure 4. Then $2\varphi_1+\theta_1=\pi$, and $\varphi_1+\alpha_1=\frac{\pi}{2}$ in view of (5.21). Hence $\theta_1=2\alpha_1$

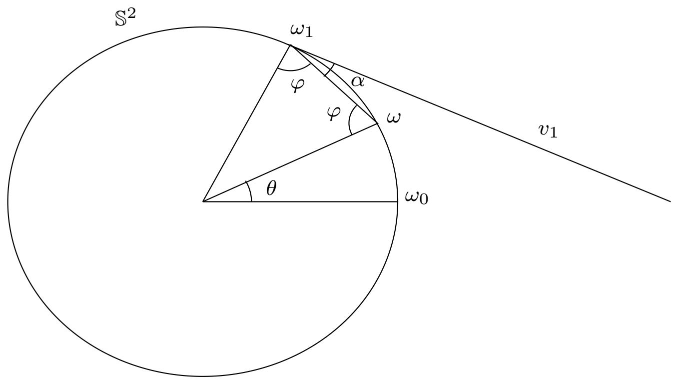

FIGURE 5. Definition of the angles θ and α

ω_0 and ω and let α denote the angle between v_1 and $\omega_1 - \omega$. In view of (5.21), simple considerations on angles imply [13] (see Figure 5)

$$\alpha = \frac{|\theta_1 - \theta|}{2}.$$

Together with (5.16), the definition of ϕ, and the Definition (5.17), we have either

$$\phi(t,x,s,y,\omega) = -\frac{1}{2}|t-s||\omega-\omega_1|^2(1+O(\varepsilon))+|\omega-\omega_1||v_1|\cos\left(\frac{\theta-\theta_1}{2}\right)(1+O(\varepsilon)), \tag{5.23}$$

if $0 \leq \theta \leq \theta_1$, or

$$\phi(t,x,s,y,\omega) = -\frac{1}{2}|t-s||\omega-\omega_1|^2(1+O(\varepsilon))-|\omega-\omega_1||v_1|\cos\left(\frac{\theta-\theta_1}{2}\right)(1+O(\varepsilon)), \tag{5.24}$$

if $\theta_1 \leq \theta \leq \pi$, where we have used the fact that (see Figure 5)

$$(\omega - \omega_1)\cdot v_1 \geq 0 \text{ if } 0 \leq \theta \leq \theta_1 \text{ and } (\omega - \omega_1)\cdot v_1 < 0 \text{ if } \theta_1 < \theta \leq \pi.$$

We consider the two cases in the next two sections.

5.3.1. *The case* $\theta_1 \leq \theta \leq \pi$. – We are in the case (5.24), so that we have for $\varepsilon > 0$ small enough

$$\phi(t,x,s,y,\omega) \leq -\frac{1}{4}|t-s||\omega-\omega_1|^2.$$

In particular, we obtain

$$|\phi(t,x,s,y,\omega)| \geq \frac{1}{4}|t-s||\omega-\omega_1|^2 \tag{5.25}$$

which is the desired estimate (4.15).

[13] Let φ the angle defined on Figure 5. Then $2\varphi + |\theta_1 - \theta| = \pi$, and $\varphi + \alpha = \frac{\pi}{2}$ in view of (5.21). Hence $|\theta_1 - \theta| = 2\alpha$

5.3.2. *The case* $0 \leq \theta \leq \theta_1$. – We are in the case (5.23), which together with (5.22) yields

$$
\begin{aligned}
(5.26) \quad \phi(t,x,s,y,\omega) = {} & \frac{|\omega-\omega_1|}{|\omega_0-\omega_1|}\frac{\cos\left(\frac{\theta-\theta_1}{2}\right)}{\cos\left(\frac{\theta_1}{2}\right)}m_0(1+O(\varepsilon))\\
& +\frac{1}{2}|t-s||\omega-\omega_1|A(\omega)(1+O(\varepsilon))+|t-s||\omega-\omega_1|^2O(\varepsilon)
\end{aligned}
$$

where A is given by

$$
(5.27) \qquad A(\omega) = -|\omega-\omega_1| + \frac{\cos\left(\frac{\theta-\theta_1}{2}\right)}{\cos\left(\frac{\theta_1}{2}\right)}|\omega_0-\omega_1|.
$$

Since θ is the angle between ω and ω_0, and θ_1 is the angle between ω_1 and ω_0, we have

$$
(5.28) \qquad |\omega-\omega_1| = \sqrt{2}\sqrt{1-\cos(\theta-\theta_1)},\ |\omega_1-\omega_0| = \sqrt{2}\sqrt{1-\cos(\theta_1)}.
$$

Together with (5.27), we obtain

$$
A(\omega) = \sqrt{2}\sqrt{1+\cos(\theta-\theta_1)}\left(\frac{\sqrt{1-\cos(\theta_1)}}{\sqrt{1+\cos(\theta_1)}} - \frac{\sqrt{1-\cos(\theta-\theta_1)}}{\sqrt{1+\cos(\theta-\theta_1)}}\right)
$$

which yields

$$
A(\omega) \geq 0 \text{ for all } 0 \leq \theta \leq \theta_1.
$$

Together with (5.26), we obtain

$$
\phi(t,x,s,y,\omega) \geq \frac{|\omega-\omega_1|}{|\omega_0-\omega_1|}\cos\left(\frac{\theta-\theta_1}{2}\right)m_0(1+O(\varepsilon)) + |t-s||\omega-\omega_1|^2O(\varepsilon).
$$

In view of the fact that from (5.20) we have

$$
0 \leq \theta \leq \theta_1 \leq \frac{\pi}{2} + O(\varepsilon),
$$

we deduce

$$
(5.29) \qquad \phi(t,x,s,y,\omega) \geq \frac{\sqrt{2}}{2}\frac{|\omega-\omega_1|}{|\omega_0-\omega_1|}m_0(1+O(\varepsilon)) + |t-s||\omega-\omega_1|^2O(\varepsilon).
$$

Evaluating (5.11) at $\omega=\omega_1$ and using the fact that $\omega_1 \in D$ so that the left-hand side of (5.11) vanishes, we obtain

$$
|t-s||\omega_1-\omega_0|^2 \lesssim m_0
$$

which together with (5.29) and the fact that

$$
|\omega-\omega_1| \leq |\omega_0-\omega_1|
$$

yields for $\varepsilon > 0$ small enough

$$
\phi(t,x,s,y,\omega) \gtrsim \frac{|\omega-\omega_1|}{|\omega_0-\omega_1|}m_0.
$$

In view of (5.28), we deduce

$$
(5.30) \qquad \phi(t,x,s,y,\omega) \gtrsim \sqrt{\frac{1-\cos(\theta-\theta_1)}{1-\cos(\theta_1)}}m_0
$$

which is the desired estimate (4.16).

Finally, in view of (5.2), (5.10), (5.25) and (5.30), we have obtained the desired estimates (4.13), (4.14), (4.15) and (4.16). This concludes the proof of Lemma 4.9.

Appendix A

Proof of Lemma 4.4

If $p \in S$, then, there is ω_0 such that

$$p = \gamma_{\omega_0}(s-t).$$

In view of (4.5), this yields

$$\begin{aligned} u(p,\omega) - u(t,x,\omega) &= u(\gamma_{\omega_0}(s-t),\omega) - u(t,x,\omega) \\ &= \int_0^{s-t} \mathbf{g}(\mathbf{D}u, \gamma'_{\omega_0}(\sigma))d\sigma \\ &= \int_0^{s-t} b^{-1}(\gamma_{\omega_0}(\sigma),\omega)b^{-1}(\gamma_{\omega_0}(\sigma),\omega_0)\mathbf{g}(L(\gamma_{\omega_0}(\sigma),\omega), L(\gamma_{\omega_0}(\sigma),\omega_0)d\sigma \\ &\leq 0 \end{aligned} \tag{A.1}$$

where we used in the last inequality the fact that the scalar product of 2 null vectors is negative

$$\mathbf{g}(L(\gamma_{\omega_0}(\sigma),\omega), L(\gamma_{\omega_0}(\sigma),\omega_0) \leq 0.$$

Also, (A.1) in the special case $\omega = \omega_0$ yields

$$\begin{aligned} u(p,\omega_0) - u(t,x,\omega_0) &= \int_0^{s-t} b^{-2}(\gamma_{\omega_0}(\sigma),\omega_0)\mathbf{g}(L(\gamma_{\omega_0}(\sigma),\omega_0), L(\gamma_{\omega_0}(\sigma),\omega_0))d\sigma \\ &= 0 \end{aligned} \tag{A.2}$$

since $L(\gamma_{\omega_0}(\sigma),\omega_0)$ is null. In view of (A.1) and (A.2), we finally obtain

$$S \subset \{p \in \Sigma_s,\ /\, m(p) = 0\}. \tag{A.3}$$

Conversely, let p such that $m(p) = 0$. Let $\omega_0 \in \mathbb{S}^2$ an angle where the max in the Definition (4.7) of m is attained. Then, we have at $\omega = \omega_0$:

$$u(p,\omega_0) = u(t,x,\omega_0),\ \partial_\omega u(p,\omega_0) = \partial_\omega u(t,x,\omega_0). \tag{A.4}$$

Also, in view of (2.12), we have

$$u(\gamma_{\omega_0}(s-t),\omega_0) = u(t,x,\omega_0),\ \partial_\omega u(\gamma_{\omega_0}(s-t),\omega_0) = \partial_\omega u(t,x,\omega_0)$$

which together with (A.4) implies

$$u(p,\omega_0) = u(\gamma_{\omega_0}(s-t),\omega_0),\ \partial_\omega u(p,\omega_0) = \partial_\omega u(\gamma_{\omega_0}(s-t),\omega_0).$$

Since $u(s,.,\omega_0), \partial_\omega u(s,.,\omega_0)$ forms a global coordinate system on Σ_s in view of the assumption in Section 2.4, we deduce

$$p = \gamma_{\omega_0}(s-t) \in S$$

and thus

$$\{p \in \Sigma_s,\ /\, m(p) = 0\} \subset S.$$

Together with (A.3), this concludes the proof of Lemma 4.4.

Appendix B

Proof of Lemma 4.7

Let (θ, φ) denote the spherical coordinates with axis ω_0. Note from the Definition (4.12) of D and the definition of ϕ that D is given by

(B.1) $$D = \{\omega \in \mathbb{S}^2,\ /\ \phi(t, x, s, y, \omega) = 0\}.$$

Recall that

$$\phi(t, x, s, y, \omega_0) = m_0 > 0.$$

Also, we have from (5.11)

$$\phi(t, x, s, y, -\omega_0) = -2|t - s|^2(1 + O(\varepsilon)) + m_0(-1 + O(\varepsilon)) < 0.$$

Thus, since ϕ is continuous, we deduce from the mean value theorem that

(B.2) $$\forall \varphi \in [0, 2\pi),\ \text{there exists at least one}\ \theta_1 \in (0, \pi)\ \text{such that}\ (\theta_1, \varphi) \in D.$$

Also, note in view of (5.25) and (5.30),

$$\forall \varphi \in [0, 2\pi),\ \text{there exists at most one}\ \theta_1 \in (0, \pi)\ \text{such that}\ (\theta_1, \varphi) \in D.$$

Together with (B.2), we deduce the existence of a 2π-periodic function

$$\theta_1 : [0, 2\pi) \to (0, \pi)$$

such that in the coordinate system (θ, φ), D is parametrized by

$$D = \{\theta = \theta_1(\varphi),\ 0 \leq \varphi < 2\pi\}.$$

To conclude the proof of Lemma 4.7, it remains to prove that θ_1 is C^1. Let $\omega_1 \in D$. Let (θ_1, φ_1) the coordinates of ω_1. By a slight abuse of notations, let us identify ω_1 with (θ_1, φ_1). Then, since

$$\omega \to \phi(t, x, s, y, \omega)$$

is a C^1 function from our assumptions on u, and since

$$\phi(t, x, s, y, \omega_1) = 0$$

in view of the fact that ω_1 belongs to D, we have

(B.3) $$\partial_\theta \phi(t, x, s, y, \omega_1) = \lim_{\theta \to \theta_1} \frac{\phi(t, x, s, y, \theta, \varphi_1)}{\theta - \theta_1}.$$

Now, (5.23) and (5.24) imply

$$\phi(t, x, s, y, \theta, \varphi_1) = |v_1|(1 + O(\varepsilon))(\theta - \theta_1)(1 + o(1))\ \text{as}\ \theta \to \theta_1$$

which together with (B.3) yields

(B.4) $$\partial_\theta \phi(t, x, s, y, \omega_1) \neq 0.$$

Finally, in view of (B.4) and the fact that ϕ is C^1, the implicit function theorem implies that θ_1 is a C^1 function. This concludes the proof of Lemma 4.7.

Appendix C
Proof of Lemma 5.2

Note that

$$u(\mu(\sigma),\omega_0)' = \mathbf{g}(\mathbf{D}u(\mu(\sigma),\omega_0),\mu'(\sigma))$$
$$= 1$$

by the definition of $\mathbf{D}u$, μ', the identity (2.13), and the fact that N is unitary. This implies

(C.1) $$u(\mu(\sigma),\omega_0) = \sigma + u(\gamma_{\omega_0}(t-s),\omega_0).$$

Also, we have

(C.2) $$\begin{aligned}\partial_\omega u(\mu(\sigma),\omega_0)' &= \mathbf{g}(\mathbf{D}\partial_\omega u(\mu(\sigma),\omega_0),\mu'(\sigma))\\ &= \mathbf{g}(-b^{-2}\partial_\omega bN(\mu(\sigma),\omega_0) + b^{-1}\partial_\omega N(\mu(\sigma),\omega_0),\mu'(\sigma))\\ &= 0,\end{aligned}$$

where we used in the last inequality the Definition (5.6) of μ' and the Definition (5.4) of a_0.

Recall the Definition (4.10) of m_0

(C.3) $$m_0 = u(s,y,\omega_0) - u(t,x,\omega_0).$$

In view of (5.3) and (5.6)-(C.3), we have

$$u(\mu(m_0),\omega_0) = u(s,y,\omega_0),\ \partial_\omega u(\mu(m_0),\omega_0) = \partial_\omega u(s,y,\omega_0).$$

Since $u(s,.,\omega_0),\partial_\omega u(s,.,\omega_0)$ forms a global coordinate system on Σ_s in view of the assumption in Section 2.4, we deduce

$$(s,y) = \mu(m_0)$$

which is the desired estimate. This concludes the proof of Lemma 5.2.

Appendix D
Proof of Lemma 5.4

Note that

$$u(\eta(\sigma),\omega_1)' = \mathbf{g}(\mathbf{D}u(\eta(\sigma),\omega_1),\eta'(\sigma))$$
$$= 0$$

by the definition of $\mathbf{D}u$, η' and the identity (2.13). This implies

$$u(\eta(\sigma),\omega_1) = u(\gamma_{\omega_1}(s-t),\omega_1) = u(t,x,\omega_1),$$

which together with the fact that $\omega_1\in D$ implies from the Definition (4.12) of D

(D.1) $$u(\eta(\sigma),\omega_1) = u(s,y,\omega_1) \text{ for all } \sigma\in\mathbb{R}.$$

Also, we have

$$\begin{aligned}\partial_\omega u(\eta(\sigma),\omega_1)' &= \mathbf{g}(\mathbf{D}\partial_\omega u(\eta(\sigma),\omega_1),\eta'(\sigma))\\ &= \mathbf{g}(-b^{-2}\partial_\omega bN(\eta(\sigma),\omega_1) + b^{-1}\partial_\omega N(\eta(\sigma),\omega_1),\eta'(\sigma))\\ &= \partial_\omega u(s,y,\omega_1) - \partial_\omega u(\gamma_{\omega_1}(s-t),\omega_1),\end{aligned}$$

where we used in the last inequality the Definition (5.14) of η'_ω and the Definition (5.13) of a_1. This implies

$$\partial_\omega u(\eta(\sigma),\omega_1) = \partial_\omega u(\gamma_{\omega_1}(s-t),\omega_1) + \sigma(\partial_\omega u(s,y,\omega_1) - \partial_\omega u(\gamma_{\omega_1}(s-t),\omega_1)). \tag{D.2}$$

In view of (D.1) and (D.2), we have

$$u(\eta(1),\omega_1) = u(s,y,\omega_1),\ \partial_\omega u(\eta(1),\omega_1) = \partial_\omega u(s,y,\omega_1).$$

Since $u(s,.,\omega_1), \partial_\omega u(s,.,\omega_1)$ forms a global coordinate system on Σ_s in view of the assumption in Section 2.4, we deduce

$$(s,y) = \eta(1)$$

which is the desired estimate. This concludes the proof of Lemma 5.4.

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