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*Anomalous partially hyperbolic diffeomorphisms I: dynamically coherent examples*

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## ANOMALOUS PARTIALLY HYPERBOLIC DIFFEOMORPHISMS I: DYNAMICALLY COHERENT EXAMPLES

### BY CHRISTIAN BONATTI, KAMLESH PARWANI AND RAFAEL POTRIE

ABSTRACT. – We build an example of a non-transitive, dynamically coherent partially hyperbolic diffeomorphism f on a closed 3-manifold with exponential growth in its fundamental group such that  $f^n$  is not isotopic to the identity for all  $n \neq 0$ . This example contradicts a conjecture in [13]. The main idea is to consider a well-understood time- $t$  map of a non-transitive Anosov flow and then carefully compose with a Dehn twist.

RÉSUMÉ. – Sur une 3-[vari](#page-16-0)été fermée dont le groupe fondamental est à croissance exponentielle, nous construisons un exemple de difféomorphisme f, partiellement hyperbolique, dynamiquement cohérent, non transitif, et dont aucune puissance  $f^n$ ,  $n \neq 0$ , n'est isotope à l'identité. Cet exemple infirme une conjecture de [13]. L'exemple est obtenu en composant avec soin le temps t d'un flot d'Anosov non transitif bien choisi avec un twist de Dehn.

#### **1. Introduction**

In recent years, partially hyperbolic diffeomorphisms have been the focus of considerable study. Informally, partially hyperbolic diffeomorphisms are generalization of hyperbolic maps. The simplest partially hyperbolic diffeomorphisms admit an invariant splitting into three bundles: one of which is uniformly contracted by the derivative, another which is uniformly expanded, and a center direction whose behavior is intermediate. A more formal definition will soon follow. In this paper, we shall restrict to the case of these diffeomorphisms on closed 3-dimensional manifolds.

The study of partially hyperbolic diffeomorphisms has followed two main directions. One consists of studying conditions under which a volume preserving partially hyperbolic diffeomorphism is stably ergodic. This is not the focus of this article. See [11, 14] for recent surveys on this subject.

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The second direction, initiated in [3, 4], has as a long term goal of classifying these partially hyperbolic systems, at least topologically. Even [in](#page-16-1) dimension 3, this goal seems quite ambitious but some partial progress has been made, which we briefly review below. This paper is intended to further this classification effort by providing new examples of partially hyperbolic diffeomorphisms. In a forthcoming paper ([2]) we shall provide new transitive examples (and even stably ergodic); their construction uses some of the ideas of this paper as well as some new ones. Another viewpoint might be that these new examples throw a monkey wrench into the classification program. In light of these new partially hyperbolic diffeomorphisms, is there any hope to achieve any reasonable sense of a classification?

#### **1.1. Preliminaries**

Before diving into a detailed exposition of these examples, we provide the necessary definitions and background. Let  $M$  be a closed 3-manifold, we say that a diffeomorphism  $f : M \to M$  is *partially hyperbolic* if the tangent bundle splits into three one-dimensional <sup>(1)</sup> Df-invariant continuous bundles  $TM = E^{ss} \oplus E^c \oplus E^{uu}$  such that there exists  $\ell > 0$  such that for every  $x \in M$ :

$$
||Df^{\ell}||_{E^{ss}(x)}|| < \min\{1, ||Df^{\ell}||_{E^{c}(x)}||\} \le \max\{1, ||Df^{\ell}||_{E^{c}(x)}\} < ||Df^{\ell}||_{E^{uu}(x)}||.
$$

Sometimes, the more restrictive notion of *absolute partial hyperbolicity* is used. This means that f is partially hyperbolic and there exists  $\lambda < 1 < \mu$  such that:

 $||Df^{\ell}|_{E^{ss}(x)}|| < \lambda < ||Df^{\ell}|_{E^{c}(x)}|| < \mu < ||Df^{\ell}|_{E^{uu}(x)}||.$ 

For the classification of such systems, one of the main obstacles is understanding the existence of invariant foliations tangent to the center direction  $E<sup>c</sup>$ . In general, the bundles appearing in [th](#page-16-2)e invariant splitting are not regular enough to guaranty unique integrability. In the case of the strong stable  $E^{ss}$  and strong unstable  $E^{uu}$  bundles, dynamical arguments insure the existence of unique foliations tangent to the strong stable and unstable bundle (see for example [8]). However, the other distributions need not be integrable.

The diffeomorphism f is *dynamically coherent* if there are 2-dimensional f-invariant foliations  $\mathcal{W}^{cs}$  and  $\mathcal{W}^{cu}$  tangent to the distributions  $E^{ss} \oplus E^c$  and  $E^c \oplus E^{uu}$ , respectively. These foliations, when they exist, intersect along a 1-dimensional foliation  $\mathcal{W}^c$  t[ang](#page-3-0)ent to  $E^c$ . [The](#page-16-3) diffeomorphism  $f$  is *robustly dynamically coherent* if there exists a  $C<sup>1</sup>$ -neighborhood of f comprised only of dynamically coherent partially hyperbolic diffeomorphisms. [The](#page-16-4)re is an example of non dynamically coherent partially hyperbolic diffeomorphisms<sup>(2)</sup> on  $\mathbb{T}^3$  (see [12]). This example is not transitive and it is not known whether every transitive partially hyperbolic diffeomorphisms on a compact 3 manifold is dynamically coherent. See [7] and references therein for the known results on dynamical coherence of partially hyperbolic diffeomorphisms in dimension 3.

Two dynamically coherent partially hyperbolic diffeomorphisms  $f: M \to M$  and  $g: N \to N$  are *leaf conjugate* if [ther](#page-17-0)e is a homeomorphism  $h: M \to N$  so that h maps

<span id="page-3-0"></span> $<sup>(1)</sup>$  One of the advantages of working with one-dimensional bundles is that the norm of  $Df$  along such bundles</sup> controls the contraction/expansion of every vector in the bundle. Compare with definitions of partial hyperbolicity when the bundles are not one-dimensional [14].

<sup>(2)</sup> We should remark that this example is not absolutely partially hyperbolic. Moreover, it is isotopic to one of the known models of partially hyperbolic diffeomorphisms.

the center foliation of f on the center foliation of g and for any  $x \in M$  the points  $h(f(x))$ and  $q(h(x))$  belong to the same center leaf of q.

Up to now the only known example of dynamically coherent partially hyperbolic diffeomorphisms were, up to finite lift and finite iterates, leaf conjugate to one of the following models:

1. linear Anosov automorphism of  $\mathbb{T}^3$ ;

2. skew [pr](#page-16-5)oducts over a linear Anosov map of the torus  $\mathbb{T}^2$ ;

[3](#page-16-0). time one map of an Anosov flow.

It has been conjectured, first in the transitive case (informally by Pujals in a talk and [th](#page-16-5)[en](#page-16-6) [w](#page-16-4)ritten in [3]), and later in the dynamically coherent case (in many talks and minicourses [13]) that every partially hyperbolic diffeomorphism should be, up to finite cover and iterate, leaf conjugate to one of these three models. Positive results have been obtained in [3, 4, 7] and some families of 3-manifolds are now known only to admit partially hyperbolic diffeomorphisms that are leaf conjugate to the models.

#### **1.2. Statements of results**

The aim of this paper is to provide a counter example to the conjecture stated above. Our examples are not isotopic to any of the models.

In order to present the ide[as](#page-16-7) in the simplest way, we have chosen to detail the construction of a specific example on a (possibly the simplest) manifold admitting a non-transitive Anosov flow transverse to a non-homologically trivial incompressible two-torus; the interested reader should consult [5] for more on 3-manifolds admitting such non-transitive Anosov flows. Our arguments go through directly in some other manifolds, but for treating the general case of 3-manifolds admitting non-transitive Anosov flows further work must be done.

T 1.1. – *There is a closed orientable* 3*-manifold* M *endowed with a non-transitive Anosov flow X and a diffeomorphism*  $f: M \to M$  *such that:* 

- **–** f *is absolutely partially hyperbolic;*
- **–** f *is robustly dynamically coherent;*
- **–** *the restriction of* f *to its chain recurrent set coincides with the time-one map of the Anosov flow* X*, and*
- $-$  *for [an](#page-16-8)y*  $n \neq 0$ *, f<sup>n</sup> is not isotopic to the identity.*

The manifold M on which our example [is](#page-16-0) [co](#page-16-4)nstructed also admits a transitive Anosov flow (see  $[1]$ ).

As a corollary of our main theorem, we show that  $f$  is a counter exampl[e to](#page-0-0) the conjecture stated above in the non-transitive case (see [13, 7]):

COROLLARY 1.2. – Let f be the diffeomorphism announced in Theorem 1.1. Then for all  $n$ the diffeomorphism  $f^n$  does not admit a finite lift that is leaf conjugate to any of the following:

- **–** *linear Anosov diffeomorphisms on* T 3 *;*
- **–** *partially hyperbolic skew product with circle fiber over an Anosov diffeomorphism on the torus*  $\mathbb{T}^2$ ;
- **–** *the time-one map of an Anosov flow.*

#### **1.3. Organization of the paper**

The paper is organized in the followin[g m](#page-6-1)anner. In Section 3 we describe a modified *DA* diffeomorphism of  $\mathbb{T}^2$ . This particular *DA* diffeomorphism of the [to](#page-16-9)rus may not [se](#page-9-0)em the simplest but it has the necessary properties that make our example easy to present using only elementary methods. In Section 4, we detail the construction of [a](#page-9-1) non-transitive Anosov flow, following the construction of Franks and Williams in [6]. In Section 5 [we](#page-11-0) establish coordinates in a model space in order to prepare for the ap[pro](#page-10-0)priate perturbati[on](#page-12-0) diff[eom](#page-15-0)orphism—a Dehn twist along a separating torus  $T_1$ . Then, in Section 6, we choose the length of the neighborhood of the separating torus  $T_1$ . We present the example in Section 8 after providing criteria for establishing partial hyperbolicity in Section 7. Then, in Sections 9 and 10 we show that the example is dynamically coherent and not leaf conjugate to previously known examples; it is also not isotopic to the identity. Next, in Section 2, we informally outline the construction of our specific example.

#### **2. Informal presentation of the example**

The example is constructed in [th](#page-16-9)e following manner. We begin with a DA map with two sources instead of [on](#page-16-7)e. This choice makes it possible to easily show that our partially hyperbolic diffeomorphism has no non-trivial iterate isotopic to the identity. Next, we build a non-transitive Anosov flow (après [6]) transverse to a torus  $T_1$  (this is always the case for nontransitive Anosov flows [5]). Then our example is obtained by composing the time N-map of this Anosov flow with a Dehn twist along a neighborhood of the torus  $T_1$  of the form

$$
\bigcup_{t\in[0,N]}X_t(T_1),
$$

which is diffeomorphic to  $[0, 1] \times \mathbb{T}^2$ .

The neighborhood and the time  $N$  is chosen in order to preserve partial hyperbolicity. In a nutshell, the idea is that a small  $C^1$ -perturbation always preserves partial hyperbolicity, and so, if we m[ake](#page-5-0) the perturbation in a long enough neighborhood of  $T_1$ , by insuring that the time interval  $[0, N]$  is sufficiently large, the effect of the Dehn twist can be made to appear negligible at the level of the derivative, even though the  $C<sup>0</sup>$ -distance cannot be made arbitrarily small (3). More precisely, we obtain conditions under which transversality between certain bundles are preserved under this kind of composition which allows to show partial hyperbolicity.

Since the perturbation is made in the wandering region of the time- $N$  of the Anosov flow, the properties of the chain-recurrent set are preserved and it is possible to study the integrability of the center bundle by simply defining it in the obvious way and showing that it plays well with attracting and repelling regions as it approaches them. The fact that center leaves cannot be fixed when they pass through the fundamental region where the perturbation is made becomes a matter of checking that the new intersections cannot be preserved by the altered dynamics.

<span id="page-5-0"></span><sup>(3)</sup> Note that these statements are not precise and the remarks in this section are intended to impart our intuition to the reader.

There are some reasons for which we present a specific example:

- 1. Even though most of our arguments are quite general and our Dehn twist perturbation can be applied to infinite family of Franks-Williams type Anosov flows, it is easier to first see these ideas presented for a single example. The construction of the perturbation and the fact that it preserves partial hyperbolicity is much easier to check in a specific case and we believe that this narrative makes the global argument more transparent.
- 2. It is possible to apply these techniques to other types of examples at the expense of having to check a few minor details. However, to perform the examples in any manifold admitting a non-transitive Anosov flow, some more work is required to guarantee the transversality of the foliations after perturbation. We believe this is beyond the scope of this paper a[nd](#page-15-0) relegated to a more detailed study in a forthcoming article.
- 3. Mainly, for the specific example it is quite easy to give a direct and intuitive argument to show that the resulting dynamics has no iterate isotopic to the identity. This is carried out in Section 10 where we use the fact that the torus  $T_1$  is homological[ly](#page-16-10) non-trivial and just utilize elementary algebraic topology to show that the action on homology is non-trivial. For the general case, showing that the perturbation is not isotopic to the identity requires more involved arguments. We remark that the paper [9] solves this problem in many situations, but in a less elementary manner. Again, these details are best left to be expounded in another article.

#### **3.** Modified  $DA$  map on  $\mathbb{T}^2$

<span id="page-6-0"></span>Our construction begins with a diffeomorphism of the torus. We us[e a](#page-16-11) modified *DA* map, with two sources instead of one.

We simply state the properties of the required map below. The classical construction of the Derived from Anosov (*DA*) diffeomorphism is well known—see [10] for instance. Our modified *DA* map is obtained by lifting (some iterate of) the classical DA map to some 2-folded cover of  $\mathbb{T}^2$ . Alternatively, one may begin with a linear Anosov diffeomorphism with 2 fixed points and then create two sources by "blowing up" these fixed points and their unstable manifolds.

**PROPOSITION** 3.1. – *There exists a diffeomorphism*  $\varphi: \mathbb{T}^2 \to \mathbb{T}^2$  *with the following properties:*

- **–** *the non-wandering set of* ϕ *consists in one non-trivial hyperbolic attractor* A *and two fixed sources*  $\sigma_1$ ,  $\sigma_2$ *;*
- <span id="page-6-1"></span>**–** *the stable foliation of the hyperbolic attractor coincides with a linear (for the affine structure on*  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$  *irrational foliation on*  $\mathbb{T}^2 \setminus {\sigma_1, \sigma_2}$ *.*

#### **4. Building non-transitive Anosov flows**

In this section, we briskly run through the relevant details of the classical Franks-Williams construction in ([6]) of a non-transitive Anosov flow. Note that the map  $\phi$  in [6] is the standard DA map with a single source, and so, the presentation below has been adapted to our context.

Let  $(M_0, Z)$  be the suspension of the DA-diffeomorphism  $\varphi$  given in Theorem 3.1. We denote by  $\gamma_i$  the periodic orbit of the flow of Z corresponding to the sources  $\sigma_i$ , and by  $A_Z$ the hyperbolic attractor of the flow of Z.

LEMMA 4.1. – *There is a convex map*  $\alpha$ :  $(0, \frac{1}{2}) \rightarrow \mathbb{R}$  *tending to*  $+\infty$  *at* 0 *and*  $\frac{1}{2}$ *, whose derivative vanishes exactly at*  $\frac{1}{4}$ *, and so that, for any i*  $\in$  1, 2 *there is a tubular neighborhood*  $\Gamma_i$ *of*  $\gamma_i$  whose boundary is an embedded torus  $T_i \simeq \mathbb{T}^2$  so that

- $T_i$  is transverse to  $Z$ , and therefore to the weak stable foliation  $W_Z^{cs}$  of the attractor  $A_Z$ ; We denote by  $F_i^s$  the 1-dimensional foliation induced by  $W_Z^{cs}$  on  $T_i$ ;
- $-$  *there are coordinates*  $\theta_i \colon \mathbb{T}^2 \to T_i$  *so that, in these coordinates:* 
	- $F_i^s$  has exactly 2 *compact leaves*  $\{0\} \times S^1$  and  $\{\frac{1}{2}\} \times S^1$ ;
	- $F_i$  *nas exactly* 2 compact leaves  $\{0\} \times S$  *and*  $\{2\} \times S$ *,*  $\}$ <br>• given any leaf L of  $F_i^s$  in  $(0, \frac{1}{2}) \times S^1$  there is  $t \in \mathbb{R}$  so that L is the projection on  $\mathbb{T}^2$ *of the graph of*  $\alpha(x) + t$ *;*
	- given any leaf L [of](#page-16-9)  $F_i^s$  in  $(\frac{1}{2},1) \times S^1$  there is  $t \in \mathbb{R}$  so that L is the projection on  $T^2$ *of the graph of*  $\alpha(x - \frac{1}{2}) + t$ *.*

*Proof*. – See page 165 of [6].

$$
\overline{\Box}
$$

Notice that the expression of  $F_1^s$  and  $F_2^s$  are the same in the chosen coordinates. We denote by  $F_i^u$  the image of  $F_i^s$  by the translation  $(x, y) \mapsto (x + \frac{1}{4}, y)$ .

COROLLARY 4.2.  $- F_i^s$  and  $F_j^u$  are transverse. Moreover  $F_i^s$  and  $F_i^u$  are invariant under any *translation in the second coordinates (that is*  $(x, y) \mapsto (x, y + t)$ *,*  $t \in \mathbb{R}$ *).* 



FIGURE 1. Gluing of the two foliations making them transverse. Notice that both foliations are invariant under vertical translations.

We are now ready to define our manifold  $M$  and the vector-field  $X$  on  $M$ .

Let  $M^+$  be the manifold with boundary obtained by removing to  $M_0$  the interior of  $\Gamma_1 \cup \Gamma_2$ , and we denote by  $Z^+$  he restriction of Z to  $M^+$ . We denote by  $T_i^+$  the boundary component of  $M^+$  corresponding to the boundary of  $\Gamma_i$ . Notice that  $T_i^+$  are tori transverse to  $Z^+$  and  $Z^+$  points inwards to  $M^+$ .

Let  $M^-$  be another copy of  $M^+$  and we denote by  $Z^-$  the restriction of  $-Z$  to  $M^-$ . We denote by  $T_i^-$  the boundary component of  $M^-$ , there are transverse to  $Z^-$  and  $Z^-$  is points outwards of M−.

We denote by  $\psi: \partial M^+ \to \partial M^-$  the diffeomorphisms sending  $T_i^+$  on  $T_i^-$ ,  $i = 1, 2$ , and whose expression in the  $\mathbb{T}^2$  coordinates are  $(x, y) \mapsto (x + \frac{1}{4}, y)$ .

We denote by M the manifold obtained by gluing  $M^+$  with  $M^-$  along the diffeomorphism  $\psi$ . We can now restate Franks-Williams theorem in our setting:

T 4.3. – *There is a smooth structure on* M*, coinciding with the smooth structures on* M<sup>+</sup> *and on* M<sup>−</sup> *and there is a smooth vector-field* X *on* M*, whose restrictions to* M<sup>+</sup> *and*  $M^-$  are  $Z^+$  and  $Z^-$  respectively.

*Furthermore* X *is an Anosov vector-field whose non-wandering set consists exactly in a non-trivial hyperbolic attractor*  $A_X$  *contained in*  $M^+$  *and a non-trivial hyperbolic repeller*  $R_X$ *contained in* M<sup>−</sup>*.*

A non-singular vector field X on a 3-manifold is an *Anosov vector field* if the time-t map  $X_t$ is partially hyperbolic for some  $t > 0$  (see [5, 6] for the classical definition). It is easy to show that the time t-map of an Anosov flow will always be absolutely partially hyperbolic.

REMARK 4.4. – Since the bundle generated by X must be necessarily the center bundle of  $X_t$  it follows that the center direction is always integrable for Anosov flows. Indeed, it is a classical result (see e.g., [8]) that Anosov flows are dynamically coherent <sup>(4)</sup>. We call  $W_X^{cs}$  and  $W_X^{cu}$  the center stable and center unstable foliations (sometimes called weak stable and weak unstable foliations in the context of Anosov flows) and  $W_X^{ss}$  and  $W_X^{uu}$  the strong stable and unstable foliations respectively. We denote as  $E_X^{cs}$ ,  $E_X^{cu}$ ,  $E_X^{ss}$  and  $E_X^{uu}$  to the tangent bundles of these foliations.

The above construction immediately generalizes to examples arising from DA maps with *n* sources instead of 2, where  $n \geq 1$ . We call these flows *Franks-Williams type Anosov flows.* The arguments in the next sections are very general and apply to all Franks-Williams type Anosov flows.

<sup>(4)</sup> This is not the way it is usually stated. Dynamical coherence is modern terminology.

#### **5. A perturbation on a model space**

<span id="page-9-0"></span>In this section we shall perform a perturbation in a certain model space.

Consider the torus  $T_1$  and  $X_1(T_1)$  where  $X_1$  is the time one map of the flow of X. Then  $T_1$ and  $X_1(T_1)$  bound a manifold diffeomorphic to  $[0,1] \times \mathbb{T}^2$ , which is a fundamental domain of  $X_1$ . We shall use the convention that sets of the form  $[a, b] \times \{p\}$  are *horizontal* and those of the form  $\{t\} \times \mathbb{T}^2$  are *vertical*.

The projection of the vector-field X on this coordinates is  $\frac{\partial}{\partial t}$ .

We denote by  $\mathcal{F}^{ss}$  and  $\mathcal{F}^{uu}$  the 1-dimensional foliations  $(\{t\} \times F_1^s)_{t \in [0,1]}$  and  $(\{t\}\times F_1^u)_{t\in[0,1]}$  and by  $\mathcal{F}^{cs}$  and  $\mathcal{F}^{cu}$  the 2-dimensional foliations  $[0,1]\times F_1^s$  and  $[0,1]\times F_1^u$ .

LEMMA 5.1. - Let  $\mathcal{G}$ :  $[0,1] \times \mathbb{T}^2 \rightarrow [0,1] \times \mathbb{T}^2$  be the diffeomorphism defined as  $(t, x, y) \mapsto (t, x, y + \rho(t))$ *. Where*  $\rho : [0, 1] \rightarrow [0, 1]$  *is a monotone smooth function such that it is identically zero in a neighborhood of* 0 *and ide[ntica](#page-0-0)lly* 1 *in a n[eighb](#page-0-0)orhood of* 1*.*

Then  $\mathscr{G}(\mathscr{F}^{cu})$  is transverse to  $\mathscr{F}^{ss}$  and  $\mathscr{G}(\mathscr{F}^{uu})$  is transverse to  $\mathscr{F}^{cs}$ .

<span id="page-9-1"></span>*Proof*. – This is a direct consequence of Lemma 4.1, Corollary 6.3 and the fact that  $\mathcal G$  makes translations only in the y-direction of the coordinates given by that lemma.  $\Box$ 

#### 6. The strong stable and unstable foliations on a fundamental domain of  $X_N$  for large N

For any  $N > 0$  we consider the fundamental domain  $\mathcal{U}_N$  of the diffeomorphism  $X_N$ (time N of the flow of X) restricted to  $M \setminus (A_X \cup R_X)$  bounded by  $T_1$  and  $X_N(T_1)$ .

This fundamental domain  $\mathcal{U}_N$  is canonically identified with  $[0, N] \times T_1$ : the projection of  $T_1$  is the identity map on  $\{0\} \times T_1$  and the projection of X is  $\frac{\partial}{\partial t}$ .

The intersection of the weak stable and weak unstable 2-foliations  $W_X^{cs}$ ,  $W_X^{cu}$  as well as the (1-dimensional) strong stable and strong unstable foliations  $W_X^{ss}$  and  $W_X^{uu}$  with  $\mathcal{U}_N$ ; will be denoted by  $W_N^{cs}$ ,  $W_N^{cu}$   $W_N^{ss}$ ,  $W_N^{uu}$ .

LEMMA 6.1. – *The expression of the tangent space of*  $W_N^{cs}$ ,  $W_N^{cu}$   $W_N^{ss}$ ,  $W_N^{uu}$  at a point  $(t, x, y) \in [0, N] \times T_1$  *only depends on*  $(x, y) \in T_1$  *: it depends neither of*  $t \in [0, N]$  *nor on* N*.*

*Proof*. – This follows from the fact that the bundles are invariant under the flow and the vector-field X is  $\frac{\partial}{\partial t}$  in this coordinates.  $\Box$ 

We denote by  $H_N: \mathcal{U}_N \to [0,1] \times \mathbb{T}^2$  the diffeomorphisms defined by  $H_N(t,p)$  $(\frac{t}{N}, \theta_1^{-1}(p)).$ 

We denote  $\mathcal{F}_N^{cs} = H_N(W_N^{cs}), \mathcal{F}_N^{cu} = H_N(W_N^{cu}), \mathcal{F}_N^{ss} = H_N(W_N^{ss}), \mathcal{F}_N^{uu} = H_N(W_N^{uu}).$  $\mathcal{F}_N^{cs} = H_N(W_N^{cs}), \mathcal{F}_N^{cu} = H_N(W_N^{cu}), \mathcal{F}_N^{ss} = H_N(W_N^{ss}), \mathcal{F}_N^{uu} = H_N(W_N^{uu}).$  $\mathcal{F}_N^{cs} = H_N(W_N^{cs}), \mathcal{F}_N^{cu} = H_N(W_N^{cu}), \mathcal{F}_N^{ss} = H_N(W_N^{ss}), \mathcal{F}_N^{uu} = H_N(W_N^{uu}).$ 

LEMMA 6.2. – For every  $N > 0$ ,  $\mathcal{F}_N^{cs} = \mathcal{F}^{cs}$  [an](#page-9-0)d  $\mathcal{F}_N^{cu} = \mathcal{F}^{cu}$ , where  $\mathcal{F}^{cs}$  and  $\mathcal{F}^{cu}$  are the 2-dimensional foliation on  $[0, 1] \times \mathbb{T}^2$  defined in Section 5.

The tangent bundle to  $\mathcal{F}_N^{ss}$  and to  $\mathcal{F}_N^{uu}$  converges in the  $C^0$  topology to the tangent bundle to the foliations  $\mathcal{F}^{ss}$  and  $\mathcal{F}^{uu}$ , defined in Section 5, as N goes to  $+\infty$ .

*Proof.* – The first assertion is a direct consequence of the fact that  $\mathcal{F}_N^{cs}$  and  $\mathcal{F}_N^{cu}$  are saturated by the orbits of the flow and Lemma 6.1.

The second assertion follows from t[he fa](#page-0-0)ct t[hat t](#page-0-0)he diffeomorphism is independent of  $N$ in the first coordinate and compresses the t coordinate by  $\frac{1}{N}$ .  $\Box$ 

As a direct consequence of Lemmas 5.1 and 6.2 we obtain

C 6.3. – *Let* G *be the diffeomorphism defined in Lemma 5.1. Then, there is*  $N_0 > 0$  *so that for any*  $N \geq N_0$  *one has:* 

<span id="page-10-0"></span> $\mathscr{G}(\mathcal{F}_N^{cu})$  is transverse to  $\mathcal{F}_N^{ss}$  and  $\mathscr{G}(\mathcal{F}_N^{uu})$  is transverse to  $\mathcal{F}_N^{cs}$ .

#### **7. Establishing partial hyperbolicity**

We just recall a classical criterion for partial hyperbolicity. As before, we remain in dimension 3 for simplicity.

Let g be a diffeomorphism on a compact 3-dimensional manifold  $M$ . Assume that there is a codimension 1 submanifold cutting  $M$  in two compact manifolds with boundary  $M^+$  and  $M^-$  so that  $M^+$  is an *attracting region* for f (i.e.,  $f(M^+)$  ⊂ Int $(M^+)$ ) and  $M^-$  is a *repelling region* (i.e., attracting region for  $g^{-1}$ ).

Assume that the maximal invariant set A in  $M^+$  and the maximal invariant set R in  $M^$ admit a (absolute) partially hyperbolic splitting  $E^{ss} \oplus E^c \oplus E^{uu}$ .

Then  $E^{ss}$  and  $E^{ss} \oplus E^c$  admit a unique invariant extension  $E^{ss}$  and  $E^{cs}$  on  $M \setminus R$  and symmetrically  $E^{uu}$  and  $E^c \oplus E^{uu}$  admit a unique invariant extension  $E^{uu}$  and  $E^{cu}$  on  $M \setminus A$ .

REMARK 7.1. – Assume that g coincides with an Anosov flow Z in a neighborhood U of A. Then, the bundles  $E^{cs}$  and  $E^{ss}$  in  $M \setminus R$  coincide exactly with the tangent spaces of the foliation  $f^{-n}(W_Z^{cs} \cap U)$  and  $f^{-n}(W_Z^{ss} \cap U)$ . A symmetric property holds for  $E^{cu}$  and  $E^{uu}$ . Notice that the center bundle cannot be a priori extended to these sets.

A compact set U will be called a *fundamental domain* of  $M \setminus (A \cup R)$  if every orbit of g restricted to  $M \setminus (A \cup R)$  intersects  $\mathcal U$  in at least one point.

THEOREM 7.2. – Let  $g : M \to M$  be a  $C^1$ -diffeomorphism. Assume that:

- **–** *there exists a codimension one submanifold cutting* M *into an attracting and a repelling regions with maximal invariant sets* A *and* R *which are (absolutely) partially hyperbolic;*
- $-$  *there exists a compact fundamental domain U of*  $M \setminus (A \cup R)$  *such that if*  $E^{ss}_A, E^{cs}_A$  *denote the extensions of the bundles*  $E^{ss}$  *and*  $E^{ss} \oplus E^c$  *of*  $A$  *<i>to*  $M \setminus R$  *and*  $E^{uu}_R$  ,  $E^{cu}_R$  *denote the* extensions of  $E^{uu}$ ,  $E^c \oplus E^{uu}$  of R to M \ A then  $E^{cs}_A$  is transverse to  $E^{uu}_R$  and  $E^{cu}_R$  is *transverse to*  $E_A^{ss}$  *at each point of U.*

*Then* g *is (absolutely) partially hyperbolic on* M*.*

*Proof.* – We have a well defined splitting  $E^{ss} \oplus E^c \oplus E^{uu}$  above A and R and we can define the bundles  $E^{ss}$  and  $E^{uu}$  everywhere as  $E^{ss} := E^{ss}_A$  and  $E^{uu} := E^{uu}_R$  in  $M \setminus (A \cup R)$ .

The transversality conditions we have assumed on  $\mathcal U$  allows us to define the  $E^c$  bundle in  $M \setminus (A \cup R)$  as the intersection between  $E_A^{cs}$  and  $E_R^{cu}$ . These intersect in a one-dimensional subbundle thanks to our transversality assumptions.

Now, let us show that the splitting we have defined is (absolutely) partially hyperbolic. To see this, it is enough to show that the decomposition is continuous. Indeed, if this is the case, one can use the fact that given a neighborhood U of  $A \cup R$  there exists  $N > 0$  such that every point outside U verifies that every iterate larger than  $N$  belongs to U. Together with continuity of the bundles and the partially hyperbolicity along  $A \cup R$  this allows to show that if  $g|_{A\cup R}$  is (absolutely) partially hyperbolic, then it must be (absolutely) partially hyperbolic globally.

To show continuity of the bundles, notice first that since for a point  $x \in M \setminus (A \cup$ R) the bundle  $E_R^{uu}$  is transverse to  $E_A^{cs}$  one has that as one iterates forward the point  $g^n(x)$  approaches A while the bundle  $D_x g^n(E_R^{uu})$  must approach  $E^{uu}$  by the transversality and domination. The same argument shows that  $E^{ss}$  is also continuous. The fact that  $E^c$ glues well with  $E^c$  along A follows from the fact that  $E_A^{cs}$  is invariant and  $E^c$  is transverse to  $E_A^{ss}$  in  $E_A^{cs}$ . The symmetric argument gives continuity of  $E^c$  as one approaches R and this concludes the proof.

 $\Box$ 

#### **8. The example: perturbation of the time** N **of the flow**

<span id="page-11-0"></span>In this section we construct the example announced in Theorem 1.1 and prove it is (absolute) partially hyperbolic.

We fix  $N \geq N_0$  as in Corollary 6.3.

Consider  $M \setminus (A_X \cup R_X)$ . Let  $V_1$  be the X-invariant open subset of M [co](#page-0-0)nsisting in the point whose orbit crosses  $T_1$ .

Consider the diffeomorphism  $\mathcal G$  on  $[0,1] \times \mathbb T^2$  defined in Lemma 5.1. We consider  $G: M \to M$  defined as the identity outside  $\mathcal{U}_N$  and  $G = H_N^{-1} \circ \mathcal{G} \circ H_N$  in  $\mathcal{U}_N$ . Then G is a smooth diffeomorphism from how  $\mathcal G$  was defined.

Define the diffeomorphism  $f : M \to M$ :

$$
f=G\circ X_N.
$$

We will prove that the diffeomorphism  $f$  defined above satisfies all the conclusion of Theorem 1.1. We begin by first demonstrating that  $f$  is partially hyperbolic.

#### THEOREM 8.1. – *The diffeomorphism* f *is absolutely partially hyperbolic.*

*Proof.* – First notice that the chain recurrent set of f coincides with the non-wandering set of  $X_N$  and thus of X, that is  $A_X \cup R_X$ . Furthermore the dynamics of f coincides with the one of  $X_N$  in a neighborhood of  $A_X \cup R_X$ . In particular, f is absolute partially hyperbolic in restriction to  $A_X \cup R_X$ .

According to Theorem 7.2 we must check that the extension  $E_A^{ss}$  and  $E_A^{cs}$  of the bundles  $E_X^{ss}$ ,  $E_X^{cs}$  in  $A_X$  and the extensions  $E_R^{cu}$ ,  $E_R^{uu}$  of  $E_X^{cu}$  and  $E_X^{uu}$  on  $R_X$  satisfy the transversality conditions between  $E_A^{cs}$  and  $E_R^{us}$  and  $E_A^{cs}$  and  $E_R^{cs}$  in a compact funda[menta](#page-0-0)l domain  $\mathcal{U}$ .

Recall that  $E_A^{ss}$ ,  $E_A^{cs}$  in a neighborhood of  $A_X$  are the tangent bundles to the strong stable and center stable foliations of  $A_X$  and  $E_R^{cu}$ ,  $E_R^{uu}$  coincide with the tangent bundles to the strong unstable and center unstable foliations of  $R<sub>X</sub>$  (see Remark 7.1 and notice that f coincides with an Anosov flow in neighborhoods of  $A_X$  and  $R_X$ ).

Notice that f admits a fundamental domain having two connected components, one (denoted as  $\Delta_{1,N}$ ) bounded by  $T_1$  and  $X_N(T_1)$  and the other bounded by  $T_2$  and  $X_N(T_2)$ (denoted as  $\Delta_{2,N}$ ). Therefore  $M\setminus (A_X \cup R_X)$  has two connected components  $V_1$  and  $V_2$  which are the sets of points whose orbits pass through one or the other fundamental domains.

On  $V_2$  the diffeomorphism f coincides with  $X_N$  and the bundles coincide with the invariant bundles of  $X$ , so that we get the transversality conditions for free.

Thus we just have to show the transversality in  $\Delta_{1,N} = \mathcal{U}_N$ .

Notice that f coincides with  $X_N$  on

$$
\bigcup_{t\geq 0} X_t(T_1).
$$

As  $\bigcup_{t\geq 0} X_t(T_1)$  is positively invariant and the orbits tends to  $A_X$  the stable and center stable foliations of  $f$  coincide on that set with those of  $X$ .

In particular,  $E_A^{cs}$  and  $E_A^{ss}$  in  $\mathcal{U}_N$  coincide with  $E_X^{cs}$  and  $E_X^{ss}$  respectively.

Notice also that f coincides with  $X_N$  on

$$
\bigcup_{t\leq -N} X_t(T_1).
$$

As  $\bigcup_{t\leq -N} X_t(T_1)$  is negatively invariant and th[e neg](#page-0-0)ative orbits tends to  $R_X$  the unstable and center unstable foliations of  $f$  coincide on that set with those of  $X$ .

In particular, we obtain that  $E_R^{cu} = G_*(E_X^{cu})$  and  $E_R^{uu} = G_*(E_X^{uu})$  in  $\mathcal{U}_N$  which satisfy the transversality conditions thanks to Corollary 6.3.  $\Box$ 

<span id="page-12-0"></span>The bundles of  $f$  will be denoted as  $E_f^{ss}, E_f^c$  and  $E_f^{uu}$ . As usual, we denote  $E_f^{cs} = E_f^{ss} \oplus E_f^c$ and  $E_f^{cu} = E_f^c \oplus E_f^{uu}$ .

#### **9. Dynamical coherence**

We are now ready to prove that the diffeomorphism  $f$  is robustly dynamically coherent and that it cannot be leaf conjugate to the time one map of an Anosov flow. Furthermore, the same result holds for any iterate and any finite lift of  $f$ .

LEMMA 9.1. – *There exists*  $K > 1$  so that for any unit vector  $v \in E_f^c$  and any  $n \in \mathbb{Z}$  one *has* 1

$$
\frac{1}{K} < \|Df^n(v)\| < K.
$$

*Proof.* – Notice that given any neighborhoods  $U_R$  and  $U_A$  of  $R_X$  and  $A_X$ , there is  $n_0 > 0$ so that  $f^{n_0}(M \setminus U_R)$  is contained in  $U_A$ .

We choose  $U_A$  as being a positively  $X_t$ -invariant neighborhood of  $A_X$  on which f coincides with  $X_N$ . In  $U_A$  the bundles  $E_f^{ss}$  and  $E_f^{cs}$  coincide with  $E_X^{ss}$  and  $E_X^{cs}$ . As  $E_f^c$  is transverse to  $E_f^{ss}$  on the compact manifold M, we get that any unit vector of  $E_f^c$  in  $U_A$  has a component in  $\mathbb{R}X$  uniformly bounded from below and above and a component in  $E_X^{ss}$  uniformly bounded (from above). Therefore, the positive iterates of v by  $Df$  are uniformly bounded from below and above.

The same phenomena occur in a neighborhood  $U_R$  of  $R_X$  when we consider backward iterates. Now the result is est[ablis](#page-0-0)hed by using the invariance o[f](#page-0-0)  $E_f^c$  and the fact that any orbit spends at most  $n_0$  iterates outside  $U_R \cup U_A$ .  $\Box$ 

The inequalities of Lemma 9.1 force the curves tangent to the center direction to possess a very useful dynamical property; this is the content of Corollary 9.3. First, let us recall a definition.

DEFINITION 9.2. – Consider a diffeomorphism q with an invariant bundle  $E \subset TM$ , one says that g is *Lyapunov stable* in the direction of E if given any  $\varepsilon > 0$  there is  $\delta > 0$  so that any path  $\gamma$  tangent to E of length sma[ller t](#page-0-0)han  $\delta$  verifies that the forward iterates  $g^n(\gamma)$  have length smaller than  $\varepsilon$ .

As a direct consequence of Lemma 9.1 one gets:

COROLLARY 9.3. - The diffeomorphism  $f$  is Lyapunov stable in the direction  $E_f^{cs}$ . Symmet*rically,*  $f^{-1}$  *is Lyapunov stable in the direction*  $E_f^{cu}$ *.* 

*Proof.* – Just notice that for any unit vector v tangent to  $E_f^{cs}$  the forward iterates  $Df^n(v)$ have uniformly bounded norm.  $\Box$ 

This permits us to demonstrate

T 9.4. – *The diffeomor[ph](#page-16-2)ism* f *is robust[ly d](#page-16-12)ynamically coherent. Moreover, the bundle*  $E_f^c$  is uniquely integrable.

*Proof.* – We [refe](#page-16-12)r the reader to [8, Secti[on](#page-16-2) 7] or [11, Section 7] for precise definitions of some notions which will appear in this proof (which are classical in the theory of p[artia](#page-0-0)lly hyperbolic systems). Related arguments appear in [4].

It is shown in [11, Theorem 7.5] (see also [8, Theorem 7.5]) that  $E_f^{cs}$  is tangent to a unique fol[iat](#page-16-2)ion provided f is Lyapunov stable with respect to  $E_f^{cs}$ . Therefore, by Corollary 9.3 the bundle  $E_f^{cs}$  is tangent to a unique foliation tangent to  $E_f^{cs}$ . Moreover, it is also established [th](#page-16-2)at under this assumptions, the unique foliation  $W_f^{cs}$  must be *plaque-expansive* in the sense of [8, Section 7].

Applying the same result for  $f^{-1}$  and  $E_f^{cu}$  we deduce dynamical coherence, and using [8, Theorem 7.1], we may conclude that the center-stable and center-unstable foliations are *structurally stable* (in particular, they exist for small  $C<sup>1</sup>$  perturbations of f). This concludes the proof of robust dynamical coherence.

Finally, unique integrability of  $E_f^c$  follows by classical arguments using the fact that curves tangent to  $E_f^c$  are Lyapunov stable for f and  $f^{-1}$  because of Lemma 9.1 (see also [11, Corollary 7.6]).  $\Box$ 

Now we are ready to prove that the example cannot be leaf conjugate to the time-one map of an Anosov flow—this is the result advertised in Corollary 1.2. However, first we must establish an important property about the leaves of the center-foliation  $W_f^c$  of f. Recall from the previous section that  $\mathcal{U}_N = \bigcup_{0 \le t \le N} X_t(T_1)$ .

LEMMA 9.5. – The connected components of  $W_f^c \cap \mathcal{U}_N$  are arcs which join  $T_1$  with  $X_N(T_1)$ *with uniformly bounded length.*

*Proof*. – This follows immediately from the fact that the perturbation preserves the  $\mathbb{T}^2 \times \{t\}$  coordinates (in the coordinates given by  $H_N$ ) and the original center direction was positively transverse to those fibers. Since  $\mathbb{T}^2 \times [0,1]$  is compact, one obtains that these arcs are of bounded length and join both boundaries of  $\mathcal{U}_N$ .  $\Box$ 

We can now show:

THEOREM 9.6. – *There are center leaves which are not fixed for no iterate of f. Consequently, there is no finite lift or finite iterate of* f *which is leaf conjugate to the time-one map of an Anosov flow.*

*Proof.* – Using unique integrability, one knows that the f-invariant foliation  $W_f^c$  tangent to  $E_f^c$  is obtained by intersecting the preimages of  $W_X^{cs}$  with the forward images of  $W_X^{cu}$ , where  $W_X^{cs}$  and  $W_X^{cu}$  are defined in n[eigh](#page-0-0)borhoods of  $A_X$  and  $R_X$  respectively.

In particular, we get that restricted to  $\mathcal{U}_N$  which is a fundamental domain for f in the complement of  $A_X \cup R_X$ , we have that  $W_f^c$  consists of  $W_X^{cs} \cap G(W_X^{cu})$  which is an arc joining  $T_1$  and  $X_N(T_1)$  as proved in Lemma 9.5.

Notice that f coincides with  $X_N$  in a neighborhood of  $T_1$  (and  $G = id$  in a neighborhood of  $T_1$ ) so that  $W_f^c$  consists of horizontal lines (i.e., of the form  $[0, \varepsilon) \times \{p\}$  or  $(1 - \varepsilon, 1] \times \{p\}$ in the coordinates given by  $H_N$ ) in neighborhoods of  $T_1$  and  $f(T_1)$ . Moreover, one has that the image of the center line which is of the form  $[0, \varepsilon) \times \{p_0\}$  in a neighborhood of  $T_1$  is sent by f to the arc of  $W_f^c$  which is of the form  $(1 - \varepsilon, 1] \times \{p_0\}$  (again because  $G = id$  in a neighborhood of  $T_1$  and  $X_N(T_1)$ ).

On the other hand, those arcs cannot be joined inside  $W_X^{cs} \cap G(W_X^{cu})$  if they are not the leaves corresponding to circles in  $T_1$ , and so, it follows that the center leaves cannot be fixed by f. The same argument implies that this is not possi[ble](#page-16-5) for  $f^k$  for any  $k \geq 0$  since f coincides with  $X_N$  once it leaves  $\mathcal{U}_N$ .  $\Box$ 

REMARK 9.7. – This in stark contrast with the results of [3] where it is shown that when f is transitive, if certain center leaves are fixed, then all center leaves must be. It is also important to note that in this example the leaves of both the center-stable and center-unstable foliations are fixed by  $f$  but the connected components of their intersections—the center leaves—are not.

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Since the 3-manifold M admits an Ano[sov](#page-16-4) flow,  $\pi_1(M)$  has exponential growth, and so, the manifol[d](#page-16-5)  $M$  does not support a partially hyperbolic diffeomorphisms isotopic to an Anosov diffeomorphism or a skew product. This is because Anosov diffeomorphisms only exist on  $\mathbb{T}^3$  in dimension 3 (see for example [7] and references therein) and skew products are only defined (in [3]) on circle bundles over  $\mathbb{T}^2$ -both these types of 3-manifolds are associated with polynomial growth in their fundamental groups. These growth properties are immune to taking finite covers, and so, the same holds true for any finite cover of  $M$ . Therefore, if a finite li[ft or](#page-0-0) iterate of  $f$  were [leaf](#page-0-0) conjugate to one of the three models, it would have to be the time-one map of an Anosov flow. In this case, there would exist an iterate of  $f$  on a finite cover that would fix every center leaf. We have shown that this is not the case, and thus, Theorem 9.6 implies Corollary 1.2.

#### **10. Non-trivial isotopy class**

<span id="page-15-0"></span>In this section we will complete the proof of Theorem 1.1 by showing that no iterate of  $f$ is isotopic to the identity. It is at this point only that we shall use the fact that the original DA-diffeomorphism has at least two sources since this provides a curve intersecting the torus where the modification is made which is homologically non trivial. So we restrict ourselves to the case where there are exactly two sources, but it should be clear that all we have done until now works with any number of such sources.

**THEOREM** 10.1. – For every  $k \neq 0$ ,  $f^k$  is not isotopic to the identity. More precisely, the *action of* f <sup>k</sup> *on homology is non-trivial.*

*Proof*. – It suffices to show that the action on homology is not trivial.

To establish this, first note that for the suspension manifold  $M_0$  the periodic orbits where we did the DA-construction are homologicall[y n](#page-15-1)on-trivial. This implies that after removing the solid torus, the circles in the same direction in the boundary are still homologically non-trivial. The gluing we have performed preserves this homology class, so that it remains homologically non-trivial after the gluing too<sup>(5)</sup>.

Notice moreover that there is a representative  $\gamma_1$  of this homology class which does not intersect  $T_1$  simply by making a small homotopy of this loop which makes it disj[oin](#page-15-2)t from  $T_1$ .

Consider a closed curve  $\gamma_2$  intersecting  $T_1$  only once (it enters by  $T_1$  and then comes back by  $T_2$ ) we know that  $\gamma_2$  is not homologous to  $\gamma_1$ . This can be shown by considering a small tubular neighborhood U of  $T_1$  and a closed 1-form which is strictly positive <sup>(6)</sup> in U and vanishes outside U. It is clear that such a 1-form has a non-vanishing integral along  $\gamma_2$  and a vanishing one along  $\gamma_1$  proving the desired claim.

From how G was chosen it is clear that  $G^k_*([\gamma_2]) = [\gamma_2] + k[\gamma_1]$ . Since  $X_N$  is isotopic to the identity, we also have that  $(G \circ X_N)_* = G_*$ .

<span id="page-15-2"></span><span id="page-15-1"></span>This implies that for every  $k \neq 0$ , the map  $(f^k)_* = (G \circ X_N)_*^k = G^k_*$  is different from the identity.  $\Box$ 

<sup>(5)</sup> A way to see this is by constructing a closed 1-form in each piece which integrates one in the desired circle. These 1-forms glue well to give a closed 1-form integrating one in the same curve.

<sup>(6)</sup> If  $U \sim T_1 \times [-1, 1]$  and we call  $\theta$  to the variable on  $[-1, 1]$  it suffices to choose  $f(\theta) d\theta$  with f smooth, positive on  $(-1, 1)$  and vanishing at  $\pm 1$ .

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