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Pierre BERGER & Romain DUJARDIN

*On stability and hyperbolicity for polynomial automorphisms of  $\mathbb{C}^2$*

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# ON STABILITY AND HYPERBOLICITY FOR POLYNOMIAL AUTOMORPHISMS OF $\mathbb{C}^2$

BY PIERRE BERGER AND ROMAIN DUJARDIN

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**ABSTRACT.** – Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a holomorphic family of polynomial automorphisms of  $\mathbb{C}^2$ . Following previous work of Dujardin and Lyubich, we say that such a family is weakly stable if saddle periodic orbits do not bifurcate. It is an open question whether this property is equivalent to structural stability on the Julia set  $J^*$  (that is, the closure of the set of saddle periodic points).

In this paper we introduce a notion of regular point for a polynomial automorphism, inspired by Pesin theory, and prove that in a weakly stable family, the set of regular points moves holomorphically. It follows that a weakly stable family is probabilistically structurally stable, in a very strong sense. Another consequence of these techniques is that weak stability preserves uniform hyperbolicity on  $J^*$ .

**RÉSUMÉ.** – Soit  $(f_\lambda)_{\lambda \in \Lambda}$  une famille holomorphe d'automorphismes polynomiaux de  $\mathbb{C}^2$ . En accord avec un travail précédent de Dujardin et Lyubich, nous disons qu'une telle famille est faiblement stable si ses points périodiques ne bifurquent pas. La question est ouverte de savoir si cette notion équivaut à celle de stabilité structurelle sur l'ensemble de Julia  $J^*$  (qui est par définition l'adhérence de l'ensemble des points périodiques selles).

Dans cet article nous introduisons une notion de point régulier pour un tel automorphisme, inspirée par la théorie de Pesin, et montrons que dans une famille faiblement stable, l'ensemble des points réguliers se déplace selon un mouvement holomorphe. Nous en déduisons qu'une famille faiblement stable est structurellement stable en un sens probabiliste. Une autre conséquence de cette étude est que la stabilité faible préserve l'hyperbolicité uniforme sur  $J^*$ .

## 1. Introduction

Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a holomorphic family of polynomial automorphisms of  $\mathbb{C}^2$ , with non-trivial dynamics<sup>(1)</sup>, parameterized by a connected complex manifold  $\Lambda$ . A basic stability/bifurcation dichotomy in this setting was introduced by M. Lyubich and the second author in [10]. In that paper it was proved in particular that under a moderate

<sup>(1)</sup> A necessary and sufficient condition for this is that the *dynamical degree*  $d = \lim(\deg(f^n))^{1/n}$  satisfies  $d \geq 2$ , see §2 for more details.

dissipativity assumption <sup>(2)</sup>, weakly stable parameters together with parameters exhibiting a homoclinic tangency form a dense subset of  $\Lambda$ . This confirms in this setting a (weak version of a) well-known conjecture of Palis. The notion of stability into consideration here is the following: a family is said to be *weakly stable* if periodic orbits do not bifurcate. Specifically, this means that the eigenvalues of the differential do not cross the unit circle.

In one-dimensional holomorphic dynamics, this seemingly weak notion of stability actually leads to the usual one of structural stability (on the Julia set or on the whole sphere) thanks to the theory of *holomorphic motions* developed independently by Mañé, Sad and Sullivan and Lyubich [18, 15, 16].

As it is well-known, the basic theory of holomorphic motions breaks down in dimension 2, and a corresponding notion of *branched holomorphic motion* (where collisions are allowed), was designed in [10]. To be more specific, let  $J^*$  be the closure of the set of saddle periodic orbits. It was shown by Bedford, Lyubich and Smillie that  $J^*$  contains all homoclinic and heteroclinic intersections of saddle points, and conversely, if  $p$  is any saddle point, then  $W^s(p) \cap W^u(p)$  is dense in  $J^*$ . It was proved in [10] that if  $(f_\lambda)_{\lambda \in \Lambda}$  is weakly stable, then there is an equivariant branched holomorphic motion of  $J^*$ , that is unbranched over the set of periodic points and homoclinic (resp. heteroclinic) intersections. This means that such points have a unique holomorphic continuation in the family, and furthermore, this continuation cannot collide with other points in  $J^*$  (see below §2.1 for more details). The underlying idea is that the motion is unbranched on sets satisfying a local (uniform) expansivity property.

Still, it remains an open question whether a weakly stable family is structurally stable on  $J^*$ . A weaker version of this question, which is natural in view of the above analysis, is whether the unbranching property holds generically with respect to hyperbolic invariant probability measures.

The first main goal in this paper is to answer this second question. We introduce a notion of *regular point*, simply defined as follows:  $p \in J^*$  is regular if there exists a sequence of saddle points  $(p_n)_{n \geq 1}$  converging to  $p$  such that  $W_{\text{loc}}^u(p_n)$  and  $W_{\text{loc}}^s(p_n)$  are of size uniformly bounded from below as  $n \rightarrow \infty$  and do not asymptotically coincide (see below §4 for the formal definition, and §3.2 for the notion of the local size of a manifold). The set  $\mathcal{R}$  of regular points is invariant and dense in  $J^*$  since it contains saddle points and homoclinic intersections. More interestingly, Katok's closing lemma [14] implies that  $\mathcal{R}$  is of full mass relative to any hyperbolic invariant probability measure (see below Proposition 4.4 for a precise statement and a sketch of proof). Observe however that our definition of regular point makes no reference to any invariant measure. Notice also that in our context, thanks to the Ruelle inequality, any invariant measure with positive entropy is hyperbolic.

Our first main result is the following.

**THEOREM A.** – *Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a substantial family of polynomial automorphisms of  $\mathbb{C}^2$  of dynamical degree  $d \geq 2$ , that is weakly stable.*

*Then the set of regular points moves holomorphically and without collisions. More precisely, for every  $\lambda \in \Lambda$ , every regular point of  $f_\lambda$  admits a unique continuation under the branched*

<sup>(2)</sup> That is, the complex Jacobian  $\text{Jac}(f)$  satisfies  $|\text{Jac}(f)| < d^{-2}$ .

*motion of  $J_\lambda^*$ , which remains regular in the whole family. In particular, the restrictions  $f_\lambda|_{\mathcal{R}_\lambda}$  are topologically conjugate.*

The meaning of the word “substantial” will be explained in §2.1 below; it will be enough for the moment to note that any dissipative family is substantial by definition. By “topologically conjugate” we mean that there exists a homeomorphism  $h : \mathcal{R}_\lambda \rightarrow \mathcal{R}_{\lambda'}$  such that  $h \circ f_\lambda = f_{\lambda'} \circ h$  in restriction to  $\mathcal{R}_\lambda$ .

Let us say that a polynomial automorphism  $f$  is *probabilistically structurally stable* (in some given family  $(f_\lambda)$ ) if for every  $f'$  sufficiently close to  $f$ , there exists a set  $\mathcal{R}_f$  (resp.  $\mathcal{R}_{f'}$ ) which is of full measure with respect to any hyperbolic invariant probability measure for  $f$  (resp.  $f'$ ) together with a continuous conjugacy  $\mathcal{R}_f \rightarrow \mathcal{R}_{f'}$ .

Recall also from the work of Friedland and Milnor [12] that every dynamically non-trivial polynomial automorphism is conjugate to a composition of Hénon mappings.

Theorem A and Corollary 4.5 in [10] suggest a description of structurally stable dissipative mappings as those far from displaying infinitely many sinks or far from displaying a homoclinic tangency. The above Theorem A enables us to go one step further in this direction.

**COROLLARY B.** – *Let  $f$  be a composition of Hénon mappings in  $\mathbb{C}^2$ . Then:*

- *$f$  can be approximated in the space of polynomial automorphisms of degree  $d$  either by a probabilistically structurally stable map, or by one possessing infinitely many sinks or sources.*
- *If  $f$  is moderately dissipative and not probabilistically structurally stable, then  $f$  is a limit of automorphisms displaying homoclinic tangencies.*

The main step of the proof of Theorem A consists in studying how the size of local stable and unstable manifolds of a given saddle point varies in a weakly stable family. More precisely, assume that for some  $\lambda_0 \in \Lambda$ ,  $p(\lambda_0)$  is a saddle point such that  $W^s(p(\lambda_0))$  has bounded geometry at scale  $r_0$  at  $p(\lambda_0)$ . Since  $(f_\lambda)$  is weakly stable,  $p(\lambda_0)$  persists as a saddle point  $p(\lambda)$  for  $\lambda \in \Lambda$ . In §3, we give estimates on the geometry of  $W_{loc}^s(p(\lambda))$  which depend only on  $r_0$ , based on the extension properties of the branched holomorphic motion of  $J^*$  along unstable manifolds devised in [10].

These estimates are used to control the geometry of the local “center stable manifold” of  $\{(\lambda, p(\lambda)), \lambda \in \Lambda\}$ , which is of codimension 1 in  $\Lambda \times \mathbb{C}^2$ . With this codimension 1 subset at hand, we can prevent collisions between the motion of points in  $J^*$  using classical tools from complex geometry (like the persistence of proper intersections and the Hurwitz Theorem).

We actually prove a more general version of Theorem A, which involves only regularity in one of the stable or the unstable directions (see Theorem 4.10 below). One motivation for this is that in the dissipative setting it is possible in certain situations to take advantage of dissipativity to obtain a good control on the geometry of stable manifolds (see Example 4.9).

If  $f$  is uniformly hyperbolic on  $J^*$ , then it is well known that  $f|_{J^*}$  is structurally stable. In particular, if  $(f_\lambda)_{\lambda \in \Lambda}$  is any family of polynomial automorphisms, and  $\lambda_0 \in \Lambda$  is such that  $f_{\lambda_0}$  is uniformly hyperbolic on  $J_{\lambda_0}^*$ , then  $(f_\lambda)$  is (weakly) stable in some neighborhood of  $\lambda_0$ . Thus,  $\lambda_0$  belongs to a *hyperbolic component* in  $\Lambda$ , where this uniform hyperbolicity is

preserved, which is itself contained in a possibly larger *weak stability component*. Our next main result asserts that these two components actually coincide.

**THEOREM C.** – *Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a substantial family of polynomial automorphisms of  $\mathbb{C}^2$  of dynamical degree  $d \geq 2$ , that is weakly stable. Assume that there exists  $\lambda_0 \in \Lambda$  such that  $f_{\lambda_0}$  is uniformly hyperbolic on  $J_{\lambda_0}^*$ . Then for every  $\lambda \in \Lambda$ ,  $f_\lambda$  is uniformly hyperbolic on  $J_\lambda^*$ .*

As a result, this theorem enables to identify the phenomena responsible for the breakdown of uniform hyperbolicity in a family of polynomial automorphisms: hyperbolicity can only be destroyed by the bifurcation of some saddle orbit to a sink or a source (which by [10] implies the creation of homoclinic tangencies, in the moderately dissipative setting).

The proof of Theorem C relies on the techniques of Theorem A, together with a geometric criterion for hyperbolicity due to Bedford and Smillie [5].

It may be useful to close this introduction with a few comments on the relationship between this paper and [10]. As explained above, in a weakly stable family, every periodic point has a well-defined holomorphic continuation and generalizing the Lambda Lemma of [18] to this setting amounts to proving that the cluster values of these motions define a unique holomorphic continuation of all points in  $J^*$ . The methods of [10] only allow to prove that such a continuation exists for points in a uniformly hyperbolic invariant subset.

The key new concept in this paper is that of a *regular point*, which in a sense generalize Pesin regular points in the holomorphic context. We then implement new techniques to control the *extrinsic* geometry of their stable and unstable manifolds in a weakly stable family. This ultimately allows to prove probabilistic structural stability, and explains what the obstruction to structural stability might be.

The plan of the paper is the following. In §2 we discuss the notion of weak stability, following [10]. We also establish some preliminary results on sequences of subvarieties in  $\mathbb{C}^d$ . In §3 we study how the geometry of unstable manifolds varies in a weakly stable family. In §4 we introduce the notion of regular point and prove Theorem A, and finally §5 is devoted to the proof of Theorem C.

Throughout the paper, we make the standing assumption that the parameter space  $\Lambda$  is the unit disk. In view of Theorems A and C this is not a restriction since we can always connect any two parameters in  $\Lambda$  by a chain of holomorphic disks. We also use the classical convention  $C(a, b, \dots)$  to denote a constant which depends only on the previously defined quantities  $a, b$ , etc.

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## 2. Preliminaries

In this section we collect some basic facts on polynomial automorphisms of  $\mathbb{C}^2$ , and give a brief account on the notion of weak stability introduced in [10]. We also establish some preliminary results on sequences of analytic subsets.

**2.1. Families of polynomial automorphisms of  $\mathbb{C}^2$**

Let us start with some standard facts about the iteration of an individual polynomial automorphism  $f$  of  $\mathbb{C}^2$  (see [3, 2] for more details and references). The *dynamical degree* is an integer  $d$  defined by  $d = \lim_{n \rightarrow \infty} (\deg(f^n))^{1/n}$ , and  $f$  has non-trivial dynamics if and only if  $d \geq 2$ . It is then conjugate to a composition of generalized Hénon mappings  $(x, y) \mapsto (p(x) + ay, ax)$ . Here are some dynamically defined subsets:

- $K^\pm$  is the set of points with bounded forward orbits under  $f^{\pm 1}$ .
- $K = K^+ \cap K^-$  is the filled Julia set.
- $J^\pm = \partial K^\pm$  are the forward and backward Julia sets. Stable (resp. unstable) manifolds of saddle periodic points are dense in  $J^+$  (resp.  $J^-$ ).
- $J^* \subset J = J^+ \cap J^-$  is the closure of the set of saddle periodic points. Saddle points and homoclinic and heteroclinic intersections are contained (and dense) in  $J^*$ .

The Green functions  $G^\pm$  are defined by  $G^\pm(z) = \lim d^{-n} \log^+ \|f^n(z)\|$ , and are non-negative continuous plurisubharmonic functions. They are pluriharmonic whenever positive and  $K^\pm$  coincides with  $\{G^\pm = 0\}$ . The associated currents are  $T^\pm = dd^c G^\pm$  whose supports are  $J^\pm$ . If  $\Delta \subset \mathbb{C}^2$  is a holomorphic disk, then  $G^+|_\Delta$  is harmonic if and only if  $T^+ \wedge [\Delta] = 0$  or equivalently  $(f^n|_\Delta)_{n \geq 1}$  is a normal family (equivalently  $\Delta \subset K^+$  or  $\Delta \subset \mathbb{C}^2 \setminus K^+$ ).

Let now  $(f_\lambda)_{\lambda \in \Lambda}$  be a holomorphic family of polynomial automorphisms with fixed dynamical degree  $d \geq 2$ , parameterized by the unit disk in  $\mathbb{C}$ , as said before. We will use the notation  $K_\lambda, J_\lambda^*$ , etc. to denote the corresponding dynamical objects.

To the family  $(f_\lambda)$  is associated a fibered dynamical system in  $\Lambda \times \mathbb{C}^2$  defined by  $\widehat{f} : (\lambda, z) \mapsto (\lambda, f_\lambda(z))$ . Then we mark with a hat the corresponding fibered objects, e.g.,  $\widehat{K} = \bigcup_{\lambda \in \Lambda} \{\lambda\} \times K_\lambda$ , etc.

Such a family is always conjugate to a family of compositions of Hénon mappings (see [10, Prop. 2.1]). It follows that the sets  $K_\lambda$  are locally uniformly bounded in  $\mathbb{C}^2$ .

From now on we report on some results from [10]. A family of polynomial automorphisms of dynamical degree  $d \geq 2$  is said to be *substantial* if: either all its members are dissipative or for any periodic point with eigenvalues  $\alpha_1$  and  $\alpha_2$ , no relation of the form  $\alpha_1^a \alpha_2^b = c$  holds persistently in parameter space, where  $a, b, c$  are complex numbers and  $|c| = 1$ . From now on, we assume without further notice that all families have constant dynamical degree  $d \geq 2$  and are substantial.

A *branched holomorphic motion*  $\mathcal{C}$  is a family of holomorphic graphs over  $\Lambda$  in  $\Lambda \times \mathbb{C}^2$ . All branched holomorphic motions considered in this paper are locally uniformly bounded, so in particular they form normal families (we then say that  $\mathcal{C}$  is *normal*). A branched holomorphic motion  $\mathcal{C}$  is *unbranched* along  $\gamma$  if  $\gamma$  does not cross any other graph in the family. If it is unbranched along any graph  $\gamma$ , then it is by definition a *holomorphic motion*. If  $\mathcal{C}$  is normal, closed and unbranched at  $\gamma$ , and if  $(\gamma_n)_{n \geq 0} \in \mathcal{C}^\mathbb{N}$  is any sequence such that for some  $\lambda_0 \in \Lambda$ ,  $\gamma_n(\lambda_0) \rightarrow \gamma(\lambda_0)$ , then  $\gamma_n \rightarrow \gamma$ . We thus see that unbranching along  $\gamma$  is a form of continuity of the motion. We can make this precise as follows: if  $\mathcal{C}$  is a (non-necessarily closed) normal holomorphic motion and  $\overline{\mathcal{C}}$  is unbranched at  $\gamma_0$ , then  $\mathcal{C}$  is continuous at  $\gamma_0$ .

A substantial family  $(f_\lambda)_{\lambda \in \Lambda}$  of polynomial automorphisms is said to be *weakly stable* if every periodic point stays of constant type (attracting, saddle, indifferent, repelling) in the family. Equivalently,  $(f_\lambda)$  is weakly stable if the sets  $J_\lambda^*$  move under an equivariant branched holomorphic motion. A central theme in this paper will be to show that this motion is unbranched at certain points. In this respect, the following result is essential.

**THEOREM 2.1** (see Lemma 4.10, Cor. 4.12 and Prop. 4.14 in [10])

*Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a weakly stable substantial family of polynomial automorphisms of dynamical degree  $d \geq 2$ . If for  $\lambda_0 \in \Lambda$ ,  $p_0$  is a saddle point (resp. a homoclinic or a heteroclinic intersection), then there exists a unique holomorphic map  $p : \Lambda \rightarrow \mathbb{C}^2$  such that  $p(\lambda_0) = p_0$  and for every  $\lambda \in \Lambda$ ,  $p(\lambda) \in K_\lambda$ . In addition  $p(\lambda)$  is a saddle point (resp. a homoclinic or a heteroclinic intersection) for every  $\lambda$  and the branched holomorphic motion of  $J^*$  is unbranched along  $p$ .*

The motion of  $J^*$  can be extended to a branched holomorphic motion of  $J^+ \cup J^-$ , using the density of stable and unstable manifolds of saddles. The details are as follows (for concreteness we deal with unstable manifolds, of course analogous results hold in the stable direction). The global unstable manifold of a saddle point is parameterized by  $\mathbb{C}$ . More precisely, in our situation, for every  $\lambda$  there exists an injective holomorphic immersion  $\psi_\lambda^u : \mathbb{C} \rightarrow \mathbb{C}^2$  such that  $\psi_\lambda^u(0) = p_\lambda$  and for  $\zeta \in \mathbb{C}$ ,  $f_\lambda \circ \psi_\lambda^u(\zeta) = \psi_\lambda^u(u_\lambda \zeta)$ , where  $u_\lambda$  denotes the unstable multiplier. Such a  $\psi^u$  is unique up to pre-composition with a linear map, and will be referred to as an *unstable parameterization*. In addition, the normalization of  $\psi_\lambda^u$  may be chosen so that  $(\lambda, \zeta) \mapsto \psi_\lambda^u(\zeta)$  is holomorphic. The precise way to do it is irrelevant for the moment; we shall have to discuss this issue more carefully later on.

Thanks to these parameterizations, we can use the theory of holomorphic motions in  $\mathbb{C}$  to derive information about the motion of unstable manifolds in  $\mathbb{C}^2$ . More specifically, the Bers-Royden theorem [7] asserts that if  $(h_\lambda)$  is a holomorphic motion of a subset  $X \subset \mathbb{C}$  defined over the unit disk with  $h_0 = \text{id}$ , then it admits a canonical extension to a holomorphic motion of the whole  $\mathbb{C}$ , defined over  $D(0, 1/3)$ .

The following is a combination of Proposition 5.2 and Lemma 5.10 in [10].

**PROPOSITION 2.2.** – *Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a weakly stable substantial family of polynomial automorphisms of dynamical degree  $d \geq 2$ . Then if  $(p_\lambda)$  is a holomorphically moving saddle point, there exists a natural equivariant holomorphic motion  $h_\lambda : W^u(p_0) \rightarrow W^u(p_\lambda)$ , defined for  $\lambda \in D(0, 1/3)$  with  $h_0 = \text{id}$ , that respects the decomposition*

$$W^u(p) = (W^u(p) \cap U^+) \sqcup (W^u(p) \cap K^+).$$

Beware that we are not claiming that the points in  $W^u(p)$  have a unique continuation, only that they have a *natural* one.

Using automorphisms of the unit disk, a similar result holds if we choose a different base point  $\lambda_0$ . Since  $D(0, 1/3)$  is the disk of radius  $\ln 3$  about 0 for the Poincaré metric, replacing 0 by  $\lambda_0$  we get a natural motion of  $W^u(p_\lambda)$  over the Poincaré disk  $D_{\text{Poin}}(\lambda_0, \ln 3)$ .

Notice that the notation  $h_\lambda$  here refers to the motion of points in  $\mathbb{C}^2$ . Given a holomorphic family of parameterizations  $\psi_\lambda^u$  of  $W^u(p_\lambda)$ , it will also be of interest in some situations to consider the corresponding holomorphic motion  $(\psi_\lambda^u)^{-1} \circ h_\lambda$  in  $\mathbb{C}$ , which we will denote by  $h_\lambda^u$ .



These holomorphic motions need not preserve the levels of the function  $G^+$ : indeed this is already the case<sup>(3)</sup> for  $J$ -stable families in dimension 1. The following easy lemma asserts that  $G^+$  admits locally uniform distortion along the motion. It thus provides a link between the intrinsic (i.e., inside unstable manifolds) and the extrinsic properties of the motion, and will play an important role in the paper.

LEMMA 2.3. – For every compact subset  $\tilde{\Lambda} \Subset D(0, 1/3) \Subset \Lambda$  there exists a constant  $C = C(\tilde{\Lambda}) \geq 1$  such that for every  $z \in W^u(p_0)$

$$\frac{1}{C} G_0^+(z) \leq G_\lambda^+(h_\lambda(z)) \leq C G_0^+(z).$$

*Proof.* – Recall that for any holomorphic family of polynomial automorphisms of degree  $d$ , the function  $(\lambda, z) \mapsto G_\lambda^+(z)$  is plurisubharmonic in  $\Lambda \times \mathbb{C}^2$ , jointly continuous in  $(\lambda, z)$ , and pluriharmonic where it is positive (see [3, §3]).

If  $z \in K_0^+$ , then  $h_\lambda(z) \in K_\lambda^+$  and  $G_\lambda^+(h_\lambda(z)) \equiv 0$  so there is nothing to prove. If  $z \notin K_0^+$  then  $\lambda \mapsto G_\lambda^+(h_\lambda(z))$  is a positive harmonic function, so the result follows from the Harnack inequality [13, Thm 3.1.7].  $\square$

## 2.2. Sequences of analytic subsets

To make the paper accessible to readers potentially not so familiar with complex geometry, let us first recall a few classical facts on complex analytic sets, and sequences of such objects. The reader is referred to the book of Chirka [8] for more details.

Let  $\Omega \subset \mathbb{C}^d$  be a connected open set. A (complex) *analytic subset* or *subvariety*  $A$  of  $\Omega$  is a subset of  $\Omega$  that is covered by open sets  $U$  of  $\mathbb{C}^n$ , for which there exist  $p \geq 0$  and a holomorphic map  $\phi: U \mapsto \mathbb{C}^p$ , such that  $A \cap U = \{z \in U, \phi(z) = 0\}$ .

A point  $a \in A$  is *regular* if there exists a neighborhood  $U$  of  $a$  so that  $A \cap U$  is a (complex) submanifold. The set of regular points of  $A$  is denoted by  $\text{Reg}(A)$ , and its complement  $\text{Sing}(A) = A \setminus \text{Reg}(A)$  is the *singular set*. A subvariety is *smooth* if its singular set is empty.

An *irreducible component* of  $A$  is the closure of a connected component of  $\text{Reg}(A)$ . It is itself an analytic subset. The *dimension* of an analytic subset  $A$  is the maximal dimension of its irreducible components. It is said of *pure dimension* if all its irreducible components have the same dimension.

A *hypersurface* (resp. a *curve*) is an analytic subset of pure codimension (resp. dimension) 1, possibly singular.

Let us also recall that the Hausdorff distance  $d_{HD}$  between two closed subsets  $E$  and  $F$  of a metric space is defined as the infimum of those  $m \in [0, \infty]$  such that  $E$  is included in the  $m$ -neighborhood of  $F$  and *vice-versa*. Let  $\Omega$  be an open subset of  $\mathbb{C}^d$ . A sequence of closed subsets  $(A_j)_j$  of  $\Omega$  converges to a closed subset  $A \subset \Omega$ , if for every compact set  $K$  of  $\Omega$ , it holds  $d_{HD}(K \cap A_j, A \cap K) \rightarrow 0$ . The set of closed (resp. closed and connected) subsets of  $\Omega$  endowed with the Hausdorff distance is relatively compact.

A key ingredient to study the convergence of analytic subsets sequences is the following classical result known as Bishop's Theorem:

<sup>(3)</sup> For instance the value of the Green function at critical points is in general not invariant in a  $J$ -stable family of polynomials.

**THEOREM 2.4** (see [8] p. 203). – *Let  $(A_j)_j$  be a sequence of pure  $p$ -dimensional subvarieties of an open subset  $\Omega \subset \mathbb{C}^d$ , converging to a (closed) subset  $A \subset \Omega$  and such that the  $2p$ -dimensional Hausdorff measure (that is, the  $2p$ -dimensional volume)  $m_{2p}(A_j)$  is locally uniformly bounded:*

$$\forall K \Subset \Omega, \exists M_K > 0, \forall j, m_{2p}(A_j \cap K) < M.$$

*Then  $A$  is also a pure  $p$ -dimensional subvariety of  $\Omega$ .*

In particular the set of subvarieties with locally uniformly bounded volume is compact. We can actually be more precise about the convergence in Bishop's Theorem. Let  $A_n$  be a sequence of analytic sets with uniformly bounded volumes converging in the Hausdorff topology to an irreducible analytic set  $A$ . Then there exists a positive integer  $m$ , the *multiplicity of convergence*, which can be described as follows. If  $p \in \text{Reg}(A)$  is any regular point of  $A$ , and  $N$  is a compact neighborhood of  $p$  in which  $(A \cap N, N)$  is biholomorphic to  $(\mathbb{D}^k \times \{0\}, \mathbb{D}^d)$ , then for  $n$  large enough,  $A_n \cap N$  is a branched cover over  $A \cap N$  of degree  $m$ . In particular if  $m = 1$ ,  $A_n \cap N$  is a graph over  $A \cap N$ .

Let us now state a few results which will be used many times in the paper. The following result can be interpreted as a kind of abstract version of the  $\Lambda$ -lemma of [18].

**PROPOSITION 2.5.** – *Let  $\Omega \subset \mathbb{C}^d$  be a connected open set. Let  $(V_n)$  be a sequence of analytic subsets of codimension 1 in  $\Omega$  with uniformly bounded volumes.*

*Assume that:*

- *the  $V_n$  are disjoint;*
- *there exists  $p_n \in V_n$  such that  $p_n \rightarrow p \in \Omega$ ;*
- *every cluster value of  $(V_n)$  is locally irreducible at  $p$ .*

*Then the sequence  $(V_n)$  converges.*

The irreducibility assumption is necessary in this result, as shown by the sequence of curves in the unit bidisk in  $\mathbb{C}^2$  defined for  $n \geq 1$  by  $V_{2n} = \{x = 1/n^2\}$  and  $V_{2n+1} = \{xy = 1/n\}$ .

*Proof.* – Assume that  $V = \lim V_{n_j}$  and  $W = \lim V_{n'_k}$  are distinct cluster limits of  $(V_n)$ . Then  $V$  and  $W$  are irreducible and contain  $p$ , therefore they must intersect non-trivially at  $p$  (i.e.,  $V \cap W$  is locally different from  $V$  and  $W$ ). Since  $V$  and  $W$  are of codimension 1, they intersect properly, that is  $\dim(V \cap W) = d - 2$ . Now, proper intersections are robust under perturbations (see prop. 2 p. 141 and cor. 4 p. 145 in [8]), so we infer that  $V_{n_j}$  and  $V_{n'_k}$  intersect non-trivially for large  $j$  and  $k$ , which is contradictory.  $\square$

In general the limit of a sequence of smooth hypersurfaces can be singular. The smoothness of the limit can be ensured in certain circumstances (compare [17, Prop. 11]).

**PROPOSITION 2.6.** – *Let  $(V_n)$  be a sequence of curves with uniformly bounded area in the unit ball of  $\mathbb{C}^2$ , which converges to  $V$ . Assume that for every  $n$ ,  $V_n$  is biholomorphic to a disk. Then  $V$  is irreducible. If in addition the multiplicity of convergence is 1, then  $V$  is smooth.*

*Proof.* – Fix  $p \in V$ . Let us first show that  $V$  is locally irreducible at  $p$ . Fix a small ball  $B$  about  $p$  and let  $V \cap B = V^1 \cup \dots \cup V^q$  be the decomposition into (local) irreducible components. Shrinking  $B$  slightly if necessary, we may assume that each  $V^i$  contains  $p$  and that  $\partial B \cap \overline{V^i}$  is not empty. Likewise, we may assume that  $V$  is smooth near  $\partial B$  and transverse to it. Hence  $V \cap \partial B$  is a union of disjoint smooth (real) curves  $(C_j)_j$  and each  $C_j$  is contained in a unique irreducible component  $V^i$ .

For every  $n$ , let  $V_n^i$  be a connected component of  $V_n \cap B$ . We can further assume that  $V_n^i$  intersects a slightly smaller ball  $B' \Subset B$ . By the uniform bound on the area and Bishop’s Theorem, we can extract a subsequence  $(V_{n'}^i)_{n'}$  converging to a curve  $A$ . Since  $A$  is included in  $V \cap B$ , it is an union of irreducible components of  $V \cap B$ .

Observe that each  $V_n^i$  is a holomorphic disk, as follows from the maximum principle applied to the subharmonic function  $z \mapsto \|\varphi_n(z) - p\|$ , where  $\varphi_n$  is a parametrization of  $V_n$ . The boundary  $C_n^i = \overline{V_n^i} \cap \partial B$  is homeomorphic to a circle for every  $n$ , hence  $(C_{n'}^i)_{n'}$  converges to a connected compact set. Our assumptions on  $\partial B \cap V$  imply that  $C_{n'}^i$  is close to a unique component  $C^i$  of  $V \cap \partial B$ . On the other hand the loop  $C^i$  is in the boundary of a unique irreducible component of  $V \cap B$ , say  $V^i$ . Thus  $A = V^i$  and  $\overline{A} \cap \partial B = \overline{V^i} \cap \partial B = C_j$ .

For every  $n'$ , consider any connected component  $V_{n'}^j$  of  $V_{n'} \cap B$ , so that  $V_{n'}^j$  converges to a certain irreducible component  $V^j$  of  $V$ . If  $V^i$  and  $V^j$  are not equal, they intersect properly, hence the same occurs for  $V_{n'}^i$  and  $V_{n'}^j$ , a contradiction. This proves that  $V$  is locally irreducible and  $\overline{V} \cap \partial B$  is a single loop  $C_j$ .

Now assume that the multiplicity of convergence is 1 and let us show that  $V$  is smooth. Assume by contradiction that  $V$  is singular at  $p$ . By the multiplicity 1 convergence hypothesis and the transversality of  $V$  and  $\partial B$ , for large  $n$  the loop  $C_{n'}^i = \overline{V_{n'}^i} \cap \partial B$  is smooth, close to  $C^i = \overline{V} \cap \partial B$  and (smoothly) isotopic to it.

The (smooth) *genus* of  $C^i$  is by definition the smallest genus of a smooth surface in  $B$  bounded by  $C^i$ . It is invariant under smooth isotopy. It is known that if  $V$  is singular at  $p$ , then for a sufficiently small ball  $B = B(p, r)$  around  $p$ , the genus of  $V \cap \partial B$  is positive (see [19, Cor 10.2]). Since  $V_{n'}^i$  is a holomorphic disk, we arrive at a contradiction, which finishes the proof.  $\square$

### 3. Uniform geometry of (un)stable manifolds

In this section we consider a weakly stable substantial family  $(f_\lambda)_{\lambda \in \Lambda}$  of polynomial automorphisms of dynamical degree  $d \geq 2$ . Let  $\lambda_0 \in \Lambda$ , and fix a saddle periodic point  $p_0$  for  $f_0$ . By weak stability,  $p_\lambda$  persists as a saddle point throughout the family. Our purpose is to give uniform estimates on the geometry of  $W_{\text{loc}}^{s/u}(p_\lambda)$ , depending only on that of  $W_{\text{loc}}^{s/u}(p_0)$ . For concreteness, from now on we deal with unstable manifolds. Recall that by assumption  $\Lambda$  is the unit disk.

We present two types of results, which will both be used afterwards. In § 3.1 we show that the area of the local unstable manifold of  $p_\lambda$  can be controlled throughout  $\Lambda$ . The techniques here are reminiscent of the results of [5, § 3]. We introduce a notion of size of a manifold at a point in § 3.2, and show that unstable parameterizations can be controlled in term of the size

of  $W^u(p)$  at  $p$ . Finally in § 3.3 we show that the size of  $W^u(p_\lambda)$  at  $p_\lambda$  is uniformly bounded from below in a neighborhood of  $\lambda_0$  that depends only on the size of  $W^{s/u}(p_0)$  at  $p_0$ .

### 3.1. Areas of local unstable manifolds

By definition we say that  $D \subset \mathbb{C}^2$  is a *holomorphic disk* if there is a holomorphic map  $\phi : \mathbb{D} \rightarrow \mathbb{C}^2$  with  $\phi(\mathbb{D}) = D$ , that extends to a homeomorphism  $\overline{\mathbb{D}} \rightarrow \overline{D}$ .

In the next lemma we give a basic estimate on the geometry of a holomorphic disk in  $\mathbb{C}^2$ , relying on simple ideas from conformal geometry. The modulus of an annulus, will be denoted by  $\text{mod}(A)$ .

LEMMA 3.1. – *Let  $D_1 \Subset D_2$  be a pair of holomorphic disks in  $\mathbb{C}^2$  with  $0 \in D_1$  and let  $d_1, d_2, m$  be positive real numbers such that*

$$\sup_{z \in D_2} \|z\| \leq d_2, \quad \sup_{z \in \partial D_1} \|z\| \geq d_1, \quad \text{and} \quad \text{mod}(D_2 \setminus \overline{D_1}) \geq m.$$

*Then there exist positive constants  $A$  and  $r$  depending only on  $d_1, d_2$  and  $m$  such that the connected component of  $D_2 \cap B(0, r)$  containing  $0$  is a properly embedded submanifold in  $B(0, r)$ , of area not greater than  $A$ .*

Notice that if  $D$  is a holomorphic disk,  $\partial D$  refers to the boundary of  $D$  relative to its intrinsic topology. Notice also that the maximum principle applied to the subharmonic function  $z \mapsto \|z\|^2$  on  $D$  implies that  $\sup_{z \in D} \|z\| = \sup_{z \in \partial D} \|z\|$ .

*Proof.* – Fix a biholomorphism  $\phi : \mathbb{D} \rightarrow D_2$  with  $\phi(0) = 0$ . We claim that there exists  $\delta > 0$  depending only on  $m$  such that  $\phi^{-1}(D_1) \subset D(0, 1 - \delta)$ . Indeed by assumption  $\text{mod}(\mathbb{D} \setminus \overline{\phi^{-1}(D_1)}) \geq m$ . Now it follows from a classical result of Grötzsch that if  $U$  is a connected and simply connected open subset of  $\mathbb{D}$  containing  $0$  and  $z$  with  $|z| = 1 - x \in (0, 1)$  then  $\text{mod}(\mathbb{D} \setminus \overline{U}) \leq \text{mod}(\mathbb{D} \setminus [0, 1 - x])$  (see Ahlfors [1, Thm 4-6]). In addition, the map  $x \mapsto \rho(x) := \text{mod}(\mathbb{D} \setminus [0, 1 - x])$  is increasing and continuous. Taking the contrapositive, we see that if  $\text{mod}(\mathbb{D} \setminus \overline{\phi^{-1}(D_1)}) \geq m$ , then  $\phi^{-1}(D_1)$  is contained in  $D(0, 1 - \delta)$  with  $\delta := \rho^{-1}(m)$ , and we conclude that  $\phi$  satisfies  $\sup_{\mathbb{D}} \|\phi\| \leq d_2$  and  $\sup_{\partial D(0, 1 - \delta)} \|\phi\| \geq d_1$ . The result then follows from Lemma 3.2 below.  $\square$

LEMMA 3.2. – *Let  $\phi : \mathbb{D} \rightarrow \mathbb{C}^2$  be a holomorphic mapping fixing  $0$  and such that  $\sup_{\mathbb{D}} \|\phi\| \leq d_2$  and  $\sup_{\partial D(0, 1 - \delta)} \|\phi\| \geq d_1$ . Then there exist constants  $A$  and  $r$  depending only on  $d_1$  and  $d_2$  such that the connected component of  $\phi(D(0, 1 - \delta)) \cap B(0, r)$  containing  $0$  is a properly embedded subvariety in  $B(0, r)$ , of area not greater than  $A$ .*

Note that in the application of this statement to the proof of Lemma 3.1,  $D_2$  is smooth by definition so we automatically get a submanifold instead of a mere subvariety.

*Proof.* – This is an elementary compactness argument. Indeed let us show that for every such  $\phi$  there exists a uniform  $r$  such that the connected component  $C$  of  $\phi^{-1}(B(0, r))$  containing  $0$  is relatively compact in  $D(0, 1 - \delta)$ , that is  $\overline{C} \subset D(0, 1 - \delta)$ . The area bound in turns follows from the Cauchy inequality. To prove that such a  $r$  exists, for the sake of contradiction we suppose the existence of a sequence  $(\phi_n)$  of such functions which violate this property for  $r_n \rightarrow 0$ . Hence there exists for every  $n$  a connected compact set  $C_n \subset \overline{D}(0, 1 - \delta)$

of diameter  $\geq 1 - \delta$  sent into  $\overline{B}(0, r_n)$  by  $\phi_n$ . We can suppose that  $(C_n)_n$  converges to a connected compact set  $C_\infty$  of diameter  $\geq 1 - \delta$  and that  $(\phi_n)_n$  converges uniformly to a certain  $\phi_\infty$  on  $\overline{D}(0, 1 - \delta)$ . Then  $\phi_\infty$  vanishes on  $C_\infty$  hence on  $\mathbb{D}$ . This contradicts the fact that  $\sup_{\partial D(0, 1 - \delta)} \|\phi_\infty\| \geq d_1$ .  $\square$

Lemmas 3.1 and 3.2 can be combined to estimate how the geometry of an unstable manifold varies in a weakly stable family. For a saddle point  $p$  and a positive real number  $r$ , we denote by  $W_r^u(p)$  the connected component of  $W^u(p) \cap B(p, r)$  containing  $p$ , which by the maximum principle is a holomorphic disk.

**PROPOSITION 3.3.** – *Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a weakly stable substantial family of polynomial automorphisms of  $\mathbb{C}^2$  of dynamical degree  $d \geq 2$ . Fix a saddle periodic point  $p_0$  for  $f_0$ , and denote by  $(p_\lambda)_{\lambda \in \Lambda}$  its continuation.*

*Consider a pair  $D_1 \Subset D_2$  of holomorphic disks in  $W^u(p_0)$ , with  $p_0 \in D_1$ , and let*

$$g_1 = \sup(G_0^+|_{D_1}), \quad g_2 = \sup(G_0^+|_{D_2}) \text{ and } m = \text{mod}(D_2 \setminus \overline{D_1}).$$

*Then there exist positive constants  $r$ ,  $g$  and  $A$  depending only on  $g_1$ ,  $g_2$  and  $m$  such that for every  $\lambda \in D(0, 1/4)$ ,  $W_r^u(p_\lambda)$  is a properly embedded submanifold into  $B(p_\lambda, r)$ , contained in  $h_\lambda(D_1)$ , whose area is not greater than  $A$ , and such that  $\sup(G_\lambda^+|_{W_r^u(p_\lambda)}) \geq g$ .*

Observe that  $G^+$  does not vanish identically in any neighborhood of  $p$  in  $W_{\text{loc}}^u(p)$  so  $g_1, g_2$  are indeed positive.

Notice also that the same result holds by replacing 0 by an arbitrary  $\lambda_0 \in \Lambda$ , in which case  $D(0, 1/4)$  has to be replaced by the disk of radius  $\ln 2$  about  $\lambda_0$  in the Poincaré metric ( $\ln 2$  is the Poincaré radius of  $D(0, 1/4) \Subset D(0, 1/3)$ ).

*Proof.* – By Proposition 2.2 for  $\lambda \in D(0, 1/3)$  we can consider the disks  $h_\lambda(D_1)$  and  $h_\lambda(D_2)$ . The quasiconformality of holomorphic motions implies that for  $\lambda \in D(0, 1/4)$ ,

$$\frac{1}{C}m \leq \text{mod}\left(h_\lambda(D_2) \setminus \overline{h_\lambda(D_1)}\right) \leq Cm,$$

where  $C$  is a universal constant. In addition, it follows from Lemma 2.3 that

$$\sup(G_\lambda^+|_{h_\lambda(D_2)}) \leq C'g_2 \text{ and } \sup(G_\lambda^+|_{h_\lambda(\partial D_1)}) = \sup(G_\lambda^+|_{h_\lambda(D_1)}) \geq (C')^{-1}g_1,$$

where again  $C'$  is universal.

Now recall that  $G_\lambda^-(z)$  is jointly continuous in  $(\lambda, z)$  and that for every  $\lambda$ ,  $G^+|_{K_\lambda^-}$  is proper. Since unstable manifolds are contained in  $K^-$  we infer that there exists  $d_2$  depending only on  $g_2$  and the family  $(f_\lambda)$  such that for  $\lambda \in D(0, 1/4)$ ,

$$\sup_{z \in h_\lambda(D_2)} \|z - p_\lambda\| \leq d_2.$$

Also it is known that the Green function  $G^+$  is Hölder continuous (see [11, Thm 1.2]). Moreover the proof of [11] easily shows that the modulus of continuity of  $G^+$  is locally uniform in  $\Lambda$ . Therefore,  $G_\lambda^+(p_\lambda) = 0$  implies the existence of a constant  $d_1$  depending only on  $g_1$  and the family  $(f_\lambda)$  such that for  $\lambda \in D(0, 1/4)$ ,

$$\sup_{z \in h_\lambda(\partial D_1)} \|z - p_\lambda\| \geq d_1.$$

Applying Lemma 3.1 then finishes the proof.  $\square$

**COROLLARY 3.4.** – *With notation as in Proposition 3.3, for every  $\tilde{\Lambda} \in \Lambda$ , there exist positive constants  $r$ ,  $g$  and  $A$  depending only on  $\tilde{\Lambda}$ ,  $g_1$ ,  $g_2$  and  $m$  such that for every  $\lambda \in \tilde{\Lambda}$ ,  $W_r^u(p_\lambda)$  is a properly embedded submanifold into  $B(p_\lambda, r)$ , contained in  $h_\lambda(D_1)$ , whose area is not greater than  $A$ .*

*Proof.* – Indeed, for  $\lambda \in \tilde{\Lambda}$  there exists a finite chain of Poincaré disks of radii  $\ln 2$  connecting 0 to  $\lambda$ , whose length depends only on  $\tilde{\Lambda}$ . Then to get the result it is enough to apply inductively the previous proposition to this chain of disks.  $\square$

### 3.2. Estimates on unstable parameterizations and applications

Endow  $\mathbb{C}^2$  with its natural Hermitian structure. A *bidisk of size  $r$*  is the image of  $D(0, r)^2$  by some affine isometry. The image of the unit bidisk under a general affine map will be referred to as an *affine bidisk*. A curve  $V$  in  $\mathbb{C}^2$  is a graph over an affine line  $L$  if the orthogonal projection onto  $L$  is injective when restricted to  $V$ . Then we have a well-defined notion of slope of a holomorphic curve with respect to  $L$ .

**DEFINITION 3.5.** – *A curve  $V$  through  $p$  is said to have bounded geometry at scale  $r$  at  $p$  (we also simply say that  $V$  has size  $r$  at  $p$ ) if there exists a neighborhood of  $p$  in  $V$  that is a graph of slope at most 1 over a disk of radius  $r$  in the tangent space  $T_p V$ .*

Let  $V$  be a disk of size  $r$  at  $p$ , and fix orthonormal coordinates  $(x, y)$  so that  $p = 0$  and  $T_p V = \{y = 0\}$ . Then the connected component of  $V$  through  $p$  in the bidisk  $D(0, r)^2$  is a graph  $\{y = \varphi(x)\}$  over the first coordinate with  $|\varphi'| \leq 1$  and  $\varphi'(0) = 0$ .

**REMARK 3.6.** – The Schwarz lemma implies that for every  $x \in D(0, r)$ ,  $|\varphi'(x)| \leq |x|/r$ .

It will be a key fact for us that the Koebe Distortion Theorem provides estimates on unstable parameterizations in terms of the size of local unstable manifolds (see also Lemma 3.11 below).

**LEMMA 3.7.** – *Let  $f$  be a polynomial automorphism of  $\mathbb{C}^2$  and  $p$  a saddle periodic point. Assume that  $W^u(p)$  is of size  $r$  at  $p$ . Normalize the coordinates so that  $p = (0, 0)$  and  $W^u(p)$  is tangent to the  $x$ -axis at  $p$ . Denote by  $\pi$  the first coordinate projection and let  $\Gamma^u(p)$  be the component of  $\pi^{-1}(D(0, r)) \cap W^u(p)$  containing  $p$ .*

*Let  $\psi^u : \mathbb{C} \rightarrow \mathbb{C}^2$  be an unstable parameterization, such that  $\psi^u(0) = p$ , and  $\|(\psi^u)'(0)\| = 1$ . Then  $\psi^u(D(0, \frac{r}{4})) \subset \Gamma^u(p) \subset D(0, r)^2$ . Moreover for every  $|z| \leq \frac{r}{8}$ ,*

$$(1) \quad D\left(0, \frac{|z|}{4}\right) \subset \pi \circ \psi^u(D(0, |z|)) \subset D(0, 4|z|).$$

*Proof.* – Without loss of generality, rotate the first coordinate so that  $(\pi \circ \psi^u)'(0) = 1$ . Under the assumptions of the lemma,  $\pi \circ \psi^u$  is univalent from some unknown domain  $\Omega \subset \mathbb{C}$  onto  $D(0, r)$ . Now recall the Koebe Distortion Theorem (see [1, Thm 5-3]): if  $g : \mathbb{D} \rightarrow \mathbb{C}$  is a univalent mapping, with  $g'(0) = 1$ , then for  $z \in \mathbb{D}$ ,

$$(2) \quad \frac{|z|}{4} \leq \frac{|z|}{(1+|z|)^2} \leq |g(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

Applying this to  $g(z) = r^{-1}(\pi \circ \psi^u)^{-1}(rz)$ , we first deduce that  $(\pi \circ \psi^u)^{-1}(D(0, r)) \supset D(0, \frac{r}{4})$ , thus  $\psi^u(D(0, \frac{r}{4})) \subset D(0, r) \times \mathbb{C}$  and so  $\psi^u(D(0, \frac{r}{4})) \subset \Gamma^u(p)$ . It follows that

the function  $h$  in  $\mathbb{D}$  defined by  $\zeta \mapsto h(\zeta) = \frac{4}{r} \pi \circ \psi^u \left( \frac{r\zeta}{4} \right)$  is univalent and satisfies  $h'(0) = 1$ . Applying (2) to  $h$  yields (1), as desired.  $\square$

Another important idea in this paper is that of the *natural continuation* of an unstable parameterization. Let us explain what this is about. Fix a parameter  $\lambda_0 \in \Lambda$ , a saddle point  $p_{\lambda_0}$  for  $f_{\lambda_0}$ , and an unstable parameterization  $\psi_{\lambda_0}^u : \mathbb{C} \rightarrow \mathbb{C}^2$  (in practice we often choose it so that  $\|(\psi_{\lambda_0}^u)'(0)\| = 1$ ). We want to find a well-adapted holomorphic family of parameterizations  $\psi_\lambda^u$  of  $W^u(p_\lambda)$ , with  $\psi_\lambda^u(0) = p_\lambda$ . This construction is based on the Bers-Royden theorem so in this discussion we work over the Poincaré disk  $D_{\text{Poin}}(\lambda_0, \ln 3)$ .

Since the Bers-Royden extension is canonical, the motion in  $\mathbb{C}^2$  of a given point  $q_{\lambda_0} \in W^u(p_{\lambda_0})$  (denoted by  $q_\lambda$ ) does not depend on this choice of parameterizations. Fix such a point  $q_{\lambda_0}$ , say  $q_{\lambda_0} = \psi_{\lambda_0}^u(1)$ . We now fix the parameterization of  $W^u(p_\lambda)$  by declaring that  $(\psi_\lambda^u)^{-1}(q_\lambda) = (\psi_0^u)^{-1}(q_0) = 1$ , or equivalently  $\psi_\lambda^u(1) = q_\lambda$ . This is by definition the *natural continuation* of  $\psi_0^u$  in the family.

Such a holomorphic family of parameterizations can be constructed from any given holomorphic family  $\tilde{\psi}_\lambda^u$  by the formula

$$\psi_\lambda^u(z) = \tilde{\psi}_\lambda^u \left( (\tilde{\psi}_\lambda^u)^{-1}(q_\lambda)z \right).$$

The advantage is now that the holomorphic motion  $h_\lambda^u$  in  $\mathbb{C}$  defined by looking at the motion of points in the coordinate  $\psi_\lambda^u$ , that is,  $h_\lambda^u(z) = (\psi_\lambda^u)^{-1}(h_\lambda(\psi_0^u(z)))$ , is normalized by  $h_\lambda^u(0) = 0$  and  $h_\lambda^u(1) = 1$ . It is well known that such a normalized holomorphic motion in  $\mathbb{C}$  satisfies uniform bounds : for every  $\tilde{\Lambda} \Subset D_{\text{Poin}}(\lambda_0, \ln 3)$ , there exist constants  $A, B$  and  $\alpha$  depending only on  $\tilde{\Lambda}$  such that if  $\lambda, \lambda' \in \tilde{\Lambda}$ ,

$$(3) \quad |h_\lambda^u(z) - h_{\lambda'}^u(z)| \leq A\rho(z, z')^\alpha + B|\lambda - \lambda'|,$$

where  $\rho$  denotes the spherical metric (see [7, Cor. 2]).

We now use these techniques to give an estimate on parameterizations which supplements Proposition 3.3.

**PROPOSITION 3.8.** – *Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a weakly stable substantial family of polynomial automorphisms of  $\mathbb{C}^2$  and let  $(p_\lambda)_{\lambda \in \Lambda}$  be a holomorphically moving saddle point. Assume that for  $\lambda = \lambda_0$ ,  $W^u(p_{\lambda_0})$  is of size  $r_0$  at  $p_0$ . Let  $\psi_{\lambda_0}^u$  be an unstable parameterization of  $W^u(p_{\lambda_0})$  and  $(\psi_\lambda^u)_{\lambda \in D_{\text{Poin}}(\lambda_0, \ln 3)}$  be its natural continuation.*

*Then for every  $\tilde{\Lambda} \subset D_{\text{Poin}}(\lambda_0, \ln 3)$  there exist constants  $c$  and  $M$  depending only on  $\tilde{\Lambda}$  and  $r_0$  such that if  $\lambda \in \tilde{\Lambda}$ ,  $\|\psi_\lambda^u\| \leq M$  on  $D(0, cr_0)$ .*

*Proof.* – By the Hölder continuity property of  $G^+$ , there exists  $g = g(r_0) > 0$  depending only on  $r_0$  such that  $\sup(G_{\lambda_0}^+|_{W_{r_0\sqrt{2}}^u(p_{\lambda_0})}) \leq g$  (recall that a bidisk of radius  $r_0$  is contained in a ball of radius  $r_0\sqrt{2}$ ). Hence by Lemma 3.7 we deduce that  $G_{\lambda_0}^+|_{\psi_{\lambda_0}^u(D(0, r_0/4))} \leq g$ , thus Lemma 2.3 implies that  $G_\lambda^+|_{\psi_\lambda^u(h_\lambda^u(D(0, r_0/4)))} \leq Cg$  where  $C$  depends only on  $\tilde{\Lambda}$ . By the properness of  $G^+|_{K^-}$ , we deduce that  $\psi_\lambda^u(h_\lambda^u(D(0, r_0/4)))$  is uniformly bounded by  $M(\tilde{\Lambda}, r_0)$  in  $\mathbb{C}^2$ . Finally, since  $h_\lambda^u(0) = 0$ , by (3) we infer that if  $\lambda \in \tilde{\Lambda}$ , then  $h_\lambda^u(D(0, r_0/4))$  contains  $D(0, cr_0)$  for some  $c = c(r_0, \tilde{\Lambda})$ , and we conclude that  $\|\psi_\lambda^u(D(0, cr_0))\| \leq M$  for  $\lambda \in \tilde{\Lambda}$ , which was the desired result.  $\square$

### 3.3. Local persistence of the size of unstable manifolds

Recall the notation  $\widehat{p} = \{(\lambda, p_\lambda), \lambda \in \Lambda\}$  for a holomorphically moving saddle point  $p_\lambda$ . Also, let us denote by  $\text{Tub}(\widehat{p}, r)$  the fibered tubular neighborhood of  $\widehat{p}$  of radius  $r$  in  $\Lambda \times \mathbb{C}^2$ , defined by

$$\text{Tub}(\widehat{p}, r) = \{(\lambda, z) \in \Lambda \times \mathbb{C}^2, \|z - p(\lambda)\| < r\}.$$

Let us first isolate a geometric lemma. For notational ease, we put  $\mathbb{C}_\lambda^2 = \{\lambda\} \times \mathbb{C}^2$ .

LEMMA 3.9. – *Let  $\widehat{p}$  be the graph of a holomorphic mapping  $p : \Lambda \rightarrow \mathbb{C}^2$  that is uniformly bounded by  $M$  on  $\Lambda$ . Fix a direction  $e \in \mathbb{P}^1(\mathbb{C})$ . Fix domains  $\widetilde{U}$  and  $U$  such that  $\widetilde{U} \Subset U \Subset \Lambda$  and assume that  $\Sigma$  is a hypersurface that is closed in  $\text{Tub}(\widehat{p}, r) \cap (U \times \mathbb{C}^2)$  and such that for each  $\lambda \in U$ ,  $\Sigma \cap \mathbb{C}_\lambda^2$  is a graph of slope at most 1 over  $p(\lambda) + e$ . Then  $\Sigma$  is smooth and the volume of  $\Sigma \cap (\widetilde{U} \times \mathbb{C}^2)$  is bounded by a constant depending only on  $\widetilde{U}$ ,  $U$  and  $r$ .*

*Proof.* – Identify  $e$  and the corresponding line through 0 in  $\mathbb{C}^2$ , and similarly for its orthogonal complement  $e^\perp$ . For  $\lambda \in U$ , let  $\phi_\lambda$  be the unique holomorphic map from an open subset of  $p(\lambda) + e$  to its orthogonal complement, whose graph is  $\Sigma \cap \mathbb{C}_\lambda^2$ . Let  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$  be the orthogonal projection on the line  $e$ , and let  $\widehat{\pi}(\lambda, z) = (\lambda, \pi(z))$ .

By continuity of the map  $(\lambda, z) \mapsto \phi_\lambda(z)$ , the following is an open subset of  $U \times e \approx U \times \mathbb{C}$ .

$$\widehat{\pi}(\Sigma) = \{(\lambda, z) \in U \times e : |\phi_\lambda(z)|^2 + |z|^2 < r^2\}.$$

By the graph property, the map  $\widehat{\pi}|_\Sigma$  is one-to-one from the subvariety  $\Sigma$  onto the open subset  $\widehat{\pi}(\Sigma)$ . Under these conditions it is classical that  $\widehat{\pi}|_\Sigma$  is a biholomorphism (see Prop. 3 p. 32 in [8]). In particular  $\Sigma$  is smooth and the map  $(\lambda, z) \in \widehat{\pi}(\Sigma') \mapsto \phi_\lambda(z)$  is holomorphic in both variables (being the inverse of  $\widehat{\pi}$ ).

To get the volume bound, we remark that the set  $\widehat{\pi}(\Sigma)$  is bounded (specifically, it is contained in  $U \times B(0, r)$ ). The derivative  $\partial_z \phi_\lambda$  is bounded by 1, and the image of  $\phi_\lambda$  is bounded by  $r$ . The Cauchy estimate implies that the derivative  $\partial_\lambda \phi_\lambda$  is bounded on  $\widetilde{U}$ . Consequently the volume of  $\Sigma \cap (\widetilde{U} \times \mathbb{C}^2)$  is bounded by a constant depending only on  $\widetilde{U}$ ,  $U$  and  $r$ .  $\square$

The main result in this subsection is that in a weakly stable family, the size of a holomorphically moving unstable manifold is locally uniformly bounded from below.

PROPOSITION 3.10. – *Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a weakly stable substantial family of polynomial automorphisms of  $\mathbb{C}^2$  and let  $\widehat{p} = (p_\lambda)_{\lambda \in \Lambda}$  be a holomorphically moving saddle point. Let  $\widetilde{\Lambda} \Subset \Lambda$  be a relatively compact open subset and fix  $\lambda_0 \in \widetilde{\Lambda}$ . Assume that for  $\lambda = \lambda_0$ ,  $W^u(p_0)$  is of size  $r_2$  at  $p_0$ . Then, for every  $r_1 < r_2$ , there exists  $\delta = \delta(r_1, r_2, \widetilde{\Lambda})$  depending only on  $r_1$ ,  $r_2$  and  $\widetilde{\Lambda}$  such that if  $|\lambda - \lambda_0| < \delta$ ,  $W^u(p_\lambda)$  is of size  $r_1$  at  $p_\lambda$ , and  $W_{r_1}^u(p_\lambda)$  is a graph of slope at most 1 over  $p_\lambda + E^u(p_0)$  (where  $E^u(p_0)$  denotes the unstable direction at  $p_0$ ).*

*Furthermore, there exists a submanifold  $\widehat{W}_{r_1}^u$  in  $\text{Tub}(\widehat{p}, r_1) \cap (D(\lambda_0, \delta) \times \mathbb{C}^2)$ , such that for every  $\lambda \in D(\lambda_0, \delta)$ ,  $\widehat{W}_{r_1}^u \cap \mathbb{C}_\lambda^2 = W_{r_1}^u(p_\lambda)$ , whose volume is bounded by a constant  $V(r_1, r_2, \widetilde{\Lambda})$  depending only on  $r_1$ ,  $r_2$  and  $\widetilde{\Lambda}$ .*



*Proof.* – Without loss of generality assume  $\lambda_0 = 0$ . Start with an unstable parameterization  $\psi_0^u$  of  $W^u(p_0)$  satisfying  $\|(\psi_0^u)'(0)\| = 1$ , and let  $\pi_0$  be the orthogonal projection onto  $E^u(p_0)$ . For  $i = 1, 2$ , we denote by  $D_i = (\pi_0 \circ \psi_0^u)^{-1}(D(0, r_i))$ , and let  $D_{12} := (\pi_0 \circ \psi_0^u)^{-1}(D(0, r_{12}))$ , with  $r_{12} := (r_1 + r_2)/2$ . These are simply connected domains in  $\mathbb{C}$  containing the origin, satisfying  $D_1 \Subset D_{12} \Subset D_2$ .

The following lemma will be proved afterwards.

LEMMA 3.11. – *For every  $z \in D(0, r_{12})$ , the following derivative estimate holds:*

$$(4) \quad \frac{1 - r_1/r_2}{16} \leq \left| \left( (\pi_0 \circ \psi_0^u)^{-1} \right)'(z) \right| \leq \frac{16}{(1 - r_1/r_2)^3}.$$

Moreover the distance between  $D_1$  and  $\partial D_2$  is greater than  $r_2(1 - r_1/r_2)^2/32$ .

Let now  $(\psi_\lambda^u)_{\lambda \in D(0, 1/3)}$  be the natural continuation of  $\psi_0^u$ . The second assertion of Lemma 3.11 together with (3) imply that there exists  $\delta = \delta(r_1, r_2, \tilde{\Lambda}) < 1/3$  such that if  $|\lambda| < \delta$ ,  $h_\lambda^u(D_1)$  stays uniformly far from  $\partial(h_\lambda(D_2))$  (farther than  $r_2(1 - r_1/r_2)^2/50$ , say) relative to the Euclidean metric on  $\mathbb{C}$ . Furthermore, arguing exactly as in Proposition 3.8, we see that for  $|\lambda| < \delta$ ,  $\psi_\lambda^u(h_\lambda^u(D_2))$  is uniformly bounded in  $\mathbb{C}^2$ .

Let  $\Psi : \Lambda \times \mathbb{C} \rightarrow \Lambda \times \mathbb{C}^2$  be defined by  $\Psi(\lambda, z) = (\lambda, \psi_\lambda^u(z))$ , and put

$$\widehat{D}_i = \bigcup_{\lambda \in B(0, \delta)} \{\lambda\} \times h_\lambda^u(D_i), \text{ for } i = 1, 2.$$

Since  $h_\lambda^u(D_1)$  stays far from  $\partial(h_\lambda^u(D_2))$ , and  $\Psi(\widehat{D}_2)$  is uniformly bounded in  $B(0, \delta) \times \mathbb{C}^2$ , by the Cauchy estimates, reducing  $\delta$  again slightly if necessary the derivatives of  $\Psi$  are uniformly bounded on  $\widehat{D}_1$ , with bounds depending only on  $\tilde{\Lambda}$ ,  $r_1$  and  $r_2$ .

We are now ready to conclude the proof. Let  $\pi_\lambda$  (resp.  $\pi_\lambda^\perp$ ) be the orthogonal projection onto  $E^u(p_\lambda)$  (resp.  $(E^u(p_\lambda))^\perp$ ). The curve  $W^u(p_\lambda)$  is of size  $r_1$  at  $p_\lambda$  if for every  $z$  in  $D_1^\lambda := \{z \in \mathbb{C}, |\pi_\lambda \circ \psi_\lambda^u(z)| < r_1\}$  the following estimate holds:

$$(5) \quad |\partial_z(\pi_\lambda^\perp \circ \psi_\lambda^u)(z)| \leq |\partial_z(\pi_\lambda \circ \psi_\lambda^u)(z)|.$$

By the Cauchy estimate on  $\partial_\lambda \partial_z \Psi$ ,  $(\psi_\lambda^u)'(0)$  is close to  $(\psi_0^u)'(0)$  for  $|\lambda| < \delta$ . In particular choosing  $\delta = \delta(r_1, r_2, \tilde{\Lambda})$  small enough we can ensure that  $\|\pi_0 - \pi_\lambda\| \leq \varepsilon$  (resp.  $\|\pi_0^\perp - \pi_\lambda^\perp\| \leq \varepsilon$ ), where  $\varepsilon$  is as small as we wish. By the Cauchy estimate on  $\partial_\lambda \Psi$ , for  $\delta = \delta(r_1, r_2, \tilde{\Lambda})$  sufficiently small, when  $|\lambda| < \delta$ , the set  $D_1^\lambda$  is included in  $D_{12} = (\pi_0 \circ \psi_0^u)^{-1}(D(0, r_{12}))$ , with  $r_{12} := (r_1 + r_2)/2$ . By Remark 3.6, for every  $z \in D_{12}$ , we have that

$$|\partial_z(\pi_0^\perp \circ \psi_0^u)(z)| \leq \frac{r_1 + r_2}{2r_2} |\partial_z(\pi_0 \circ \psi_0^u)(z)| = \left(1 - \frac{r_2 - r_1}{2r_2}\right) |\partial_z(\pi_0 \circ \psi_0^u)(z)|.$$

From this we infer that with  $\varepsilon$  as above and  $z \in D_{12}$ ,

$$(6) \quad \begin{aligned} |\partial_z(\pi_\lambda^\perp \circ \psi_\lambda^u)(z)| &\leq \left(1 - \frac{r_2 - r_1}{2r_2}\right) |\partial_z(\pi_\lambda \circ \psi_\lambda^u)(z)| \\ &\quad + \varepsilon \|\partial_z \psi_0^u(z)\| + \varepsilon \|\partial_z \psi_\lambda^u(z)\| + 2 \|\partial_z(\psi_0^u - \psi_\lambda^u)(z)\|. \end{aligned}$$

In addition, the right hand inequality in (4) implies that for  $z \in D_{12}$ ,

$$|\partial_z(\pi_0 \circ \psi_0^u)(z)| \geq \frac{(1 - r_1/r_2)^3}{16}.$$

By the Cauchy estimate on  $\partial_\lambda \partial_z \Psi$ , for  $\delta = \delta(r_1, r_2, \tilde{\Lambda})$  sufficiently small, a similar estimate holds for  $\partial_z(\pi_\lambda \circ \psi_\lambda^u)(z)$  (with 16 replaced by 32, say) for  $|\lambda| < \delta$  and  $z \in D_{12}$ . Recall that under our assumptions  $D_{12}$  contains  $D_1^\lambda$ . Thus, by choosing  $\varepsilon = \varepsilon(r_1, r_2, \tilde{\Lambda})$  appropriately and reducing  $\delta$  again if necessary, we can ensure that for  $z \in D_1^\lambda$ ,

$$\varepsilon \|\partial_z \psi_0^u(z)\| + \varepsilon \|\partial_z \psi_\lambda^u(z)\| + 2 \|\partial_z(\psi_0^u - \psi_\lambda^u)(z)\| \leq \frac{r_2 - r_1}{2r_2} |\partial_z(\pi_\lambda \circ \psi_\lambda^u)(z)|,$$

which by (6) yields (5).

Finally, we define  $\widehat{W}_{r_1}^u$  to be the connected component of  $\Psi(\widehat{D}_1)$  in  $\text{Tub}(\widehat{p}, r_1) \cap (D(0, \delta) \times \mathbb{C}^2)$  containing  $\widehat{p}$ , which is a surface with the desired properties (its smoothness follows from Lemma 3.9). □

*Proof of Lemma 3.11.* – By the Koebe Distortion Theorem (see [1, Thm 5-3]), if  $g : \mathbb{D} \rightarrow \mathbb{C}$  is a univalent mapping with  $g'(0) = 1$ , then for every  $r < 1$  and every  $z \in D(0, (1+r)/2)$  we have that

$$\frac{1-r}{16} \leq \frac{1-|z|}{(1+|z|)^3} \leq |g'(z)| \leq \frac{1+|z|}{(1-|z|)^3} \leq \frac{16}{(1-r)^3}.$$

Applying this to  $g(z) = r_2^{-1}(\pi \circ \psi_0^u)^{-1}(r_2 z)$  and  $r = r_1/r_2$ , we deduce the desired bound on  $((\pi \circ \psi_0^u)^{-1})'$ . The estimate on the distance from  $D_1$  to  $\partial D_2$  immediately follows. □

REMARK 3.12. – One may wonder why we did not conclude to the existence of such a submanifold  $\widehat{W}^u$  in  $D(0, 1/3) \times \mathbb{C}^2$  straight after Proposition 3.3, using a “fibered” compactness argument in the style of Lemma 3.2.

The trouble is that in this general situation, the information about the area of  $W_r^u(p_\lambda)$  is not sufficient to control the geometry (say, the volume) of  $\widehat{W}^u$  because  $\widehat{W}^u \cap (\{\lambda\} \times B(p_\lambda, r))$  can get disconnected for some values of  $\lambda$ , and the geometry of components other than  $W_r^u(p_\lambda)$  can go out of control (see Figure 1 below for a graphical illustration). Proposition 3.10 shows that this phenomenon does not occur in a neighborhood of  $\lambda_0$ , whose size depends only on that of  $W_{\text{loc}}^u(p)$ .

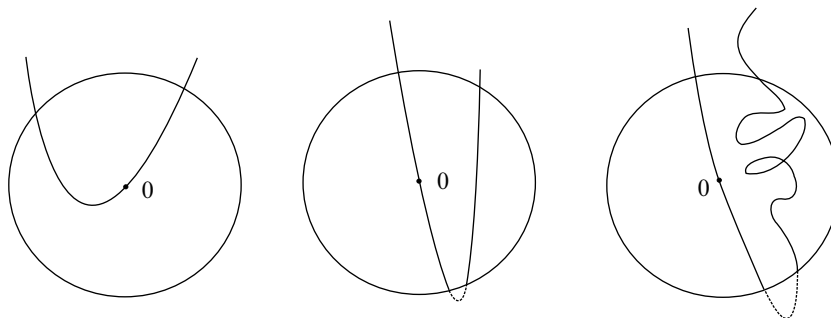


FIGURE 1. Evolution of  $\widehat{W}^u \cap (\{\lambda\} \times B(p_\lambda, r))$  in function of  $\lambda$ .

Let us briefly describe an explicit example where such a scenario happens.

Let  $\phi : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}^2$  be defined by  $\phi(\lambda, z) = z(z - 2\lambda)(z, g(z))$ , where  $g$  is a holomorphic function on  $\mathbb{D}$  such that  $|g| < 1$  but  $\int_{\mathbb{D}} |g'|^2 = \infty$ .

By Lemma 3.2, there exist positive constants  $r$  and  $A$  such that for every  $\lambda \in D(0, 3/4)$ , the connected component  $W_\lambda$  of  $\phi(\lambda, \mathbb{D}) \cap B(0, r)$  containing 0 is properly embedded and of area at most  $A$ .

Now let  $\Phi : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D} \times \mathbb{D}^2$  be defined by  $\Phi(\lambda, z) = (\lambda, \varphi(\lambda, z))$ . We see that  $\Phi(\mathbb{D} \times \{0\}) = \mathbb{D} \times \{0\}$ . With  $r$  as above, consider the component  $\widehat{W}$  of  $\Phi(\mathbb{D}^2) \cap \text{Tub}(\mathbb{D} \times \{0\}, r)$  containing  $\mathbb{D} \times \{0\}$ . Put  $V = \{(\lambda, z), z(z - 2\lambda) = 0\} \subset \mathbb{D}^2$ , and observe that  $\Phi(V) = \mathbb{D} \times \{0\}$ . Note that if  $|z| < r/4$  or  $|z - 2\lambda| < r/4$  then  $|\varphi(\lambda, z)| < r$ , thus  $\widehat{W}$  contains  $\Phi(\text{Tub}(V, r/4))$ . So we infer that when  $|\lambda|$  is close to  $1/2$ ,  $\widehat{W} \cap \mathbb{D}_\lambda^2$  is made of at least two irreducible components, and for the values of  $\lambda$  such that  $\int_{D(\lambda, r/4)} |g'|^2 = \infty$ , one of these is of infinite volume.  $\square$

#### 4. Holomorphic motion of regular points

In this section we introduce the concept of regular point for a polynomial automorphism of  $\mathbb{C}^2$ , and prove Theorem A, in a slightly more general form.

##### 4.1. Definitions and main statements

DEFINITION 4.1. – We say that  $p \in J^*$  is *u-regular* (resp. *s-regular*) if there exists  $r > 0$  and a sequence of saddle periodic points  $p_n$  converging to  $p$ , with the property that  $W^u(p_n)$  (resp.  $W^s(p_n)$ ) has bounded geometry at scale  $r$  at  $p_n$ .

If necessary, we make the size appearing in the definition explicit by speaking of “u-regular point of size  $r$ ”. The key property of u-regular (resp. s-regular) points is that they possess “local unstable (resp. stable) manifolds,” as the following proposition shows.

PROPOSITION 4.2. – Let  $f$  be a polynomial automorphism of  $\mathbb{C}^2$  with dynamical degree  $d \geq 2$ . Let  $p$  be a u-regular point of size  $r$ . Then there exists a unique submanifold  $W_r^u(p)$  of size  $r$  at  $p$  such that if  $(p_n)$  is any sequence of saddle points converging to  $p$ , such that  $W^u(p_n)$  is of size  $r$  at  $p_n$ , the sequence of disks  $(W_r^u(p_n))$  converges to  $W_r^u(p)$  with multiplicity 1 in  $B(p, r)$ . In particular the unstable directions converge as well.

By definition  $W_r^u(p)$  will be referred to as the *local unstable manifold* of  $p$  (and likewise for s-regular points). If the size  $r$  is not relevant (i.e., if we think of the local unstable manifold as a germ) we simply refer to it as  $W_{\text{loc}}^u(p)$ . Let us stress that we do *not* claim that  $W_{\text{loc}}^u(p)$  is an unstable manifold in the usual sense.

*Proof.* – Fix  $r' < r$ . Then for  $n \geq N(r')$ ,  $W_r^u(p_n) \cap B(p, r')$  is a closed submanifold in  $B(p, r')$ . Given any subsequence  $W_r^u(p_{n_j})$ , up to further extraction we may assume that the  $W_r^u(p_{n_j})$  are graphs of slope at most 2 over a fixed direction. It follows that all cluster values of the sequence  $(W_r^u(p_n))$  are smooth, irreducible and of multiplicity 1. From Proposition 2.5 we infer that this sequence actually converges and the proof is complete.  $\square$

DEFINITION 4.3. – We say that  $p \in J^*$  is regular if it is both  $s$ - and  $u$ -regular and if its local stable and unstable manifolds do not coincide at  $p$ . If in addition these local stable and unstable manifolds are transverse, we say that  $p$  is transverse regular.

Examples of transverse regular points include saddle periodic points, as well as transverse homoclinic intersections (due to Smale's horseshoe construction).

We recall that an ergodic invariant measure  $\nu$  is said to be hyperbolic if its Lyapunov exponents satisfy  $\chi^-(\nu) < 0 < \chi^+(\nu)$ . The following proposition is an immediate consequence of the so-called Katok Closing Lemma.

PROPOSITION 4.4. – Let  $\nu$  be a hyperbolic ergodic invariant measure. Then  $\nu$ -a.e. point is transverse regular.

*Proof.* – By the Main Lemma (in § 3 p.154) and Corollary 2.1 in [14] the following holds. For every  $\eta > 0$ , there exist  $r, \theta > 0$  and a set  $\Lambda_\eta$  of measure at least  $1 - \eta$ , made of points whose Pesin stable and unstable manifolds are of size  $r$  and transverse, with angle bounded from below by  $\theta$ . Moreover, for every  $\delta > 0$ , there exists a number  $\psi = \psi(\delta, \eta) > 0$  so that for every point  $z \in \Lambda_\eta$  satisfying  $f^n(z) \in \Lambda_\eta \cap B(z, \psi)$  for some  $n \geq 1$ , there exists an  $n$ -periodic point  $z' \in B(z, \delta)$  whose stable and unstable manifolds are of size  $r$  with a uniform angle at  $z'$ .

By the Poincaré Recurrence Theorem,  $\nu$ -almost every point returns arbitrarily close to itself. Hence the above statement implies that for  $\nu$ -almost every point in  $\Lambda_\eta$ , there exists a sequence of periodic points converging to  $z$ , whose stable and unstable manifolds are of size  $r$  and uniformly transverse. Hence  $z$  is transverse regular. The result finally follows by letting  $\eta$  decrease to 0.  $\square$

Let us now introduce a weaker notion of regularity, which involves the stable direction only.

DEFINITION-PROPOSITION 4.5. – Let  $p \in J^*$  be a  $s$ -regular point. We say that  $p$  is  $s$ -exposed if one of the following equivalent properties is satisfied:

- (i)  $W_{\text{loc}}^s(p)$  is not contained in  $K$ ;
- (ii)  $G^-|_{W_{\text{loc}}^s(p)} \neq 0$ ;
- (iii)  $T^- \wedge [W_{\text{loc}}^s(p)] > 0$ ;
- (iv) for every saddle point  $q$ , the manifold  $W^u(q)$  admits transverse intersections with  $W_{\text{loc}}^s(p)$ .

*Proof.* – The equivalence between (i), (ii) and (iii) is clear. To see that (iv) implies (i), it suffices to notice that by the inclination lemma, a small neighborhood of  $W^u(q) \cap W_{\text{loc}}^s(p)$  in  $W_{\text{loc}}^s(p)$  cannot be included in  $K$ . The fact that (iii) implies (iv) readily follows from Lemma 4.6 below.  $\square$

LEMMA 4.6. – Let  $f$  be a polynomial automorphism of  $\mathbb{C}^2$  of dynamical degree  $d \geq 2$ . Let  $\Delta$  be a holomorphic disk in  $\mathbb{C}^2$  such that  $T^- \wedge [\Delta] > 0$ . Then for any saddle point  $q$ ,  $W^u(q)$  admits transverse intersections with  $\Delta$ .

This result is implicit in [2, § 9] and was explained in detail in the proof of [10, Lemma 5.1]. Let us sketch the proof for the reader's convenience.

*Proof (sketch).* – Let  $\psi$  be a smooth cutoff function on  $\Delta$  such that  $\int \psi[\Delta] \wedge T^- > 0$ . It follows from [4, Thm 1.6] that  $d^{-n}(f^n)^*(\psi[\Delta])$  converges in the sense of currents to  $cT^+$ , with  $c = \int \psi[\Delta] \wedge T^-$ . Let now  $P$  be a Pesin set of positive  $\mu$ -measure and  $S^-$  be the uniformly laminar current made of the local Pesin unstable manifolds  $W_{\text{loc}}^u(z)$  for  $z \in P$ , with transverse measure given by the stable conditionals of  $\mu$ . By [2, §8], we have that  $0 < S^- \leq T^-$ ,  $S^-$  has continuous potentials, and  $S^- \wedge T^+ \geq \mu|_P > 0$ . Therefore for large enough  $n$ ,  $(f^n)^*(\psi[\Delta]) \wedge S^- > 0$ , so  $f^{-n}(\Delta)$  admits non-trivial intersections with  $W_{\text{loc}}^u(z)$  for some  $z \in P$ . Applying the inclination lemma and Poincaré recurrence, we infer that increasing  $n$  further if necessary,  $f^{-n}(\Delta)$  admits transverse intersections with  $W_{\text{loc}}^u(z)$  for every  $z \in P$ .

To conclude the proof, we apply the same argument to a neighborhood of  $q$  in  $W^u(q)$  to get that  $W^u(q)$  contains disks arbitrary  $C^1$ -close (i.e., with multiplicity 1) to  $W_{\text{loc}}^u(z)$  for some  $z \in P$ . Thus  $f^{-n}(\Delta)$  intersects  $W^u(q)$  transversally and we are done.  $\square$

PROPOSITION 4.7. – *If  $p \in J^*$  is regular, then it is s- and u-exposed.*

*Proof.* – It is enough to prove that  $p$  is s-exposed. Let  $(p_n)$  be a sequence of saddle points with  $W^u(p_n)$  of size  $r$  at  $p_n$  converging to  $p$ . We assume that the sequence  $(p_n)$  takes pairwise distinct values; the remaining case—which occurs when  $p$  is saddle periodic point—is easy and left to the reader. Then removing at most one term from this sequence we may assume that for every  $n$ ,  $p \notin W^u(p_n)$ . We claim that for large  $n$ ,  $W_r^u(p_n)$  intersects transversally  $W_{\text{loc}}^s(p)$  at a point close to  $p$ . If  $W_{\text{loc}}^u(p)$  and  $W_{\text{loc}}^s(p)$  are transverse this is clear. If not, since  $W_r^u(p_n) \cap W_r^u(p) = \emptyset$ , this follows from [2, Lemma 6.4]. In any case, arguing as in the implication (iv)  $\Rightarrow$  (i) of Proposition 4.5 we conclude that  $W_{\text{loc}}^s(p)$  is not contained in  $K$  and we are done.  $\square$

Here is a basic example:

EXAMPLE 4.8. – *If  $p$  is a saddle point and  $q$  belongs to the boundary of  $W^s(p) \cap K^-$  relative to the intrinsic topology of  $W^s(p)$ , then  $q$  is s-regular and exposed. Indeed it is shown in [10, Lemma 5.1] that  $q$  is the limit of a sequence of homoclinic intersections  $(t_n)$ , thus  $q$  is exposed inside  $W^s(p)$ . Furthermore if  $\Delta \subset W^s(p)$  is any disk containing  $p$  and  $q$ , it follows from Smale’s horseshoe construction that for every  $n$ ,  $t_n$  is a limit of a sequence of saddle points  $(p_{n,k})_k$  whose stable manifolds are graphs over  $\Delta$ . By considering the diagonal sequence  $p_{n,n}$  we conclude that  $q$  is s-regular, as desired.  $\square$*

Also there are examples of points which are s-regular and exposed but a priori not regular:

EXAMPLE 4.9. – *Let  $f$  be a dissipative polynomial automorphism. Let  $m$  be an ergodic probability measure supported on  $J^*$  with the property that  $m = \lim m_n$ , where for each  $n$ ,  $m_n$  is a probability measure equidistributed on a set of non-attracting periodic orbits (that is, saddle or semi-neutral). Since  $f$  is dissipative, the negative Lyapunov exponent of  $m_n$  satisfies  $\chi^-(m_n) \leq \log |\text{Jac}(f)| < 0$ , and likewise for  $m$ . On the other hand we make no assumption on the remaining (non-negative) Lyapunov exponent.*

Then it is possible<sup>(4)</sup> to adapt the techniques of Wang-Young [20, § 2] (see also Benedicks-Carleson [6]) to show that if the Jacobian is sufficiently small, by the Pliss Lemma there exists a set of periodic points  $A_r$  such that  $m_n(A_r) \geq 1/2$  for each  $n$ , and such that for every  $p \in A_r$  the local stable manifold of  $p$  is of size  $r$ . Thus the same holds for  $m$ , and by ergodicity we conclude that  $m$ -a.e. point is  $s$ -regular. Furthermore, since in this case the local stable manifolds obtained by Proposition 4.2 coincide with Pesin stable manifolds, it follows that  $m$ -a.e. point is  $s$ -exposed.

An interesting example of such a situation is given by the unique invariant probability measure supported on the attractor of an infinitely renormalizable Hénon map (see [9]).  $\square$

Here is a more precise version of Theorem A.

**THEOREM 4.10.** – *Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a weakly stable substantial family of polynomial automorphisms of  $\mathbb{C}^2$  of dynamical degree  $d \geq 2$ . If for some parameter  $\lambda_0$ ,  $p_0 \in J_{\lambda_0}^*$  is  $s$ -regular and exposed for  $f_{\lambda_0}$ , then there exists a unique holomorphic mapping  $\Lambda \ni \lambda \mapsto p(\lambda) \in \mathbb{C}^2$  such that for every  $\lambda$ ,  $p(\lambda) \in K_\lambda$  and  $p(\lambda_0) = p_0$ . Moreover, for every  $\lambda \in \Lambda$ ,  $p(\lambda)$  is  $s$ -regular and exposed. In particular the branched holomorphic motion of  $J^*$  is unbranched along the curve  $\{(\lambda, p(\lambda)), \lambda \in \Lambda\}$ .*

Using the terminology introduced in [10, § 3], we can reformulate this by saying that the set  $\mathcal{R}^s$  of  $s$ -regular and exposed points moves under a strongly unbranched, hence continuous, holomorphic motion. In particular for  $\lambda_1, \lambda_2 \in \Lambda$ ,  $f_{\lambda_1}|_{\mathcal{R}_{\lambda_1}^s}$  is topologically conjugate to  $f_{\lambda_2}|_{\mathcal{R}_{\lambda_2}^s}$ , that is, the induced conjugacy  $\mathcal{R}_{\lambda_1}^s \rightarrow \mathcal{R}_{\lambda_2}^s$  is a homeomorphism.

Since regular points are  $u$ - and  $s$ -exposed we obtain the following corollary, which contains Theorem A. The conclusion about transversality is not obvious and will be proved afterwards.

**COROLLARY 4.11.** – *Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a weakly stable substantial family of polynomial automorphisms of  $\mathbb{C}^2$  of dynamical degree  $d \geq 2$ . Then regular points move under a strongly unbranched holomorphic motion. Furthermore, transverse regular points remain transverse throughout the family.*

The following corollary is a first step towards Theorem C.

**COROLLARY 4.12.** – *If  $(f_\lambda)_{\lambda \in \Lambda}$  is a weakly stable substantial family of polynomial automorphisms of  $\mathbb{C}^2$  and if for some  $\lambda_0 \in \Lambda$ ,  $f_{\lambda_0}$  is uniformly hyperbolic on  $J_{\lambda_0}^*$ , then for every  $\lambda \in \Lambda$ ,  $f_\lambda|_{J_\lambda^*}$  is topologically conjugate to  $f_{\lambda_0}|_{J_{\lambda_0}^*}$ .*

Indeed, just observe that for a hyperbolic map, all points in  $J^*$  are regular.

<sup>(4)</sup> Details will appear in subsequent work.

**4.2. Proofs of Theorem 4.10 and Corollary 4.11**

*Proof of Theorem 4.10.* – The plan of the proof is the following: we start by treating the particular case of points belonging to stable manifolds of saddle points. Using the results of § 3.3, we work locally in  $\Lambda$  to show that the branched holomorphic motion of  $J^*$  is unbranched at  $s$ -regular and exposed points. Then, using the global area bounds from § 3.1 we show that regular points remain regular in the family, which allows to conclude the proof.

*Step 0.* – A particular case.

Here we prove the following lemma, which is essentially contained in [10].

LEMMA 4.13. – *Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a weakly stable substantial family of polynomial automorphisms of  $\mathbb{C}^2$ . Let  $p : \Lambda \rightarrow \mathbb{C}^2$  be such that for every  $\lambda$ ,  $p(\lambda) \in K_\lambda$ . Assume that for some  $\lambda_0 \in \Lambda$ ,  $p(\lambda_0)$  belongs to the stable manifold of a saddle point  $m(\lambda_0)$  (which necessarily persists as  $m(\lambda)$  in the family). Then for every  $\lambda \in \Lambda$ ,  $p(\lambda) \in W^s(m(\lambda))$ .*

*If in addition,  $p(\lambda_0)$  is exposed inside  $W^s(m(\lambda_0))$ , then the branched motion of  $J^*$  is unbranched along  $\widehat{p}$  and  $p(\lambda)$  remains exposed throughout the family.*

Of course, the same result holds for unstable manifolds. Recall that  $p(\lambda)$  is exposed inside  $W^s(m(\lambda))$  if and only if  $p(\lambda)$  is a limit of homoclinic or heteroclinic intersections for the intrinsic topology of  $W^s(m(\lambda))$ .

*Proof.* – The sequence of iterates  $\widehat{f}^n(\widehat{p})$  is locally uniformly bounded in  $\Lambda \times \mathbb{C}^2$ . Pick a cluster value  $\widehat{r}$  of this sequence. Then  $r(\lambda_0) = m(\lambda_0)$  and  $\widehat{r} \subset \widehat{K}$ . Then by Theorem 2.1,  $r \equiv m$ , so we conclude that for every  $\lambda$ ,  $p(\lambda) \in W^s(m(\lambda))$ .

To get the second conclusion, note that for  $\lambda = \lambda_0$ ,  $p(\lambda_0) = \lim t_k(\lambda_0)$  is a limit of homoclinic intersections, in the intrinsic topology of  $W^s(m(\lambda_0))$ . By Theorem 2.1,  $t_k(\lambda_0)$  admits a unique continuation  $t_k$  to  $\Lambda$  as a homoclinic intersection. Let  $\Delta_{\lambda_0} \subset W^s(m(\lambda_0))$  be a disk containing  $p(\lambda_0)$ . By the persistence of stable manifolds of saddle points, there exists a neighborhood  $N$  of  $(\lambda_0, p(\lambda_0))$  in  $\Lambda \times \mathbb{C}^2$  and a smooth surface  $\widehat{W}$  in  $N$  such that  $\widehat{W} \cap \mathbb{C}_{\lambda_0}^2 = \Delta_{\lambda_0}$  and  $\widehat{W} \subset \widehat{W}^s(\widehat{m})$ . Now there are two cases: either  $p(\lambda_0)$  is itself a homoclinic intersection, and we conclude by Theorem 2.1. Otherwise  $p(\lambda)$  is always distinct from  $t_k(\lambda)$ , and applying the Hurwitz Theorem inside  $\widehat{W}$ , we conclude that when  $k \rightarrow \infty$ ,  $t_k \rightarrow p$  in a neighborhood of  $\lambda_0$ , hence everywhere by analytic continuation. In both cases we conclude that  $p(\lambda_0)$  admits a unique continuation  $p$  staying in  $K$ .

To conclude that the branched motion of  $J^*$  is unbranched along  $p$  at all parameters, it suffices to show that  $p(\lambda)$  remains exposed inside  $W^s(m(\lambda))$ . For this, it is enough to show that  $G^-|_{W^s(m(\lambda))} \neq 0$  in any neighborhood of  $p(\lambda)$ , which follows directly from Lemma 2.3. The proof is complete. □

Let us note for future reference the following consequence of this lemma.

COROLLARY 4.14. – *Let  $p(\lambda_0)$  be a  $s$ -regular point of size  $r$  and  $t(\lambda_0)$  be an intersection between  $W_r^s(p(\lambda_0))$  and  $W^u(m(\lambda_0))$ , where  $m(\lambda_0)$  is a saddle point. Then there exists a unique continuation  $t$  of  $t(\lambda_0)$  such that  $\widehat{t} \subset \widehat{K}$ , and the branched motion of  $J^*$  is unbranched along  $\widehat{t}$ .*

*Proof.* – In virtue of Lemma 4.13, it is enough to show that  $t(\lambda_0)$  is exposed inside  $W^u(m(\lambda_0))$ . For this, recall that  $W_r^s(p(\lambda_0))$  is the limit of a sequence  $W_r^s(p_n(\lambda_0))$  of local stable manifolds of saddle points. Therefore, by the persistence of proper intersections,  $t(\lambda_0)$  is the limit in the intrinsic topology of  $W^u(m(\lambda_0))$  of a sequence of heteroclinic intersections with  $W_r^s(p_n(\lambda_0))$ , and we are done.  $\square$

*Step 1.* – The branched motion is unbranched at s-regular and exposed points.

Let  $\lambda_0 \in \tilde{\Lambda} \Subset \Lambda$  and  $p(\lambda_0)$  be s-regular and exposed for  $f_{\lambda_0}$ . We want to construct a natural continuation of  $p(\lambda_0)$ . Let  $r_0 > 0$  be such that there exists a sequence of distinct saddle points  $p_n \rightarrow p$  with local stable manifolds of size  $r'_0 := 2r_0$ . Extracting a subsequence we assume that  $(\hat{p}_n)$  converges to some  $\hat{p}$  in  $\Lambda \times \mathbb{C}^2$  (later on we will see that this limit is unique). It follows from Proposition 3.10 that for  $|\lambda - \lambda_0| < \delta = \delta(r_0, \tilde{\Lambda})$ ,  $W_{\text{loc}}^s(p_n(\lambda))$  is of size  $r_0$ , therefore  $p(\lambda)$  is s-regular<sup>(5)</sup>.

Our goal here is to show that if  $q : \Lambda \rightarrow \mathbb{C}^2$  is such that  $q(\lambda_0) = p(\lambda_0)$  and  $q(\lambda) \in K_\lambda$  for every  $\lambda$ , then  $q(\lambda) = p(\lambda)$  for every  $\lambda$ .

CLAIM 4.15. – *There exists a neighborhood  $N = \text{Tub}(\hat{p}, r_0) \cap (D(\lambda_0, \delta(r_0)) \times \mathbb{C}^2)$  of  $(\lambda_0, p(\lambda_0))$  in  $\Lambda \times \mathbb{C}^2$  and a smooth hypersurface  $\widehat{W}_{r_0}^s(\hat{p})$  in  $N$  such that the sequence of hypersurfaces  $\widehat{W}_{r_0}^s(\hat{p}_n)$  given by Proposition 3.10 converges to  $\widehat{W}_{r_0}^s(\hat{p})$  with multiplicity 1 in  $N$ .*

*Proof.* – Indeed the volumes of  $\widehat{W}_{r_0}^s(\hat{p}_n)$  are uniformly bounded in  $N$ , so we may extract converging subsequences by Bishop's Theorem 2.4. Fix such a subsequence  $\widehat{W}_{r_0}^s(\hat{p}_{n_j}) \rightarrow W$ . Since the unstable directions  $E^u(p_n(\lambda_0))$  converge, for  $|\lambda - \lambda_0| < \delta$ ,  $W_{r_0}^s(p_n(\lambda))$  is a graph of slope at most 2 over a fixed direction for large  $n$ . In particular the convergence is of multiplicity 1. By Lemma 3.9,  $W$  is smooth, and from Proposition 2.5 we get that the sequence  $\widehat{W}_{r_0}^s(\hat{p}_n)$  actually converges. Notice that by construction, for  $|\lambda - \lambda_0| < \delta$ , we have that  $W_{r_0}^s(p(\lambda)) = \widehat{W}_{r_0}^s(\hat{p}) \cap \mathbb{C}_\lambda^2$ .  $\square$

As a consequence of this claim and the Hurwitz Theorem we get the following:

CLAIM 4.16. – *There exists  $\eta = \eta(r_0, \tilde{\Lambda}) > 0$  such that if  $q$  is a holomorphic map  $\Lambda \rightarrow \mathbb{C}^2$  such that  $\hat{q} \subset \widehat{K}$  and  $q(\lambda_0) \in W_{r_0/2}^s(p(\lambda_0))$ , then for  $|\lambda - \lambda_0| < \eta$ ,  $q(\lambda) \in W_{r_0}^s(p(\lambda))$ .*

*Proof.* – Discarding at most one value of  $n$  if needed, we may assume that  $\hat{q}$  is disjoint from  $\widehat{W}_{r_0}^s(\hat{p}_n)$ . Since  $q(\lambda_0) \in \widehat{W}_{r_0/2}^s(\hat{p})$ , by the Cauchy estimates, there exists  $\eta = \eta(r_0) > 0$  so that if  $|\lambda - \lambda_0| < \eta$ , then the point  $q(\lambda)$  stays in  $B(p(\lambda), r_0)$ .

Now we have that  $\hat{q} \subset \widehat{W}_{r_0}^s(\hat{p})$ . Indeed otherwise these two manifolds would have a proper intersection at  $(\lambda_0, q(\lambda_0))$ , and by persistence of proper intersections we would get that  $\hat{q}$  intersects  $\widehat{W}_{r_0}^s(\hat{p}_n)$ , a contradiction.  $\square$

<sup>(5)</sup> We see that it is crucial here that  $\delta(r_1, r_2)$  in Proposition 3.10 depends only on  $r_1$  and  $r_2$ .



If  $p(\lambda_0)$  is a transverse regular point, this is enough to conclude. Indeed, applying the same reasoning in the unstable direction we get that  $q(\lambda) \subset W_{r_0}^u(p(\lambda))$  for  $\lambda$  close to  $\lambda_0$ . Now the intersection  $\widehat{W}_{r_0}^s(\widehat{p}) \cap \widehat{W}_{r_0}^u(\widehat{p})$  is transverse near  $(\lambda_0, p(\lambda_0))$ , therefore it coincides with  $\widehat{p}$ . We conclude that  $\widehat{p} = \widehat{q}$  near  $\lambda_0$ , hence everywhere, which was the desired result.

Let us now deal with the general case.

CLAIM 4.17. – *Let  $p(\lambda_0)$  be  $s$ -regular and exposed. If  $q : \Lambda \rightarrow \mathbb{C}^2$  is such that  $q(\lambda_0) = p(\lambda_0)$  and  $q(\lambda) \in K_\lambda$  for every  $\lambda$ , then  $q(\lambda) = p(\lambda)$  for every  $\lambda$ .*

*Proof.* – By Definition-Proposition 4.5, for a given saddle point  $m(\lambda_0)$ , there exist a sequence of transverse intersection points  $(t_k(\lambda_0))$  between  $W_{r_0/2}^s(p(\lambda_0))$  and  $W^u(m(\lambda_0))$  such that  $t_k(\lambda_0) \rightarrow p(\lambda_0) = q(\lambda_0)$ . By Corollary 4.14, there exists a unique holomorphic continuation  $t_k$  of  $t_k(\lambda_0)$  with the property that for every  $\lambda \in \Lambda$ ,  $t_k(\lambda) \in W^u(m(\lambda)) \cap K(\lambda)$ . In particular if  $p(\lambda_0)$  itself belongs to  $W^u(m(\lambda_0))$  we are done, so let us assume that  $p(\lambda_0) \notin W^u(m(\lambda_0))$ . By the previous Claim 4.16, for every  $\lambda$  so that  $|\lambda - \lambda_0| < \eta(r_0)$ , the point  $t_k(\lambda)$  belongs to  $W^u(m(\lambda)) \cap W_{r_0}^s(p(\lambda))$  and  $q(\lambda)$  belongs to  $W_{r_0}^s(p(\lambda))$ . To conclude, we observe that since  $p(\lambda_0) = q(\lambda_0) \notin W^u(m(\lambda_0))$ , applying Corollary 4.14 again, we deduce that  $t_k(\lambda)$  is disjoint from  $q(\lambda)$  for every  $\lambda \in \Lambda$ . Working inside  $\widehat{W}_{r_0}^s(\widehat{p})$ , which is a smooth complex surface, we can apply the Hurwitz Theorem to conclude that the sequence  $\widehat{t}_k$  converges to  $\widehat{q}$ . Therefore the continuation  $p$  is unique, which was the result to be proved.  $\square$

Step 2. – The branched motion preserves  $s$ -regularity and exposure.

Fix  $\lambda_0 \in \Lambda$ . Throughout Step 2, we denote by  $\widetilde{\Lambda}$  the disk of radius  $\ln 2$  around  $\lambda_0$  relative to the hyperbolic metric, therefore we can apply the results of §3.1 and §3.2.

Let  $p(\lambda_0) \in J_{\lambda_0}^*$  be  $s$ -regular and exposed. Thus there exists a sequence of saddle points  $(p_n)$  with stable manifolds of size  $r_0$ , such that  $p_n(\lambda_0) \rightarrow p(\lambda_0)$ . By Proposition 4.2,  $W_{r_0}^s(p_n(\lambda_0))$  converges to  $W_{r_0}^s(p(\lambda_0))$  with multiplicity 1, that is,  $W_{r_0}^s(p_n(\lambda_0))$  is a graph over  $W_{r_0}^s(p(\lambda_0))$  for large  $n$ . Since  $p(\lambda_0)$  is  $s$ -exposed, for every  $r$  we get that  $G^-|_{W_r^s(p(\lambda_0))}$  is not identically 0.

Let  $\psi_{\lambda_0,n}^s$  be a stable parameterization of  $W^s(p_n(\lambda_0))$  with  $\|(\psi_{\lambda_0,n}^s)'(0)\| = 1$ , and  $(\psi_{\lambda,n}^s)$  be the natural continuation of  $\psi_{\lambda_0,n}^s$ , as defined in §3.2. Recall the notation  $h_\lambda^s$  for the holomorphic motion inside stable manifolds, as viewed from the parameterizations  $(\psi_{\lambda,n}^s)$ . Recall that  $h_\lambda^s$  satisfies  $h_\lambda^s(0) = 0$  and  $h_\lambda^s(1) = 1$  (hence also the estimate (3)). By (3), there exists  $c > 0$  such that for every  $\lambda \in \widetilde{\Lambda}$ ,  $(h_\lambda^s)^{-1}(D(0, cr_0)) \subset D(0, r_0/4)$ . Without loss of generality, we can assume that  $c < 1/8$ . Choose  $c' < c$  so small that for every  $\lambda \in \widetilde{\Lambda}$ ,  $h_\lambda^s(D(0, c'r_0)) \Subset D(0, cr_0)$

Fix a pair of holomorphic disks  $D_1 \Subset D_2$  in  $W_{c'r_0/4}^s(p(\lambda_0))$ , with  $p(\lambda_0) \in D_1$ . Set  $m = \text{mod}(D_2 \setminus \overline{D_1})$ , and for  $i = 1, 2$ , put  $g_i = \sup G_{\lambda_0}^-|_{D_i}$ . By the continuity of  $G_{\lambda_0}^-$  and the multiplicity 1 convergence, for large  $n$  we can lift  $D_1$  and  $D_2$  to holomorphic disks  $D_{1,n}$  and  $D_{2,n}$  in  $W_{c'r_0/4}^s(p_n(\lambda_0))$ , such that  $\text{mod}(D_{2,n} \setminus \overline{D_{1,n}}) \rightarrow m$  and  $\sup G_{\lambda_0}^-|_{D_{i,n}} \rightarrow g_i$ . From Lemma 3.7 we infer that  $\psi_{\lambda_0,n}^s(D(0, c'r_0)) \supset W_{c'r_0/4}^s(p_n(\lambda_0))$ , so in particular  $\psi_{\lambda_0,n}^s(D(0, c'r_0))$  contains  $D_{1,n}$  and  $D_{2,n}$ .

Now fix another parameter  $\lambda_1 \in \tilde{\Lambda}$ . By the first step of the proof we know that  $p_n(\lambda_1) \rightarrow p(\lambda_1)$ . Applying Proposition 3.3 we infer that there exist positive constants  $r_1$ ,  $g$  and  $A$  such that  $W_{r_1}^s(p_n(\lambda_1))$  is a submanifold properly embedded into  $B(p_n(\lambda_1), r_1)$ , contained in  $h_{\lambda_1}(D_{1,n})$ , with area at most  $A$ , and  $\sup(G_{\lambda_1}^-|_{W_{r_1}^s(p_n(\lambda_1))}) \geq g$ . Using Bishop's Theorem, we extract a subsequence  $n_j$  so that  $(W_{r_1}^s(p_{n_j}(\lambda_1)))_j$  converges to some analytic set  $W \ni p(\lambda_1)$  with  $\sup(G_{\lambda_1}^-|_W) \geq g$ .

The main step of the proof is the following lemma.

LEMMA 4.18. – *The multiplicity of convergence of  $W_{r_1}^s(p_{n_j}(\lambda_1))$  to  $W$  is equal to 1.*

Before establishing the lemma let us show how to conclude the proof of Step 2. Recall that by the Maximum Principle  $W_{r_1}^s(p_{n_j}(\lambda_1))$  is a holomorphic disk. Since the multiplicity of convergence is 1, we deduce from Proposition 2.6 that  $W$  is smooth. Also Proposition 2.5 implies that the sequence  $W_{r_1}^s(p_n(\lambda_1))$  actually converges.

Since  $W$  is smooth at  $p(\lambda_1)$  and the multiplicity of convergence is 1, we see that in any small neighborhood of  $p(\lambda_1)$ ,  $W_{r_1}^s(p_n(\lambda_1))$  is a graph over  $W$  for large  $n$ . In particular  $W_{r_1}^s(p_n(\lambda_1))$  has size uniformly bounded from below, and therefore  $p(\lambda_1)$  is regular. We already observed that  $\sup(G_{\lambda_1}^-|_W) \geq g > 0$ , and the same holds in any neighborhood of  $p(\lambda_1)$  by choosing a smaller  $r_0$  at the beginning. Hence  $p(\lambda_1)$  is s-exposed, which finishes the proof of Step 2.

*Proof of Lemma 4.18.* – For notational ease we put  $n_j = n$ . Let  $k$  be the multiplicity of convergence of  $W_{r_1}^s(p_n(\lambda_1))$  to  $W$ . By Proposition 3.8, we know that for every  $\lambda \in \tilde{\Lambda}$ ,  $\|\psi_{\lambda,n}^s\| \leq M$  on  $\tilde{\Lambda} \times D(0, cr_0)$ , so we can extract a converging subsequence (still denoted by  $n$ ) to a limiting map  $\varphi_\lambda(\cdot)$ . Notice that for  $\lambda = \lambda_0$ ,  $(\psi_{\lambda,n}^s)_n$  converges on  $D(0, r_0/4)$  to an injective map  $D(0, r_0/4) \rightarrow W_r^s(p(\lambda_0))$

We recall that  $D_{2,n} \subset \psi_{\lambda_0,n}^s(D(0, c'r_0))$ , so by definition of  $c'$  we get that for every  $\lambda$ ,  $\psi_{\lambda,n}^s(D(0, cr_0))$  contains  $h_\lambda(D_{2,n})$ . It follows that for  $\lambda = \lambda_1$ ,  $W_{r_1}^s(p_n(\lambda_1)) \subset \psi_{\lambda_1,n}^s(D(0, cr_0))$ .

Furthermore, since  $\text{mod}(D_{2,n} \setminus D_{1,n}) \rightarrow m > 0$ , there exists a uniform  $c'' < c$  such that  $W_{r_1}^s(p_n(\lambda_1)) \subset \psi_{\lambda_1,n}^u(D(0, c''r_0))$ . It follows that  $\varphi_{\lambda_1}$  is non-constant and that the component  $\Omega$  of 0 in  $\varphi_{\lambda_1}^{-1}(W)$ , is such that  $\varphi_{\lambda_1} : \Omega \rightarrow W$  is proper. Its degree is equal to the multiplicity of convergence  $k$ .

Since  $G_{\lambda_1}^+|_W$  (resp.  $G_{\lambda_1}^+ \circ \varphi_{\lambda_1}$ ) is continuous and not harmonic, its Laplacian is nonzero and gives no mass to points. By the s-exposure assumption, Definition-Proposition 4.5 ensures the existence of a saddle point  $m(\lambda_1)$  whose unstable manifold intersects transversally  $W$  at a certain point  $q(\lambda_1)$  that is a regular value of  $\varphi_{\lambda_1}$ .

Now if  $k > 1$ , there exist two distinct points  $a$  and  $b$  in  $\Omega \subset D(0, cr_0)$  such that  $\varphi_{\lambda_1}(a) = \varphi_{\lambda_1}(b) = q(\lambda_1)$ . Thus there exists  $a_n \rightarrow a$  (resp.  $b_n \rightarrow b$ ) such that  $\psi_{\lambda_1,n}^s(a_n)$  (resp.  $\psi_{\lambda_1,n}^s(b_n)$ ) are intersection points of  $W^u(m(\lambda_1))$  and  $W_{r_1}^s(p_n(\lambda_1))$  converging to  $q(\lambda_1)$ .

To conclude the proof, we flow back to  $\lambda_0$  using the holomorphic motion to obtain a contradiction with Corollary 4.14. The details are as follows. Consider the continuations of the heteroclinic intersections  $\psi_{\lambda_1,n}^s(a_n)$  and  $\psi_{\lambda_1,n}^s(b_n)$  for  $\lambda \in \tilde{\Lambda}$ . Notice that they stay in a compact piece of  $W^u(m(\lambda))$ . For  $\lambda = \lambda_0$ , the corresponding points are  $\psi_{\lambda_0,n}^s((h_{\lambda_1}^s)^{-1}(a_n))$  and  $\psi_{\lambda_0,n}^s((h_{\lambda_1}^s)^{-1}(b_n))$ , which converge respectively to  $\varphi_{\lambda_0}((h_{\lambda_1}^s)^{-1}(a))$

and  $\varphi_{\lambda_0}((h_{\lambda_1}^s)^{-1}(b))$ . Now  $(h_{\lambda_1}^s)^{-1}(a)$  and  $(h_{\lambda_1}^s)^{-1}(b)$  are distinct and by definition of  $c$ , they belong to  $D(0, r_0/4)$ . On this disk,  $\varphi_{\lambda_0}$  is injective therefore  $\varphi_{\lambda_0}((h_{\lambda_1}^s)^{-1}(a))$  and  $\varphi_{\lambda_0}((h_{\lambda_1}^s)^{-1}(b))$  are distinct intersection points between  $W_r^s(p(\lambda_0))$  and  $W^u(m(\lambda_0))$  with continuations colliding at  $\lambda_1$ . This contradicts Corollary 4.14, and concludes the proof of the lemma.  $\square$

REMARK 4.19. – The proof does not give any estimate on the size  $r_1$  of the stable manifold  $W^s(p(\lambda_1))$  at  $p(\lambda_1)$ . Indeed,  $r_1$  depends on the size of  $W$  at  $p(\lambda_1)$ , upon which we have no control (the only information we have is a local area bound). In particular it is unclear whether  $r_1$  depends only on  $r_0$ .

REMARK 4.20. – As opposed to Step 1 of the proof, Step 2 does not become significantly easier if we assume that  $p(\lambda_0)$  is regular instead of s-regular and exposed. Indeed, the whole point is to prove that  $p(\lambda)$  remains s-regular throughout  $\Lambda$ .

Step 3. – Conclusion.

We have shown in Steps 1 and 2 that if  $p(\lambda_0)$  is s-regular and exposed, then it admits a unique holomorphic continuation  $\widehat{p} \subset \widehat{K}$  to  $D_{\text{Poin}}(\lambda_0, \ln 2)$  such that for every  $\lambda$ ,  $p(\lambda)$  is s-regular and exposed, too. Thus the branched motion of  $J^*$  must be unbranched, in particular continuous, at  $(\lambda, p(\lambda))$ .

Taking a locally finite cover of  $\Lambda$  by such disks, we see that this property holds throughout  $\Lambda$ . In particular  $p(\lambda)$  cannot collide with the continuation of any other point in  $J^*$  and we indeed have a continuous holomorphic motion of  $\mathcal{R}^s$  over  $\Lambda$ . This finishes the proof of Theorem 4.10.  $\square$

Let us note for future reference the following consequence of the proof.

PROPOSITION 4.21. – *Let  $(f_\lambda)$  be a weakly stable holomorphic family of polynomial automorphisms. Assume that for  $\lambda = \lambda_0$ ,  $p(\lambda_0)$  is a regular point and let  $(p_n(\lambda_0))$  be a sequence of saddle points converging to  $p(\lambda_0)$  such that  $W^s(p_n(\lambda_0))$  (resp.  $W^u(p_n(\lambda_0))$ ) is of size  $r_0$  at  $p_n(\lambda_0)$ . Then for every  $\lambda_1 \in \Lambda$ , there exists  $r_1 > 0$  such that  $p_n(\lambda_1) \rightarrow p(\lambda_1)$  and  $W^s(p_n(\lambda_1))$  (resp.  $W^u(p_n(\lambda_1))$ ) is of size  $r_1$  at  $p_n(\lambda_1)$ .*

We now show that regular points (resp. transverse regular points) remain regular (resp. transverse).

*Proof of Corollary 4.11.* – It follows from Theorem 4.10 that the regular points move without collision, and remain s- and u-regular and exposed in both directions. Let  $p(\lambda_0)$  be regular relative to  $f_{\lambda_0}$ . Then for every  $\lambda \in \Lambda$ ,  $p(\lambda)$  is s- and u-regular, so it possesses local stable and unstable manifolds. If  $W_{\text{loc}}^u(p(\lambda_1))$  was to coincide with  $W_{\text{loc}}^s(p(\lambda_1))$ , we would get that  $W_{\text{loc}}^u(p(\lambda_1)) = W_{\text{loc}}^s(p(\lambda_1)) \subset K_{\lambda_1}$ , thus contradicting s- and u-exposure. Therefore  $p(\lambda_1)$  is regular.

To show that transverse regular points stay transverse, recall from the proof of Theorem 4.10 that if  $p(\lambda_0)$  is regular, then  $W_{\text{loc}}^s(p(\lambda_0))$  and  $W_{\text{loc}}^u(p(\lambda_0))$  can be locally continued as smooth surfaces  $\widehat{W}_{\text{loc}}^s(\widehat{p})$  and  $\widehat{W}_{\text{loc}}^u(\widehat{p})$  in  $N(\lambda_0) \times B(p(\lambda_0), r)$ . Now assume that  $W_{\text{loc}}^s(p(\lambda_0))$  and  $W_{\text{loc}}^u(p(\lambda_0))$  are tangent at  $p(\lambda_0)$ , that is, their intersection multiplicity at  $p(\lambda_0)$  is  $m > 1$ .

If this tangency does not persist for nearby parameters, by the persistence of proper intersections, we get that for nearby  $\lambda$ ,  $W_{\text{loc}}^s(p(\lambda))$  and  $W_{\text{loc}}^u(p(\lambda))$  intersect at  $m$  points counting multiplicities, not all identical to  $p(\lambda)$ .

Consider the intersection  $\widehat{C} = \widehat{W}_{\text{loc}}^s(\widehat{p}) \cap \widehat{W}_{\text{loc}}^u(\widehat{p})$ . This is a curve in  $N(\lambda_0) \times B(p(\lambda_0), r)$  such that  $\widehat{C} \cap \mathbb{C}_{\lambda_0}^2 = \{p(\lambda_0)\}$ . One irreducible component of  $\widehat{C}$  is given by the continuation  $\widehat{p}$ , and by assumption there exists another irreducible component  $\widehat{C}'$  of  $\widehat{C}$ .

Assume first that  $\widehat{C}'$  is a graph  $\widehat{q}$  over  $N(\lambda_0)$ . Then, since for every  $\lambda \in N(\lambda_0)$ ,  $q(\lambda) \in W_{\text{loc}}^s(p(\lambda)) \cap W_{\text{loc}}^u(p(\lambda)) \subset K_\lambda$ , we get a collision between  $p$  and a holomorphically moving point  $q$  staying in  $K$ , which contradicts Theorem 4.10.

We will reduce the general case to this one by a classical trick: replacing  $\Lambda$  by a well-suited branched cover  $M \rightarrow \Lambda$ . We detail the argument for the convenience of the reader. Consider a local irreducible component of  $\widehat{C}'$  at  $(\lambda_0, p(\lambda_0))$ , still denoted by  $\widehat{C}'$  for simplicity. Denote by  $\varpi : \mathbb{D} \rightarrow \widehat{C}'$  a normalization of  $\widehat{C}'$  such that  $\varpi(0) = (\lambda_0, p(\lambda_0))$ . Denote respectively by  $\pi_\Lambda$  and  $\pi_{\mathbb{C}^2}$  the projection onto the first and second factors in  $\Lambda \times \mathbb{C}^2$  and put  $M = \mathbb{D}$  and  $\lambda(\mu) = \pi_\Lambda \circ \varpi(\mu)$ . Then we can consider the holomorphic family of polynomial automorphisms defined by  $(\widetilde{f}_\mu) := (f_{\lambda(\mu)})$ , which is weakly stable (of course, nothing has changed from the dynamical point of view). For  $\mu = 0$ , the point  $p(\lambda(0))$  is regular and can be continued as a regular point  $\mu \mapsto p(\lambda(\mu))$  as before. But now we have another holomorphic continuation of  $(0, p(\lambda(0)))$  in  $\widehat{K} \subset M \times \mathbb{C}^2$  given by  $\mu \mapsto (\mu, \pi_{\mathbb{C}^2} \circ \varpi(\mu))$ . Thus we arrive at a contradiction and the proof is complete.  $\square$

REMARK 4.22. – A similar argument shows that more generally the order of tangency between local stable and unstable manifolds of regular points is preserved in weakly stable families.

## 5. Propagation of hyperbolicity

In this section we establish Theorem C, that is, we prove that uniform hyperbolicity on  $J^*$  is preserved in weakly stable families. Let us start with a variation on Definition 4.1.

DEFINITION 5.1. – *We say that  $p \in J^*$  is uniformly u-regular (resp. uniformly s-regular) if there exists  $r > 0$  such that for every sequence of saddle points  $(p_n)$  converging to  $p$ ,  $W^u(p_n)$  (resp.  $W^s(p_n)$ ) is of size  $r$  at  $p_n$ .*

*Likewise,  $p$  is uniformly (resp. transverse) regular if it is (resp. transverse) regular and uniformly regular in both stable and unstable directions.*

If necessary we will specify the size appearing in the definition by saying that “ $p \in J^*$  is uniformly u-regular of size  $r$ ”. Recall from Proposition 4.2 that if  $p$  is uniformly u-regular of size  $r$ , then it has a well defined local unstable manifold  $W_r^u(p)$  and that if  $p_n \rightarrow p$  is any sequence of saddle points,  $W_r^u(p_n)$  converges to  $W_r^u(p)$  with multiplicity 1.

If  $f$  is uniformly hyperbolic on  $J^*$ , then every  $p \in J^*$  is uniformly regular and transverse. Interestingly enough, the converse is true, as shown by the following result which is essentially contained in [5].

PROPOSITION 5.2. – *Let  $f$  be a polynomial automorphism of  $\mathbb{C}^2$  with dynamical degree  $d \geq 2$ . If every point in  $J^*$  is uniformly regular and transverse then  $f$  is uniformly hyperbolic on  $J^*$ .*

The main step of the proof is the following lemma.

LEMMA 5.3. – *Let  $f$  be a polynomial automorphism of  $\mathbb{C}^2$ , such that every point in  $J^*$  is uniformly u-regular. Then there exists a neighborhood  $N$  of  $J^*$  such that the restriction to  $N$  of  $\bigcup_{p \in J^*} W_{\text{loc}}^u(p)$  forms a lamination.*

*Proof.* – Let us start by showing that the size of unstable manifolds is uniformly bounded from below. For this, notice that Definition 5.1 may be reformulated as follows:  $p$  is uniformly u-regular if there exists  $r > 0$  and  $\varepsilon > 0$  such that if  $q \in B(p, \varepsilon)$  is any saddle point, then  $W^u(q)$  is of size  $r$  at  $q$ . Then by compactness of  $J^*$ , we can cover  $J^*$  with finitely many such balls, and deduce that if every point in  $J^*$  is uniformly u-regular, then the size of unstable manifolds of saddle points is uniformly bounded from below, as claimed.

From this point, the remainder of the proof is classical. As observed above, for every  $p \in J^*$  there exists  $r > 0$  and  $\varepsilon > 0$  such that if  $q \in B(p, \varepsilon)$  is any saddle point, then  $W_r^u(q)$  is of size  $r$  at  $q$ . Taking  $\varepsilon$  smaller if needed, we may assume that  $W_r^u(q)$  is closed in  $B(p, \varepsilon)$ . Furthermore, any two such local unstable manifolds of saddle points are disjoint or coincide. Thus taking the closure, we get that  $\bigcup W_r^u(q) \cap B(p, \varepsilon)$  is a lamination in  $B(p, \varepsilon)$ , where the union ranges over all saddle points lying in  $B(p, \varepsilon)$ . The result follows.  $\square$

*Proof of Proposition 5.2.* – The result is essentially a direct consequence of Theorem 8.3. in [5], which asserts that if there exist laminations of  $J^+$  and  $J^-$  in a neighborhood of  $J^*$ , which are transverse at every point of  $J^*$  then  $f$  is uniformly hyperbolic on  $J^*$ .

In our situation, the existence of stable and unstable laminations  $\mathcal{L}^s$  and  $\mathcal{L}^u$  is guaranteed by Lemma 5.3, while these laminations are transverse at every point of  $J^*$  by assumption.

Unfortunately, this is slightly different from the hypotheses of [5, Thm. 8.3] because we do not know that the lamination  $\mathcal{L}^u$  fills up the whole  $J^-$  in a neighborhood of  $J^*$ . However, the reader will easily check that the only place in the proof of [5] where this assumption is used is to ensure that for every  $p \in J^*$ ,  $W_{\text{loc}}^u(p)$  is contained in a leaf of  $\mathcal{L}^u$ , which is trivially satisfied in our case. Hence the result applies and we are done.  $\square$

We now have all the necessary ingredients for Theorem C.

*Proof of Theorem C.* – By assumption, every point in  $J_{\lambda_0}^*$  is uniformly regular and transverse. From Corollary 4.11 we deduce that  $J^*$  moves holomorphically and all points remain regular and transverse. Proposition 4.21 implies that strong s- and u-regularity are preserved as well. Therefore, for every  $\lambda \in \Lambda$ , every point in  $J_\lambda^*$  is uniformly regular and transverse, so the result follows from Proposition 5.2.  $\square$

The concept of quasi-expansion, developed in [5] has been a source of inspiration for the techniques in this paper. A polynomial automorphism of  $\mathbb{C}^2$  of dynamical degree  $d \geq 2$  is *quasi-expanding* if there exists positive constants  $r$  and  $A$  such that for every saddle point  $p$ ,  $W_r^u(p)$  is properly embedded in  $B(p, r)$ , of area at most  $A$  and for every  $\delta > 0$  there exists

$\eta > 0$  such that  $\sup(G^+|_{W_r^u(p)}) \geq \eta$  (see [5, Cor 3.5] for this definition). There is a parallel notion of *quasi-contraction* in the stable direction.

It is worthwhile to state the following result of independent interest.

**PROPOSITION 5.4.** – *Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a weakly stable and substantial holomorphic family of polynomial automorphisms. If there exists  $\lambda_0 \in \Lambda$  such that  $f_{\lambda_0}$  is quasi-expanding, then  $f_\lambda$  is quasi-expanding for every  $\lambda \in \Lambda$ .*

*Proof.* – For  $\lambda = \lambda_0$ , let  $r, A$  be the uniform constants provided by the definition of quasi-expansion. Let  $p(\lambda_0)$  be a saddle point. By [5, Thm 3.1], the modulus of the annulus  $W_r^u(p(\lambda_0)) \setminus W_{r/2}^u(p(\lambda_0))$  is bounded from below by a constant  $m$  depending only on  $A$  and  $r$ . By the Hölder continuity property of  $G^+$  we get that  $\sup(G^+|_{W_r^u(p)}) \leq g_2(r)$  and by definition of quasi-expansion,  $\sup(G^+|_{W_{r/2}^u(p)}) \geq g_1 > 0$ . Therefore applying Corollary 3.4 we obtain for every  $\lambda \in \Lambda$  positive constants  $r', A'$  and  $g'$  such that for every saddle point  $p'$  for  $f_{\lambda'}$ ,  $W_{r'}^u(p')$  is properly embedded in  $B(p', r')$ , of area at most  $A'$  and  $\sup(G^+|_{W_{r'}^u(p')}) \geq g'$ . Finally, Theorem 3.4 in [5] implies that  $f_\lambda$  is quasi-expanding.  $\square$

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