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VERSIONS OF INJECTIVITY AND EXTENSION THEOREMS

BY YOSHINORI GONGYO AND SHIN-ICHI MATSUMURA

ABSTRACT. – We give an analytic version of the injectivity theorem by using multiplier ideal sheaves of singular hermitian metrics, and prove extension theorems for the log canonical bundle of dlt pairs. Moreover we obtain partial results related to the abundance conjecture in birational geometry and the semi-ampleness conjecture for hyperKähler manifolds.

RÉSUMÉ. – Nous donnons une version analytique du théorème d'injectivité en utilisant les idéaux multiplicateurs, et démontrons des théorèmes d'extension pour le faisceau adjoint d'une paire dlt. De plus nous obtenons des résultats de semi-amplitude liés à la conjecture d'abondance en géométrie birationnelle et la conjecture de semi-amplitude pour les variétés hyperkählériennes.

1. Introduction

The following conjecture, the so-called abundance conjecture, is one of the most important problems in the classification theory of algebraic varieties. In this paper, we give an analytic version of the injectivity theorem, and study the extension problem for (holomorphic) sections of the pluri-log canonical bundle and its applications to the abundance conjecture.

CONJECTURE 1.1 (Generalized abundance conjecture). – *Let X be a normal projective variety and Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is a klt pair. Then $\kappa(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta)$. In particular, if $K_X + \Delta$ is nef, then it is semi-ample.* (*See* [38] *for the definition of $\kappa(\cdot)$ and $\kappa_\sigma(\cdot)$.*)

Throughout this paper, we work over $\mathbb{C}$, the complex number field, and freely use the standard notation in [4], [7], [27], and [32]. Further we interchangeably use the words "Cartier divisors," "line bundles," and "invertible sheaves".

Toward the abundance conjecture, we need to solve the non-vanishing conjecture and the extension conjecture (for example, see [8], [12, Introduction], and [20, Section 5]). One of

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the purposes of this paper is to study the following extension conjecture formulated in [8, Conjecture 1.3]:

CONJECTURE 1.2 (Extension conjecture for dlt pairs). – *Let X be a normal projective variety and $S + B$ be an effective $\mathbb{Q}$-divisor with the following assumptions*:

- $(X, S + B)$ *is a dlt pair.*
- $\lfloor S + B \rfloor = S$.
- $K_X + S + B$ *is nef.*
- $K_X + S + B$ *is $\mathbb{Q}$-linearly equivalent to an effective divisor D such that*

$$S \subseteq \operatorname{Supp} D \subseteq \operatorname{Supp}(S + B).$$

Then the restriction map

$$H^0(X, \mathcal{O}_X(m(K_X + S + B))) \to H^0(S, \mathcal{O}_S(m(K_X + S + B)))$$

is surjective for all sufficiently divisible integers $m \geq 2$.

When S is a normal irreducible variety (that is, $(X, S + B)$ is a plt pair), Demailly-Hacon-Păun have already proved the above conjecture in [8] by using techniques based on a version of the Ohsawa-Takegoshi L^2 extension theorem. However, the extension theorem for plt pairs is not enough for an inductive proof of the abundance conjecture.

In this paper, we study the extension conjecture for *dlt pairs* by giving an analytic version of the injectivity theorem instead of the Ohsawa-Takegoshi extension theorem. Thanks to our injectivity theorem, we can obtain extension theorems for not only plt pairs but also dlt pairs. This is one of the advantages of our approach. The following result is our injectivity theorem.

THEOREM 1.3 (Analytic version of the injectivity theorem: Theorem 3.1)
Let (F, h_F) and (L, h_L) be (possibly) singular hermitian line bundles with semi-positive curvature on a compact Kähler manifold X. Assume that there exists an effective $\mathbb{R}$-divisor Δ with

$$h_F = h_L^a \cdot h_\Delta,$$

where a is a positive real number and h_Δ is the singular (hermitian) metric defined by the effective divisor Δ.

Then, for a non-zero (holomorphic) section s of L satisfying $\sup_X |s|_{h_L} < \infty$, the multiplication map induced by s

$$H^q(X, K_X \otimes F \otimes \mathcal{J}(h_F)) \xrightarrow{\otimes s} H^q(X, K_X \otimes F \otimes L \otimes \mathcal{J}(h_F h_L))$$

is (well-defined and) injective for every q. Here $\mathcal{J}(h)$ denotes the multiplier ideal sheaf associated to a singular (hermitian) metric h.

In the last decades, the injectivity theorem has been studied by several authors, for example, Tankeev [40], Kollár [29], Enoki [10], Ohsawa [39], Esnault-Viehweg [11], Fujino [18], [14], [16], [17], and Ambro [1], [2]. See [19] and [35] for recent developments. Theorem 1.3 can be seen as a generalization of [10], [16], [29], [34], [37], and [40]. In [37], the second author established an injectivity theorem with multiplier ideal sheaves of singular (hermitian) metrics with arbitrary singularities, which corresponds to the case $\Delta = 0$ of Theorem 1.3.

By applying the above injectivity theorem to the extension problem, we obtain the following extension theorem. Even if $K_X + \Delta$ is semi-positive (namely, it admits a smooth hermitian metric with semi-positive curvature), it seems to be rather difficult to obtain the extension theorem for dlt pairs by the Ohsawa-Takegoshi extension theorem, at least in its present forms. This is because there exists a counterexample to the Ohsawa-Takegoshi extension theorem for dlt pairs (see [39, page 576]). For this reason, we need our injectivity theorem.

THEOREM 1.4 (Extension theorem: Theorem 4.1). – *Let X be a compact Kähler manifold and $S + B$ be an effective $\mathbb{Q}$-divisor with the following assumptions*:

- *$S + B$ is a simple normal crossing divisor with $0 \leq S + B \leq 1$ and $\lfloor S + B \rfloor = S$.*
- *$K_X + S + B$ is $\mathbb{Q}$-linearly equivalent to an effective divisor D with $S \subseteq \operatorname{Supp} D$.*
- *$K_X + S + B$ admits a singular (hermitian) metric h with semi-positive curvature.*

Then, for an integer $m \geq 2$ with $m(K_X + S + B)$ Cartier and a section

$$u \in H^0(S, \mathcal{O}_S(m(K_X + S + B)))$$

that comes from $H^0(S, \mathcal{O}_S(m(K_X + S + B)) \otimes \mathcal{J}(h^{m-1}h_B))$, the section u can be extended to a section in $H^0(X, \mathcal{O}_X(m(K_X + S + B)))$.

By this theorem, we can solve the extension problem for dlt pairs if there exists a singular (hermitian) metric with mild singularities on S (see Corollary 4.2 and Corollary 4.4). When we consider the extension problem, we first construct a (possibly singular) hermitian metric with "good" properties by taking the limit of a family of suitable metrics. In the second step, we extend sections by using the metric constructed in the first step. Currently we do not know the first step to construct a suitable metric. However we can solve the second step by Theorem 1.4 and its corollaries.

Moreover, assuming the non-vanishing conjecture, we can prove the abundance conjecture if $K_X + \Delta$ admits a singular (hermitian) metric h whose curvature is semi-positive and Lelong number is identically zero. This assumption is stronger than the assumption that $K_X + \Delta$ is nef, but weaker than the assumption that $K_X + \Delta$ is semi-positive. To investigate the Lelong number is much easier than to check the regularity (smoothness) of the metric constructed by taking the limit. Therefore it is worth formulating our extension theorem for a singular (hermitian) metric h whose Lelong number is identically zero (see Corollary 4.2 and Corollary 4.4).

As compared with Conjecture 1.2, one of our advantages is to remove the condition $\operatorname{Supp} D \subseteq \operatorname{Supp}(S + B)$ in Conjecture 1.2, which is needed in [8]. (Such an extension conjecture was given in [20, Conjecture 5.8]). Thanks to removing this condition, we can apply the extension theorem more directly than [8, Section 8] and [20, Theorem 5.9], and construct a (non-klt) dlt birational model whose log canonical divisor is a pullback of the original canonical divisor up to positive multiples. Therefore we finally obtain the following theorem related to the abundance conjecture:

THEOREM 1.5 (Partial result of the abundance conjecture, cf. Theorem 5.1)

Assume that Conjecture 1.1 holds in dimension $(n - 1)$. Let X be an n-dimensional normal projective variety and Δ be an effective $\mathbb{Q}$-divisor with the following assumptions:

- (X, Δ) *is a klt pair.*
- *There exists a projective birational morphism* $\varphi : Y \to X$ *such that* Y *is smooth and* $\varphi^*(m(K_X+\Delta))$ *admits a singular (hermitian) metric whose curvature is semi-positive and Lelong number is identically zero. Here* m *is a positive integer with* $m(K_X + \Delta)$ *Cartier.*

If $\kappa(K_X + \Delta) \geq 0$*, then* $K_X + \Delta$ *is semi-ample.*

In [41], Verbitsky proved the non-vanishing conjecture on hyperKähler manifolds (holomorphic symplectic manifolds) under the same assumption. By combining Verbitsky's non-vanishing theorem with our results, we obtain a result for semi-ampleness on 4-dimensional projective hyperKähler manifolds, which is closely related to the Strominger-Yau-Zaslow conjecture for hyperKähler manifolds (see [41], and see also [3] for recent related topics and [5] for non-algebraic cases).

Theorem 1.6 (Semi-ampleness theorem for hyperKähler manifolds : Corollary 5.5)
Let X *be a* 4*-dimensional projective hyperKähler manifold and* L *be a (holomorphic) line bundle admitting a singular (hermitian) metric whose curvature is semi-positive and Lelong number is identically zero. Then* L *is semi-ample.*

Recently, Lazić-Peternell proved several results for the non-vanishing conjecture, and obtained stronger results for the abundance conjecture by combining with our results (see [33, Theorem B] for more details).

We summarize the contents of this paper. In Section 2, we collect the basic notions and facts needed later. In Section 3, we prove our injectivity theorem (Theorem 1.3). In Section 4, we give applications of the injectivity theorem to the extension problem (Theorem 1.4 and its corollaries). In Section 5, we prove some results for semi-ampleness related to the abundance conjecture (Theorem 1.5 and Theorem 1.6).

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2. Preliminaries

2.1. Singular hermitian metrics and multiplier ideal sheaves

In this subsection, let X be a compact complex manifold and F be a (holomorphic) line bundle on X. For simplicity, we fix a smooth (hermitian) metric g on F.

DEFINITION 2.1 (Singular hermitian metrics and Curvatures). – (1) For an L^1-function φ on X, the hermitian metric h defined by

$$h := ge^{-2\varphi}$$

is called a *singular hermitian metric* on F. Further φ is called the *weight* of h with respect to the fixed smooth metric g.

(2) The *curvature* $\sqrt{-1}\Theta_h(F)$ associated to h is defined by

$$\sqrt{-1}\Theta_h(F) = \sqrt{-1}\Theta_g(F) + 2\sqrt{-1}\partial\overline{\partial}\varphi,$$

where $\sqrt{-1}\Theta_g(F)$ is the Chern curvature of g.

In this paper, we often abbreviate the singular hermitian metric to the singular metric or the metric. Here $\sqrt{-1}\partial\overline{\partial}\varphi$ is taken in the sense of distributions, and thus the curvature is a $(1,1)$-current but not always a smooth $(1,1)$-form. The curvature $\sqrt{-1}\Theta_h(F)$ is said to be *semi-positive* if $\sqrt{-1}\Theta_h(F) \geq 0$ in the sense of currents (that is, the local potential function of $\sqrt{-1}\Theta_h(F)$ is a psh function).

DEFINITION 2.2 (Multiplier ideal sheaves). – For a singular metric h on F, the ideal sheaf $\mathcal{J}(h)$ defined to be

$$\mathcal{J}(h)(B) := \mathcal{J}(\varphi)(B) := \{f \in \mathcal{O}_X(B) \mid |f|e^{-\varphi} \in L^2_{\mathrm{loc}}(B)\}$$

for every open set $B \subseteq X$, is called the *multiplier ideal sheaf* associated to h.

A theorem of Nadel states that the multiplier ideal sheaf $\mathcal{J}(h)$ is coherent if $\sqrt{-1}\Theta_h(F) \geq \gamma$ holds for some smooth $(1,1)$-form γ on X. The following example is a typical example of singular metrics that often appears in algebraic geometry.

EXAMPLE 2.3. – For given sections $\{s_i\}_{i=1}^N$ of the m-th tensor power F^m of F, the metric $ge^{-2\varphi}$ is defined by

$$\varphi := \frac{1}{2m}\log\Big(\sum_{i=1}^{N}|s_i|^2_{g^m}\Big).$$

Then the metric $ge^{-2\varphi}$ is independent of the choice of g, and its curvature is semi-positive. Further the multiplier ideal sheaf can be algebraically computed (see [7]). For example, for the metric h_D defined by the natural section of an effective $\mathbb{R}$-divisor D, we can easily check $\mathcal{J}(h_D) = \mathcal{O}_X(-\lfloor D \rfloor)$ if D is a simple normal crossing divisor.

We recall the definition of the Lelong number of singular metrics and Skoda's lemma which gives a relation between the multiplier ideal sheaf and the Lelong number.

DEFINITION 2.4 (Lelong numbers). – Let φ be a (quasi-)psh function on an open set B in $\mathbb{C}^n$. The *Lelong number* $\nu(\varphi, x)$ of φ at $x \in B$ is defined by

$$\nu(\varphi, x) = \liminf_{z \to x} \frac{\varphi(z)}{\log|z-x|}.$$

For a singular metric h such that $\sqrt{-1}\Theta_h(F) \geq \gamma$ holds for some smooth $(1,1)$-form γ, we define the Lelong number $\nu(h, x)$ of h at $x \in X$ by $\nu(h, x) := \nu(\varphi, x)$, where φ is a weight of h.

THEOREM 2.5 (Skoda's lemma). – *Let φ be a (quasi)-psh function on an open set B in $\mathbb{C}^n$.*

- *If $\nu(\varphi, x) < 1$, then we have $\mathcal{J}(\varphi)_x = \mathcal{O}_{B,x}$.*
- *If $\nu(\varphi, x) \geq n + s$ for some integer $s \geq 0$, then we have $\mathcal{J}(\varphi)_x \subseteq \mathfrak{M}_{B,x}^{s+1}$, where $\mathfrak{M}_{B,x}$ is the maximal ideal of $\mathcal{O}_{B,x}$.*

From the above example, it is easy to see that a semi-ample line bundle is always semi-positive (namely, it admits a smooth hermitian metric with semi-positive curvature). From the regularization theorem for singular metrics in [6], it follows that F is nef if F admits a singular metric h such that $\sqrt{-1}\Theta_h(F) \geq 0$ and $\nu(h, x) \equiv 0$ on X.

2.2. Singularities of pairs

In this subsection, we recall the definition of singularities of pairs.

DEFINITION 2.6 (Klt, lc, dlt, plt pairs). – Let X be a normal variety and Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. For a log resolution $\varphi : Y \to X$ of (X, Δ), we have

$$K_Y = \varphi^*(K_X + \Delta) + \sum a_i E_i,$$

where $a_i \in \mathbb{Q}$ and E_i is a prime divisor on Y for every i.

The pair (X, Δ) is called

- *kawamata log terminal* (*klt*, for short) if $a_i > -1$ for all i,
- *log canonical* (*lc*, for short) if $a_i \geq -1$ for all i.

Let (X, Δ) be an lc pair. If there is a log resolution $\varphi : Y \to X$ of (X, Δ) such that

- $\mathrm{Exc}(\varphi)$ is a divisor and
- $a_i > -1$ for every φ-exceptional divisor E_i in the above formula,

then the pair (X, Δ) is called *divisorial log terminal* (*dlt*, for short). Moreover if (X, Δ) is a dlt pair and $\lfloor \Delta \rfloor$ is a prime divisor, then the pair (X, Δ) is called *purely log terminal* (*plt*, for short).

DEFINITION 2.7 (Slc and dslt pairs, [12, Def. 1.1]). – Let X be a reduced S_2-scheme of pure dimension n and normal crossing in codimension 1, and let Δ be an effective $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Let $X = \bigcup X_i$ be the decomposition into irreducible components and

$$\nu : X^\nu := \coprod X_i^\nu \to X = \bigcup X_i$$

be the *normalization*. Here the *normalization* $\nu : X^\nu = \coprod X_i^\nu \to X = \bigcup X_i$ means that $\nu|_{X_i^\nu} : X_i^\nu \to X_i$ is the usual normalization for every i. The scheme X is called a *normal scheme* if ν is an isomorphism.

Define the $\mathbb{Q}$-divisor Θ on X^ν by $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$ and Θ_i by $\Theta_i := \Theta|_{X_i^\nu}$. The pair (X, Δ) is called *semi-log canonical* (*slc*, for short) if (X_i^ν, Θ_i) is an lc pair for every i. Moreover the pair (X, Δ) is called a *divisorial semi-log terminal* (*dslt*, for short) if X_i is normal (that is, X_i^ν is isomorphic to X_i) and (X^ν, Θ) is a dlt pair.

For the reader's convenience, we give simple examples of singularities of pairs:

EXAMPLE 2.8. – For two different lines l_1 and l_2 in the affine plane $\mathbb{A}^2$, we consider the pair $(\mathbb{A}^2, l_1 + al_2)$ for a positive number a. If $a < 1$, $(\mathbb{A}^2, l_1 + al_2)$ is a plt pair and $(l_1, al_1 \cap l_2)$ is klt. On the other hand, in the case of $a = 1$, $(\mathbb{A}^2, l_1 + l_2)$ is a dlt pair and $(l_1, l_1 \cap l_2)$ is dlt. For no boundary examples, any singularities of normal quotient surfaces by finite groups are klt (see [32, Corollary 5.21]), and the cone singularities by abelian varieties are always lc but not klt (see [18, Proposition 4.38]).

In Section 5, we use the dlt blow-up. The following theorem was originally proved by Hacon (for example, see [14, Theorem 10.4], [31, Theorem 3.1], and see [15, Section 4] for a simpler proof).

THEOREM 2.9 (Dlt blow-up). – *Let X be a normal quasi-projective variety and Δ be an effective $\mathbb{R}$-divisor on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier and (X, Δ) is an lc pair. Then there exists a projective birational morphism $\varphi : Y \to X$ from a normal quasi-projective variety Y with the following properties:*

– *Y is $\mathbb{Q}$-factorial.*
– *$a(E, X, \Delta) = -1$ for every φ-exceptional divisor E on Y.*
– *For Γ defined by*

$$\Gamma := \varphi_*^{-1}\Delta + \sum_{E:\varphi\text{-exceptional}} E,$$

the pair (Y, Γ) is dlt and $K_Y + \Gamma = \varphi^*(K_X + \Delta)$.

3. An analytic version of the injectivity theorem

The purpose of this section is to prove an analytic version of the injectivity theorem.

THEOREM 3.1 (Theorem 1.3). – *Let (F, h_F) and (L, h_L) be singular hermitian line bundles with semi-positive curvature on a compact Kähler manifold X. Assume that there exists an effective $\mathbb{R}$-divisor Δ with*

$$h_F = h_L^a \cdot h_\Delta,$$

where a is a positive real number and h_Δ is the singular metric defined by the effective divisor Δ.

Then, for a non-zero section s of L satisfying $\sup_X |s|_{h_L} < \infty$, the multiplication map

$$H^q(X, K_X \otimes F \otimes \mathcal{J}(h_F)) \xrightarrow{\otimes s} H^q(X, K_X \otimes F \otimes L \otimes \mathcal{J}(h_F h_L))$$

is injective for every q.

REMARK 3.2. – (1) *The multiplication map is well-defined thanks to the assumption $\sup_X |s|_{h_L} < \infty$. When h_L is a metric with minimal singularities on L, this assumption is always satisfied for any section s of L (see* [7] *for the definition of metrics with minimal singularities).*

(2) *The case $\Delta = 0$ of Theorem 3.1 corresponds to* [37, Theorem 1.3]. *To prove our extension theorems, we need to consider the case $\Delta \neq 0$.*

(3) *If h_L and h_F are smooth on a Zariski open set, the same conclusion holds under the weaker assumption $\sqrt{-1}\Theta_{h_F}(F) \geq a\sqrt{-1}\Theta_{h_L}(L)$ (see* [16] *and* [34]).

Proof. – The proof is a slight generalization of the proof of [37, Theorem 1.3]. The case $q = 0$ is obvious, and thus we assume $q > 0$. In [10], Enoki proved the special case that all metrics are smooth and $\Delta = 0$ by using the theory of harmonic integrals. In our situation, we can not (at least directly) apply the theory of harmonic integrals since we have to consider singular metrics with transcendental (non-algebraic) singularities. It is quite difficult to directly handle transcendental singularities, and thus we approximate a given singular metric h_F in Step 1.

STEP 1 (Equisingular approximation of h_F). – Throughout the proof, we fix a Kähler form ω on X. For the proof, we want to apply the theory of harmonic integrals, but the metric h_F may not be smooth. For this reason, We first approximate h_F by a family of metrics $\{h_\varepsilon\}_{\varepsilon>0}$ that are smooth on a Zariski open set. By [9, Theorem 2.3], we obtain singular metrics $\{h_\varepsilon\}_{1\gg\varepsilon>0}$ on F with the following properties:

(a) h_ε is smooth on $X \setminus Z_\varepsilon$, where Z_ε is a proper subvariety on X.
(b) $h_{\varepsilon_2} \leq h_{\varepsilon_1} \leq h_F$ holds for any $0 < \varepsilon_1 < \varepsilon_2$.
(c) $\mathcal{J}(h_F) = \mathcal{J}(h_\varepsilon)$.
(d) $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq -\varepsilon\omega$.

Since the point-wise norm $|s|_{h_L}$ is bounded on X and $h_F = h_L^a h_\Delta$, the set $\{x \in X \mid \nu(h_F, x) > 0\}$ is contained in the subvariety Z defined by $Z := s^{-1}(0) \cup \operatorname{Supp}\Delta$. Therefore we may assume a stronger property than property (a) (for example, see [37, Theorem 2.3]), namely

(e) h_ε is smooth on $Y := X \setminus Z$, where $Z = s^{-1}(0) \cup \operatorname{Supp}\Delta$.

Next we construct a "complete" Kähler form on Y with suitable potential function. Take a quasi-psh function ψ on X such that ψ has a logarithmic pole along Z and ψ is smooth on Y. Since quasi-psh functions are upper semi-continuous, we may assume $\psi \leq -e$. We define the $(1,1)$-form $\widetilde{\omega}$ on Y by

$$\widetilde{\omega} := k\omega + \sqrt{-1}\partial\overline{\partial}\Psi,$$

where k is a positive real number and $\Psi := 1/\log(-\psi)$. Then we can show that the $(1,1)$-form $\widetilde{\omega}$ satisfies the following properties for a sufficiently large $k > 0$:

(A) $\widetilde{\omega}$ is a complete Kähler form on Y.
(B) Ψ is bounded on X.
(C) $\widetilde{\omega} \geq \omega$.

Indeed, properties (B), (C) follow from the definition of $\Psi, \widetilde{\omega}$ and property (A) follows from straightforward computations (see [16, Lemma 3.1]).

Let $L^{n,q}_{(2)}(Y, F)_{h_\varepsilon,\widetilde{\omega}}$ be the space of L^2-integrable F-valued (n,q)-forms α with respect to the inner product $\|\cdot\|_{h_\varepsilon,\widetilde{\omega}}$ defined by

$$\|\alpha\|^2_{h_\varepsilon,\widetilde{\omega}} := \int_Y |\alpha|^2_{h_\varepsilon,\widetilde{\omega}}\, dV_{\widetilde{\omega}},$$

where $dV_{\widetilde{\omega}} := \widetilde{\omega}^n/n!$ and $n := \dim X$. Note that the $\overline{\partial}$-operator determines the densely defined closed operator $\overline{\partial}$ between the L^2-spaces $L^{n,\bullet}_{(2)}(Y, F)_{h_\varepsilon,\widetilde{\omega}}$. Since $\operatorname{Im}\overline{\partial}$ is closed in $L^{n,q}_{(2)}(Y,F)_{h_\varepsilon,\widetilde{\omega}}$ (for example, see [16, Claim 1] and [37, Proposition 5.8]), we obtain the orthogonal decomposition:

$$L^{n,q}_{(2)}(Y,F)_{h_\varepsilon,\widetilde{\omega}} = \operatorname{Im}\overline{\partial} \oplus \mathcal{H}^{n,q}_{h_\varepsilon,\widetilde{\omega}}(F) \oplus \operatorname{Im}\overline{\partial}^*_{h_\varepsilon}.$$

Here the operator $\overline{\partial}^*_{h_\varepsilon}$ denotes the closed extension of the formal adjoint of $\overline{\partial}$. We remark that it agrees with the Hilbert space adjoint of $\overline{\partial}$ since $\widetilde{\omega}$ is complete. Further $\mathcal{H}^{n,q}_{h_\varepsilon,\widetilde{\omega}}(F)$ denotes the space of harmonic forms with respect to h_ε and $\widetilde{\omega}$, namely

$$\mathcal{H}^{n,q}_{h_\varepsilon,\widetilde{\omega}}(F) := \{\alpha \mid \alpha \text{ is an } F\text{-valued } (n,q)\text{-form on } Y \text{ with } \overline{\partial}\alpha = 0 \text{ and } \overline{\partial}^*_{h_\varepsilon}\alpha = 0.\}.$$

Harmonic forms in $\mathcal{H}^{n,q}_{h_\varepsilon,\widetilde{\omega}}(F)$ are smooth by elliptic regularity. These results are known to specialists. The precise proof can be found in [16] and [37, Section 5].

Take an arbitrary cohomology class $\{u\} \in H^q(X, K_X \otimes F \otimes \mathcal{J}(h_F))$ represented by an F-valued (n,q)-form u with $\|u\|_{h_F,\omega} < \infty$. We assume that the cohomology class of su is zero in $H^q(X, K_X \otimes F \otimes L \otimes \mathcal{J}(h_F h_L))$. Our goal is to show that the cohomology class of u is actually zero under this assumption.

We have $|\beta|^2_{\widetilde{\omega}}\, dV_{\widetilde{\omega}} \le |\beta|^2_{\omega}\, dV_{\omega}$ for every (n,q)-form β since the inequality $\widetilde{\omega} \ge \omega$ holds by property (C). From this inequality and property (b) of h_ε, we obtain

$$\|u\|_{h_\varepsilon,\widetilde{\omega}} \le \|u\|_{h_\varepsilon,\omega} \le \|u\|_{h_F,\omega} < \infty. \tag{1}$$

By the above inequality, we have $\|u\|_{h_\varepsilon,\widetilde{\omega}} < \infty$ for any $\varepsilon > 0$. Therefore, by the above orthogonal decomposition, there exist $u_\varepsilon \in \mathcal{H}^{n,q}_{h_\varepsilon,\widetilde{\omega}}(F)$ and $w_\varepsilon \in \operatorname{Dom}\overline{\partial} \subseteq L^{n,q-1}_{(2)}(Y,F)_{h_\varepsilon,\widetilde{\omega}}$ such that

$$u = u_\varepsilon + \overline{\partial} w_\varepsilon.$$

We remark that the component of $\operatorname{Im}\overline{\partial}^*_{h_\varepsilon}$ is zero since u is $\overline{\partial}$-closed.

At the end of this step, we explain the strategy of the proof. In Step 2, by generalizing Enoki's proof, we show that the L^2-norm $\|\overline{\partial}^*_{h_\varepsilon h_{L,\varepsilon}} s u_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}$ converges to zero as ε goes to zero. Here $h_{L,\varepsilon}$ is the singular metric on L defined by

$$h_{L,\varepsilon} := h_\varepsilon^{1/a}\, h_\Delta^{-1/a}.$$

In Step 3, we construct solutions v_ε of the $\overline{\partial}$-equation $\overline{\partial} v_\varepsilon = s u_\varepsilon$ such that the norm $\|v_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}$ is uniformly bounded in ε. By Step 2 and Step 3, we can easily see that

$$\|s u_\varepsilon\|^2_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} \le \langle\!\langle s u_\varepsilon, \overline{\partial} v_\varepsilon \rangle\!\rangle_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} \le \|\overline{\partial}^*_{h_\varepsilon h_{L,\varepsilon}} s u_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} \|v_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} \to 0 \quad \text{as } \varepsilon \to 0.$$

In Step 4, from this convergence, we prove that u_ε converges to zero in a suitable sense.

REMARK 3.3. – *The weight of $h_{L,\varepsilon}$ may not be a quasi-psh function. By the definition of $h_{L,\varepsilon}$, we have*

$$\sqrt{-1}\Theta_{h_{L,\varepsilon}}(L) = \frac{1}{a}(\sqrt{-1}\Theta_{h_\varepsilon}(F) - [\Delta]),$$

where $[\Delta]$ is the current of integration over Δ. Hence the weight of $h_{L,\varepsilon}$ can be written as the difference of the weights of h_ε and h_Δ which are quasi-psh functions. In [8] *we already dealt with such an singular metric.*

STEP 2 (A generalization of Enoki's argument for the injectivity theorem)

The purpose of this step is to prove the following proposition, whose proof can be seen as a generalization of Enoki's proof (see [37] and [36]).

PROPOSITION 3.4. – *When ε goes to zero, the norm $\|\overline{\partial}^{*}_{h_\varepsilon h_{L,\varepsilon}} s u_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}$ converges to zero.*

Proof. – The following inequality plays an important role in the proof.

$$\|u_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}} \leq \|u\|_{h_\varepsilon,\widetilde{\omega}} \leq \|u\|_{h_F,\omega} < \infty. \tag{2}$$

The first inequality follows from the definition of u_ε and the second inequality follows from inequality (1). We remark that the right hand side does not depend on ε. By applying the Bochner-Kodaira-Nakano identity and the density lemma to u_ε (for example, see [37, Proposition 2.4]), we obtain

$$0 = \langle\langle \sqrt{-1}\Theta_{h_\varepsilon}(F)\Lambda_{\widetilde{\omega}} u_\varepsilon, u_\varepsilon \rangle\rangle_{h_\varepsilon,\widetilde{\omega}} + \|D'^{*}_{h_\varepsilon} u_\varepsilon\|^2_{h_\varepsilon,\widetilde{\omega}}. \tag{3}$$

Here $\Lambda_{\widetilde{\omega}}$ is the adjoint operator of the wedge product $\widetilde{\omega} \wedge \bullet$, and $D'^{*}_{h_\varepsilon}$ is the closed extension of the formal adjoint of the $(1,0)$-part D'_{h_ε} of the Chern connection $D_{h_\varepsilon} = D'_{h_\varepsilon} + \overline{\partial}$. Let A_ε be the first term and B_ε be the second term of the right hand side of equality (3). We first show that the first term A_ε and the second term B_ε converge to zero. For simplicity let g_ε be the integrand of A_ε, namely

$$g_\varepsilon := \langle \sqrt{-1}\Theta_{h_\varepsilon}(F)\Lambda_{\widetilde{\omega}} u_\varepsilon, u_\varepsilon \rangle_{h_\varepsilon,\widetilde{\omega}}.$$

Then there exists a positive constant $C > 0$ (independent of ε) such that

$$g_\varepsilon \geq -\varepsilon C |u_\varepsilon|^2_{h_\varepsilon,\widetilde{\omega}}. \tag{4}$$

It is easy to check this inequality. Indeed, let $\lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \cdots \leq \lambda_n^\varepsilon$ be the eigenvalues of $\sqrt{-1}\Theta_{h_\varepsilon}(F)$ with respect to $\widetilde{\omega}$. Then, for every point $y \in Y$, there exists a local coordinate $(z_1, z_2, \ldots, z_n)$ centered at y such that

$$\sqrt{-1}\Theta_{h_\varepsilon}(F) = \frac{\sqrt{-1}}{2}\sum_{j=1}^{n} \lambda_j^\varepsilon dz_j \wedge d\overline{z_j} \quad \text{and} \quad \widetilde{\omega} = \frac{\sqrt{-1}}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z_j} \quad \text{at } y.$$

When we locally write u_ε as $u_\varepsilon = \sum_{|K|=q} f_K^\varepsilon \, dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_K$, we have

$$g_\varepsilon = \sum_{|K|=q} \Big(\sum_{j\in K} \lambda_j^\varepsilon\Big) |f_K^\varepsilon|^2_{h_\varepsilon}$$

by straightforward computations. On the other hand, from property (C) of $\widetilde{\omega}$ and property (d) of h_ε, we have $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq -\varepsilon\omega \geq -\varepsilon\widetilde{\omega}$. This implies $\lambda_j^\varepsilon \geq -\varepsilon$, and thus we obtain inequality (4).

From equality (3) and inequality (4), we obtain

$$0 \geq A_\varepsilon = \int_Y g_\varepsilon \, dV_{\widetilde{\omega}} \geq -\varepsilon C \int_Y |u_\varepsilon|^2_{h_\varepsilon,\widetilde{\omega}} \, dV_{\widetilde{\omega}} \geq -\varepsilon C \|u\|^2_{h_F,\omega}.$$

The last inequality follows from inequality (2). Therefore A_ε converges to zero, and we can conclude that B_ε also converges to zero by equality (3).

To apply the Bochner-Kodaira-Nakano identity to su_ε again, we first check whether $su_\varepsilon \in L^{n,q}_{(2)}(Y, F \otimes L)_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}$. By the assumption, the point-wise norm $|s|_{h_L}$ with respect to h_L is bounded, and further we have $|s|_{h_{L,\varepsilon}} \leq |s|_{h_L}$ from property (b) of h_ε. Therefore we obtain

$$\|su_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} \leq \sup_X |s|_{h_{L,\varepsilon}} \|u_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}} \leq \sup_X |s|_{h_L} \|u\|_{h_F,\omega} < \infty.$$

Observe that the right hand side does not depend on ε. By applying the Bochner-Kodaira-Nakano identity to su_ε, we obtain

(5)
$$\|\bar{\partial}^*_{h_\varepsilon h_{L,\varepsilon}} su_\varepsilon\|^2_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} = \langle\!\langle \sqrt{-1}\Theta_{h_\varepsilon h_{L,\varepsilon}}(F\otimes L)\Lambda_{\widetilde{\omega}} su_\varepsilon, su_\varepsilon\rangle\!\rangle_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} + \|D'^*_{h_\varepsilon h_{L,\varepsilon}} su_\varepsilon\|^2_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}.$$

Here we used $\bar{\partial} su_\varepsilon = s\bar{\partial} u_\varepsilon = 0$. We see that the second term of the right hand side converges to zero. Since s is a holomorphic $(0,0)$-form, we can easily see that $D'^*_{h_\varepsilon h_{L,\varepsilon}} su_\varepsilon = sD'^*_{h_\varepsilon} u_\varepsilon$. Therefore we have

$$\|D'^*_{h_\varepsilon h_{L,\varepsilon}} su_\varepsilon\|^2_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} \leq \sup_X |s|^2_{h_{L,\varepsilon}} \int_Y |D'^*_{h_\varepsilon} u_\varepsilon|^2_{h_\varepsilon,\widetilde{\omega}}\, dV_{\widetilde{\omega}} \leq \sup_X |s|^2_{h_L} B_\varepsilon.$$

Since $|s|^2_{h_L}$ is bounded and B_ε converges to zero, the second term $\|D'^*_{h_\varepsilon h_{L,\varepsilon}} su_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}$ also converges to zero.

It remains to show that the first term of the right hand side of equality (5) converges to zero. It follows that $\sqrt{-1}\Theta_{h_\varepsilon h_{L,\varepsilon}}(F\otimes L) = (1+1/a)\sqrt{-1}\Theta_{h_\varepsilon}(F)$ holds on Y from $\sqrt{-1}\Theta_{h_\Delta} = 0$ on Y and the definition of $h_{L,\varepsilon}$ (see Remark 3.3). Therefore we obtain

$$\langle\!\langle \sqrt{-1}\Theta_{h_\varepsilon h_{L,\varepsilon}}(F\otimes L)\Lambda_{\widetilde{\omega}} su_\varepsilon, su_\varepsilon\rangle\!\rangle_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} = (1+1/a)\int_Y |s|^2_{h_{L,\varepsilon}} g_\varepsilon\, dV_{\widetilde{\omega}}.$$

Now we investigate A_ε in detail. By the definition of A_ε, we have

$$A_\varepsilon = \int_{\{g_\varepsilon \geq 0\}} g_\varepsilon\, dV_{\widetilde{\omega}} + \int_{\{g_\varepsilon \leq 0\}} g_\varepsilon\, dV_{\widetilde{\omega}}.$$

It is easy to see that the second term converges to zero. Indeed, by simple computations and inequality (4), we obtain

$$\begin{aligned} 0 \geq \int_{\{g_\varepsilon \leq 0\}} g_\varepsilon\, dV_{\widetilde{\omega}} &\geq -\varepsilon C \int_{\{g_\varepsilon \leq 0\}} |u_\varepsilon|^2_{h_\varepsilon,\widetilde{\omega}}\, dV_{\widetilde{\omega}} \\ &\geq -\varepsilon C \int_Y |u_\varepsilon|^2_{h_\varepsilon,\widetilde{\omega}}\, dV_{\widetilde{\omega}} \\ &\geq -\varepsilon C \|u\|^2_{h_F,\omega}. \end{aligned}$$

The first term also converges to zero. On the other hand, we have

- $$\begin{aligned} 0 \leq \int_{\{g_\varepsilon \geq 0\}} |s|^2_{h_{L,\varepsilon}} g_\varepsilon\, dV_{\widetilde{\omega}} &\leq \sup_X |s|^2_{h_{L,\varepsilon}} \int_{\{g_\varepsilon \geq 0\}} g_\varepsilon\, dV_{\widetilde{\omega}} \\ &\leq \sup_X |s|^2_{h_L} \int_{\{g_\varepsilon \geq 0\}} g_\varepsilon\, dV_{\widetilde{\omega}}, \end{aligned}$$
- $$\begin{aligned} 0 \geq \int_{\{g_\varepsilon \leq 0\}} |s|^2_{h_{L,\varepsilon}} g_\varepsilon\, dV_{\widetilde{\omega}} &\geq \sup_X |s|^2_{h_{L,\varepsilon}} \int_{\{g_\varepsilon \leq 0\}} g_\varepsilon\, dV_{\widetilde{\omega}} \\ &\geq \sup_X |s|^2_{h_L} \int_{\{g_\varepsilon \leq 0\}} g_\varepsilon\, dV_{\widetilde{\omega}}. \end{aligned}$$

Therefore the right hand side of equality (5) converges to zero. □

STEP 3 (A construction of solutions of the $\bar{\partial}$-equation via the Čech complex)

The purpose of this step is to prove the following proposition.

PROPOSITION 3.5. – *There exist F-valued $(n, q-1)$-forms w_ε on Y with the following properties:*

(1) $\overline{\partial} w_\varepsilon = u - u_\varepsilon$. (2) *The norm $\|w_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}}$ is uniformly bounded in ε.*

Proof. – The proof is the same as in the proof in [37]. We need only the case $q = 1$ for the proof of Theorem 1.4 and Theorem 1.5. For the reader's convenience, we sketch the proof for $q = 1$, in which case the computations are more easy to check but the essential arguments are still clearly visible. For the general case, see [37, Proposition 3.3, Theorem 5.9].

The main idea is to convert the $\overline{\partial}$-equation $\overline{\partial} w_\varepsilon = u - u_\varepsilon$ to the equation $\delta\gamma_\varepsilon = \beta_\varepsilon$ of the coboundary operator δ in the space of cochains $C^\bullet(K_X \otimes F \otimes \mathcal{J}(h_\varepsilon))$, by using the Čech complex and pursuing the De Rham-Weil isomorphism. Here β_ε is the 1-cochain constructed from $u - u_\varepsilon$. In this construction, we locally solve the $\overline{\partial}$-equation by the standard technique of the L^2-method for the $\overline{\partial}$-equation (for example, see [37, Lemma 5.4]). The L^2-space $L^{n,\bullet}_{(2)}(Y, F)_{h_\varepsilon,\widetilde{\omega}}$ depends on ε, but the space of cochains $C^\bullet(K_X \otimes F \otimes \mathcal{J}(h_\varepsilon))$ is independent of ε thanks to property (c). This is one of the important points. In Claim 3.6, we show that β_ε converges to some 1-coboundary β_0 in $C^1(K_X \otimes F \otimes \mathcal{J}(h_F))$ with respect to the topology defined by the local L^2-norms (see [37, Section 5] for this topology). Further we see that the coboundary operator δ is an open map by Claim 3.7. By these observations, we construct solutions γ_ε of the equation $\delta\gamma_\varepsilon = \beta_\varepsilon$ with suitable norm. Finally, by using a partition of unity, we conversely construct w_ε satisfying the properties in Proposition 3.5.

Let $\mathcal{U}$ be a finite open cover $\mathcal{U} := \{B_i\}_{i\in I}$ of X by (sufficiently small) Stein open sets B_i. For simplicity we put $U_\varepsilon := u - u_\varepsilon$. By [37, Lemma 5.4], we obtain (local) solutions $\beta_{\varepsilon,i}$ on $B_i \setminus Z$ of the $\overline{\partial}$-equation $\overline{\partial}\beta_{\varepsilon,i} = U_\varepsilon$ satisfying $\|\beta_{\varepsilon,i}\|_{B_i,h_\varepsilon,\widetilde{\omega}} \le C\|U_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}}$ for some positive constant C (independent of ε). In the proof C denotes a (possibly different) positive constant independent of ε. Inequality (2) implies

$$\|U_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}} \le \|u\|_{h_\varepsilon,\widetilde{\omega}} + \|u_\varepsilon\|_{h_\varepsilon\widetilde{\omega}} \le 2\|u\|_{h_F,\omega}.$$

In particular, the norm $\|\beta_{\varepsilon,i}\|_{B_i,h_\varepsilon,\widetilde{\omega}}$ on B_i can be estimated by a constant independent of ε. We consider the F-valued $(n,0)$-form $(\beta_{\varepsilon,j} - \beta_{\varepsilon,i})$ on $B_{ij} \setminus Z$, where $B_{ij} := B_i \cap B_j$. In the proof, we often regard $\overline{\partial}$-closed F-valued $(n,0)$-forms as holomorphic functions. In general, we have $|f|^2_{\widetilde{\omega}}\, dV_{\widetilde{\omega}} = |f|^2_\omega\, dV_\omega$ for an $(n,0)$-form f. Therefore $(\beta_{\varepsilon,j} - \beta_{\varepsilon,i})$ can be seen as a holomorphic function with bounded L^2-norm. By the Riemann extension theorem, it can be extended to the $\overline{\partial}$-closed F-valued $(n,0)$-form on B_{ij} (which is denoted by the same notation). Further it belongs to $H^0(B_{ij}, K_X \otimes F \otimes \mathcal{J}(h_F))$ by property (c).

We define the 1-cocycle β_ε by

$$\beta_\varepsilon := \delta(\{\beta_{\varepsilon,i}\}) := \{(\beta_{\varepsilon,j} - \beta_{\varepsilon,i})\},$$

where δ is the coboundary operator defined on the space of cochains $C^\bullet(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h_F))$ calculated by $\mathcal{U}$. The topology of $C^p(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h_F))$ is induced by the semi-norms $\{p_K(\cdot)\}$ defined to be

$$p_K^2(\{f_{i_0\dots i_p}\}) := \int_K |f_{i_0\dots i_p}|^2_{h_F,\omega}\, dV_\omega$$

for every $\{f_{i_0\dots i_p}\} \in C^p(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h_F))$ and $K \Subset B_{i_0\dots i_p}$. The above integral is independent of ω since $f_{i_0\dots i_p}$ is an F-valued $(n,0)$-form. Then $C^p(\mathcal{U}, K_X \otimes F \otimes \mathcal{J}(h_F))$

becomes a Fréchet space with respect to these semi-norms (see [37, Theorem 5.3]). Then we prove the following claim.

Claim 3.6. – *There exists a subsequence of the sequence $\{\beta_\varepsilon\}_{\varepsilon>0}$ that converges to some β_0 in $C^1(\mathscr{U}, K_X \otimes F \otimes \mathscr{J}(h_F))$.*

Proof of Claim 3.6. – We regard $\beta_{\varepsilon,ij} := \beta_{\varepsilon,j} - \beta_{\varepsilon,i}$ as a holomorphic function on B_{ij}. By the construction of $\beta_{\varepsilon,i}$, the norm $\|\beta_{\varepsilon,ij}\|_{B_{ij},h_\varepsilon}$ is uniformly bounded. This implies that the sup-norm $\sup_K |\beta_{\varepsilon,ij}|$ is also uniformly bounded for every $K \Subset B_{ij}$. (Recall that the local sup-norm of holomorphic functions can be estimated by the L^2-norm). By Montel's theorem, there exists a subsequence of $\{\beta_{\varepsilon,ij}\}_{\varepsilon>0}$ such that it uniformly converges to some $\beta_{0,ij}$ on every relatively compact set in B_{ij}. By [37, Lemma 5.2], this subsequence converges to $\beta_{0,ij}$ with respect to the above semi-norms $p_K(\cdot)$. From this argument, we can find a subsequence satisfying the conclusion of the claim. □

For simplicity, we continue to use the same notation for the subsequence in Claim 3.6. To apply the open mapping theorem to δ, we consider the topology of the image of δ.

Claim 3.7. – *The space of cocycles $Z^p(\mathscr{U}, K_X \otimes F \otimes \mathscr{J}(h_F)) := \operatorname{Ker}\delta$ and the space of coboundaries $B^p(\mathscr{U}, K_X \otimes F \otimes \mathscr{J}(h_F)) := \operatorname{Im}\delta$ are closed subspaces. In particular, the limit β_0 is also a 1-coboundary.*

Proof of Claim 3.7. – It is easy to see that the coboundary operator δ is continuous, and thus $Z^p(\mathscr{U}, K_X \otimes F \otimes \mathscr{J}(h_F)) = \operatorname{Ker}\delta$ is a closed subspace. The Čech cohomology group $\check{H}^p(\mathscr{U}, K_X \otimes F \otimes \mathscr{J}(h_F))$ is a finite dimensional vector space, and thus $B^p(\mathscr{U}, K_X \otimes F \otimes \mathscr{J}(h_F))$ is closed (see [37, Proposition 2.8, Lemma 5.7]). On the other hand β_ε is a 1-coboundary since $U_\varepsilon = u - u_\varepsilon$ belongs to $\operatorname{Im}\overline{\partial}$ in $L^{n,q}_{(2)}(Y, F)_{h_\varepsilon,\omega}$. Therefore the limit β_0 is also 1-coboundary. □

We construct solutions γ_ε of the δ-equation $\delta\gamma_\varepsilon = \beta_\varepsilon$ with suitable local L^2-norm. The coboundary operator

$$\delta\colon C^{p-1}(\mathscr{U}, K_X \otimes F \otimes \mathscr{J}(h_F)) \to B^p(\mathscr{U}, K_X \otimes F \otimes \mathscr{J}(h_F))$$

is continuous and surjective between Fréchet spaces, and thus it is an open map by the open mapping theorem. From the latter conclusion of Claim 3.7, there exists $\gamma_0 \in C^0(\mathscr{U}, K_X \otimes F \otimes \mathscr{J}(h_F))$ such that $\delta\gamma_0 = \beta_0$. For an arbitrary family $K := \{K_i\}_{i\in I}$ of relative compact sets $K_i \Subset B_i$, the image $\delta(\Delta_K)$ of Δ_K is an open neighborhood of β_0, where Δ_K is the open neighborhood of γ_0 defined by

$$\Delta_K := \{\gamma \in C^0(\mathscr{U}, K_X \otimes F \otimes \mathscr{J}(h_F)) \mid p_{K_i}(\gamma - \gamma_0) < 1 \text{ for any } i \in I.\}.$$

Since the image $\delta(\Delta_K)$ is an open neighborhood of β_0 and β_ε converges to β_0, there exists $\gamma_\varepsilon := \{\gamma_{\varepsilon,i}\} \in \Delta_K$ such that

$$\{\gamma_{\varepsilon,j} - \gamma_{\varepsilon,i}\} = \delta\gamma_\varepsilon = \beta_\varepsilon = \{\beta_{\varepsilon,j} - \beta_{\varepsilon,i}\}, \tag{6}$$

$$p^2_{K_i}(\gamma_\varepsilon) = \int_{K_i} |\gamma_{\varepsilon,i}|^2_{h_F,\omega}\, dV_\omega \leq C_K \tag{7}$$

for some positive constant C_K (which depends on the choice of K, γ_0, but is independent of ε).

From now on, we construct solutions w_ε with the properties in Proposition 3.5. For a partition of unity $\{\rho_i\}_{i\in I}$ of $\mathcal{U}$, we obtain

$$\{\overline{\partial}\sum_{k\in I}\rho_k(\gamma_{\varepsilon,i}-\gamma_{\varepsilon,k})\} = \{\overline{\partial}\sum_{k\in I}\rho_k(\beta_{\varepsilon,i}-\beta_{\varepsilon,k})\}$$

from equality (6). Note that the above cochain determines the global F-valued $(n,1)$-form on X. By $\overline{\partial}\gamma_{\varepsilon,i}=0$ and $\overline{\partial}\beta_{\varepsilon,i}=U_\varepsilon$ on $B_i\setminus Z$, it is easy to see that

$$\{\overline{\partial}\sum_{k\in I}\rho_k(\gamma_{\varepsilon,i}-\gamma_{\varepsilon,k})\} = -\overline{\partial}\sum_{k\in I}\rho_k\gamma_{\varepsilon,k},$$
$$\{\overline{\partial}\sum_{k\in I}\rho_k(\beta_{\varepsilon,i}-\beta_{\varepsilon,k})\} = U_\varepsilon-\overline{\partial}\sum_{k\in I}\rho_k\beta_{\varepsilon,k}.$$

Therefore $w_\varepsilon := \sum_{k\in I}\rho_k\beta_{\varepsilon,k}-\sum_{k\in I}\rho_k\gamma_{\varepsilon,k}$ satisfies $\overline{\partial}w_\varepsilon=U_\varepsilon$. It remains to estimate the L^2-norm of w_ε. By simple computations, we have

$$\int_Y \Big|\sum_{k\in I}\rho_k\beta_{\varepsilon,k}\Big|^2_{h_\varepsilon,\widetilde{\omega}}\,dV_{\widetilde{\omega}} \leq \sum_{k\in I}\int_{B_k\setminus Z}\big|\beta_{\varepsilon,k}\big|^2_{h_\varepsilon,\widetilde{\omega}}\,dV_{\widetilde{\omega}} \leq C\,\|u\|^2_{h_F,\omega}$$

for some $C>0$. On the other hand, by putting $K_i := \operatorname{Supp}\rho_i$, we may assume that the inequality

$$p^2_{K_i}(\gamma_\varepsilon)=\int_{\operatorname{Supp}\rho_i}|\gamma_{\varepsilon,i}|^2_{h_F,\omega}dV_\omega \leq C_K$$

holds by inequality (7). Hence we obtain

$$\int_X \Big|\sum_{k\in I}\rho_k\gamma_{\varepsilon,k}\Big|^2_{h_\varepsilon,\omega}\,dV_\omega \leq \sum_{k\in I}\int_{B_k\cap\operatorname{Supp}\rho_k}\big|\gamma_{\varepsilon,k}\big|^2_{h_\varepsilon,\omega}\,dV_\omega \leq C_K\,\sharp I.$$

These inequalities complete the proof. □

STEP 4 (Limit of the harmonic forms). – In this step, we investigate the limit of u_ε and complete the proof of Theorem 3.1. First we prove the following proposition.

PROPOSITION 3.8. – *There exist $F\otimes L$-valued $(n,q-1)$-forms v_ε on Y with the following properties:*

(1) $\overline{\partial}v_\varepsilon = su_\varepsilon$. (2) *The norm $\|v_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}$ is uniformly bounded in ε.*

Proof. – There exists an $F\otimes L$-valued $(n,q-1)$-form v such that $\overline{\partial}v = su$ and $\|v\|_{h_F h_L,\omega}<\infty$, since we are assuming that the cohomology class of su is zero in $H^q(X, K_X\otimes F\otimes L\otimes \mathcal{J}(h_F h_L))$. For w_ε with the properties in Proposition 3.5, we put $v_\varepsilon := -sw_\varepsilon+v$. Then it is easy to check $\overline{\partial}v_\varepsilon = su_\varepsilon$. Furthermore, an easy computation yields

$$\|v_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} \leq \|sw_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}+\|v\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} \leq \sup_X|s|_{h_L}\|w_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}}+\|v\|_{h_F h_L,\widetilde{\omega}}.$$

Since $\|v\|_{h_F h_L,\widetilde{\omega}}\leq\|v\|_{h_F h_L,\omega}<\infty$ and the norm $\|w_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}}$ is uniformly bounded, the right hand side can be estimated by a constant independent of ε. □

Next we consider the limit of the norm $\|su_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}$.

PROPOSITION 3.9. – *The norm $\|su_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}$ converges to zero when ε tends to zero.*

Proof. – For $v_\varepsilon \in L^{n,q-1}_{(2)}(Y, F \otimes L)_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}$ satisfying the properties in Proposition 3.8, we obtain

$$\begin{aligned}\|su_\varepsilon\|^2_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} &= \langle\!\langle su_\varepsilon, \overline{\partial} v_\varepsilon\rangle\!\rangle_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} \\ &= \langle\!\langle \overline{\partial}^*_{h_\varepsilon h_{L,\varepsilon}} su_\varepsilon, v_\varepsilon\rangle\!\rangle_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} \\ &\leq \|\overline{\partial}^*_{h_\varepsilon h_{L,\varepsilon}} su_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} \|v_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}.\end{aligned}$$

The norm $\|v_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}$ is uniformly bounded by Proposition 3.8. On the other hand, the norm $\|\overline{\partial}^*_{h_\varepsilon h_{L,\varepsilon}} su_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}$ converges to zero by Proposition 3.4. Therefore the norm $\|su_\varepsilon\|_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}}$ also converges to zero. □

Fix a sufficiently small number $\varepsilon_0 > 0$. Then, for every positive number ε with $0 < \varepsilon < \varepsilon_0$, by property (b) of h_ε, we obtain

$$\|u_\varepsilon\|_{h_{\varepsilon_0},\widetilde{\omega}} \leq \|u_\varepsilon\|_{h_\varepsilon,\widetilde{\omega}} \leq \|u\|_{h_F,\omega}.$$

In particular, the norm of u_ε with respect to h_{ε_0} is uniformly bounded. Therefore there exists a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ that converges to $\alpha \in L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon_0},\widetilde{\omega}}$ with respect to the weak L^2-topology in $L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon_0},\widetilde{\omega}}$. For simplicity, we use the same notation $\{u_\varepsilon\}_{\varepsilon>0}$ for this subsequence. Then we prove the following proposition.

Proposition 3.10. – *The weak limit α of $\{u_\varepsilon\}_{\varepsilon>0}$ in $L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon_0},\widetilde{\omega}}$ is zero.*

Proof. – For every positive number $\delta > 0$, we define the open subset A_δ of Y by $A_\delta := \{y \in Y \mid |s|^2_{h_{L,\varepsilon_0}} > \delta \text{ at } y.\}$. By an easy computation, we have

$$\begin{aligned}\|su_\varepsilon\|^2_{h_\varepsilon h_{L,\varepsilon},\widetilde{\omega}} &\geq \|su_\varepsilon\|^2_{h_{\varepsilon_0} h_{L,\varepsilon_0},\widetilde{\omega}} \\ &\geq \int_{A_\delta} |s|^2_{h_{L,\varepsilon_0}} |u_\varepsilon|^2_{h_{\varepsilon_0},\widetilde{\omega}}\, dV_{\widetilde{\omega}} \\ &\geq \delta \int_{A_\delta} |u_\varepsilon|^2_{h_{\varepsilon_0},\widetilde{\omega}}\, dV_{\widetilde{\omega}} \geq 0\end{aligned}$$

for every $\delta > 0$. Since the left hand side converges to zero, the norm $\|u_\varepsilon\|_{A_\delta,h_{\varepsilon_0},\widetilde{\omega}}$ on A_δ also converges to zero. Notice that $u_\varepsilon|_{A_\delta}$ converges to $\alpha|_{A_\delta}$ with respect to the weak L^2-topology in $L^{n,q}_{(2)}(A_\delta, F)_{h_{\varepsilon_0},\widetilde{\omega}}$. Here $u_\varepsilon|_{A_\delta}$ (resp. $\alpha|_{A_\delta}$) denotes the restriction of u_ε (resp. α) to A_δ. Indeed, for every $\gamma \in L^{n,q}_{(2)}(A_\delta, F)_{h_{\varepsilon_0},\widetilde{\omega}}$, the inner product $\langle\!\langle u_\varepsilon|_{A_\delta}, \gamma\rangle\!\rangle_{A_\delta} = \langle\!\langle u_\varepsilon, \widetilde{\gamma}\rangle\!\rangle_Y$ converges to $\langle\!\langle \alpha, \widetilde{\gamma}\rangle\!\rangle_Y = \langle\!\langle \alpha|_{A_\delta}, \gamma\rangle\!\rangle_{A_\delta}$, where $\widetilde{\gamma}$ denotes the zero extension of γ to Y. Since $u_\varepsilon|_{A_\delta}$ converges to $\alpha|_{A_\delta}$, we obtain

$$\|\alpha|_{A_\delta}\|_{A_\delta,h_{\varepsilon_0},\widetilde{\omega}} \leq \liminf_{\varepsilon\to 0} \|u_\varepsilon|_{A_\delta}\|_{A_\delta,h_{\varepsilon_0},\widetilde{\omega}} = 0.$$

The first inequality follows since the norm is lower semi-continuous with respect to the weak convergence. Therefore we have $\alpha|_{A_\delta} = 0$ for any $\delta > 0$. By the definition of A_δ, the union of $\{A_\delta\}_{\delta>0}$ agrees with $Y = X \setminus Z$, which asserts that the weak limit α is zero on Y. □

By using Proposition 3.10, we complete the proof of Theorem 3.1. By the definition of u_ε, we have

$$u = u_\varepsilon + \overline{\partial} v_\varepsilon.$$

By Proposition 3.10, the form $\overline{\partial} v_\varepsilon$ converges to u with respect to the weak L^2-topology. Then it is easy to see that u is a $\overline{\partial}$-exact form (that is, $u \in \operatorname{Im}\overline{\partial} \subset L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon_0}, \widetilde{\omega}}$). This is because the subspace $\operatorname{Im}\overline{\partial}$ is closed in $L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon_0}, \widetilde{\omega}}$ with respect to the weak L^2-topology. In summary, we proved that u is a $\overline{\partial}$-exact form in $L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon_0}, \widetilde{\omega}}$. This implies that the cohomology class $\{u\}$ of u is zero in $H^q(X, K_X \otimes F \otimes \mathcal{J}(h_{\varepsilon_0}))$. By property (c), we obtain the conclusion of Theorem 3.1. □

4. Theorems related to the extension conjecture

The purpose of this section is to obtain some extension theorems as applications of Theorem 3.1. For this purpose, by making use of our injectivity theorem, we first prove the following extension theorem, which can be seen as a special case of the extension conjecture for dlt pairs.

THEOREM 4.1 (Theorem 1.4). – *Let X be a compact Kähler manifold and $\Delta := S + B$ be an effective $\mathbb{Q}$-divisor with the following assumptions:*

(1) *Δ is a simple normal crossing divisor with $0 \le \Delta \le 1$ and $\lfloor \Delta \rfloor = S$.*
(2) *$K_X + \Delta$ is $\mathbb{Q}$-linearly equivalent to an effective divisor D with $S \subseteq \operatorname{Supp} D$.*
(3) *$K_X + \Delta$ admits a singular metric h with semi-positive curvature.*

Then, for an integer $m \ge 2$ with $m(K_X + \Delta)$ Cartier and a section $u \in H^0(S, \mathcal{O}_S(m(K_X + \Delta)))$ that belongs to the image of $H^0(S, \mathcal{O}_S(m(K_X + \Delta)) \otimes \mathcal{J}(h^{m-1}h_B)) \to H^0(S, \mathcal{O}_S(m(K_X + \Delta)))$, the section u can be extended to a section in $H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$.

Moreover if $h \le C h_D$ holds for some $C > 0$ and the singular metric h_D induced by D, then every cohomology class $u \in H^q(S, \mathcal{O}_S(m(K_X + \Delta)) \otimes \mathcal{J}(h^{m-1}h_B))$ can be extended to a class in $H^q(X, \mathcal{O}_X(m(K_X + \Delta)) \otimes \mathcal{J}(h^{m-1}h_B))$ for any $q \ge 0$.

Proof. – We may add the assumption of $h \le h_D$, where h_D is the singular metric on $K_X + \Delta$ defined by the effective divisor D. Indeed, for a smooth metric g on $K_X + \Delta$ and an L^1-function φ (resp. φ_D) with $h = g\, e^{-2\varphi}$ (resp. $h_D = g\, e^{-2\varphi_D}$), the metric defined by $g\, e^{-2\max(\varphi, \varphi_D)}$ satisfies assumption (3) again, and the multiplier ideal only gets larger.

For the Cartier divisor $G := m(K_X + \Delta)$, we consider the following exact sequence:

$$0 \to \mathcal{O}_X(G - S) \otimes I(h^{m-1}h_B) \to \mathcal{O}_X(G) \otimes \mathcal{J}(h^{m-1}h_B) \to \mathcal{O}_S(G) \otimes \mathcal{J}(h^{m-1}h_B) \to 0.$$

We prove that the natural homomorphism

$$+S : H^q(X, \mathcal{O}_X(G - S) \otimes I(h^{m-1}h_B)) \to H^q(X, \mathcal{O}_X(G) \otimes I(h^{m-1}h_B))$$

is injective. Then the conclusion follows from the induced long exact sequence.

By the assumption on the support of D, we can take an integer $a > 0$ such that aD is a Cartier divisor and $S \le aD$. Then we have the following commutative diagram:

$$\begin{array}{ccc} & & H^q(X, \mathcal{O}_X(G) \otimes I(h^{m-1}h_B)) \supseteq \operatorname{Im}(+S) \\ & \overset{+S}{\nearrow} & \Big\downarrow {\scriptstyle +(aD-S)} \\ H^q(X, \mathcal{O}_X(G - S) \otimes I(h^{m-1}h_B)) & \xrightarrow[+aD]{} & H^q(X, \mathcal{O}_X(G - S + aD) \otimes I(h^{a+m-1}h_B)). \end{array}$$

Our purpose is to show that the map to the upper right is injective. For this purpose, we show that the horizontal map is injective as an application of Theorem 3.1.

By the definition of G, we have

$$G - S = m(K_X + \Delta) - S = K_X + (m-1)(K_X + \Delta) + B.$$

Then the line bundle $F := \mathcal{O}_X((m-1)(K_X+\Delta)+B)$ equipped with the metric $h_F := h^{m-1}h_B$ and the line bundle $L := \mathcal{O}_X(aD)$ equipped with the metric $h_L := h^a$ satisfy the assumptions in Theorem 3.1. Indeed, we have $h_F = h_L^{(m-1)/a}h_B$ by the construction, and further the point-wise norm $|s_{aD}|_{h_L}$ is bounded on X by the inequality $h \leq h_D$, where s_{aD} is the natural section of aD. Therefore we can conclude that the horizontal map is injective by Theorem 3.1. □

To obtain some results related to the abundance conjecture (Theorem 5.1 and Corollary 5.3), we need the following corollary, which is a slight generalization of Theorem 4.1.

COROLLARY 4.2. – *Under the same situation as in Theorem 4.1, instead of assumption* (3)*, we assume the following assumption*:

(3′) *There exist effective $\mathbb{Q}$-divisors E and F and a singular metric h on $\mathcal{O}_X(F)$ with semi-positive curvature such that*
- $K_X + \Delta \sim_{\mathbb{Q}} E + F$,
- *$E + B$ is simple normal crossing,*
- *E has no common component with S,*
- $\nu(h, x) = 0$ *at every point $x \in S$.*

Let $m(\geq 2)$ be an integer such that mE, mF, $m(K_X + \Delta)$ are Cartier, and let $\widetilde{s}$ be the natural section of mE. Then, for a section $u \in H^0(S, \mathcal{O}_S(mF))$, the section $u \cdot \widetilde{s} \in H^0(S, \mathcal{O}_S(m(K_X + \Delta)))$ can be extended to a section in $H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$.

Proof. – Let h_E be the singular metric on E induced by the section $\widetilde{s} \in H^0(X, \mathcal{O}_X(mE))$. By the definition, the metric h_E satisfies $\sqrt{-1}\Theta_{h_E}(E) \geq 0$ and $\sup_X |\widetilde{s}|_{h_E^m} < \infty$. The product $h \cdot h_E$ determines the singular metric on $K_X + \Delta$ with semi-positive curvature. It is sufficient to show that $u \cdot \widetilde{s}$ belongs to $H^0(S, \mathcal{O}_S(m(K_X+\Delta)) \otimes \mathcal{J})$, where $\mathcal{J}$ is the multiplier ideal defined by $\mathcal{J} := \mathcal{J}(h^{m-1}h_E^{m-1}h_B)$.

In the first step, we see that

$$\mathcal{J}_x = \mathcal{J}(h_E^{m-1}h_B)_x$$

for every $x \in S$, where $\mathcal{J}_x$ denotes the stalk of a sheaf $\mathcal{J}$ at x. Let f be a holomorphic function on an open neighborhood U_x of $x \in S$ with $f \in \mathcal{J}(h_E^{m-1}h_B)_x$, and let φ (resp. φ_E, φ_B) be a local weight of h (resp. h_E, h_B). By taking a real number $p > 1$ with $\mathcal{J}(h_E^{p(m-1)}h_B^p) = \mathcal{J}(h_E^{m-1}h_B)$, we may assume that $|f|e^{-p(m-1)\varphi_E - p\varphi_B}$ is L^2-integrable on U_x. Then, for the positive number q with $1/p + 1/q = 1$, we obtain

$$\int_{U_x} |f|^2 e^{-2(m-1)\varphi - 2(m-1)\varphi_E - 2\varphi_B} \leq \Big(\int_{U_x} |f|^{2p} e^{-2p(m-1)\varphi_E - 2p\varphi_B}\Big)^{1/p} \cdot \Big(\int_{U_x} e^{-2q(m-1)\varphi}\Big)^{1/q}$$

by Hölder's inequality. The function $e^{-2q(m-1)\varphi}$ is locally L^2-integrable for any $q > 0$ by Skoda's lemma and the assumption on the Lelong number. On the other hand, as mentioned above, the function $|f|^p e^{-p(m-1)\varphi_E - p\varphi_B}$ is also locally L^2-integrable. Therefore we have $\mathcal{J}_x = \mathcal{J}(h_E^{m-1}h_B)_x$ for every $x \in S$.

In the second step, we prove

$$u \cdot \widetilde{s} \in H^0(S, \mathcal{O}_S(m(K_X + \Delta)) \otimes \mathcal{J}|_S),$$

where $\mathcal{J}|_S$ is the restriction of $\mathcal{J}$ defined by

$$\mathcal{J}|_S := \mathcal{J} \cdot \mathcal{O}_S = \mathcal{J}/(\mathcal{J} \cap \mathcal{I}_S).$$

Let $\widetilde{u}$ be a local extension of u on an open neighborhood U_x of $x \in S$. By the klt condition of B, we can take a real number $p > 1$ with $\mathcal{J}(h_B^p) = \mathcal{O}_X$. Then, for the holomorphic function $g := \widetilde{u} \cdot \widetilde{s}$, by taking the positive number q with $1/p + 1/q = 1$, we obtain

$$\begin{aligned}\int_{U_x} |g|^2 e^{-2(m-1)\varphi_E - 2\varphi_B} &\leq \Big(\int_{U_x} |g|^{2p} e^{-2p(m-1)\varphi_E - 2p\varphi_B} \Big)^{1/p} \cdot \Big(\int_{U_x} 1 \Big)^{1/q} \\ &\leq \sup_{U_x} |g|^{2p} e^{-2p(m-1)\varphi_E} \Big(\int_{U_x} e^{-2p\varphi_B} \Big)^{1/p} \cdot \Big(\int_{U_x} 1 \Big)^{1/q}\end{aligned}$$

by Hölder's inequality again. The point-wise norm $|g|^{2p} e^{-2p(m-1)\varphi_E}$ is bounded by the choice of h_E. It implies that $u \cdot \widetilde{s}$ belongs to $\mathcal{J}(h_E^{m-1} h_B)|_S = \mathcal{J}|_S$.

Finally we show

$$u \cdot \widetilde{s} \in H^0(S, \mathcal{O}_S(m(K_X + \Delta)) \otimes \mathcal{J}).$$

By simple computations we have $\mathcal{O}_S \otimes \mathcal{J} = \mathcal{O}_X \otimes \mathcal{J}/(\mathcal{J} \cdot \mathcal{I}_S)$, and thus, by the second step, it is sufficient to see

$$\mathcal{J} \cap \mathcal{I}_S = \mathcal{J} \cdot \mathcal{I}_S.$$

Here $\mathcal{I}_S$ denotes the ideal sheaf defined by S. By the first step and the assumption on the support of $E + B$. we have

$$\mathcal{J}_x = \mathcal{J}(h_E^{m-1} h_B)_x = \mathcal{O}_X(-\lfloor (m-1)E + B \rfloor)_x$$

for every $x \in S$. Therefore we can easily see $\mathcal{J} \cap \mathcal{I}_S = \mathcal{J} \cdot \mathcal{I}_S$ since S and $E + B$ have no common component by the assumption. The section $u \cdot \widetilde{s}$ actually belongs to $H^0(S, \mathcal{O}_S(m(K_X + \Delta)) \otimes \mathcal{J})$. The conclusion follows from Theorem 4.1. □

REMARK 4.3. – *When we apply the injectivity theorem in order to extend sections, we need to handle $\mathcal{O}_S \otimes \mathcal{J}(\varphi)$ (not $\mathcal{J}(\varphi)|_S$). On the other hand, when we apply the Ohsawa-Takegoshi extension theorem, we usually use the restriction of multiplier ideal sheaves $\mathcal{J}(\varphi)|_S$. It is relatively difficult to handle $\mathcal{O}_S \otimes \mathcal{J}(\varphi)$. However the support condition* (*the second assumption of the above corollary*) *fortunately appears in the proof of the applications related to the abundance conjecture, which asserts $\mathcal{O}_S \otimes \mathcal{J}(\varphi) = \mathcal{J}(\varphi)|_S$.*

The following corollary is the the special case that $E = \mathcal{O}_X$ and $\widetilde{s} = 1 \in H^0(X, \mathcal{O}_X)$ of the above corollary.

COROLLARY 4.4. – *Under the same situation as in Theorem 4.1, instead of assumption* (3)*, we assume the following assumption*:

(3″) $K_X + \Delta$ *admits a singular metric h such that $\sqrt{-1}\Theta_h \geq 0$ and $\nu(h, x) = 0$ at every point $x \in S$.*

Then, for an integer $m \geq 2$ with $m(K_X + \Delta)$ Cartier, a section $u \in H^0(S, \mathcal{O}_S(m(K_X + \Delta)))$ can be extended to a section in $H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$.

For further applications of the above results, we prepare the following lemma.

LEMMA 4.5. – *Let φ be a (quasi)-psh function on a complex manifold X and $\pi : Y \to X$ be a modification. The Lelong number $\nu(\varphi, x_0)$ is zero at $x_0 \in X$ if and only if the Lelong number $\nu(\pi^*\varphi, y)$ is zero at every point $y \in \pi^{-1}(x_0)$.*

Proof. – The "if" part follows from the inequality

$$\nu(\varphi, x_0) \leq \nu(\pi^*\varphi, y).$$

Now we show the "only if" part. For a contradiction, we assume that $\nu(\pi^*\varphi, y_0) > 0$ for some point $y_0 \in \pi^{-1}(x_0)$. By Skoda's lemma (Theorem 2.5), we can take a sufficiently large number $m > 0$ such that $\pi^* dV_X e^{-2m\pi^*\varphi}$ is not integrable on a neighborhood of y_0, where dV_X is a standard volume form on a neighborhood B of x_0. By the change of variable formula, we have

$$\int_B e^{-2m\varphi}\, dV_X = \int_{\pi^{-1}(B)} e^{-2m\pi^*\varphi}\, \pi^* dV_X.$$

By the assumption of $\nu(\varphi, x_0) = 0$, the left hand side is finite for a sufficiently small B. It is a contradiction to the choice of m. Therefore we have $\nu(\pi^*\varphi, y) = 0$ at every point $y \in \pi^{-1}(x_0)$. □

5. Theorems related to the abundance conjecture

In this section, we prove some applications related to the abundance conjecture. The proof of the following theorem is based on [8, Section 8] and [20, Theorem 5.9]. In our case, we use the dlt blow-up (Theorem 2.9).

THEOREM 5.1 (cf. Theorem 1.5). – *Assume that Conjecture 1.1 holds in dimension $(n-1)$. Let X be an n-dimensional normal projective variety and Δ be an effective $\mathbb{Q}$-divisor with the following assumptions:*

- *(X, Δ) is a klt pair.*
- *There exists an effective $\mathbb{Q}$-divisor D such that $K_X + \Delta \sim_{\mathbb{Q}} D$.*
- *There exists a projective birational morphism $\varphi : Y \to X$ such that Y is smooth and $\varphi^*(m(K_X + \Delta))$ admits a singular metric h whose curvature is semi-positive and Lelong number is identically zero on $\operatorname{Supp} \varphi^* D$. Here m is a positive integer with $m(K_X + \Delta)$ Cartier.*

Then $K_X + \Delta$ is semi-ample.

Proof. – By replacing h with $g\, e^{-2\max(\psi, \psi_{\varphi^* mD})}$, we may assume that the Lelong number of h is identically zero on Y. Here g is a smooth metric on $\varphi^*(m(K_X + \Delta))$ and ψ (resp. $\psi_{\varphi^* mD}$) is the weight of h (resp. $h_{\varphi^* mD}$). In particular, we can see that D is nef. Conjecture 1.1 in dimension $(n-1)$ implies the existence of good minimal models for $(n-1)$-dimensional klt pairs (see [22, Theorem 4.3] or [8, Remark 2.6]). By Kawamata's Theorem [25, Theorem 7.3] (see also [28, Lemma 5.6]), it is enough to show that $\kappa(K_X + \Delta) = 0$ implies $D = 0$. So assume $\kappa(K_X+\Delta) = 0$ and $D \neq 0$. Put $l := \mathrm{lct}(D; X, \Delta)$ and take a dlt blow-up $\varphi' : Y' \to X$ of $(X, \Delta + lD)$ by Theorem 2.9. We write

$$K_{Y'} + S' + B' = \varphi'^*(K_X + \Delta + lD) \sim_{\mathbb{Q}} (1+l)\varphi'^* D,$$

where $\lfloor S' + B' \rfloor = S'$ and $(Y', S' + B')$ is dlt. We remark that S' can not be assumed to be a prime divisor and may be non-normal in general. We only know that S' is a union of prime divisors each mapped to lc centers of $(X, \Delta + lD)$. By taking a log resolution which is isomorphic over the generic point of every lc center of $(Y,' S' + B')$, we may assume that $\varphi : Y \to X$ factors through a log resolution $f : Y \to Y'$ of $(Y', S' + B')$ (see Lemma 4.5). We have

$$K_Y + S + B = f^*(K_{Y'} + S' + B') + E,$$

where $S (\neq 0)$ is the strict transform of S', E is φ-exceptional, and no two of S, B and E have a common component. Then we have $S \subseteq \mathrm{Supp}\, \varphi^* D$ since every lc center of $(X, \Delta + lD)$ is contained in $\mathrm{Supp}\, D$. Since $K_{Y'} + B' + S' \sim_{\mathbb{Q}} (1+l)\varphi'^* D$, we see that $K_{Y'} + B' + S'$ is nef. In particular, the restriction $K_{S'} + B'_{S'} = (K_{Y'} + B' + S')|_{S'}$ is also nef.

Claim 5.2. – *For a sufficiently divisible integer $m' \geq 2$, the restriction map*

$$H^0(Y', m'(K_{Y'} + S' + B')) \to H^0(S', m'(K_{S'} + B'_{S'}))$$

is surjective.

Proof of Claim 5.2. – Let u be a non-zero section in $H^0(S', m'(K_{S'} + B'_{S'}))$. Let $u_{m'E}$ be the natural section of $m'E$. By the assumption, it is easy to see that $h^{\frac{m'(1+l)}{m}}$ determines the singular metric on

$$m'(1+l)\varphi^* D = m'(1+l) f^* \varphi'^* D \sim_{\mathbb{Q}} m' f^*(K_{Y'} + S' + B')$$

such that the curvature is semi-positive and the Lelong number is identically zero on Y. Since S and E have no common component, Corollary 4.2 applied to $F := (1+l) f^* \varphi'^* D$ and $\widetilde{s} := u_{m'E}$ yields a a section $U \in H^0(Y, m'(K_Y + S + B))$ such that $U_{|S} = f^*_{|S} u \otimes (u_{m'E})_{|S}$ (cf. Lemma 4.5).

On the other hand, the mapping

$$H^0(Y', m'(K_{Y'} + S' + B')) \to H^0(Y, m'(K_Y + S + B)) \text{ given by } s \mapsto f^* s \otimes u_{m'E}$$

for a section $s \in H^0(Y', m'(K_{Y'} + S' + B'))$ is an isomorphism. Moreover the mapping

$$H^0(S', m'(K_{S'} + B'_{S'})) \to H^0(S, m'(K_S + B_{|S})) \text{ given by } t \mapsto f^*_{|S} t \otimes (u_{m'E})|_S$$

for a section $t \in H^0(S', m'(K_{Y'} + S' + B'))$ is injective from $f_* \mathcal{O}_S = \mathcal{O}_{S'}$ and Kollár-Shokurov's connectedness theorem (see [30, Theorem 17.4]). Hence we can conclude that

$$H^0(Y', m'(K_{Y'} + S' + B')) \to H^0(S', m'(K_{S'} + B'_{S'}))$$

is surjective. □

On the other hand, this restriction map is zero map since $\kappa(K_X + \Delta) = \kappa(K_{Y'} + B' + S') = 0$ and $S' \subseteq \mathrm{Supp}\, \varphi'^* D$. Since $(Y', S' + B')$ is a dlt pair, the pair $(S', B'_{S'})$ is divisorial semi-log terminal (see [12, Remark 1.2 (3)]). In particular $(S', B'_{S'})$ is semi-log canonical (see [13, Proposition 3.9.2]). We can apply [20, Theorem 1.5] or [24] to $(S', B'_{S'})$ since the abundance conjecture for lc pairs in dimension $(n-1)$ holds by [20, Theorem 5.5, Corollary 5.6], [21, Theorem 1.5], and [23, Theorem 1.1, Theorem 1.5]. Since furthermore $K_{S'} + B'_{S'}$ is nef, this implies that $K_{S'} + B'_{S'}$ is semi-ample. Here we need the assumption of projectivity. This is a contradiction to Claim 5.2, and thus $D = 0$. This finishes the proof. □

By using the abundance theorem in dimension 3 ([26, Theorem 1.1], [28, 1.1 Theorem], [12, Theorem 0.1]), we obtain the following results:

Corollary 5.3. – *Let (X, Δ) be a 4-dimensional projective klt pair. Assume that there exists a projective birational morphism $\varphi : Y \to X$ such that Y is smooth and $\varphi^*(m(K_X + \Delta))$ admits a singular metric whose curvature is semi-positive and Lelong number is identically zero. Here m is an integer with $m(K_X + \Delta)$ Cartier. If $\kappa(K_X + \Delta) \geq 0$, then $K_X + \Delta$ is semi-ample.*

Remark 5.4. – *The assumption of Corollary 5.3 for h is satisfied when h is smooth. In this case, we can show Corollary 5.3 by replacing Theorem 1.3 with generalized Enoki's injectivity theorem after Fujino* ([16, Theorem 1.2, Corollary1.3]).

Finally we give a result for semi-ampleness by combining with Verbitsky's non-vanishing theorem ([41, Theorem 4.1]).

Corollary 5.5 (Theorem 1.6). – *Let X be a 4-dimensional projective hyperKähler manifold and L be a line bundle admitting a singular metric whose curvature is semi-positive and Lelong number is identically zero* (*which holds, in particular, if h is smooth*). *Then L is semi-ample.*

Proof. – It is enough to show $\kappa(L) \geq 0$ by Corollary 5.3 since if there exists an effective $\mathbb{Q}$-divisor such that $D \sim_{\mathbb{Q}} L$, the pair $(X, \varepsilon D)$ is klt and $K_X + \varepsilon D \sim_{\mathbb{Q}} \varepsilon L$ for sufficiently small $\varepsilon > 0$. If $q(L, L) > 0$, then L is big, where $q(\cdot, \cdot)$ is the Bogomolov-Beauville-Fujiki form. On the other hand, if $q(L, L) = L^{\dim X} = 0$, then we have $\kappa(L) \geq 0$ from [41, Theorem 4.1]. □

By the above results, Conjecture 1.1 is reduced to the non-vanishing conjecture and the following problem:

Question 5.6. – *How can we construct a singular metric on a nef* (*log*) *canonical bundle such that the curvature is semi-positive and the Lelong number is zero everywhere?*

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