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KNOTTED STRUCTURES
IN HIGH-ENERGY BELTRAMI FIELDS
ON THE TORUS AND THE SPHERE

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and Francisco TORRES de LIZAUR

Abstract. – Let $\mathcal{S}$ be a finite union of (pairwise disjoint but possibly knotted and linked) closed curves and tubes in the round sphere $\mathbb{S}^3$ or in the flat torus $\mathbb{T}^3$. In the case of the torus, $\mathcal{S}$ is further assumed to be contained in a contractible subset of $\mathbb{T}^3$. In this paper we show that for any sufficiently large odd integer λ there exists a Beltrami field on $\mathbb{S}^3$ or $\mathbb{T}^3$ satisfying $\operatorname{curl} u = \lambda u$ and with a collection of vortex lines and vortex tubes given by $\mathcal{S}$, up to an ambient diffeomorphism.

Résumé. – Soit $\mathcal{S}$ une collection finie de courbes et de tubes fermés, disjoints deux à deux mais pouvant être noués et entrelacés, dans la sphère ronde $\mathbb{S}^3$ ou dans le tore plat $\mathbb{T}^3$. Dans le cas du tore, on suppose davantage que $\mathcal{S}$ est contenu dans un sous-ensemble contractile de $\mathbb{T}^3$. Dans cet article on montre que, pour tout entier impair λ suffisamment grand, il existe un champ de Beltrami dans $\mathbb{S}^3$ ou $\mathbb{T}^3$ satisfaisant $\operatorname{curl} u = \lambda u$ et qui a une collection de lignes et tubes de vorticité donnés par $\mathcal{S}$, modulo un difféomorphisme ambiant.

1. Introduction

An incompressible fluid flow in $\mathbb{R}^3$ is described by its velocity field $u(x, t)$, which is a time-dependent vector field satisfying the Euler equations

$$\partial_t u + (u \cdot \nabla)u = -\nabla P, \qquad \operatorname{div} u = 0$$

for some pressure function $P(x, t)$. When the velocity field does not depend on time, the fluid is said to be *stationary*. This paper concerns stationary solutions of the Euler equations, which describe equilibrium configurations of the fluid.

A central topic in topological fluid mechanics, which can be traced back to Lord Kelvin in the 19th century [20], concerns the existence of knotted stream and vortex structures in stationary fluid flows. The most relevant of these structures are the stream lines, vortex lines and vortex tubes of the fluid. We recall that a *stream line* and a *vortex line* are simply a trajectory (or integral curve) of the velocity field u and the vorticity $\omega := \operatorname{curl} u$, respectively, while a *vortex tube* is the interior domain bounded by an invariant torus of the vorticity.

The existence of topologically complicated stream and vortex lines is a central topic in the Lagrangian theory of turbulence and in magnetohydrodynamics, and has been studied extensively in the last decades (see e.g., [13, 15] for recent accounts of the subject).

Our understanding of the set of stationary states of the Euler equations in three dimensions is much more limited than in the two-dimensional situation [4, 17]. This is due to the fact that, in two dimensions, the vorticity is a scalar quantity, whereas in the three dimensional case it is a vector field, which can exhibit a much richer behavior. In particular, the existence of stationary solutions in $\mathbb{R}^3$ having stream lines, vortex lines and vortex tubes that are knotted and linked in arbitrarily complicated ways has been established only very recently [7, 8, 9]. Following a suggestion of Arnold [2, 1] related to his celebrated structure theorem, to prove these results one does not consider just any kind of solutions to the stationary Euler equations, but a very particular class that are called Beltrami fields. A *Beltrami field* in $\mathbb{R}^3$ is a vector field satisfying the equation

$$(1.1) \qquad\qquad\qquad\qquad \operatorname{curl} u = \lambda u$$

for some nonzero constant λ. Notice that stream lines and vortex lines coincide in the case of a Beltrami field, and that a Beltrami field is automatically smooth (even real analytic) by the elliptic regularity theory.

The stationary solutions in $\mathbb{R}^3$ that one can construct using the techniques in [7, 8] fall off at infinity as $1/|x|$, this decay being sharp for Beltrami fields but not fast enough for the velocity to be in the energy space $L^2(\mathbb{R}^3)$. In fact, the incompressibility condition ensures that there are no Beltrami fields in $\mathbb{R}^3$ with finite energy even if the proportionality factor λ is allowed to be nonconstant, as has been recently shown in [18, 3].

On the contrary, Beltrami fields in a closed Riemannian 3-manifold M (or a bounded domain of $\mathbb{R}^3$) are stationary solutions to the Euler equations that do have finite energy. If $\mathscr{S}$ is a union of (possibly knotted and linked) closed curves and embedded tori in the 3-sphere, in this setting one can use contact topology to show [10] that there is a Riemannian metric g on the sphere with an associated Beltrami field u having a collection of vortex lines and vortex tubes given precisely by $\mathscr{S}$. The main ideas of the proof are that the Reeb field of a contact form is in fact a Beltrami field in some adapted metric and that one can indeed construct contact forms on the sphere whose Reeb fields have the collection of periodic trajectories and invariant tori given by $\mathscr{S}$. Notice that, as it is a Reeb vector field, a Beltrami field obtained in this fashion does not vanish. Conversely, any nonvanishing Beltrami field on the sphere is the Reeb vector field of some contact form.

Our goal in this paper is to establish the existence of knotted and linked vortex structures in Beltrami fields on compact manifolds with a *fixed* Riemannian metric. Specifically, we will consider Beltrami fields in the flat 3-torus $\mathbb{T}^3$ and in the unit 3-sphere $\mathbb{S}^3$; in fact, the former is the most fundamental space considered in the fluid mechanics literature other than $\mathbb{R}^3$ and the latter is perhaps the simplest example of a closed Riemannian 3-manifold from a geometric point of view.

It is worth emphasizing that, for a fixed Riemannian structure, the problem is much more rigid than when one can freely choose a metric adapted to the geometry of the set of lines and tubes that one aims to recover from the trajectories of a Beltrami field. An

obvious reason is that, analytically, Beltrami fields in a closed Riemannian manifold arise as eigenfields of the curl operator, which defines a self-adjoint operator with discrete spectrum and a dense domain in the space of divergence-free L^2 fields. In the context of spectral theory, the proportionality constant λ, or rather its absolute value, can be thought of as the *energy* of the Beltrami field, although of course it is in no way related to the L^2 norm of the latter.

Our main theorem asserts that there are "many" Beltrami fields u in the sphere and in the torus with vortex lines and vortex tubes of any link type. Furthermore, these structures are *structurally stable* in the sense that any vector field on the torus or the sphere which is sufficiently close to u in the C^4 norm and which preserves some smooth volume measure will also have this collection of periodic trajectories and invariant tori, up to a diffeomorphism. To state this result precisely, let us call a *tube* the closure of a domain (in $\mathbb{S}^3$ or $\mathbb{T}^3$) whose boundary is an embedded torus. Throughout, diffeomorphisms are of class C^∞, curves are all assumed to be non-self-intersecting, and we will agree to say that an integer is large when it is large in absolute value.

THEOREM 1.1. – *Let $\mathcal{S}$ be a finite union of (pairwise disjoint, but possibly knotted and linked) closed curves and tubes in $\mathbb{S}^3$ or $\mathbb{T}^3$. In the case of the torus, we also assume that $\mathcal{S}$ is contained in a contractible subset of $\mathbb{T}^3$. Then for any large enough odd integer λ there exists a Beltrami field u satisfying the equation $\operatorname{curl} u = \lambda u$ and a diffeomorphism Φ of $\mathbb{S}^3$ or $\mathbb{T}^3$ such that $\Phi(\mathcal{S})$ is a union of vortex lines and vortex tubes of u. Furthermore, this set is structurally stable.*

An important observation is that the proof of this theorem yields a reasonably complete understanding of the behavior of the diffeomorphism Φ, which is, in particular, connected with the identity. Oversimplifying a little, the effect of Φ is to uniformly rescale a contractible subset of the manifold that contains $\mathcal{S}$ to have a diameter of order $1/|\lambda|$. In particular, the control that we have over the diffeomorphism Φ allows us to prove an analog of this result for quotients of the sphere by finite groups of isometries (lens spaces). Notice that $\Phi(\mathcal{S})$ is not guaranteed to contain all vortex lines and vortex tubes of the Beltrami field. It is also worth mentioning that, if $\mathcal{S}$ only consists of curves, the condition that the perturbation of the Beltrami field be volume-preserving is not necessary for the structural stability of $\Phi(\mathcal{S})$, and the smallness in C^4 can be replaced by a C^1 condition.

In $\mathbb{S}^3$ and $\mathbb{T}^3$, Theorem 1.1 proves a conjecture of Arnold [2] asserting that there should be Beltrami fields having stream lines with complicated topology. Furthermore, it should be noticed that the helicity of the vorticity, that is, the quantity [14]

$$\mathcal{H}(\operatorname{curl} u) := \int_M u \cdot \operatorname{curl} u \,,$$

is proportional to its eigenvalue λ, so the Beltrami fields constructed in the main theorem have very large helicity. More precisely, the quantity

$$\frac{\mathcal{H}(\operatorname{curl} u)}{\|u\|_{L^2}^2} \,,$$

which is given by λ in the case of a Beltrami field, becomes arbitrarily large. This is fully consistent with Moffatt's interpretation [14, 15] of helicity as a measure of the degree of knottedness of the vortex lines in the fluid flow.

The proof of the theorem involves an interplay between rigid and flexible properties of high-energy Beltrami fields. Indeed, rigidity appears because high-energy Beltrami fields in any 3-manifold behave, locally in sets of diameter $1/\lambda$, as Beltrami fields in $\mathbb{R}^3$ with parameter $\lambda = 1$ do in balls of diameter 1. The catch here is that, in general, one cannot check whether a given Beltrami field in $\mathbb{R}^3$ actually corresponds to a high-energy Beltrami field on the compact manifold. To prove a partial converse implication in this direction (Theorem 2.1), it is key to exploit some flexibility that arises in the problem as a consequence of the fact that large eigenvalues of the curl operator in the torus or in the sphere have increasingly high multiplicities. For this reason the proof does not work in a general Riemannian 3-manifold.

One should notice that the techniques introduced in [7, 8] to prove the existence of Beltrami fields in $\mathbb{R}^3$ with a prescribed set $\mathcal{S}$ of closed vortex lines and vortex tubes do not work for compact manifolds. The reason is that the proof is based on the construction of a local Beltrami field in a neighborhood of $\mathcal{S}$, which is then approximated by a global Beltrami field in $\mathbb{R}^3$ using a Runge-type global approximation theorem. For compact manifolds the complement of the set $\mathcal{S}$ is precompact, so we cannot apply the global approximation theorem obtained in [7, 8]. In fact, as is well known, this is not just a technical issue, but a fundamental obstruction in any approximation theorem of this sort. This invalidates the whole strategy followed in [7, 8] and makes it apparent that new tools are needed to prove the existence of Beltrami fields with geometrically complex vortex lines and vortex tubes in compact manifolds.

The paper is organized as follows. In Section 2 we will prove the main theorem assuming that Theorem 2.1 holds. Theorem 2.1 will be proved in Section 3 in the case of the sphere, with the proof of some technical results relegated to Sections 4–6, and in Section 7 in the case of the torus. The paper concludes with some remarks that we present in Section 8, where in particular we prove an analog of the main theorem for lens spaces.

2. Proof of the main theorem

For the ease of notation, we shall write $\mathbb{M}^3$ to denote either $\mathbb{T}^3$ (the standard flat 3-torus, $(\mathbb{R}/2\pi\mathbb{Z})^3$) or $\mathbb{S}^3$ (the unit sphere in $\mathbb{R}^4$). A Beltrami field u in $\mathbb{M}^3$ is an eigenfield of the curl operator, which satisfies

$$\operatorname{curl} u = \lambda u \,,$$

for some nonzero constant λ. It is known (see e.g., [11]) that the spectrum of curl in the sphere consists of the integers of absolute value greater than or equal to 2. In the case of $\mathbb{T}^3$ it is easy to check using Fourier series that the spectrum consists of the real numbers of the form

$$\lambda = \pm|k|$$

for some $k \in \mathbb{Z}^3$. In particular, the spectrum of curl in $\mathbb{T}^3$ contains the set of integers. Here and in what follows, $|\cdot|$ denotes the usual Euclidean norm of a vector.

The following theorem, whose proof is presented in Section 3, shows that a Beltrami field v in $\mathbb{R}^3$ can be approximated, up to a suitable rescaling, by a high-energy Beltrami field u in $\mathbb{M}^3$. This fact is key to the proof of Theorem 1.1 as it implies that the dynamics of any Beltrami field of $\mathbb{R}^3$ in compact sets can be reproduced in a small ball of $\mathbb{M}^3$ by a high-energy Beltrami field on the manifold, provided that the dynamical properties under consideration are robust under suitably small perturbations. For concreteness, we will henceforth assume that λ is positive; the case of negative λ is completely analogous.

For the precise statement of the theorem, let us fix an arbitrary point $p_0 \in \mathbb{M}^3$ and take a patch of normal geodesic coordinates $\Psi : \mathbb{B} \to B$ centered at p_0. Here and in what follows, B_ρ (resp. $\mathbb{B}_\rho$) denotes the ball in $\mathbb{R}^3$ (resp. the geodesic ball in $\mathbb{M}^3$) centered at the origin (resp. at p_0) and of radius ρ, and we shall drop the subscript when $\rho = 1$. The theorem will be then stated in terms of the vector field $\Psi_* u$ on B, which is just the expression of the Beltrami field u in local normal coordinates. If $u^i(x)$ are the three components of $\Psi_* u$ in the Cartesian basis $\{e_i\}_{i=1}^3$ of $\mathbb{R}^3$, i.e.,

$$\Psi_* u(x) = \sum_{i=1}^{3} u^i(x)\, e_i \,,$$

we will make use of the rescaled vector field

$$\Psi_* u\left(\frac{\cdot}{\lambda}\right) := \sum_{i=1}^{3} u^i\left(\frac{\cdot}{\lambda}\right) e_i \,.$$

THEOREM 2.1. – *Let v be a Beltrami field in $\mathbb{R}^3$, satisfying $\operatorname{curl} v = v$. Let us fix any positive numbers ε and m. Then for any large enough odd integer λ there is a Beltrami field u, satisfying $\operatorname{curl} u = \lambda u$ in $\mathbb{M}^3$, such that*

$$(2.1) \qquad \left\| \Psi_* u\left(\frac{\cdot}{\lambda}\right) - v \right\|_{C^m(B)} < \varepsilon \,.$$

Let us now show how this result can be exploited to prove the main theorem. For this, let Φ' be a diffeomorphism of $\mathbb{M}^3$ mapping the set $\mathcal{S}$ into the ball $\mathbb{B}_{1/2}$, and the ball $\mathbb{B}_{1/2}$ into itself. (In $\mathbb{S}^3$, the existence of such a diffeomorphism is trivial, while in the case of $\mathbb{T}^3$ it follows from the assumption that $\mathcal{S}$ is contained in a contractible set.) We can now define a set $\mathcal{S}'$ of finitely many closed curves and tubes in the ball $B_{1/2}$ as

$$\mathcal{S}' := (\Psi \circ \Phi')(\mathcal{S}) \,.$$

The following result is a straightforward consequence of the main theorem in [8]:

THEOREM 2.2. – *There is a Beltrami field v in $\mathbb{R}^3$ satisfying $\operatorname{curl} v = v$ and an orientation-preserving diffeomorphism Φ_0 of $\mathbb{R}^3$, which coincides with the identity in the complement of $B_{1/2}$, such that $\Phi_0(\mathcal{S}')$ is a union of vortex lines and vortex tubes of v. Furthermore, this set is structurally stable.*

Proof. – It was shown in [8] that there is a Beltrami field $\tilde{v}$ in $\mathbb{R}^3$, satisfying

$$\operatorname{curl} \tilde{v} = \tilde{\lambda} \tilde{v}$$

for some small positive constant $\tilde{\lambda} < 1$, and an orientation-preserving diffeomorphism $\widetilde{\Phi}$ of $\mathbb{R}^3$ that is the identity in the complement of $B_{1/2}$ such that $\widetilde{\Phi}(\mathcal{S}')$ is a set of closed

vortex lines and vortex tubes of $\tilde{v}$. The closed vortex lines are elliptic trajectories of $\tilde{v}$ and the boundaries of the vortex tubes are KAM-nondegenerate invariant tori of $\tilde{v}$. Given a positive number Λ let us denote the rescaling with factor Λ by $\Theta_\Lambda(x) := \Lambda x$. The theorem follows setting $v(x) := \tilde{v}(x/\tilde{\lambda})$, which satisfies the equation $\operatorname{curl} v = v$ in $\mathbb{R}^3$, and noticing that $(\Theta_{\tilde{\lambda}} \circ \widetilde{\Phi})(\mathcal{S}')$ is a set of closed vortex lines and vortex tubes of v. Since this set is contained in $B_{1/2}$ because $\tilde{\lambda} < 1$, it is standard that there exists a diffeomorphism Φ_0 of $\mathbb{R}^3$ mapping $\mathcal{S}'$ onto $\Theta_{\tilde{\lambda}} \circ \widetilde{\Phi}(\mathcal{S}')$ which is the identity in the complement of $B_{1/2}$. The closed vortex lines in the set $\Phi_0(\mathcal{S}')$ are structurally stable under C^1-small perturbations because they are elliptic [16, Section 2.1], while the vortex tubes are structurally stable under C^4-small volume-preserving perturbations by the KAM theorem. $\qquad\square$

Let us now combine Theorems 2.1 and 2.2 to conclude the proof of Theorem 1.1. Theorem 2.1 guarantees that, for any large enough odd integer λ, the Beltrami field v constructed in Theorem 2.2 can be approximated in the sense of Eq. (2.1) by a Beltrami field u defined on $\mathbb{M}^3$. Then it is not hard to see that the structural stability of the set $\Phi_0(\mathcal{S}')$ of closed vortex lines and vortex tubes of v implies the existence of a diffeomorphism Φ_1 of $\mathbb{R}^3$, which is the identity in the complement of $B_{1/2}$, such that $\Phi_1(\mathcal{S}') \subset B_{1/2}$ is a set of structurally stable closed vortex lines and vortex tubes of the rescaled field

$$(2.2) \qquad \Psi_* u\left(\frac{\cdot}{\lambda}\right).$$

Indeed, because of the ellipticity of the trajectories, this claim is immediate in the case of closed vortex lines provided that the number m appearing in the approximation estimate (2.1) is at least 1. For the case of vortex tubes one can use that the Beltrami field u is divergence-free in $\mathbb{M}^3$, which ensures that the field (2.2) preserves a smooth volume 3-form in B that is a small perturbation of the Euclidean one, namely

$$(\Psi_* \mu)\left(\frac{\cdot}{\lambda}\right) = \mu_0 + O(\lambda^{-1}).$$

Here μ and μ_0 respectively denote the canonical volume 3-forms of $\mathbb{M}^3$ and $\mathbb{R}^3$. Hence, taking $m \geqslant 4$ in the approximation estimate (2.1), this enables us to apply the KAM theorem for volume-preserving fields in $\mathbb{R}^3$, which ensures the existence of the aforementioned diffeomorphism Φ_1 yielding the desired set of vortex tubes of the rescaled field (2.2). (For the benefit of the reader let us recall that, in order to prove this KAM result, one takes a Poincaré section transversal to the tube of v under consideration, thereby reducing the problem to perturbations of a nondegenerate twist map of the annulus with the intersection property. It is then standard that one can apply a Moser-type twist theorem to guarantee the preservation of the invariant tori. The details, which go as in [8, Section 7.4], are omitted.)

It follows from the above discussion that the diffeomorphism Φ of $\mathbb{M}^3$ can be then defined as

$$\Phi(x) := \begin{cases} \Phi'(x) & \text{if } x \notin \Phi'^{-1}(\mathbb{B}), \\ (\Psi^{-1} \circ \widetilde{\Theta}_{1/\lambda} \circ \Phi_1 \circ \Psi \circ \Phi')(x) & \text{if } x \in \Phi'^{-1}(\mathbb{B}), \end{cases}$$

where $\widetilde{\Theta}_{1/\lambda}$ is a smooth diffeomorphism of $\mathbb{R}^3$ which is equal to the rescaling $\Theta_{1/\lambda}$ in the ball $B_{1/2}$ and is the identity in the complement of the ball $B_{3/4}$. This ensures that Φ is a

smooth diffeomorphism of $\mathbb{M}^3$ such that the set $\Phi(\mathcal{S})$ is the union of structurally stable closed vortex lines and vortex tubes of the Beltrami field u, so the main theorem follows.

3. Proof of Theorem 2.1 in the sphere

In this section we show that for any Beltrami field v in $\mathbb{R}^3$ satisfying $\operatorname{curl} v = v$ there exists a Beltrami field u in $\mathbb{S}^3$ satisfying $\operatorname{curl} u = \lambda u$ whose dynamics in a ball of radius λ^{-1} is very close to the dynamics of v in the unit ball. The proof is divided in three steps. In the first step we show that the Beltrami field v can be approximated in B by a field w that is a finite sum of spherical Bessel functions $j_0(|x - x_n|)$ centered at different points $x_n \in \mathbb{R}^3$ (Proposition 3.1). The field w is not generally a Beltrami field, however. In the second step we show that one can take three spherical harmonics Y_1, Y_2, Y_3 in $\mathbb{S}^3$ of energy $\lambda(\lambda - 2)$ whose behaviors in a ball of radius $1/\lambda$ respectively correspond to those of the three components of the field w in a ball of radius 1, provided that λ is large enough (Proposition 3.2). Finally, in the third step we construct a Beltrami field u in $\mathbb{S}^3$ of energy λ, using as key ingredients the spherical harmonics Y_k and a basis of Hopf fields, so that u approximates the field v in the sense of Eq. (8.1) (Proposition 3.3).

For notational convenience, in this section we will write $\Lambda := \lambda - 2$. Notice that Λ is then a large integer.

Step 1: Approximating the Beltrami field v by sums of shifted spherical Bessel functions. – The first step of the proof of Theorem 2.1 consists in showing that there is a finite sum w of spherical Bessel functions j_0 centered at different points that approximates the Beltrami field v in the unit ball of $\mathbb{R}^3$. The field w is not a Beltrami field but, just as v, it satisfies the Helmholtz equation

$$\Delta w + w = 0 \,.$$

PROPOSITION 3.1. – *For any $\delta > 0$, there is a finite radius R and finitely many constants $\{c_n\}_{n=1}^N \subset \mathbb{R}^3$ and $\{x_n\}_{n=1}^N \subset B_R$ such that the field*

$$w := \sum_{n=1}^N c_n \, j_0(|x - x_n|)$$

approximates the Beltrami field v in the ball B as

$$\|v - w\|_{C^{m+2}(B)} < \delta \,.$$

The proof of this proposition will be presented in Section 4.

Step 2: Approximating the field w by high-energy spherical harmonics. – Let us write the vector field w in terms of its components w^i in the Cartesian basis $\{e_i\}_{i=1}^3$ of $\mathbb{R}^3$:

$$w = \sum_{i=1}^3 w^i e_i \,.$$

Each component w^i is a solution of the Helmholtz equation $\Delta w^i + w^i = 0$ in $\mathbb{R}^3$. We now show that for any large enough integer Λ, there exists a spherical harmonic Y_i on $\mathbb{S}^3$ with

energy $\Lambda(\Lambda + 2)$ that behaves in the ball $\mathbb{B}_{1/\Lambda}$ as w^i does in the unit ball. The proof of this result is based on the asymptotic expressions for the ultraspherical polynomials, which are the building blocks for any spherical harmonic on $\mathbb{S}^3$, and exploits in a crucial way the expression for w as a finite sum of spherical Bessel functions that we obtained in Step 1:

PROPOSITION 3.2. – *Given any positive constant δ, for any large enough integer Λ there is a spherical harmonic Y_i on $\mathbb{S}^3$ with energy $\Lambda(\Lambda + 2)$ such that*

$$\left\| w^i - Y_i \circ \Psi^{-1}\left(\frac{\cdot}{\Lambda}\right) \right\|_{C^{m+2}(B)} < \delta \,.$$

The proof of this proposition is given in Section 5.

Step 3: Construction of the Beltrami field on $\mathbb{S}^3$ using spherical harmonics and Hopf fields. – Let us consider the three positively oriented orthonormal Hopf vector fields in $\mathbb{S}^3$ that, in terms of the Cartesian coordinates of $\mathbb{R}^4$, are explicitly given by

$$h_1 := (-x_4, x_3, -x_2, x_1) \,,$$
$$h_2 := (-x_3, -x_4, x_1, x_2) \,,$$
$$h_3 := (-x_2, x_1, x_4, -x_3) \,.$$

It is well known that they are curl eigenfields with eigenvalue 2, that is,

$$\operatorname{curl} h_i = 2h_i \,.$$

We have taken the Cartesian basis e_i of $\mathbb{R}^3$ so that $\Psi_* h_i(0) = e_i$.

In the following proposition we show how to construct a Beltrami field on $\mathbb{S}^3$ using the spherical harmonics Y_i obtained in Proposition 3.2 and the Hopf fields h_i so that it approximates the Beltrami field v in a suitable sense.

PROPOSITION 3.3. – *The vector field on the sphere*

$$u := \frac{1}{2\Lambda^2} \operatorname{curl}(\operatorname{curl} + \Lambda) (Y_1 h_1 + Y_2 h_2 + Y_3 h_3)$$

is a Beltrami field satisfying $\operatorname{curl} u = (\Lambda + 2)u$ and approximates v as

$$\left\| \Psi_* u\left(\frac{\cdot}{\Lambda}\right) - v \right\|_{C^m(B)} < C\delta \,,$$

provided that Λ is sufficiently large.

Here C is a constant depending on m but not on δ. Since rescaling $\Psi_* u$ by Λ is essentially equivalent to rescaling it by λ because

$$\frac{1}{\Lambda} = \frac{1}{\lambda}\left(1 + \frac{2}{\Lambda}\right),$$

Theorem 2.1 then follows from Proposition 3.3 provided Λ is sufficiently large and δ is chosen small enough for $C\delta$ not to be larger than $\varepsilon/2$. The proof of Proposition 3.3 is given in Section 6.

4. Proof of Proposition 3.1

Since the Beltrami field v satisfies the Helmholtz equation $\Delta v + v = 0$, upon expanding the components of v in a series of spherical harmonics it is elementary to realize that v can be written in the ball B_2 as a Fourier-Bessel series of the form

$$(4.1) \qquad v = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_{lm} \, j_l(r) \, Y_{lm}(\omega)$$

that converges in $L^2(B_2)$. Here $r := |x| \in \mathbb{R}^+$ and $\omega := x/r \in \mathbb{S}^2$ are spherical coordinates, j_l is the spherical Bessel function, Y_{lm} are the spherical harmonics and $b_{lm} \in \mathbb{R}^3$ are constant vectors.

Since the series (4.1) converges in $L^2(B_2)$, for any δ' there is an integer l_0 such that the finite sum

$$v_1 := \sum_{l=0}^{l_0} \sum_{m=-l}^{l} b_{lm} \, j_l(r) \, Y_{lm}(\omega)$$

approximates the field v in an L^2 sense, that is,

$$(4.2) \qquad \|v_1 - v\|_{L^2(B_2)} < \delta'.$$

Next, let us observe that the properties of the spherical Bessel functions imply that the field v_1 falls off at infinity as $|v_1(x)| < C/|x|$. In particular, it then follows from Herglotz's theorem (see e.g., [12, Theorem 7.1.27]) that v_1 can be written as the Fourier transform of a distribution supported on the unit sphere of the form

$$(4.3) \qquad v_1(x) = \int_{\mathbb{S}^2} f_1(\xi) \, e^{ix\cdot\xi} \, d\sigma(\xi) \, ,$$

where $d\sigma$ is the area measure induced on the unit sphere $\mathbb{S}^2 := \{\xi \in \mathbb{R}^3 : |\xi| = 1\}$ and f_1 is an $\mathbb{R}^3$-valued function in $L^2(\mathbb{S}^2)$.

By the density of smooth functions in $L^2(\mathbb{S}^2)$, we can approximate f_1 by a smooth function $f_2 : \mathbb{S}^2 \to \mathbb{R}^3$ so that their difference is bounded as

$$\|f_1 - f_2\|_{L^2(\mathbb{S}^2)} < \delta'.$$

Therefore the field

$$(4.4) \qquad v_2(x) := \int_{\mathbb{S}^2} f_2(\xi) \, e^{ix\cdot\xi} \, d\sigma(\xi) \, ,$$

approximates v_1 uniformly, as for any $x \in \mathbb{R}^3$ the Cauchy-Schwarz inequality yields

$$(4.5) \qquad |v_2(x) - v_1(x)| = \left| \int_{\mathbb{S}^2} (f_2(\xi) - f_1(\xi)) \, e^{ix\cdot\xi} \, d\sigma(\xi) \right| \leqslant C \|f_2 - f_1\|_{L^2(\mathbb{S}^2)} < C\delta'.$$

Our next objective is to show that for any δ' there is a radius $R > 0$ and finitely many constants $\{c_n\}_{n=1}^{N} \subset \mathbb{R}^3$ and $\{x_n\}_{n=1}^{N} \subset B_R$ such that the restriction to the unit sphere of the smooth field in $\mathbb{R}^3$

$$f(\xi) := \sum_{n=1}^{N} c_n \, e^{-ix_n\xi}$$

approximates the field f_2 in the C^0 norm, that is,

$$(4.6) \qquad\qquad \|f - f_2\|_{C^0(\mathbb{S}^2)} < \delta'.$$

To prove this claim, we first extend f_2 to a smooth vector field $g : \mathbb{R}^3 \to \mathbb{R}^3$ with compact support. This can be done by setting

$$g(\xi) := \chi(|\xi|) \, f_2\!\left(\frac{\xi}{|\xi|}\right),$$

with $\chi(s)$ a smooth function which is equal to 1, say, if $|s-1| < \frac{1}{4}$ and vanishes for $|s-1| > \frac{1}{2}$. Since the Fourier transform $\widehat{g}$ of g is Schwartz, we easily infer that there is a radius R such that the L^1 norm of $\widehat{g}$ is essentially contained in B_R in the sense that

$$\int_{\mathbb{R}^3 \setminus B_R} |\widehat{g}(x)| \, dx < \delta'.$$

It then follows that the Fourier integral representation of g can be essentially truncated to an integral over the ball B_R, i.e., one has the uniform bound

$$(4.7) \qquad\qquad \sup_{\xi \in \mathbb{R}^3} \left| g(\xi) - \int_{B_R} \widehat{g}(x) \, e^{-ix\cdot\xi} \, dx \right| < \delta'.$$

Now, an easy continuity argument allows us to uniformly approximate the integral

$$\int_{B_R} \widehat{g}(x) \, e^{-ix\cdot\xi} \, dx$$

by a finite sum

$$(4.8) \qquad\qquad f(\xi) := \sum_{n=1}^{N} c_n \, e^{-ix_n\cdot\xi}$$

with constants $c_n \in \mathbb{R}^3$ and $x_n \in B_R$ in such a way that the error introduced in the approximation is bounded by

$$(4.9) \qquad\qquad \sup_{\xi \in \mathbb{S}^2} \left| \int_{B_R} \widehat{g}(x) \, e^{-ix\cdot\xi} \, dx - f(\xi) \right| < \delta'.$$

Indeed, let us cover the ball B_R by finitely many closed sets $\{U_n\}_{n=1}^{N}$ with piecewise smooth boundaries and pairwise disjoint interiors such that the diameter of each set is at most δ''. The function $e^{-ix\cdot\xi} \, \widehat{g}(x)$ being smooth, it then follows that for each $x, y \in U_n$ one has

$$\sup_{\xi \in \mathbb{S}^2} \left| \widehat{g}(x) \, e^{-ix\cdot\xi} - \widehat{g}(y) \, e^{-iy\cdot\xi} \right| < C\delta'',$$

where the constant C depends on $\widehat{g}$ (and therefore on δ') but not on δ''. It is then straightforward that if x_n is any point in U_n and we set $c_n := \widehat{g}(x_n) \, |U_n|$ in (4.8), one has

$$\sup_{\xi \in \mathbb{S}^2} \left| \int_{B_R} \widehat{g}(x) \, e^{-ix\cdot\xi} \, dx - f(\xi) \right| \leqslant \sum_{n=1}^{N} \int_{U_n} \sup_{\xi \in \mathbb{S}^2} \left| \widehat{g}(x) \, e^{-ix\cdot\xi} - \widehat{g}(x_n) \, e^{-ix_n\cdot\xi} \right| dx$$

$$\leqslant C\delta'',$$

where again C depends on δ' and R but not on δ'' or N. Hence one can take δ'' small enough so that $C\delta'' < \delta'$, thereby proving the estimate (4.9).

Putting together the estimates (4.7) and (4.9) we infer that

$$\|f - g\|_{C^0(\mathbb{S}^2)} < C\delta',$$

with a constant independent of δ'. Since the restriction to $\mathbb{S}^2$ of the function g is precisely f_2, the estimate (4.6) then follows.

Finally, if we define the vector field

$$w(x) := \int_{\mathbb{S}^2} f(\xi)\, e^{ix\cdot\xi}\, d\sigma(\xi) = \sum_{n=1}^{N} c_n \int_{\mathbb{S}^2} e^{i(x-x_n)\cdot\xi}\, d\sigma(\xi) = \sum_{n=1}^{N} c_n\, j_0(|x - x_n|),$$

we conclude from Eq. (4.6) that

$$\|w - v_2\|_{C^0(\mathbb{R}^3)} \leqslant \int_{\mathbb{S}^2} |f(\xi) - f_2(\xi)|\, d\sigma(\xi) < C\delta',$$

so we readily infer from Eqs. (4.2) and (4.5) the L^2 bound

$$(4.10) \quad \|v - w\|_{L^2(B_2)} \leqslant C\|w - v_2\|_{C^0(\mathbb{R}^3)} + C\|v_2 - v_1\|_{C^0(\mathbb{R}^3)} + \|v_1 - v\|_{L^2(B_2)} < C\delta'.$$

Furthermore, as the Fourier transform of w is supported on $\mathbb{S}^2$, w satisfies the Helmholtz equation

$$\Delta w + w = 0$$

in $\mathbb{R}^3$. Since the Beltrami field v also satisfies the Helmholtz equation $\Delta v + v = 0$, standard elliptic estimates enable us to promote the L^2 bound (4.10) to the C^{m+2} estimate

$$\|v - w\|_{C^{m+2}(B)} \leqslant C\|v - w\|_{L^2(B_2)} < C\delta',$$

so the proposition follows upon choosing $C\delta' < \delta$.

5. Proof of Proposition 3.2

For any positive integer Λ, let $C_\Lambda(t)$ be the ultraspherical (also called Gegenbauer) polynomial of dimension 4 and degree Λ, which can be defined in terms of the Jacobi polynomials $P_\Lambda^{(\alpha,\beta)}$ as

$$(5.1) \qquad\qquad C_\Lambda(t) := \frac{\sqrt{\pi}}{2} \frac{\Gamma(\Lambda + 1)}{\Gamma(\Lambda + \frac{3}{2})}\, P_\Lambda^{(\frac{1}{2},\frac{1}{2})}(t),$$

where we are using the normalization $C_\Lambda(1) = 1$ for all Λ (see e.g [19, Chapter IV, Section 4.7]).

If $p, q \in \mathbb{S}^3$ are two points in the 3-sphere, understood as the subset $\{|p| = 1\}$ of $\mathbb{R}^4$, the addition theorem for ultraspherical polynomials shows that $C_\Lambda(p \cdot q)$ can be written as a linear combination of spherical harmonics. Specifically,

$$(5.2) \qquad\qquad C_\Lambda(p \cdot q) = \frac{2\pi^2}{(\Lambda + 1)^2} \sum_{j=1}^{(\Lambda+1)^2} Y_{\Lambda j}(p)\, Y_{\Lambda j}(q),$$

where $\{Y_{\Lambda j}\}_{j=1}^{(\Lambda+1)^2}$ is an arbitrary orthonormal basis of spherical harmonics of energy $\Lambda(\Lambda + 2)$ and $p \cdot q$ denotes the scalar product in $\mathbb{R}^4$ of the unit vectors p, q. Notice that $(\Lambda + 1)^2$ is

precisely the multiplicity of the eigenvalue $\Lambda(\Lambda + 2)$ of the Laplacian on $\mathbb{S}^3$ (or, equivalently, the dimension of the space of spherical harmonics of energy $\Lambda(\Lambda + 2)$).

Let us write the i-th Cartesian component of the vector field w as

$$w^i(x) = \sum_{n=1}^{N} c_n^i \, j_0(|x - x_n|),$$

where c_n^i is the i-th component of the constant $c_n \in \mathbb{R}^3$ and the points x_n are contained in the ball B_R. Let us set, for any $p \in \mathbb{S}^3$,

$$Y_i(p) := \sum_{n=1}^{N} c_n^i \, C_\Lambda(p \cdot p_n),$$

with

$$p_n := \Psi^{-1}\left(\frac{x_n}{\Lambda}\right).$$

Note that p_n is well defined provided Λ is bigger than R. It is obvious from Eq. (5.2) that Y_i is a spherical harmonic of energy $\Lambda(\Lambda + 2)$.

In order to study the asymptotic properties of the spherical harmonic Y_i we first observe that, if we restrict our attention to points of the form

$$p := \Psi^{-1}\left(\frac{x}{\Lambda}\right)$$

with $x \in B_R$ and $\Lambda > R$, we then have

$$(5.3) \qquad p \cdot p_n = \cos\left(\mathrm{dist}_{\mathbb{S}^3}(p, p_n)\right) = \cos\left(\frac{|x - x_n| + O(\Lambda^{-1})}{\Lambda}\right),$$

as $\Lambda \to \infty$. Here $\mathrm{dist}_{\mathbb{S}^3}(p, p_n)$ denotes the distance between the points p and p_n as measured on the sphere $\mathbb{S}^3$ and the last equality stems from the fact that $\Psi : \mathbb{B} \to B$ is a patch of normal geodesic coordinates. We will henceforth use the notation

$$(5.4) \qquad \widetilde{Y}_i(x) := Y_i \circ \Psi^{-1}\left(\frac{x}{\Lambda}\right).$$

Since for Λ large we have the asymptotic behavior

$$\frac{\Gamma(\Lambda + 1)}{\Gamma(\Lambda + \frac{3}{2})} = \frac{1}{\sqrt{\Lambda}} + O(\Lambda^{-\frac{3}{2}}),$$

we conclude from Eq. (5.3) that

$$(5.5) \qquad C_\Lambda(p \cdot p_n) = \left(\frac{\sqrt{\pi}}{2\sqrt{\Lambda}} + O(\Lambda^{-\frac{3}{2}})\right) P_\Lambda^{(\frac{1}{2}, \frac{1}{2})}\left(\cos\left(\frac{|x - x_n| + O(\Lambda^{-1})}{\Lambda}\right)\right).$$

Now Darboux's asymptotic formula for the Jacobi polynomials [19, Theorem 8.1.1] implies

$$\frac{1}{\sqrt{\Lambda}} P_\Lambda^{(\frac{1}{2}, \frac{1}{2})}\left(\cos\frac{t}{\Lambda}\right) = \frac{2}{\sqrt{\pi}} \, j_0(t) + O(\Lambda^{-1}),$$

which holds uniformly for compact sets (e.g., for $|t| \leqslant 2R$). Therefore Eq. (5.4) can be written by virtue of Eq. (5.5) as

$$
\begin{aligned}
\widetilde{Y}_i(x) &= \sum_{n=1}^{N} c_n^i C_\Lambda \left(\cos \left(\frac{|x - x_n| + O(\Lambda^{-1})}{\Lambda} \right) \right) \\
&= \sum_{n=1}^{N} c_n^i \, j_0(|x - x_n|) + O(\Lambda^{-1}),
\end{aligned}
$$

provided that Λ is sufficiently large and $x, x_n \in B_R$. This proves that for any $\delta' > 0$ and all Λ large enough we have the uniform bound

$$
(5.6) \qquad \qquad \| w^i - \widetilde{Y}_i \|_{C^0(B)} < \delta' .
$$

To get the C^{m+2} bound stated in the proposition, we notice that the eigenvalue equation

$$
\Delta Y_i + \Lambda(\Lambda + 2)Y_i = 0
$$

for the spherical harmonic Y_i in $\mathbb{S}^3$ can be written in terms of the rescaled function $\widetilde{Y}_i$ as

$$
\Delta_0 \widetilde{Y}_i + \widetilde{Y}_i = \frac{1}{\Lambda} A \widetilde{Y}_i ,
$$

where the coordinates x are assumed to take values in B, $\Delta_0 := \sum_i \partial_{x_i}^2$ is the flat space Laplacian acting on the x coordinates and A is a scalar second-order operator of the form

$$
A \widetilde{Y}_i := -2\widetilde{Y}_i + G_1 \, D \widetilde{Y}_i + G_2 \, D^2 \widetilde{Y}_i .
$$

Here the functions $G_i(x, \Lambda)$ are (possibly matrix-valued) functions that depend smoothly on all their variables and whose derivatives are bounded independently of Λ for $x \in B$, i.e.,

$$
(5.7) \qquad \qquad \sup_{x \in B} |D_x^\alpha G_i(x, \Lambda)| < C_\alpha .
$$

Here the constant C_α depends on the multiindex α but not on Λ.

By construction, the function w^i satisfies the Helmholtz equation

$$
\Delta_0 w^i + w^i = 0 ,
$$

and hence the difference $w^i - \widetilde{Y}_i$ satisfies the equation

$$
\Delta_0(w^i - \widetilde{Y}_i) + (w^i - \widetilde{Y}_i) = \frac{1}{\Lambda} A \widetilde{Y}_i .
$$

Therefore, in view of the uniform bounds (5.6) and (5.7), standard elliptic estimates yield

$$
\begin{aligned}
\| w^i - \widetilde{Y}_i \|_{C^{m+2,\alpha}(B)} &< C \| w^i - \widetilde{Y}_i \|_{C^0(B)} + \frac{C}{\Lambda} \| A \widetilde{Y}_i \|_{C^{m,\alpha}(B)} \\
&< C\delta' + \frac{C}{\Lambda} \| w^i - \widetilde{Y}_i \|_{C^{m+2,\alpha}(B)} + \frac{C}{\Lambda} \| w^i \|_{C^{m+2,\alpha}(B)} ,
\end{aligned}
$$

which implies that

$$
\| w^i - \widetilde{Y}_i \|_{C^{m+2}(B)} \leqslant C\delta' + \frac{C \| w^i \|_{C^{m+2,\alpha}}}{\Lambda} < \delta
$$

provided that Λ is large enough (which in turn implies that δ' is small). This completes the proof of the proposition.

6. Proof of Proposition 3.3

We start by defining a vector field $\widetilde{u}$ on $\mathbb{S}^3$ using the Hopf fields h_i as

$$\widetilde{u} := Y_1\, h_1 + Y_2\, h_2 + Y_3\, h_3\,,$$

where the functions Y_i are the spherical harmonics obtained in Proposition 3.2. In the following lemma we compute the action of the Laplacian on the vector field $\widetilde{u}$ using the properties of the Hopf fields. The Laplacian on vector fields that we need to consider is defined as the dual of the Hodge Laplacian on 1-forms, and can be computed as

$$\Delta := -\operatorname{curl}\operatorname{curl} + \nabla \operatorname{div}\,,$$

where ∇ and div are the gradient and divergence operators, respectively.

LEMMA 6.1. – *The Laplacian of the vector field $\widetilde{u}$ is*

$$-\Delta\widetilde{u} = \Lambda(\Lambda + 2)\,\widetilde{u} + 2\operatorname{curl}\widetilde{u}\,.$$

Proof. – The proof is simpler if we work with differential forms, so let us denote by $\widetilde{\beta}$ and α_i the 1-forms that are dual to $\widetilde{u}$ and h_i, respectively, with respect to the canonical metric on $\mathbb{S}^3$. We recall that the dual of $\operatorname{curl}\widetilde{u}$ is the 1-form $\star d\widetilde{\beta}$, with $\star$ being the Hodge star operator. The 1-form $\widetilde{\beta}$ is given by $\widetilde{\beta} = Y_i\,\alpha_i$, where summation over repeated indices is understood throughout. The Laplacian of $\widetilde{\beta}$ is then

$$-\Delta\widetilde{\beta} := dd^*\widetilde{\beta} + d^*d\widetilde{\beta} = -d \star d \star (Y_i\,\alpha_i) + \star d \star d(Y_i\,\alpha_i)\,.$$

Using that $\star d\alpha_i = 2\alpha_i$ because α_i is the dual 1-form of the Hopf field h_i, and that the differential of Y_i can be written as $dY_i = h_j(Y_i)\,\alpha_j$, where $h_j(Y_k)$ denotes the action of the vector field h_j on the scalar function Y_k, we readily obtain

$$d \star d \star (Y_i\,\alpha_i) = \frac{1}{2}d \star (h_j(Y_i)\,\alpha_j \wedge d\alpha_i)\,.$$

Observing that $\alpha_j \wedge d\alpha_i = 2\alpha_j \wedge \star\alpha_i = 2\delta_{jk}\,\mu$, where μ stands for the Riemannian volume 3-form on $\mathbb{S}^3$, it follows that

$$(6.1) \qquad\qquad d \star d \star (Y_i\,\alpha_i) = d(h_i(Y_i)) = h_j h_i(Y_i)\,\alpha_j\,.$$

Analogously, a straightforward computation using that $\star(\alpha_j \wedge \alpha_i) = \varepsilon_{jil}\alpha_l$, where ε_{jil} stands for the Levi-Civita permutation symbol, and the identity $\varepsilon_{iml}\varepsilon_{jkl} = \delta_{ij}\delta_{mk} - \delta_{ik}\delta_{mj}$ yields

$$(6.2) \qquad \star d(Y_i\,\alpha_i) = \varepsilon_{jil}h_j(Y_i)\,\alpha_l + 2Y_i\,\alpha_i\,,$$

$$(6.3) \qquad \star d \star d(Y_i\,\alpha_i) = -h_j h_j(Y_i)\,\alpha_i + h_i h_j(Y_i)\,\alpha_j + 4\varepsilon_{jil}h_j(Y_i)\,\alpha_l + 4Y_i\,\alpha_i\,.$$

Finally, adding Eqs. (6.1) and (6.3) we obtain

$$-\Delta\widetilde{\beta} = -h_j h_i(Y_i)\,\alpha_j + h_i h_j(Y_i)\,\alpha_j - h_j h_j(Y_i)\,\alpha_i + 4\varepsilon_{jil}h_j(Y_i)\,\alpha_l + 4Y_i\,\alpha_i$$
$$= \Lambda(\Lambda + 2)Y_i\,\alpha_i + 2\varepsilon_{jil}h_j(Y_i)\,\alpha_l + 4Y_i\,\alpha_i\,,$$

where we have used that $\Delta Y_i = -\Lambda(\Lambda + 2)Y_i$ and that the commutator of Hopf fields is $[h_i, h_j] = -2\varepsilon_{ijl}h_l$. The lemma then follows upon noticing that

$$2\varepsilon_{jil}h_j(Y_i)\,\alpha_l + 4Y_i\,\alpha_i = 2 \star d\widetilde{\beta}$$

by Eq. (6.2). $\qquad\square$

Using this lemma, it is easy to check that

$$u := \frac{1}{2\Lambda^2} \operatorname{curl}(\operatorname{curl} + \Lambda)\widetilde{u}$$

is a Beltrami field with eigenvalue $\Lambda + 2$. Indeed, a straightforward computation shows that

$$\operatorname{curl} u = \frac{1}{2\Lambda^2} \operatorname{curl}\operatorname{curl}(\operatorname{curl} + \Lambda)\widetilde{u} = \frac{1}{2\Lambda^2} \operatorname{curl}(-\Delta + \Lambda \operatorname{curl})\widetilde{u}$$

$$= \frac{\Lambda + 2}{2\Lambda^2} \operatorname{curl}(\operatorname{curl} + \Lambda)\widetilde{u} = (\Lambda + 2)u\,.$$

To prove the C^m estimate of the proposition, it is convenient to introduce the following auxiliary vector field in the unit ball B of $\mathbb{R}^3$

$$\bar{u}(x) := \widetilde{Y}_1(x)\, e_1 + \widetilde{Y}_2(x)\, e_2 + \widetilde{Y}_3(x)\, e_3\,,$$

where $x \in B$ and $\widetilde{Y}_i$ was defined in (5.4). There is no loss of generality in choosing the orthonormal basis e_i of $\mathbb{R}^3$ compatible with the Hopf fields h_i in the sense that $\Psi_*(h_i)(0) = e_i$. It is then easy to check that for $x \in B$ one has:

$$\Psi_*\widetilde{u}\Big(\frac{\cdot}{\Lambda}\Big) = \bar{u} + \frac{G_1}{\Lambda}\bar{u}\,,$$

$$\Psi_*(\operatorname{curl}\widetilde{u})\Big(\frac{\cdot}{\Lambda}\Big) = \Lambda\Big(\operatorname{curl}_0 \bar{u} + \frac{G_2}{\Lambda}\bar{u} + \frac{G_3}{\Lambda}D\bar{u}\Big)\,,$$

$$\Psi_*(\operatorname{curl}\operatorname{curl}\widetilde{u})\Big(\frac{\cdot}{\Lambda}\Big) = \Lambda^2\Big(\operatorname{curl}_0\operatorname{curl}_0 \bar{u} + \frac{G_4}{\Lambda}\bar{u} + \frac{G_5}{\Lambda}D\bar{u} + \frac{G_6}{\Lambda}D^2\bar{u}\Big)\,.$$

Here curl_0 denotes the Euclidean curl operator, acting on the variables x, and the functions $G_i(x, \Lambda)$ are (possibly matrix-valued) functions that depend smoothly on all their variables and whose derivatives are uniformly bounded as

$$(6.4) \qquad\qquad \sup_{x \in B} |D_x^\alpha G_i(x, \Lambda)| < C_\alpha\,.$$

Here the constant C_α depends on the multiindex α but not on Λ.

These identities and the fact that $(\operatorname{curl}_0\operatorname{curl}_0 + \operatorname{curl}_0)v = 2v$ then permits us to write

$$\Big\|\Psi_* u\Big(\frac{\cdot}{\Lambda}\Big) - v\Big\|_{C^m(B)} \leqslant \Big\|\frac{1}{2}(\operatorname{curl}_0\operatorname{curl}_0 + \operatorname{curl}_0)(\bar{u} - v)\Big\|_{C^m(B)} + \frac{C}{\Lambda}\|\bar{u}\|_{C^{m+2}(B)}$$

$$\leqslant C\|\bar{u} - v\|_{C^{m+2}(B)} + \frac{C}{\Lambda}\|\bar{u} - w\|_{C^{m+2}(B)}$$

$$(6.5) \qquad\qquad + \frac{C}{\Lambda}\|v - w\|_{C^{m+2}(B)} + \frac{C}{\Lambda}\|v\|_{C^{m+2}(B)}\,.$$

To conclude, notice that it stems from Propositions 3.1 and 3.2 that

$$\|v - w\|_{C^{m+2}(B)} < \delta$$

$$\|\bar{u} - w\|_{C^{m+2}(B)} < 3\delta\,,$$

so in particular

$$\|\bar{u} - v\|_{C^{m+2}(B)} \leqslant \|\bar{u} - w\|_{C^{m+2}(B)} + \|v - w\|_{C^{m+2}(B)} < 4\delta\,.$$

Hence the proposition follows from the estimate (6.5) upon noticing that v is a fixed vector field (so its norm is independent of Λ) and choosing Λ large enough, which also allows us to take δ as small as one wishes.

7. Proof of Theorem 2.1 in the torus

Arguing as in the proof of Proposition 3.1 we can readily show that for any $\delta > 0$, there exists a vector field v_1 on $\mathbb{R}^3$ that approximates the Beltrami field v in the ball B as

$$(7.1) \qquad \qquad \|v_1 - v\|_{C^0(B)} < \delta,$$

and that can be represented as the Fourier transform of a distribution supported on the unit sphere of the form

$$v_1(x) = \int_{\mathbb{S}^2} f(\xi)\, e^{i\xi \cdot x}\, d\sigma(\xi).$$

Again $\mathbb{S}^2$ denotes the unit sphere $\{\xi \in \mathbb{R}^3 : |\xi| = 1\}$ and f is a smooth $\mathbb{R}^3$-valued function on $\mathbb{S}^2$.

Let us now cover the sphere $\mathbb{S}^2$ by finitely many closed sets $\{U_n\}_{n=1}^N$ with piecewise smooth boundaries and pairwise disjoint interiors such that the diameter of each set is at most δ'. We can then repeat the argument used in the proof of Proposition 3.1 to infer that, if ξ_n is any point in U_n and we set

$$c_n := f(\xi_n)\, |U_n|\,,$$

the field

$$w(x) := \sum_{n=1}^N c_n\, e^{i\xi_n \cdot x}$$

approximates the field v_1 uniformly with an error proportional to δ':

$$\|w - v_1\|_{C^0(B)} < C\delta'.$$

The constant C depends on δ but not on δ', so one can choose the maximal diameter δ' small enough so that

$$(7.2) \qquad \qquad \|w - v_1\|_{C^0(B)} < \delta.$$

In turn, the uniform estimate

$$\|w - v\|_{C^0(B)} \leqslant \|w - v_1\|_{C^0(B)} + \|v - v_1\|_{C^0(B)} < 2\delta$$

can be readily promoted to the C^{m+2} bound

$$(7.3) \qquad \qquad \|w - v\|_{C^{m+2}(B)} < C\delta.$$

This follows from standard elliptic estimates as both w (whose Fourier transform is supported on $\mathbb{S}^2$) and v satisfy the Helmholtz equation:

$$\Delta v + v = 0\,, \qquad \Delta w + w = 0\,.$$

Furthermore, replacing w by its real part if necessary, we can safely assume that the field w is real-valued.

Let us now observe that for any large enough odd integer Λ one can choose the points $\xi_n \in U_n \subset \mathbb{S}^2$ so that they have rational components (i.e., $\xi_n \in \mathbb{Q}^3$) and the rescalings $\Lambda \xi_n$

are actually integer vectors (i.e., $\Lambda \xi_n \in \mathbb{Z}^3$). This is because rational points $\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$ with $\Lambda \xi \in \mathbb{Z}^3$ are uniformly distributed on the unit sphere as $\Lambda \to \infty$ through odd values [6].

Choosing ξ_n as above, we are now ready to prove Theorem 2.1 in the torus. Without loss of generality, we will take the origin as the base point p, so that we can identify the ball $\mathbb{B}$ with B through the canonical 2π-periodic coordinates on the torus. In particular, the diffeomorphism $\Psi : \mathbb{B} \to B$ that appears in the statement of Theorem 2.1 can be understood to be the identity.

Since $\Lambda \xi_n \in \mathbb{Z}^3$, it follows that the vector field

$$\widetilde{u}(x) := \sum_{n=1}^{N} c_n e^{i \Lambda \xi_n \cdot x}$$

is 2π-periodic (that is, invariant under the translation $x \to x + 2\pi\, a$ for any vector $a \in \mathbb{Z}^3$). Therefore it descends to a well-defined vector field on the torus $\mathbb{T}^3 := \mathbb{R}^3/(2\pi\mathbb{Z})^3$, which we will still denote by $\widetilde{u}$.

Since the Fourier transform of $\widetilde{u}$ if now supported on the sphere of radius Λ, $\widetilde{u}$ then satisfies the Helmholtz equation on the flat torus $\mathbb{T}^3$ with energy Λ^2,

$$\Delta \widetilde{u} + \Lambda^2 \widetilde{u} = 0 \,.$$

A straightforward calculation then reveals that the vector field on the torus

$$u := \frac{\operatorname{curl}\operatorname{curl}\widetilde{u} + \Lambda \operatorname{curl}\widetilde{u}}{2\Lambda^2}$$

satisfies the equation

$$\operatorname{curl} u = \Lambda u \,,$$

so it is a Beltrami field on $\mathbb{T}^3$ with eigenvalue $\lambda := \Lambda$.

Let us now notice that, with some abuse of notation,

$$\widetilde{u}\left(\frac{x}{\Lambda}\right) = w(x)$$

for all points x, say, in the ball B. In particular, as the derivatives of the rescaled vector field $\widetilde{u}(\cdot/\Lambda)$ behave as

$$\operatorname{curl}\widetilde{u}\left(\frac{\cdot}{\Lambda}\right) = \Lambda \operatorname{curl} w \,,$$

$$\operatorname{curl}\operatorname{curl}\widetilde{u}\left(\frac{\cdot}{\Lambda}\right) = \Lambda^2 \operatorname{curl}\operatorname{curl} w \,,$$

it then follows that

$$\begin{aligned}
\left\| u\left(\frac{\cdot}{\Lambda}\right) - v \right\|_{C^m(B)} &= \left\| \frac{\Lambda^2 \operatorname{curl}\operatorname{curl} w + \Lambda^2 \operatorname{curl} w}{2\Lambda^2} - v \right\|_{C^m(B)} \\
&= \left\| \frac{\operatorname{curl}\operatorname{curl}(w - v) + \operatorname{curl}(w - v)}{2} \right\|_{C^m(B)} \\
&\leqslant C \left\| w - v \right\|_{C^{m+2}(B)} \\
&< C\delta \,,
\end{aligned}$$

where we have used the identity $\operatorname{curl} \operatorname{curl} v + \operatorname{curl} v = 2v$ to pass to the second equality and the estimate (7.3) to derive the last inequality. The theorem then follows provided that δ is chosen small enough for $C\delta < \varepsilon$.

8. Concluding remarks

To conclude, let us make a few simple observations about our main result that follow from its proof:

There are many Beltrami fields with closed vortex lines and tubes of a given link type. – Indeed, since our construction works for any large enough odd integer λ and Beltrami fields corresponding to different eigenvalues are L^2 orthogonal, there are many non-proportional Beltrami fields with closed vortex lines and tubes realizing any given link.

In the sphere, the result holds true for any large enough eigenvalue λ. – Indeed, the fact that Λ is odd was never used in the proof of Theorem 2.1 in $\mathbb{S}^3$ (cf. Section 3), so it stems that, given any finite union of closed curves and tubes $\mathcal{S}$, for any integer λ with $|\lambda|$ greater than certain constant $\Lambda_0(\mathcal{S})$ there is a Beltrami field with eigenvalue λ having a structurally stable set of vortex lines and vortex tubes diffeomorphic to $\mathcal{S}$.

In our Beltrami fields on the sphere, knots and links appear in pairs. – In fact, using the Hopf basis $\{h_i\}_{i=1}^3$ introduced in Section 3, any Beltrami field u on $\mathbb{S}^3$ with eigenvalue $\lambda := \Lambda + 2$, with Λ a nonnegative integer, can be written as

$$u = F_1 h_1 + F_2 h_2 + F_3 h_3,$$

where F_i are smooth functions on the sphere. It is then easy to check using Eq. (6.2) that F_i must be a spherical harmonic of energy $\Lambda(\Lambda + 2)$. Since such a spherical harmonic is known to have parity $(-1)^\Lambda$, in the sense that

$$F_i(-p) = (-1)^\Lambda F_i(p)$$

for all points p in the unit sphere $\mathbb{S}^3$, and the Hopf fields h_i are odd (i.e., $h_i(-p) = -h_i(p)$), we conclude that a Beltrami field on the sphere with eigenvalue λ has parity $(-1)^{\lambda+1}$, so it is either even or odd. Therefore, the fact that $\Phi(\mathcal{S})$ is a set of vortex lines and vortex tubes of the Beltrami field u diffeomorphic to $\mathcal{S}$ and contained in a ball of small radius $1/\lambda$ automatically implies that so is the antipodal set $-\Phi(\mathcal{S})$.

The result carries over to lens spaces. – In order to see why, the key is that in the sphere the statement of Theorem 2.1 can be refined to include localizations around different points of the sphere. More precisely, let us fix l points $P_1, \ldots, P_l$ in $\mathbb{S}^3$, none of which are antipodal to another (that is, $P_j \neq -P_k$), and denote by $\Psi_j : \mathbb{B}(P_j, R_0) \to B_{R_0}$ a patch of normal geodesic coordinates centered at the point P_j. Here $\mathbb{B}(P_j, R_0)$ denotes the geodesic ball in the sphere of center P_j and radius

$$R_0 := \frac{1}{2} \min_{j \neq k} \operatorname{dist}_{\mathbb{S}^3}(P_j, P_k).$$

The approximation theorem can then be stated as follows:

THEOREM 8.1. – *Let $\{v_j\}_{j=1}^l$ be Beltrami fields in $\mathbb{R}^3$, satisfying* $\operatorname{curl} v_j = v_j$. *Let us fix any positive numbers ε and m. Then for any large enough integer λ there is a Beltrami field u, satisfying* $\operatorname{curl} u = \lambda u$ *in $\mathbb{S}^3$, such that*

$$(8.1) \qquad \left\| (\Psi_j)_* u\left(\frac{\cdot}{\lambda}\right) - v_j \right\|_{C^m(B)} < \varepsilon$$

for all $1 \leqslant j \leqslant l$.

Proof. – Arguing as in Proposition 3.2 we infer that for any large enough integer Λ there are spherical harmonics $\hat{Y}_{ij}$ of energy $\Lambda(\Lambda + 2)$ such that

$$\left\| w_j^i - \hat{Y}_{ij} \circ \Psi_j^{-1}\left(\frac{\cdot}{\Lambda}\right) \right\|_{C^{m+2}(B)} < \delta \,,$$

where w_j is a vector field of the form

$$w_j = \sum_{n=1}^N c_{jn}\, j_0(|x - x_{jn}|)$$

that approximates the Beltrami field v_j in $C^{m+2}(B)$ as in Proposition 3.1, and w_j^i $(1 \leqslant i \leqslant 3)$ denotes its i-th Cartesian component. Noticing that the Jacobi polynomial behaves as

$$\Lambda^{-\frac{1}{2}}\, P_\Lambda^{(\frac{1}{2},\frac{1}{2})}(\cos t) = \frac{O(\Lambda^{-1})}{t}$$

uniformly for $\Lambda^{-1} < t < \pi - \Lambda^{-1}$ [19, Theorem 7.32.2], it stems that the ultraspherical polynomial C_Λ is uniformly bounded as

$$|C_\Lambda(p \cdot q)| \leqslant \frac{C_\rho}{\Lambda}$$

for any points p, q in $\mathbb{S}^3$ such that

$$\operatorname{dist}_{\mathbb{S}^3}(p, q) \geqslant \rho \quad \text{and} \quad \operatorname{dist}_{\mathbb{S}^3}(p, -q) \geqslant \rho \,,$$

with a constant C_ρ that only depends on the positive constant ρ.

Using the formulas of Section 5 it is now easy to show that for any j and any fixed positive radius ρ we have

$$\|\hat{Y}_{ij}\|_{C^0(\mathbb{S}^3 \setminus (\mathbb{B}(P_j, \rho) \cup \mathbb{B}(-P_j, \rho)))} \leqslant \frac{C_\rho}{\Lambda}$$

for large Λ, with a constant that depends on ρ (and, of course, on v and δ). If we now define

$$Y_i := \sum_{j=1}^l \hat{Y}_{ij} \,,$$

and choose ρ small enough so that the sets $\mathbb{B}(P_j, \rho) \cup \mathbb{B}(-P_j, \rho)$ are disjoint for all j, the same reasoning that we employed in the proof of Proposition 3.2 shows that

$$\left\| w_j^i - Y_i \circ \Psi_j^{-1}\left(\frac{\cdot}{\Lambda}\right) \right\|_{C^{m+2}(B)} < C\delta$$

for all $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant l$, which plays a role completely analogous to that of Proposition 3.2 in the generalized context that we are now considering. The rest of the argument remains exactly as in Section 3, so the result follows. $\qquad \square$

In particular, this yields the existence of Beltrami fields in the sphere having prescribed sets of closed vortex lines and tubes (modulo diffeomorphism) around any finite number of points $P_1, \ldots, P_l$. These lines and tubes are contained in balls of radius $1/\lambda$. This line of reasoning also allows us to prove an analog of Theorem 1.1 in any lens space $L(p, q)$:

THEOREM 8.2. – *Let $\mathcal{S}$ be a finite union of (pairwise disjoint, but possibly knotted and linked) closed curves and tubes contained in a contractible subset of a three-dimensional lens space $L(p, q)$. Then for any large enough even integer λ there exists a Beltrami field u satisfying the equation $\operatorname{curl} u = \lambda u$ and a diffeomorphism Φ of $L(p, q)$ such that $\Phi(\mathcal{S})$ is a union of vortex lines and vortex tubes of u. Furthermore, this set is structurally stable.*

Proof. – The lens space can be written as

$$L(p, q) = \mathbb{S}^3/G,$$

where G is a finite isometry group isomorphic to $\mathbb{Z}_p$. We can assume that G is generated by certain isometry g. Let us now fix a point $p_0 \in \mathbb{S}^3$ and set

$$P_j := g^j \cdot p_0$$

for $0 \leqslant j \leqslant p - 1$. If Ψ is a patch of normal geodesic coordinates around p_0, we will also set $\Psi_j(x) := \Psi(g^{-j} \cdot x)$. Notice that if p is odd there are not any points in the set $\{P_j\}_{j=0}^{p-1}$ that are antipodal to each other, while for p even P_j and P_k are antipodal if and only if $|j - k| = \frac{p}{2}$.

Let us fix a Beltrami field v in $\mathbb{R}^3$ as in Theorem 2.2. Theorem 8.1 then ensures the existence of a Beltrami field $\widetilde{u}$ in $\mathbb{S}^3$ such that

$$\left\| (\Psi_j)_* \widetilde{u}\left(\frac{\cdot}{\lambda}\right) - v_j \right\|_{C^m(B)} < \varepsilon,$$

where $0 \leqslant j \leqslant p' - 1$ with $p' := p$ if p is odd and $p' := \frac{p}{2}$ if p is even. Here $v_0 := v$ and $v_j := 0$ for $1 \leqslant j \leqslant p' - 1$. Notice that, as λ is even, we saw in the previous remark that $\widetilde{u}$ is odd, i.e., $\widetilde{u}(x) = -\widetilde{u}(-x)$, so that $\widetilde{u}$ is equivariant under the isometry $x \mapsto -x$. Hence, by construction, the vector field

$$u := \sum_{j=0}^{p'-1} (g^j)_* \widetilde{u}$$

is G-equivariant, and therefore it defines a vector field in the quotient space $L(p, q) = \mathbb{S}^3/G$ that we still denote by u with some abuse of notation. Arguing exactly as in the proof of the main theorem one can show that the vector field u on $L(p, q)$ indeed has the desired properties, so the statement then follows.   □

In the torus, the distribution of rational points on the 2-sphere is key. – The proof that we have given holds provided that the eigenvalue λ is an odd integer of sufficiently large absolute value. It does not say anything about even integers, or about eigenvalues that are not integers. This assertion can be refined a little, however. We have seen that for any eigenvalue λ of the curl operator in $\mathbb{T}^3$ there is a set of points $\{\xi_n\}_{n=1}^N$ lying on the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$ such that $\lambda \xi_n \in \mathbb{Z}^3$ (this is obvious from the fact that one can write $\lambda = |k|$ with $k \in \mathbb{Z}^3$). Therefore, in the proof of Theorem 2.1 for the torus (cf. Section 7) one can substitute the collection of odd integers Λ by any subset of eigenvalues λ for which there is a set of points

$\{\xi_n\}_{n=1}^N \subset \mathbb{S}^2$ (depending on λ and such that the rescalings $\lambda\xi_n$ are in $\mathbb{Z}^3$) that becomes dense in the sphere as $|\lambda| \to \infty$ along this subset of eigenvalues. In particular, replacing the density condition by the more stringent assumption that $\{\xi_n\}$ becomes equidistributed on the sphere, it turns out that the characterization of the numbers λ that satisfy this property is somehow related to the celebrated Linnik problem in number theory. In particular, since the aforementioned equidistribution property holds for any eigenvalue for which the integer λ^2 is square-free [5], we immediately infer that the statement of Theorem 1.1 also holds for any large enough eigenvalue λ of curl (possibly even or non-integer) for which λ^2 is square-free.

Another interesting point to consider is whether the above methods apply to tori of the form $\mathbb{T}_{\mathscr{L}}^3 := \mathbb{R}^3/(2\pi\,\mathscr{L})$, where $\mathscr{L}$ is a general lattice in $\mathbb{R}^3$. As above, the key in this case is to have a density or equidistribution result at disposal, but this time not for rational points on the sphere and integer eigenvalues, but for a set of points $\{\xi_n\}_{n=1}^N \subset \mathbb{S}^2$ and real eigenvalues λ such that $\lambda\xi_n \in \mathscr{L}'$, where $\mathscr{L}'$ is the so called reciprocal lattice to $\mathscr{L}$, which is defined as the set of points $k \in \mathbb{R}^3$ such that $k \cdot x \in \mathbb{Z}$ for all $x \in \mathscr{L}$. In some very particular cases, e.g., for lattices of the form $\mathscr{L} := a\mathbb{Z}^3$ with $a \in \mathbb{R}\backslash\{0\}$, the previous equidistribution results directly hold for a sequence of eigenvalues of the form λ/a with λ an odd integer. In general, however, the authors are not aware of any results in this direction.

BIBLIOGRAPHY

[1] V. I. ARNOLD, B. A. KHESIN, *Topological methods in hydrodynamics*, Applied Mathematical Sciences **125**, Springer, New York, 1998.

[2] V. ARNOL'D, Sur la topologie des écoulements stationnaires des fluides parfaits, *C. R. Acad. Sci. Paris* **261** (1965), 17–20.

[3] D. CHAE, P. CONSTANTIN, Remarks on a Liouville-type theorem for Beltrami flows, *Int. Math. Res. Not.* **2015** (2015), 10012–10016.

[4] A. CHOFFRUT, V. ŠVERÁK, Local structure of the set of steady-state solutions to the 2D incompressible Euler equations, *Geom. Funct. Anal.* **22** (2012), 136–201.

[5] W. DUKE, Hyperbolic distribution problems and half-integral weight Maass forms, *Invent. math.* **92** (1988), 73–90.

[6] W. DUKE, Rational points on the sphere, *Ramanujan J.* **7** (2003), 235–239.

[7] A. ENCISO, D. PERALTA-SALAS, Knots and links in steady solutions of the Euler equation, *Ann. of Math.* **175** (2012), 345–367.

[8] A. ENCISO, D. PERALTA-SALAS, Existence of knotted vortex tubes in steady Euler flows, *Acta Math.* **214** (2015), 61–134.

[9] A. Enciso, D. Peralta-Salas, Knotted vortex lines and vortex tubes in stationary fluid flows, *Eur. Math. Soc. Newsl.* **96** (2015), 26–33.

[10] J. Etnyre, R. Ghrist, Contact topology and hydrodynamics. III. Knotted orbits, *Trans. Amer. Math. Soc.* **352** (2000), 5781–5794.

[11] G. B. Folland, Harmonic analysis of the de Rham complex on the sphere, *J. reine angew. Math.* **398** (1989), 130–143.

[12] L. Hörmander, *The analysis of linear partial differential operators. I*, Classics in Mathematics, Springer, Berlin, 2003.

[13] B. Khesin, Topological fluid dynamics, *Notices Amer. Math. Soc.* **52** (2005), 9–19.

[14] H. Moffatt, The degree of knottedness of tangled vortex lines, *J. Fluid Mech.* **35** (1969), 117–129.

[15] H. Moffatt, Helicity and singular structures in fluid dynamics, *Proc. Natl. Acad. Sci.* **111** (2014), 3663–3670.

[16] J. Moser, E. J. Zehnder, *Notes on dynamical systems*, Courant Lecture Notes in Math. **12**, New York University, Courant Institute of Mathematical Sciences, New York; Amer. Math. Soc., Providence, RI, 2005.

[17] N. Nadirashvili, On stationary solutions of two-dimensional Euler equation, *Arch. Ration. Mech. Anal.* **209** (2013), 729–745.

[18] N. Nadirashvili, Liouville theorem for Beltrami flow, *Geom. Funct. Anal.* **24** (2014), 916–921.

[19] G. Szegő, *Orthogonal polynomials*, fourth ed., Amer. Math. Soc., Providence, R.I., 1975, American Mathematical Society, Colloquium Publications, Vol. XXIII.

[20] W. L. K. Thomson, On vortex atoms, *Proc. Roy. Soc. Edin.* **6** (1867), 94–105, reprinted in: *Mathematical and physical papers IV*, Cambridge University Press, Cambridge, 2011.

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