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CANONICAL SMOOTHING OF COMPACT ALEKSANDROV SURFACES VIA RICCI FLOW

BY Thomas RICHARD

Abstract. – In this paper, we show existence and uniqueness of Ricci flow whose initial condition is a compact Aleksandrov surface with curvature bounded from below. This requires a weakening of the notion of initial condition which is able to deal with a priori non-Riemannian metric spaces. As a by-product, we obtain that the Ricci flow of a surface depends smoothly on Gromov-Hausdorff perturbations of the initial condition.

Résumé. – Dans cet article, on montre l'existence et l'unicité du flot de Ricci avec pour condition initiale une surface d'Aleksandrov compacte à courbure minorée. Cela nécessite un affaiblissement de la notion de condition initiale permettant de considérer des espaces métriques a priori non riemanniens. Comme corollaire, on montre que le flot de Ricci d'une surface compacte dépend lissement des perturbations de sa condition initiale au sens de Gromov-Hausdorff.

Introduction

Ricci flow of smooth manifolds has had strong applications to the study of smooth Riemannian manifolds. It is therefore natural to ask if Ricci flow can be helpful in the study of non-smooth geometric objects. A reasonable assumption to make on a metric space (X, d) that we want to deform by the Ricci flow is to require (X, d) to be approximated in some sense by a sequence (M_i, g_i) of smooth Riemannian manifolds. In [13] and [14], M. Simon studied a class of 3-dimensional metric spaces by this method. An important feature of such "Ricci flows of metric spaces" is that the notion of initial condition has to be weakened. In the work of M. Simon [13] and [14], and of the author [12], a weak notion of initial condition has been used, which we call "metric initial condition":

Definition 0.1. – *A Ricci flow $(M, g(t))_{t\in(0,T)}$ on a compact manifold M is said to have the metric space (X, d) as metric initial condition if the Riemannian distances $d_{g(t)}$ uniformly converge as t goes to 0 (as functions $M \times M \to \mathbb{R}$) to a distance $\tilde{d}$ on M such that $(M, \tilde{d})$ is isometric to (X, d).*

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REMARK 0.2. – The compactness assumption in the definition gives that (X, d) is homeomorphic to M with its manifold topology. This follows from the fact that $\tilde{d}$ is continuous on M, which implies that the identity of M is continuous as a map from M with its usual topology to M with the topology defined by $\tilde{d}$, compactness of M then gives that the identity is a homeomorphism.

The existence of such flows for some classes of metric spaces (X, d) has been proved in [13, 14] and [12]. An interesting class of spaces for which existence holds is the class of compact Aleksandrov surfaces whose curvature is bounded from below.

DEFINITION 0.3. – *A compact Aleksandrov surface whose curvature is bounded from below is a geodesic metric space (X, d) which is at the same time a compact topological surface (without boundary) and a metric space with curvature bounded from below in the sense of Aleksandrov.*

REMARK 0.4. – A geodesic metric space has curvature greater than $k \in \mathbb{R}$ in the sense of Aleksandrov if its geodesic triangles are bigger than the geodesic triangles in the complete simply connected surface $\mathbb{S}^2_k$ with curvature k.

To be more precise, a geodesic metric space (X, d) has curvature greater than k in the sense of Aleksandrov if and only if the following condition is satisfied:

Let a, b, c be any three points in (X, d), and m be any point on a shortest path from b to c. Let $\tilde{a}, \tilde{b}, \tilde{c}$ be points in $\mathbb{S}^2_k$ such that $d_k(\tilde{a}, \tilde{b}) = d(a, b)$, $d_k(\tilde{a}, \tilde{c}) = d(a, c)$ and $d_k(\tilde{b}, \tilde{c}) = d(b, c)$ where d_k is the usual distance in $\mathbb{S}^2_k$, and $\tilde{m}$ be a point on a shortest path from $\tilde{b}$ to $\tilde{c}$ such that $d_k(\tilde{b}, \tilde{m}) = d(b, m)$. Then $d(a, m) \geq d_k(\tilde{a}, \tilde{m})$.

By Toponogov's Theorem, every complete smooth surface (M, g) with Gauss curvature K_g satisfying $K_g(x) \geq k$ for every $x \in M$ is an Aleksandrov surface with curvature bounded from below by k. Another example is the boundary X of a convex set in $\mathbb{R}^n$ (resp. $\mathbb{H}^n$), endowed with its intrinsic metric d coming from the ambient metric. It can be shown (see [4], Theorem 10.2.6) that X has curvature bounded from below by 0 (resp. -1).

A metric space (X, d) will be said to have curvature bounded from below in the sense of Aleksandrov if it has curvature greater than some $k \in \mathbb{R}$ in the sense of Aleksandrov.

For more on Aleksandrov spaces, see [4], Chapters 4 and 10.

In this paper we prove uniqueness for the Ricci flow with such surfaces as metric initial condition, more precisely:

THEOREM 0.5. – *Let $(M_1, g_1(t))_{t\in(0,T]}$ and $(M_2, g_2(t))_{t\in(0,T]}$ be two smooth Ricci flows which admit a compact Aleksandrov surface (X, d) as metric initial condition. Assume furthermore that one can find $K > 0$ such that:*

$$\forall (x, t) \in M_i \times (0, T] \quad K_{g_i(t)}(x) \geq -K.$$

where $K_{g_i(t)}(x)$ is the Gauss curvature of $(M_i, g_i(t))$ at the point x.

Then there exists a diffeomorphism $\varphi : M_1 \to M_2$ such that $g_2(t) = \varphi^* g_1(t)$.

Note that the required bounds on the Ricci flow are provided by the existence proof outlined in Section 1.1.

In the next few lines, we outline the proof of Theorem 0.5. Both of the two Ricci flows $(M_i, g_i(t))$ stay in a fixed conformal class, and thus can be written $g_i(t) = w_i(x,t)h_i(x)$ for some fixed background metric h_i which can be chosen to have constant curvature. We first show that the metric initial condition prescribes the conformal class of the flow, thus we can assume that $h_1 = h_2 = h$. The proof of this fact uses deep results from the theory of singular surfaces introduced by A. D. Aleksandrov. This implies that our two Ricci flows can be seen as solutions of the following nonlinear PDE on (M, h):

$$\frac{\partial w_i}{\partial t} = \Delta_h \log(w_i) - 2K_h.$$

One then shows that w_1 and w_2 share the same L^1 initial condition as t goes to 0 and uses standard techniques to show uniqueness.

Our result can be stated in two other ways:

Proposition 0.6. – *Let M be a compact topological surface, and d be a distance on M which induces on M its manifold topology and such that (M, d) is an Aleksandrov surface with curvature bounded from below.*

Let $g_1(t)_{t\in(0,T)}$ and $g_2(t)_{t\in(0,T)}$ be two Ricci flows on M which are smooth with respect to some differential structures on M. Assume furthermore that one can find $K > 0$ such that:

$$\forall (x,t) \in M \times (0,T) \quad K_{g_i(t)}(x) \geq -K$$

and that for $i = 1, 2$ the distances $d_{g_i(t)}$ uniformly converge to d as t goes to 0.

Then the two a priori different smooth structures on M agree and $g_1(t) = g_2(t)$ for $t \in (0,T)$.

This proposition is not a consequence of Theorem 0.5, but just requires a minor adjustment in its proof, which will be indicated in Section 2.

Proposition 0.7. – *Let $(M_1, g_1(t))_{t\in(0,T]}$ and $(M_2, g_2(t))_{t\in(0,T]}$ be two smooth Ricci flows such that for $i = 1, 2$ $(M_i, g_i(t))$ Gromov-Hausdorff converges to a compact Aleksandrov surface (X, d) with curvature bounded from below as t goes to 0. Assume furthermore that one can find $K > 0$ such that:*

$$\forall (x,t) \in M_i \times (0,T] \quad K_{g_i(t)}(x) \geq -K.$$

Then there exists a diffeomorphism $\varphi : M_1 \to M_2$ such that $g_2(t) = \varphi^* g_1(t)$.

Proof. – We just have to show that if $(M^2, g(t))_{t\in(0,T)}$ is a smooth Ricci flow on a surface M^2 such that for all $t \in (0,T)$ $K_{g(t)} \geq -K$ and such that $(M^2, g(t))$ Gromov-Hausdorff converges to (X, d) as t goes to 0, then (X, d) is the metric initial condition for the Ricci flow $(M^2, g(t))$.

Since the diameter and the volume are continuous with respect to Gromov-Hausdorff convergence with sectional curvature bounded from below, we have bounds on the diameter and the volume of $(M, g(t))$ which are independent of t. Thanks to the lower bound on the curvature, the upper bound on the diameter and the lower bound on the volume, the Bishop-Gromov inequality implies that we have some $v_0 > 0$ such that:

$$\forall t \in (0,T)\ \forall x \in M \quad \mathrm{vol}_{g(t)}(B_{g(t)}(x,1)) \geq v_0.$$

Thanks to Lemma 4.2 in [14], we then have that, for some constant $C > 0$ and all $t \in (0, T)$ (for some possibly smaller $T > 0$):

$$\forall t \in (0, T) \quad |K_{g(t)}| \leq \frac{C}{T}.$$

One can then argue as in the proof of Theorem 9.2 of [14] to show that, as t goes 0, the Riemannian distances uniformly converge to a distance $\tilde{d}$ on M such that $(M, \tilde{d})$ is isometric to (X, d). Thus (X, d) is the metric initial condition of the Ricci flow $(M, g(t))$. □

As a corollary, we obtain the following statement, which says that for surfaces with curvature bounded from below Gromov-Hausdorff convergence of the initial conditions implies smooth convergence of the Ricci flows:

Corollary 0.8. – *Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a sequence of compact surfaces with Gaussian curvature greater than -1 which converges to a compact Aleksandrov surface (X, d) with curvature bounded from below, then there exists $T > 0$ such that the Ricci flows $(M_i, g_i(t))_{i\in\mathbb{N}}$ with initial condition (M_i, g_i) exist at least for $t \in (0, T)$ and converge (as smooth Ricci flows on $(0, T)$) to the unique Ricci flow with metric initial condition satisfying the bounds of Theorem 0.5.*

Proof of Corollary 0.8. – Let $(M_i, g_i)_{i\in\mathbb{N}}$ be a sequence satisfying the assumptions of Corollary 0.8. By continuity of the volume and the diameter with respect to Gromov-Hausdorff convergence of Aleksandrov surfaces, we have constants V and D such that for any $i \in \mathbb{N}$:

- $K_{g_i} \geq -1$,
- $\mathrm{diam}(M_i, g_i) \leq D$,
- $\frac{V}{2} \leq \mathrm{vol}(M_i, g_i) \leq V$.

The existence theory (Theorem 1.1) implies the Ricci flows $(M_i, g_i(t))$ exist at least for $t \in [0, T)$ and form a precompact sequence whose accumulation points can only be Ricci flows with metric initial condition (X, d) satisfying the bounds of Theorem 0.5. The uniqueness theorem then implies that there is only one accumulation point. □

Uniqueness and non-uniqueness issues have been previously considered for the Ricci flow of surfaces with "exotic" initial conditions in the works of Giesen and Topping ([5, 15]) and Ramos [10].

The paper is organized as follows, in the first section, we sketch M. Simon's existence proof in dimension 2. In the second section, we show that the metric initial condition uniquely specifies the conformal class. The last section completes the proof. In the appendix, we quickly summarize the results we need from the theory of Aleksandrov surfaces.

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1. Existence

Here we briefly review the work of Miles Simon which shows the existence of a Ricci flow for compact Aleksandrov surfaces with curvature bounded from below. Without loss of generality, we will assume that all Aleksandrov surfaces with curvature bounded from below have curvature bounded from below by -1.

What allows us to flow these surfaces is that they can be approximated by smooth surfaces in a controlled way. This is what Theorem A.1 in the appendix says.

We will now construct a Ricci flow with metric initial condition (X, d) as limit of the Ricci flows of the (M_i, g_i). In order to do this, we use the following estimates due to M. Simon:

THEOREM 1.1. – *For any $V > 0$ and $D > 0$, there exist $\kappa > 0$ and $T > 0$ such that if (M, g) is compact Riemannian surface satisfying:*

- $K_g \geq -1$,
- $\mathrm{diam}(M, g) \leq D$,
- $\frac{V}{2} \leq \mathrm{vol}(M, g) \leq V$,

then the Ricci flow $(M, g(t))$ with (classic) initial condition (M, g) exists at least for $t \in [0, T)$ and satisfies:

- $-1 \leq K_{g(t)} \leq \frac{\kappa}{t}$ *for* $t \in [0, T)$,
- $\mathrm{diam}(M, g(t)) \leq 2D$ *for* $t \in [0, T)$,
- $\frac{V}{4} \leq \mathrm{vol}(M, g(t)) \leq 2V$ *for* $t \in [0, T)$,
- $d_{g(s)} - \kappa(\sqrt{t} - \sqrt{s}) \leq d_{g(t)} \leq e^{\kappa(t-s)} d_{g(s)}$ *for* $0 < s < t \leq T$.

REMARK 1.2. – Note that in dimension 2 a lot of the arguments used by M. Simon to prove these estimates in dimension 3 become very simple. Only the existence of T and the κ/t bound require a delicate blowup analysis.

Using these estimates on each Ricci flow $(M_i, g_i(t))$ with classic initial condition (M_i, g_i), we have, using the compactness theorem of Hamilton for flows, a subsequence which converges to a Ricci flow $(M, g(t))$ defined for $t \in (0, T)$ which satisfies the estimates of Theorem 1.1. Using the estimate on the distances, we can argue as in [14] to show that $(M, g(t))$ has (X, d) as metric initial condition.

2. Uniqueness of the conformal class

In this section, we prove that the metric initial condition determines the conformal class of the flow under the geometric estimates we have assumed.

PROPOSITION 2.1. – *Let (M_1, h_1) (resp. (M_2, h_2)) be compact Riemannian surfaces of constant curvature, $g_1(x, t) = w_1(x, t)h_1(x)$ (resp. $g_2(x, t) = w_2(x, t)h_2(x)$) a smooth Ricci flow on $M_1 \times (0, T]$ (resp. $M_2 \times (0, T]$).*

Assume that:

1. $-1 \leq K_{g_1}(x, t)$ *and* $-1 \leq K_{g_2}(x, t)$,
2. *$(M_1, g_1(t))$ and $(M_2, g_2(t))$ have the same compact Aleksandrov surface (X, d) as metric initial condition.*

Then there exists a conformal diffeomorphism $\varphi : (M_1, h_1) \to (M_2, h_2)$.

Set $u_i(x,t) = \frac{1}{2}\log w_i(x,t)$, so that $g_i = e^{2u_i}h_i$. In the following lemmas, u (resp. w) denotes either u_1 or u_2 (resp. w_1 or w_2).

LEMMA 2.2. – *When t goes to 0, $u(.,t)$ (resp. $w(.,t)$) converges in L^1 norm to an integrable function $u_0(.)$ (resp. $w_0(.)$).*

Proof. – Since $\partial_t u = -K_g \leq 1$, we have that $u(x,t)-t$ increases as t decreases to 0. This allows us to define the pointwise limit $u_0(x)$ of $u(t,x)$ as t goes to 0. If we fix $t_0 > 0$, this also gives us that, using the smoothness of u, for $t \in (0,t_0)$, $u(x,t) \geq u(x,t_0)-(t_0-t)$. Thus u is uniformly bounded from below.

Moreover, by Jensen's inequality:

$$\exp\left(2\int_M u(x,t)\frac{dv_h}{\mathrm{vol}(M,h)}\right) \leq \int_M e^{2u(x,t)}\frac{dv_h}{\mathrm{vol}(M,h)} = \frac{\mathrm{vol}(M,g(t))}{\mathrm{vol}(M,h)}.$$

Now it is easy to see using the Gauss-Bonnet formula that

$$\tfrac{d}{dt}\,\mathrm{vol}(M,g(t)) = -2\int_M K_{g(t)}dv_{g(t)} = -4\pi\chi(M),$$

hence $\mathrm{vol}(M,g(t))$ is equal to $\mathrm{vol}(M,g(t_0)) - 4\pi\chi(M)(t-t_0)$ and $\mathrm{vol}(M,g(t))$ is uniformly bounded above on any interval $(0,t_0]$. This gives that $u(.,t)$ is uniformly bounded in L^1. Using Lebesgue's monotone convergence theorem, we get that u_0 is in L^1 and $u(.,t)$ converges to u_0 in L^1 norm. This settles the convergence of $u(x,t)$.

For w, since $w = e^{2u}$ and $\int w dv_h = \mathrm{vol}(M,g(t))$, a similar monotonicity argument can be applied. □

LEMMA 2.3. – *u_0 belongs to the space* $\mathrm{Pot}(M,h)$ *defined in the appendix.*

Proof. – The previous lemma shows that u_0 is an L^1 function. We just need to check that the distributional Laplacian of u_0 is a signed measure.

To see this, we write, for a smooth function $\eta : M \to \mathbb{R}$:

$$\begin{aligned}\int_M \eta(x)\Delta_h u(x,t)dv_h(x) &= \int_M \eta(x)(K_h - K_{g(t)}e^{2u(x,t)})dv_h\\ &= \int_M \eta(x)K_h dv_h - \int_M \eta(x)d\omega_{g(t)}\end{aligned}$$

where $d\omega_{g(t)} = K_{g(t)}e^{2u(x,t)}dv_h$ is the curvature measure of $(M_i, g_i(t))$. By Theorem A.15, since the distance $d_{g(t)}$ uniformly converges to the distance d, the curvature measures weakly converge to the curvature measure of (M,d) which we call $d\omega$. We integrate by parts on the left side of the previous equality and let t go to 0, we get:

$$\int_{M_i} u_0(x)\Delta_h\eta(x)dv_h = \int_M \eta(x)K_h dv_h - \int_{M_i}\eta(x)d\omega.$$

This tells us that the distributional Laplacian of u_0 is the measure $d\mu = K_h dv_h - d\omega$. □

As in the appendix, we define a new distance on M by $d_0 = d_{h,u_0}$. Since (M,d) has curvature bounded from below, the condition $d\mu^+(\{x\}) < 2\pi$ is satisfied (see Remark A.18), and d_0 is a distance on M whose induced topology is the usual manifold topology of M.

LEMMA 2.4. – *For any x and y in M, $d(x,y) = d_0(x,y)$.*

Proof. – For $t > 0$, consider the curvature measures:

$$d\omega_t = K_{g(t)} e^{2u(x,t)} dv_h.$$

By Theorem A.15, the curvature measures weakly converge to the curvature measure $d\omega$ of (M, d). Moreover, since the curvature of (M, d) is bounded from below by -1, $d\omega \geq -e^{2u_0} dv_h$. Set:

$$d\mu_t = K_h dv_h - d\omega_t.$$

As t goes to 0, $d\mu_t$ weakly converges to $d\mu$, since $d\mu$ is bounded from below by an integrable function, we have that $d\mu_t^+$ and $d\mu_t^-$ weakly converge to $d\mu^+$ and $d\mu^-$. We also have convergence of the volumes. We can then apply Theorem A.17 to get that $d_{g(t)}$ uniformly converges to d_{h,u_0}. This gives the claimed result. □

We will write $(M, e^{2u_0} h)$ for M equipped with the distance d_0.

We are now ready to prove Proposition 2.1:

Proof (of Proposition 2.1). – For each Ricci flow $(M_i, e^{2u_i(x,t)} h_i(x))$, we have constructed a $u_{i,0}(x)$ such that $(M_i, e^{2u_{i,0}} h_i)$ is isometric to (X, d). Thus there exists an isometry φ from $(M_1, e^{2u_{1,0}(x)} h_2(x))$ to $(M_2, e^{2u_{2,0}(x)} h_2(x))$. Theorem A.19 exactly gives that φ is conformal from (M, h_1) to (M, h_2). □

3. End of the proof

Thanks to the results of the previous section, we can now assume that $g_1(x,t) = w_1(x,t)h(x)$ and $g_2(x,t) = w_2(x,t)h(x)$ are two Ricci flows on a surface (M, h) with metric initial condition (M, d) defined for t in $(0, T]$.

It is a standard fact that w_1 and w_2 satisfy the following equation of $M \times (0, T]$:

$$\frac{\partial w_i}{\partial t} = \Delta_h \log(w_i) - 2K_h. \tag{1}$$

The next lemma relates the metric initial condition with the behavior of w_i as t goes to 0:

LEMMA 3.1. – *$w_i(.,t) dv_h$ weakly converges to the 2-dimensional area measure $d\sigma$ associated with d.*

This is Theorem A.15 in the appendix.

First we prove some estimates on w_i:

LEMMA 3.2. – *One can find $C > 0$ depending on K, w_1 and w_2 only, such that:*

$$Ce^t \leq w_i(x,t)$$

for all x in $M \times (0, T]$.

Proof. – We set $w = w_1$, the proof is the same for w_2. We have:

$$\partial_t g = \partial_t w h = -2K_g g = -2K_g w h$$

which gives $\partial_t w = -2K_g w$. Using the geometric estimates on the curvature, we get:

$$\frac{\partial_t w}{w} \leq 2.$$

Let $0 < t_1 < t_2 < T$, compute at some fixed $x \in M$, then:

$$[\log(w(x,t))]_{t_1}^{t_2} \leq 2(t_2 - t_1)$$

and:

$$\frac{w(x,t_2)}{w(x,t_1)} \leq e^{2(t_2-t_1)}$$

thus:

$$\frac{w(x,t_1)}{w(x,t_2)} \geq e^{2(t_1-t_2)}.$$

Let $t_1 = t$ and $t_2 > 0$ be some fixed time in $(0,T)$ and use that $w(.,t_2)$ is smooth on M compact, we get the required estimate. □

The weak convergence of $w_i(.,t)dv_h$ to $d\sigma$ is not really pleasant to work with when dealing with uniqueness issues. In fact, the following lemma shows that the convergence is strong in L^1.

LEMMA 3.3. – *As t goes to* 0*, $w(.,t)$ converges in L^1 norm to a function w_0 which satisfies $w_0 dv_h = d\sigma$.*

Proof. – Let $\tilde{w}(x,t) = e^{-2t} w(x,t)$, then:

$$\partial_t \tilde{w}(x,t) = -2e^{-2t} w(x,t) + e^{-2t} \partial_t w(x,t).$$

As in the proof of the previous lemma: $\partial_t w \leq 2w$. So $\partial_t \tilde{w} \leq 0$ and $\tilde{w}(x,t)$ increases as t decreases to 0. Let w_0 be the pointwise limit of $\tilde{w}(.,t)$ as t goes to 0. Since $\int_M \tilde{w}(x,t)dv_h = e^{-2t}\operatorname{vol}(M,g(t))$ is bounded, Lebesgue's monotone convergence theorem gives that w_0 is in L^1 and $\tilde{w}(.,t)$ (and $w(.,t)$) converges in L^1 norm to w_0. Since L^1 convergence implies weak convergence, $w_0 dv_h = d\sigma$. □

We now prove the uniqueness statement.

PROPOSITION 3.4. – *$w_1(x,t) = w_2(x,t)$ for any $x \in M$ and $t \in (0,T]$.*

Proof. – We will prove that for any smooth nonnegative function η on M and any $T' \in (0,T]$:

$$\int_M (w_1(x,T') - w_2(x,T'))\eta(x)dv_h(x) = 0.$$

Let ψ be a smooth function on $M \times (0,T']$ and $0 < s < T'$, then:

$$\int_M (w_2(x,T') - w_1(x,T'))\psi(x,T')dv_h - \int_M (w_2(x,s) - w_1(x,s))\psi(x,s)dv_h$$
$$= \int_s^{T'} \int_M (w_2(x,\tau) - w_1(x,\tau))(A(x,\tau)\Delta_h \psi(x,\tau) + \partial_t \psi(x,\tau))dv_h d\tau$$

where $A(x,\tau) = \frac{\log(w_2(x,\tau)) - \log(w_1(x,\tau))}{w_2(x,\tau) - w_1(x,\tau)}$. Since w_1 and w_2 are smooth and positive on $M \times (0,T]$, A is smooth too. Moreover, by the mean value theorem and Lemma 3.2, we have, for $(x,t) \in M \times (0,T]$:

$$\varphi(t) \leq A(x,t) \leq \frac{1}{C_1}$$

where φ is the positive continuous function defined by:

$$\varphi(t) = \inf_{x \in M} \min\left(\frac{1}{w_1(x,t)}, \frac{1}{w_2(x,t)}\right) > 0.$$

We now choose $\psi(x,t)$ to be the solution of the following backward linear heat equation:

$$\begin{cases} \frac{\partial \psi}{\partial t}(x,t) = -A(x,t)\Delta_h \psi(x,t), \\ \psi(x,T') = \eta(x). \end{cases}$$

Thanks to the properties of smoothness and positivity of A, ψ exists and is smooth on $M \times (0,T']$ and the maximum principle shows that: $0 \leq \psi(x,t) \leq \sup_{x \in M} \eta(x)$. We also get:

$$\int_M (w_2(x,T') - w_1(x,T'))\eta(x)dv_h = \int_M (w_2(x,s) - w_1(x,s))\psi(x,s)dv_h.$$

We now let s go to 0. Since $w_1(.,s) - w_2(.,s)$ goes to 0 in L^1 norm and $\psi(x,s)$ is bounded, the right hand side of the previous equality goes to 0 and :

$$\int_M (w_2(x,T') - w_1(x,T'))\eta(x)dv_h = 0.$$

Since this equality is true for any η and any $T' > 0$, we have that w_1 and w_2 are equal almost everywhere, since these functions are smooth, we get equality everywhere. □

Appendix

Facts from the theory of Aleksandrov surfaces

This appendix gathers the results from the theory of Aleksandrov surfaces with bounded integral curvature or curvature bounded from below that have been used in the paper. All these results can be found in the works of Aleksandrov and Reshetnyak (see [3], [1] and [11]). A survey in a more modern language can be found in [16].

We use two notions of surfaces with special curvature properties in this work that we will present in the two subsections of this appendix.

A.1. Surfaces with curvature bounded from below

Our main objects of interest are compact surfaces with curvature bounded from below by $-k$, which are surfaces with an intrinsic metric (X,d) whose geodesic triangles are "fatter" than those in the complete simply-connected surface of constant curvature $-k$. These were defined in the introduction, but see also [4], Chapters 4 and 10. For $k = 0$ the theory has been developed by A.D. Aleksandrov. All the results we mention here are shown in [3] in the $k = 0$ case. The generalization of these results to nonzero curvature bounds can usually be reduced to simple exercises in non-Euclidean plane geometry, however we will give references when available or sketch the proofs.

First we need a theorem on the approximation of compact Aleksandrov surfaces with curvature bounded from below by smooth surfaces:

THEOREM A.1. – *For any compact Aleksandrov surface with curvature bounded from below by* $-k$ (X,d)*, there exist a sequence of smooth compact Riemannian surfaces* $(M_i,g_i)_{i\in\mathbb{N}}$ *satisfying:*

- $K_{g_i} \geq -k$,
- $\operatorname{diam}(M_i,g_i) \leq D$,
- $\frac{V}{2} \leq \operatorname{vol}(M_i,g_i) \leq V$,

which Gromov-Hausdorff converges to (X,d).

Proof. – We first remark that we just need to build a sequence (M_i,g_i) satisfying $K_{g_i} \geq -k$ which Gromov-Hausdorff converges to (X,d): the other bounds will follow from the continuity of the diameter and the two-dimensional Hausdorff with respect to Gromov-Hausdorff convergence with curvature bounded from below, see Exercise 7.3.14 and Theorem 10.10.10 in [4].

This theorem seems to have belonged to the folklore of the metric geometry of surfaces for several decades, but a proof of it is surprisingly hard to locate. To the knowledge of the author, the only written proof that can be found is Lemma 2.4 in [7]. □

We will also need some properties of angles in a metric space.

Let a,b,c be three points in a metric space (X,d); we define the k-comparison angle $\tilde{\angle}_k a_b^c$ as the angle at $\tilde{a}$ of the comparison triangle $\tilde{a}\tilde{b}\tilde{c}$ in $\mathbb{S}_k^2$ whose sides have length $d_k(\tilde{a},\tilde{b}) = d(a,b)$, $d_k(\tilde{a},\tilde{c}) = d(a,c)$ and $d_k(\tilde{b},\tilde{c}) = d(b,c)$. This notion gives us another characterization of spaces with curvature bounded below:

PROPOSITION A.2 ([4], Proposition 10.1.1). – *Let* (X,d) *be a geodesic metric space.* (X,d) *has curvature bounded from below by* $k \in \mathbb{R}$ *if and only if for any points* $a,b,c,d \in X$*, we have the following inequality:*

$$\tilde{\angle}_k a_b^c + \tilde{\angle}_k a_c^d + \tilde{\angle}_k a_d^b \leq 2\pi.$$

Let $(\gamma_1(s))_{s\in[0,T]}$ and $(\gamma_2(t))_{t\in[0,T]}$ be two shortest paths parametrized by arc length in (X,d) such that $\gamma_1(0) = \gamma_2(0) = m \in X$. We define the upper angle $\overline{\angle}(\gamma_1,\gamma_2)$ between γ_1 and γ_2 by:

$$\overline{\angle}(\gamma_1,\gamma_2) = \limsup_{t\to 0, s\to 0} \tilde{\angle}_0 m_{\gamma_2(t)}^{\gamma_1(s)}.$$

Since the cosine laws in Euclidean, spherical and hyperbolic geometry are equivalent when the sides of the triangle go to 0, using $\tilde{\angle}_k$ for $k \neq 0$ instead of $\tilde{\angle}_0$ in the definition above will not change the upper angle.

If the limsup above is actually a limit, $\overline{\angle}(\gamma_1,\gamma_2)$ is called the angle between γ_1 and γ_2 and is denoted by $\angle(\gamma_1,\gamma_2)$. In an Aleksandrov space with curvature bounded from below, the angle between two shortest paths is always defined: see Proposition 4.3.2 in [4]. If T is a geodesic triangle and a,b,c are the vertices of T, then the angle $\angle a = \angle a_b^c$ at a is the angle between the two edges of T which emanate from a.

Angles allow us to give an alternative characterization of compact surfaces with curvature bounded from below in the sense of Aleksandrov.

PROPOSITION A.3 ([4], Theorem 4.3.5). – *A compact topological surface X with an intrinsic distance d is a compact surface with curvature bounded greater than k in the sense of Aleksandrov if and only if the following two conditions are satisfied:*

1. *for any geodesic triangle T the angles at the vertices of T are bigger than the corresponding angle in the comparison triangle $\overline{T} \subset \mathbb{S}^2_k$ whose sides have the same length than the sides of T;*
2. *for any four points a, b, c, d, for any shortest paths γ from a to b and η from c to d, if c lies on γ and is different from a and b, then $\angle c_a^d + \angle c_b^d = \pi$.*

In the end of this section, we will state two properties of Aleksandrov surfaces with curvature bounded from below which are specific to dimension 2 and play an important role in linking surfaces with curvature bounded from below with surfaces with bounded integral curvature.

First we need to define some concepts. These discussions are extracted from [3], Chapter IV and [1] Chapter II for convenience of the reader.

In a surface (X, d) with curvature bounded from below, any two distinct shortest paths $(\gamma(s))_{s\in[0,\varepsilon)}$ and $(\eta(t))_{t\in[0,\varepsilon)}$ such that $\gamma(0) = \eta(0) = m \in X$ will not meet for some finite time $\varepsilon' > 0$, by this we mean that $\gamma([0, \varepsilon')) \cap \eta([0, \varepsilon')) = m$. It follows that the same is true for any finite collection $\gamma_1, \dots, \gamma_n$ of shortest paths emanating from m. Intersecting these shortest paths with a very small topological disk D containing m, this gives a partition of D into n sectors, each of them bounded by two of the γ_i and a portion of the boundary of D. If γ_k and γ_l are two distinct shortest paths from m, they divide a small enough disk D around m into two connected components. If D is endowed with an orientation, we can pick the component which lies on the right side of the path going from γ_k to m to γ_l. We call this component the sector inside D bounded by γ_k and γ_l.

Two shortest paths which are part of the boundary of one of the sectors are said to be adjacent. A shortest path γ_i is said to be between γ_k and γ_l if for any $\epsilon > 0$ and any topological disk D around m, $\gamma_i([0, \varepsilon))$ intersects the sector inside D bounded by γ_k and γ_l. A finite list $\gamma_1, \dots, \gamma_n$ of shortest paths emanating from a point $m \in X$ is said to be ordered if for any i, γ_i is between γ_{i-1} and γ_{i+1} (where the indices are understood modulo n).

With these notions, we have the following result:

PROPOSITION A.4. – *Let $m \in X$ where (X, d) is a compact surface with curvature bounded from below by k. Let $\gamma_1, \dots, \gamma_n$ be an ordered list of n shortest paths emanating from $m \in X$. Let $\alpha_i = \angle(\gamma_i, \gamma_{i+1})$. Then:*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n \leq 2\pi.$$

REMARK A.5. – The author has not been able to locate an elementary proof of this result. For the $k = 0$ case, Aleksandrov proves this via approximation of the surface by convex polyhedra in [3]. In [8], this fact is used for arbitrary k but a proof is not provided. One can probably argue as follows: the tangent cone to (X, d) at any point m is a cone over a dimension 1 Aleksandrov space of curvature greater than 1 (see [4], Corollary 10.9.6). Since an Aleksandrov space of dimension 1 is either an interval of length not greater than π or a circle of length not greater than 2π, this implies that the proposition is true for the tangent cone. Since the proposition is infinitesimal in nature, it should hold for the original surface. For convenience of the reader we sketch an elementary proof below.

Proof (Sketch). – We prove the result by induction on n. The statement is empty for $n = 0, 1$ and trivial for $n = 2$ since any angle is less than π by definition. For $n = 3$, the result follows by applying Proposition A.2 with $a = m$, $b = \gamma_1(t)$, $c = \gamma_2(t)$ and $d = \gamma_3(t)$ and letting t go to 0.

Now the induction step is done as follows. If we have $n \geq 4$ shortest paths, we can find two shortest paths γ_k and γ_l which are not adjacent. There are two cases to consider here, either the broken path formed by γ_k and γ_l is a shortest path, in which case we can apply Theorem 1* on p. 130 of [3] to both sides of this shortest path and conclude, or it is not. In this case, consider the sequence of shortest paths c_i joining $\gamma_k(\frac{1}{i})$ and $\gamma_l(\frac{1}{i})$. Because of the Jordan curve theorem, we can assume after passing to a subsequence that c_i intersects all the shortest paths γ_i lying on a fixed side of the broken path made by γ_k and γ_l in an order compatible with the order on the shortest path. Without loss of generality, we can assume that $k < l$ and c_i intersects every γ_i for $k < i < l$. In this situation we can apply Theorem 3 on p. 128 of [3], and conclude that the angle between γ_k and γ_l is the sum $\alpha_k + \cdots + \alpha_{l-1}$. We can thus discard all the shortest paths γ_i for $k < i < l$ without modifying the sum, which will then be less than 2π by the induction hypothesis. This ends the induction step. □

We will also need the following elementary observation:

LEMMA A.6. – *For any point m in a compact surface (X, d) with curvature bounded from below by k, one can find two shortest paths γ_1, γ_2 emanating from m such that $\angle(\gamma_1, \gamma_2) > 0$.*

Proof. – Since (X, d) is a topological surface and a geodesic metric space, we can find two points p_1 and p_2 such that:

$$\begin{aligned} d(p_1, p_2) &< d(p_1, m) + d(m, p_2), \\ d(p_1, m) &< d(p_1, p_2) + d(p_2, m), \\ d(p_2, m) &< d(p_2, p_1) + d(p_1, m). \end{aligned}$$

Let γ_i be a shortest path from m to p_i; by the angle characterization of lower curvature bounds, we have that $\angle(\gamma_1, \gamma_2)$ is greater than the angle in the comparison triangle in $\mathbb{S}^2_k$. In particular:

$$\angle(\gamma_1, \gamma_2) > 0.$$ □

These two previous propositions can be summarized using the concept of complete angle at a point $m \in X$.

DEFINITION A.7. – *The complete angle $\theta(m)$ at a point $m \in (X, d)$ is the supremum of*

$$\sum_{i=1}^{n} \angle(\gamma_i, \gamma_{i+1})$$

over all ordered finite lists of shortest paths $(\gamma_1, \ldots, \gamma_n)$ emanating from m.

The two previous propositions show:

PROPOSITION A.8. – *The complete angle $\theta(m)$ at a point m in a surface (X,d) with curvature bounded from below satisfies:*

$$\theta(m) \in (0, 2\pi].$$

A.2. Surfaces with bounded integral curvature

A wider class of surfaces is the class of surfaces with bounded integral curvature in the sense of Aleksandrov. The references for this topic are [1] and [11]. Before defining what it means for a surface to have bounded integral curvature, we first need some definitions.

Let (X,d) be a topological surface endowed with an intrinsic metric d.

DEFINITION A.9. – *A filled geodesic triangle $T \subset X$ is an open set homeomorphic to a disk whose boundary consists of three consecutive shortest paths. We will call the endpoints of these shortest paths the vertices of T, the shortest paths themselves the edges of T.*

DEFINITION A.10. – *A filled geodesic triangle T is said to be simple if for any pair of points e, f in the boundary of T, any curve joining e to f in $X \backslash T$ is longer than the shortest path joining e to f in the boundary of T.*

REMARK A.11. – It is easy to see that a simple triangle is convex in the sense that any shortest path between points in T remains in T.

The upper angle at a vertex a of a geodesic triangle T is the upper angle between the two edges of T emanating from a and is denoted by $\overline{\angle}a$. The excess of a triangle is defined by $e(T) = \overline{\angle}a + \overline{\angle}b + \overline{\angle}c - \pi$ where a, b and c are the vertices of T.

We can now define the concept of bounded integral curvature.

DEFINITION A.12. – *A compact surface with an intrinsic metric (X,d) is said to have bounded integral curvature if there is a constant C such that for any finite family (T_i) of disjoint simple triangles, $\sum_i |e(T_i)| \leq C$.*

We then have the following result:

PROPOSITION A.13. – *A compact surface with curvature bounded from below by some $k \in \mathbb{R}$ in the sense of Aleksandrov has bounded integral curvature.*

A proof of this fact for $k = 0$ follows easily from the work of Aleksandrov. For the case of general k, it can be extracted from [8]. For convenience of the reader, we provide a proof.

Proof. – We first treat the $k = 0$ case. Let us first notice that the excess of a geodesic triangle in a surface of nonnegative curvature is always nonnegative by Proposition A.3. Consider a finite collection $\mathcal{T}$ of disjoint simple triangles. It follows from the Theorem on p. 88 of [3] that we can add finitely many geodesic triangles to $\mathcal{T}$ and get a triangulation $\mathcal{T}'$ of X. Let V be the number of vertices of the triangulation induced by $\mathcal{T}'$ and F be the number of triangles of $\mathcal{T}'$. Then the Euler-Poincaré characteristic of X satisfies $2\chi(X) = 2V - F$.

On the other hand we have:

$$\sum_{T \in \mathcal{T}'} e(T) = \sum_{T \in \mathcal{T}'} (\alpha_T + \beta_T + \gamma_T - \pi) \leq 2\pi V - \pi F = 2\pi\chi(X), \tag{2}$$

where we have denoted by α_T, β_T and γ_T the three angles of a triangle T and used Proposition A.4 to bound the sum of all angles at a given vertex by 2π. Now since the excesses of each triangle are nonnegative we get:

$$\sum_{T\in\mathcal{T}} |e(T)| \leq \sum_{T\in\mathcal{T}'} |e(T)| = \sum_{T\in\mathcal{T}'} e(T) \leq 2\pi\chi(X).$$

Which completes the proof for the $k = 0$ case.

We now investigate the case when $k \neq 0$. It is actually enough to only consider the case $k = -1$. The bound (2) still holds. Let $\mathcal{A}$ denote the area measure on (X, d), it can be defined as the 2-dimensional Hausdorff measure on Borel subsets of (X, d). The area of a filled geodesic triangle is by definition $\mathcal{A}(T)$, moreover the total area of a compact surface with curvature bounded below is finite. We will need the following fact:

If T is a simple triangle in a compact surface with curvature greater than -1, then $e(T) \geq -\mathcal{A}(T)$.

We postpone the proof of this fact and first see how it implies our result.

Decompose $\mathcal{T}$ into $\mathcal{T}_+ = \{T \in \mathcal{T} | e(T) \geq 0\}$ and $\mathcal{T}_- = \{T \in \mathcal{T} | e(T) \leq 0\}$. Then:

$$\sum_{T\in\mathcal{T}} |e(T)| = \sum_{T\in\mathcal{T}_+} e(T) + (-\sum_{T\in\mathcal{T}_-} e(T)).$$

The second term can be bounded above by $\sum_{T\in\mathcal{T}_-} \mathcal{A}(T) \leq \mathcal{A}(X)$. To give an upper bound for the first term we write, using Equation (2):

$$\sum_{T\in\mathcal{T}_+} e(T) \leq 2\pi\chi(X) - \sum_{T\in\mathcal{T}'\setminus\mathcal{T}_+} e(T) \leq 2\pi\chi(X) + \sum_{T\in\mathcal{T}_+} \mathcal{A}(T) \leq 2\pi\chi(X) + d\,\mathcal{A}(X).$$

In the end we get:

$$\sum_{T\in\mathcal{T}} |e(T)| \leq 2\pi\chi(X) + 2\mathcal{A}(X),$$

which shows that (X, d) has bounded integral curvature.

Let us now show the estimate $e(T) \geq -\mathcal{A}(T)$. To this end, denote by $\tilde{T}$ the comparison triangle in $\mathbb{S}^2_{-1} = \mathbb{H}^2$ whose sides have the same length than the sides of T; it follows from Proposition A.3 that $e(T) \geq e(\tilde{T})$. Moreover, $e(\tilde{T}) = -\mathcal{A}(\tilde{T})$ since $\tilde{T}$ is a hyperbolic triangle and $\mathcal{A}$ is just the usual area in $\mathbb{H}^2$.

We thus just need to show that $\mathcal{A}(\tilde{T}) \leq \mathcal{A}(T)$. This can be proved along the following lines: first isometrically identify each of the sides of T with the corresponding side of $\tilde{T}$, using the definition of curvature bounded from below given in the introduction, this gives a 1-Lipschitz map f from the boundary of T to the boundary of $\tilde{T}$. We then use the Kirszbraun theorem from [2], which asserts that f can be extended as a 1-Lipschitz map from T to $\tilde{T}$. This map is surjective for topological reasons. Thus the area of T is bigger than the area of $\tilde{T}$. □

Aleksandrov surfaces with bounded integral curvature have well-defined notions of area and curvature, which are measures on the surface (signed measure for the curvature). Following the notations of [1], we will denote the area measure by $d\sigma$ and the curvature measure by $d\omega$. The curvature measure $d\omega$ is defined in [1] p. 156, on a simple triangle T, the curvature $d\omega(T)$ is just the excess $e(T)$ of this triangle.

In the case of compact smooth surfaces (M, g), these measures coincide with the usual notions of volume form dv_g and curvature measure $K_g dv_g$, see [1], Chapters 5 and 8.

REMARK A.14. – In the case of surfaces with curvature bounded below we used in the last proof as an area measure the 2-dimensional Hausdorff measure. The area measure is defined in [1] in a different way; however it is said on p. 266 of the previous reference that these two notions of area coincide on Borel sets. Unfortunately the author has not been able to locate the reference to the proof that is given in the book. Anyway, we will not need this result.

The next theorem shows that, within the class of Aleksandrov surfaces with bounded integral curvature, the curvature measure and the area measure depend continuously on the distance, this is Theorem 6, p. 240 and Theorem 9 p. 269 in [1].

THEOREM A.15. – *Let $(d_i)_{i\in\mathbb{N}}$ and d be distances on a compact surface M such that:*

- *(M, d) and each of the (M, d_i) are Aleksandrov surfaces of bounded integral curvature;*
- *as functions on $M \times M$, the distances d_i uniformly converge to d.*

Then the curvature measures ω_i of (M, d_i) weakly converges to the curvature measure ω of (M, d), that is, for any continuous φ function on M:

$$\int_M \varphi d\omega_i \xrightarrow{i\to\infty} \int_M \varphi d\omega.$$

Moreover, the area measure σ_i of d_i weakly converges to the area measure σ of d.

Our aim now is to present a partial converse of the previous theorem. In the sequel, h is a fixed smooth Riemannian metric on M. We consider the space $\mathrm{Pot}(M, h)$ of L^1 functions u on M whose distributional Laplacian with respect to h is a signed measure $d\mu$ on M, we say that u is the potential of $d\mu$. Such a u is the difference of two subharmonic functions and has a representative which is well-defined outside a set of Hausdorff dimension 0 in M.

The volume of u is defined by $V(u) = \int_M e^{2u} dv_h$. Given a zero mass signed measure $d\mu$ and $V > 0$, $d\mu$ has a unique potential $u_{d\mu,V}$ of volume V. We will denote by $d\mu = d\mu^+ - d\mu^-$ the Jordan decomposition of $d\mu$. Reshetnyak has studied the non-smooth Riemannian metric $e^{2u}h$. We have ([11] Theorem 7.1.1 on p. 100, [16] Proposition 5.3):

THEOREM A.16. – *Let $u \in \mathrm{Pot}(M, h)$ be a potential of $d\mu$. Assume that $d\mu^+(\{x\}) < 2\pi$ for any $x \in M$. Define:*

$$d_{h,u}(x, y) = \inf_{\gamma\in\Gamma(x,y)} \int_0^1 e^{u(\gamma(\tau))} |\dot{\gamma}(\tau)|_h d\tau$$

where $\Gamma(x, y)$ is the space of C^1 paths γ from $[0, 1]$ to M with $\gamma(0) = x$ and $\gamma(1) = y$. Then $d_{h,u}$ is a distance on M such that $(M, d_{h,u})$ has bounded integral curvature. The curvature measure of this surface is given by:

$$d\omega = K_h dv_h - d\mu.$$

We are now ready to state the converse of Theorem A.15. This is Theorem 7.3.1 in [11], p. 112, see also [16], Theorem 6.2.

Theorem A.17. – *Let (M,h) be a smooth Riemannian surface and $(d\mu_i^+)_{i\in\mathbb{N}}$ $(d\mu_i^-)_{i\in\mathbb{N}}$ be two sequences of (nonnegative) measures which weakly converge to $d\mu^+$ and $d\mu^-$ and such that $d\mu_i^+(M)$ and $d\mu_i^-(M)$ are equal and bounded independently of i.*

Let V_i be a sequence of positive numbers converging to $V > 0$. Let u_i be the potential of $d\mu_i = d\mu_i^+ - d\mu_i^-$ of volume V_i and u be the potential of $d\mu = d\mu^+ - d\mu^-$ of volume V.

Assume that $d\mu(\{x\}) < 2\pi$ for all $x \in M$. Then the distances d_{h,u_i} uniformly converge as i goes to infinity to the distance $d_{h,u}$.

Remark A.18. – When $d\mu$ is the curvature measure of a surface with curvature bounded from below, the condition $d\mu(\{x\}) < 2\pi$ is automatically fulfilled. In fact, it follows from the discussion on "complete angles at a point" in [1] (Chapter 2, Section 5 and Chapter 4, Section 4) that the curvature $d\omega(\{x\})$ of a point x in (M,d) is equal to $2\pi - \theta(x)$, where $\theta(x)$ is the complete angle at x from Definition A.7. Proposition A.8 thus implies that $d\mu(\{x\}) \in [0, 2\pi)$.

The next theorem, due to Huber ([6], Satz A.), says that the distance $d_{h,u}$ determines the conformal class of h, see [11] Theorem 7.1.3 or [16] Theorem 6.4.

Theorem A.19. – *Let (M,h) and (M',h') be two compact Riemannian surfaces, $u \in \mathrm{Pot}(M,h)$ and $u' \in \mathrm{Pot}(M',h')$. Assume f is an isometry from $(M,d_{h,u})$ to $(M',d_{h',u'})$, then f is a conformal diffeomorphism from (M,h) to (M',h').*

The proof of this theorem is related to some ideas from the theory of quasiconformal maps. Recall that a homeomorphism f between two open sets of $\mathbb{C}$ is said to be H-quasiconformal ($H \geq 1$) if

$$\limsup_{r\to 0} \frac{\max_{|z-z_0|=r} |f(z) - f(z_0)|}{\min_{|z-z_0|=r} |f(z) - f(z_0)|} \leq H$$

for any z_0 in the domain of f. Menchoff proved in [9] that a 1-quasiconformal map is actually a conformal diffeomorphism (and hence is holomorphic or anti-holomorphic), he actually showed that it is enough to require that f is 1-quasiconformal on its domain except maybe at a finite number of points.

In [6], Huber shows that, around every point p in M except a finite number, one can find complex charts U and U' on M and M', and at any point $z_0 \in U$:

$$\lim_{r\to 0} \frac{\max_{|z-z_0|=r} |f(z) - f(z_0)|}{\min_{|z-z_0|=r} |f(z) - f(z_0)|} = 1.$$

This is Equation (3.20) in [6]. It is proved using delicate potential theoretic estimates on u and u'.

One can then use a theorem of Menchoff [9] to conclude that f is conformal.

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