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A FUNCTIONAL ANALYSIS PROOF OF GROMOV'S POLYNOMIAL GROWTH THEOREM

BY Narutaka OZAWA

Abstract. – The celebrated theorem of Gromov asserts that any finitely generated group with polynomial growth contains a nilpotent subgroup of finite index. Alternative proofs have been given by Kleiner and others. In this note, we give yet another proof of Gromov's theorem, along the lines of Shalom and Chifan-Sinclair, which is based on the analysis of reduced cohomology and Shalom's property H_{FD}.

Résumé. – Un résultat célèbre de Gromov affirme que tout groupe finiment engendré de croissance polynomiale contient un sous-groupe nilpotent d'indice fini. Des preuves alternatives de ce résultat ont été données par Kleiner, entre autres. Dans cette note, nous donnons une nouvelle preuve du théorème de Gromov, dans l'esprit de résultats de Shalom et Chifan-Sinclair, reposant sur l'analyse de la cohomologie réduite et la propriété H_{FD} de Shalom.

1. Introduction

The celebrated theorem of Gromov ([10, 7]) asserts that any finitely generated group with weakly polynomial growth contains a nilpotent subgroup of finite index. Here a group G is said to have *weakly polynomial growth* if $\liminf \log|S^n|/\log n < \infty$ for any finite generating subset S such that $1 \in S = S^{-1}$. Alternative proofs have been given by Kleiner and others ([13, 18, 12, 3]). In this note, we give yet another proof of Gromov's theorem, along the lines of Shalom ([17]) and Chifan-Sinclair ([5]), which is based on the analysis of reduced cohomology and Shalom's property H_{FD}.

Let $\pi\colon G \curvearrowright \mathcal{H}$ be a unitary representation. Recall that a 1-*cocycle* of G with coefficients in π is a map $b\colon G \to \mathcal{H}$ which satisfies

$$\forall g, x \in G \quad b(gx) = b(g) + \pi_g b(x).$$

A 1-*coboundary* is a 1-cocycle of the form $b(g) = \xi - \pi_g \xi$ for some $\xi \in \mathcal{H}$, and an *approximate* 1-*coboundary* is a 1-cocycle that is a pointwise limit of 1-coboundaries. The

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spaces of 1-cocycles, 1-coboundaries, approximate 1-coboundaries are written respectively by $Z^1(G,\pi)$, $B^1(G,\pi)$, and $\overline{B^1(G,\pi)}$, and so the *reduced cohomology* space $\overline{H^1}(G,\pi)$ is $Z^1(G,\pi)/\overline{B^1(G,\pi)}$. It is proved by Mok and Korevaar-Schoen ([15, 14], see also [16] and Theorem A in Appendix) that any finitely generated group G without Kazhdan's property (T) admits a unitary representation π with $\overline{H^1}(G,\pi) \neq 0$. A group G is said to have *Shalom's property* H_{FD} if $\overline{H^1}(G,\pi) \neq 0$ implies that π is not weakly mixing. Here π is said to be *weakly mixing* if $\mathcal{H}$ admits no nonzero finite-dimensional $\pi(G)$-invariant subspaces. We recall that infinite amenable groups, and in particular groups with weakly polynomial growth, do not have property (T) (see e.g., [4, Chapter 12]). Thus, if such a group has property H_{FD}, then it has a finite-dimensional unitary representation π with $\overline{H^1}(G,\pi) \neq 0$. Shalom has observed that a proof of property H_{FD} for a group with weakly polynomial growth implies Gromov's theorem (see [17, Section 6.7] and [19]). In this paper, we prove that a group with slow entropy growth has property H_{FD}, thus giving a new proof of Gromov's theorem. Here we say G has *slow entropy growth* if there is a non-degenerate finitely-supported symmetric probability measure μ on G with $\mu(e) > 0$ such that

$$\liminf_n n(H(\mu^{*n+1}) - H(\mu^{*n})) < \infty,$$

where H is the entropy functional. This property is formerly weaker than but probably equivalent to weakly polynomial growth (see Section 3).

THEOREM. – *A finitely generated group with slow entropy growth has property* H_{FD}.

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2. Reduced cohomology and harmonic 1-cocycles

Let G be a finitely generated group and fix a non-degenerate finitely-supported symmetric probability measure μ with $\mu(e) > 0$. Let $\pi\colon G \curvearrowright \mathcal{H}$ be a unitary representation. We first recall the fact that every element in the reduced cohomology space $\overline{H^1}(G,\pi)$ is uniquely represented by a μ-harmonic 1-cocycle (see [11, 1]). The space $Z^1(G,\pi)$ of 1-cocycles is a Hilbert space under the norm

$$\|b\|_{Z^1(G,\pi)} := \Big(\sum_x \mu(x)\|b(x)\|^2\Big)^{1/2},$$

and the space $\overline{B^1(G,\pi)}$ agrees with the closure of $B^1(G,\pi)$ in the Hilbert space $Z^1(G,\pi)$. We observe that $b \in Z^1(G,\pi)$ is orthogonal to $B^1(G,\pi)$ if and only if it is *μ-harmonic*: $\sum_x \mu(x)b(x) = 0$ or equivalently $\sum_x \mu(x)b(gx) = b(g)$ for all $g \in G$. Indeed, this follows from the identities $b(x^{-1}) + \pi_x^{-1}b(x) = b(e) = 0$ and

$$\sum_x \mu(x)\langle b(x), \xi - \pi_x\xi\rangle = 2\langle \sum_x \mu(x)b(x), \xi\rangle.$$

Since $Z^1(G,\pi) = \overline{B^1(G,\pi)} \oplus B^1(G,\pi)^\perp$ as a Hilbert space, $\overline{H^1}(G,\pi)$ can be identified with the space $B^1(G,\pi)^\perp$ of μ-harmonic 1-cocycles.

By the above discussion, we may concentrate on μ-harmonic 1-cocycles. For any μ-harmonic 1-cocycle b, one has $\sum_x \mu^{*n}(x)\|b(x)\|^2 = n\sum_x \mu(x)\|b(x)\|^2$ (by induction on n). In this section, we give a better inequality, which is inspired by the work of Chifan and Sinclair ([5]). Let $\mathcal{H} \otimes \bar{\mathcal{H}}$ denote the Hilbert space tensor product of the Hilbert space $\mathcal{H}$ and its complex conjugate $\bar{\mathcal{H}}$. We recall that π is weakly mixing if and only if the unitary representation $\pi \otimes \bar{\pi}$ on $\mathcal{H} \otimes \bar{\mathcal{H}}$ has no nonzero invariant vectors. Indeed, $\mathcal{H} \otimes \bar{\mathcal{H}}$ can be identified with the space $S_2(\mathcal{H})$ of Hilbert-Schmidt operators on $\mathcal{H}$, and under this identification $\pi_g \otimes \bar{\pi}_g$ becomes the conjugation action $\operatorname{Ad}\pi_g$ of π_g on $S_2(\mathcal{H})$ (see e.g., Section 13.5 in [4]). Since any nonzero Hilbert-Schmidt operator (which is $\operatorname{Ad}\pi_g$-invariant) is compact and has a nonzero finite-dimensional eigenspace (which is π_g-invariant), our claim follows.

LEMMA. – *Let $b\colon G \to \mathcal{H}$ be a μ-harmonic 1-cocycle with coefficients in a weakly mixing unitary representation π. Then one has*

$$\frac{1}{n}\Big\|\sum_x \mu^{*n}(x)(b(x)\otimes\bar{b}(x))\Big\|_{\mathcal{H}\otimes\bar{\mathcal{H}}} \to 0.$$

In particular,

$$\sup_{\xi\in\mathcal{H},\,\|\xi\|\le 1} \frac{1}{n}\sum_x \mu^{*n}(x)|\langle b(x),\xi\rangle|^2 \to 0.$$

Proof. – Since b is μ^{*n}-harmonic for every n, one has for every n and $g \in G$

$$\sum_x \mu^{*n}(x)(b(gx)\otimes\bar{b}(gx)) = b(g)\otimes\bar{b}(g) + (\pi_g\otimes\bar{\pi}_g)\sum_x \mu^{*n}(x)(b(x)\otimes\bar{b}(x)).$$

Thus, putting $\zeta := \sum_x \mu(x)(b(x)\otimes\bar{b}(x))$ and $T := \sum_g \mu(g)(\pi_g\otimes\bar{\pi}_g)$, one has

$$\begin{aligned}\sum_x \mu^{*n}(x)(b(x)\otimes\bar{b}(x)) &= \sum_{g,x}\mu(g)\mu^{*n-1}(x)(b(gx)\otimes\bar{b}(gx))\\ &= \zeta + T\sum_x \mu^{*n-1}(x)(b(x)\otimes\bar{b}(x))\\ &= \cdots = (1+T+\cdots+T^{n-1})\zeta.\end{aligned}$$

Since π is weakly mixing, $\pi\otimes\bar{\pi}$ admits no nonzero invariant vectors, and hence by strict convexity of a Hilbert space, 1 is not an eigenvalue of the self-adjoint contraction T. Hence, the measure $m(\cdot) := \langle E_T(\cdot)\zeta,\zeta\rangle$, associated with the spectral resolution E_T of T, is supported on $[-1,1]$ and satisfies $m(\{1\}) = 0$. Thus, one has

$$\frac{1}{n}\Big\|\sum_x \mu^{*n}(x)(b(x)\otimes\bar{b}(x))\Big\|_{\mathcal{H}\otimes\bar{\mathcal{H}}} = \Big(\int_{-1}^{1}\Big|\frac{1+t+\cdots+t^{n-1}}{n}\Big|^2\,dm(t)\Big)^{1/2} \to 0$$

by Bounded Convergence Theorem. The second statement follows from the first, because $|\langle b(x),\xi\rangle|^2 = \langle b(x)\otimes\bar{b}(x),\xi\otimes\bar{\xi}\rangle$. □

3. Concavity of the entropy functional

We recall the basic fact that the entropy functional

$$p \mapsto H(p) := -\sum_x p(x)\log p(x) = \sum_x p(x)\log\frac{1}{p(x)}$$

is concave and examine its modulus of concavity. Our discussion in this section is inspired by Erschler and Karlsson's work [8]. See also [2] for relevant information. Let p and q be any non-negative functions. For convenience, we put

$$\delta(p,q) := H(\frac{p+q}{2}) - \frac{H(p)+H(q)}{2}.$$

Since

$$\frac{1}{2}(a\log a + b\log b) - \frac{a+b}{2}\log\frac{a+b}{2} \geq \frac{|a-b|^2}{8(a+b)} \geq 0$$

for any $a, b \geq 0$ (this follows from the fact $(t\log t)'' = (1+\log t)' = t^{-1} \geq (a+b)^{-1}$ for all t between a and b), one has

$$\delta(p,q) \geq \sum_x \frac{|p(x)-q(x)|^2}{8(p(x)+q(x))} \geq 0.$$

This implies concavity of H. Moreover, for any non-negative function f, one has

$$\sum_x f(x)|p-q|(x) \leq \big(8\delta(p,q)\sum_x f(x)^2(p+q)(x)\big)^{1/2} \tag{1}$$

by the Cauchy-Schwarz inequality.In particular, $\|p-q\|_1 \leq (8\delta(p,q)\|p+q\|_1)^{1/2}$.

For any probability measures μ and ν on G and $g_0 \in G$, one has

$$H(\mu * \nu) - H(\nu) \geq 2\min\{\mu(e), \mu(g_0)\}\delta(\nu, g_0\nu). \tag{2}$$

Here $\mu * \nu = \sum_g \mu(g)(g\nu)$ and $(g\nu)(x) = \nu(g^{-1}x)$. Indeed, put $\lambda := \min\{\mu(e), \mu(g_0)\}$ and observe that $\nu' := (1-2\lambda)^{-1}(\mu * \nu - (\lambda\nu + \lambda g_0\nu))$ is a convex combination of $g\nu$'s, and that $H(g\nu) = H(\nu)$ for any g. Hence, $H(\nu') \geq H(\nu)$ by concavity and

$$\begin{aligned} H(\mu * \nu) - H(\nu) &= H(2\lambda\frac{\nu + g_0\nu}{2} + (1-2\lambda)\nu') - H(\nu) \\ &\geq 2\lambda\big(H(\frac{\nu+g_0\nu}{2}) - H(\nu)\big) + (1-2\lambda)\big(H(\nu') - H(\nu)\big) \\ &\geq 2\lambda\delta(\nu, g_0\nu). \end{aligned}$$

Here we explain that a group G with weakly polynomial growth has slow entropy growth. Let μ be any non-degenerate finitely-supported symmetric probability measure on G with $\mu(e) > 0$. By concavity of log, one has

$$H(\mu^{*n}) = \sum_x \mu^{*n}(x)\log\frac{1}{\mu^{*n}(x)} \leq \log\sum_{x\in\operatorname{supp}\mu^{*n}}\frac{\mu^{*n}(x)}{\mu^{*n}(x)} = \log|\operatorname{supp}\mu^{*n}|.$$

Since $|\operatorname{supp}\mu^{*n}| = |(\operatorname{supp}\mu)^n|$ has weak polynomial growth, this implies that

$$d := \liminf_n \frac{H(\mu^{*n})}{\log n} < \infty.$$

Now for any $d' < \liminf_n n(H(\mu^{*n+1}) - H(\mu^{*n}))$, one has

$$H(\mu^{*n}) = \sum_{k=0}^{n-1}(H(\mu^{*k+1}) - H(\mu^{*k})) \geq \text{const.} + \sum_{k=1}^{n-1}\frac{d'}{k} = \text{const.} + d' \log n.$$

Thus $d' \leq d$, that is to say, G has slow entropy growth. It is likely that the converse is also true. Indeed, suppose that G does not have weakly polynomial growth. Then, for any d, one has $\liminf_n |S^n|/n^d = \infty$ and so, by Varopoulos's inequality ([20]), $\mu^{*n}(e) = O(n^{-d/2})$. But since $\mu^{*n}(e) \geq \mu^{*n}(g)$ for every even n and every $g \in G$ by the Cauchy-Schwarz inequality, this implies

$$H(\mu^{*n}) \geq \log\frac{1}{\mu^{*n}(e)} \geq \text{const.} + \frac{d}{2}\log n$$

for even n. Since d was arbitrary, one has $\lim_n H(\mu^{*n})/\log n = \infty$.

4. Proof of theorem

Proof of theorem. – Let $b\colon G \to \mathcal{H}$ be a 1-cocycle with coefficients in a weakly mixing unitary representation π, and we will prove that $b \in \overline{B^1(G,\pi)}$. As discussed in Section 2, we may assume that b is μ-harmonic. Let $g \in \operatorname{supp}\mu$ and $\xi \in \mathcal{H}$ be given. Then, since b is μ^{*n}-harmonic for every $n \in \mathbb{N}$, one has for every n

$$\langle b(g), \xi\rangle = \langle \sum_x (b(gx) - b(x))\mu^{*n}(x), \xi\rangle = \sum_x \langle b(x), \xi\rangle (g\mu^{*n} - \mu^{*n})(x),$$

and so, by inequalities (1) and (2),

$$\begin{aligned}|\langle b(g), \xi\rangle|^2 &\leq 8\delta(\mu^{*n}, g\mu^{*n})\sum_x |\langle b(x), \xi\rangle|^2 (g\mu^{*n} + \mu^{*n})(x)\\ &\leq \lambda_g (H(\mu^{*n+1}) - H(\mu^{*n}))\sum_x |\langle b(x), \xi\rangle|^2 (g\mu^{*n} + \mu^{*n})(x),\end{aligned}$$

where $\lambda_g = 4\min\{\mu(e), \mu(g)\}^{-1}$. Since $\sum_x |\langle b(x), \xi\rangle|^2 (g\mu^{*n} + \mu^{*n})(x)$ has sublinear growth by the lemma of Section 2 (and the Cauchy-Schwarz inequality), slow entropy growth implies that $\langle b(g), \xi\rangle = 0$. Since $g \in \operatorname{supp}\mu$ and $\xi \in \mathcal{H}$ were arbitrary and μ is non-degenerate, this implies that $b = 0$. □

Note that the slow entropy growth condition implies the following ([8, Lemma 8])

$$\liminf_n n^{-1/2} \max_{|g|\leq 1} \|\mu^{*n} - g\mu^{*n}\|_1 < \infty.$$

Since the slow entropy growth condition seems too restrictive, we give here a supplementary result that the above condition yields a weaker conclusion (although the author is still not aware of any super-polynomial growth group to which the proposition applies.)

PROPOSITION. – *Let G be a finitely generated group with the word length $|\cdot|$ and let μ be a non-degenerate finitely-supported symmetric probability measure with $\mu(e) > 0$. Assume that there is $\delta > 0$ such that for any $\varepsilon > 0$ and any $N \in \mathbb{N}$ there is $n \geq N$ such that for any $g \in G$ and any $E \subset G$ if $|g| \leq \delta n^{1/2}$ and $\mu^{*n}(E) \geq 1 - \delta$ then $\mu^{*n}(gEB_{\varepsilon n^{1/2}}) \geq \delta$. Here $B_r = \{x : |x| \leq r\}$. Then, any 1-cocycle b with coefficients in a weakly mixing unitary representation π has sublinear growth in the sense that $\|b(g)\| \leq f(|g|)$ for some f with*

$f(n)/n \to 0$. In particular if G moreover has a controlled Følner sequence, then G has property H_{FD}.

Proof. – Since approximate 1-coboundaries have sublinear growth ([6, Corollary 3.3]), we may assume that b is a μ-harmonic cocycle such that $\|b(g)\| \le |g|$. Let $\gamma > 0$ be given arbitrary and put $\varepsilon := \gamma\delta^2$. By the lemma of Section 2, there is N such that $n \ge N$ implies

$$\sup_{\xi \in \mathcal{H},\, \|\xi\| \le 1} \sum_x \mu^{*n}(x) |\langle b(x), \xi \rangle|^2 < \varepsilon^2 n.$$

Take $n \ge \max\{N, 4\delta^{-2}\}$ which fulfills the statement in the proposition. Let $g \in G$ be such that $|g| \le \delta n^{1/2}$. For each unit vector $\xi \in \mathcal{H}$, put

$$E_\xi := \{x \in G : |\langle b(x), \xi\rangle| \le \varepsilon n^{1/2}/\delta\}$$

and observe that $\mu^{*n}(E_\xi) > 1 - \delta$. Hence, one has $\mu^{*n}(E_{\pi_g^*\xi} \cap g^{-1}E_\xi B_{\varepsilon n^{1/2}}) > 0$ by assumption, and so there exist $x_\xi \in E_{\pi_g^*\xi}$ and $y_\xi \in B_{\varepsilon n^{1/2}}$ such that $g x_\xi y_\xi \in E_\xi$. It follows that

$$\begin{aligned} |\langle b(g), \xi\rangle| &\le |\langle b(g x_\xi y_\xi), \xi\rangle| + |\langle b(x_\xi), \pi_g^*\xi\rangle| + |\langle b(y_\xi), \pi_{g x_\xi}^*\xi\rangle| \\ &\le 3\varepsilon n^{1/2}/\delta = 3\gamma\delta n^{1/2}. \end{aligned}$$

This means that $\|b(g)\| \le 3\gamma\delta n^{1/2}$ for all $g \in G$ such that $|g| \le \delta n^{1/2}$, and so

$$\lim_m \frac{\max\{\|b(g)\| : |g| \le m\}}{m} = \inf_m \frac{\max\{\|b(g)\| : |g| \le m\}}{m} \le \frac{3\gamma\delta n^{1/2}}{\lfloor \delta n^{1/2} \rfloor} \le 6\gamma,$$

where the first equality follows from subadditivity. Since $\gamma > 0$ was arbitrary, this proves the first statement. The second follows from [6, Corollary 3.7]. □

Appendix
Property (T) and harmonic 1-cocycles

We give a simple proof of the theorem of Mok and Korevaar-Schoen cited in the introduction ([15, 14], see also [13, Appendix A]). After submitting the first draft of this paper, the author learned that the same proof had been presented in Jesse Peterson's lecture at Vanderbilt University in Spring 2013. A more explicit construction (which still uses an ultrafilter) is provided later in [9].

THEOREM A. – *Let G be a finitely generated group without property (T). Then, for any non-degenerate finitely-supported symmetric probability measure μ on G, there is a nonzero μ-harmonic 1-cocycle with coefficients in some unitary representation.*

Proof. – Since G does not have property (T), there is a unitary representation $\pi : G \curvearrowright \mathcal{H}$ having approximate invariant vectors but having no nonzero invariant vectors (see [4, Theorem 12.1.7]). Thus the self-adjoint contraction $T = \sum_x \mu(x)\pi_x \in \mathbb{B}(\mathcal{H})$ contains 1 in the spectrum, but not as an eigenvalue. This means that 1 is a limit point of the spectrum of T. Hence there is a sequence $\varepsilon_n \searrow 0$ such that the spectral subspaces $\mathcal{H}_n := E_T([1-2\varepsilon_n, 1-\varepsilon_n])\mathcal{H}$ are nonzero. Take unit vectors $\xi_n \in \mathcal{H}_n$. One has

$$\sum_x \mu(x)\|\xi_n - \pi_x\xi_n\|^2 = 2(1 - \langle T\xi_n, \xi_n\rangle) \in [2\varepsilon_n, 4\varepsilon_n].$$

Fix a free ultrafilter $\mathcal{U}$ and consider the ultrapower unitary representation $\pi_{\mathcal{U}}$ on the ultrapower Hilbert space $\mathcal{H}_{\mathcal{U}}$ (see [4, 12.1.4]). Then the map $b\colon G \to \mathcal{H}_{\mathcal{U}}$, given by $b(x) = (\varepsilon_n^{-1/2}(\xi_n - \pi_x\xi_n))_{n\to\mathcal{U}}$, is a 1-cocycle with coefficients in $\pi_{\mathcal{U}}$ such that

$$\sum_x \mu(x)\|b(x)\|^2 = \lim_{n\to\mathcal{U}} \varepsilon_n^{-1} \sum_x \mu(x)\|\xi_n - \pi_x\xi_n\|^2 \in [2,4]$$

and

$$\begin{aligned}\|\sum_x \mu(x)b(x)\| &= \lim_{n\to\mathcal{U}} \varepsilon_n^{-1/2}\|\sum_x \mu(x)(\xi_n - \pi_x\xi_n)\| \\ &= \lim_{n\to\mathcal{U}} \varepsilon_n^{-1/2}\|\sum_x (1-T)\xi_n\| \leq \lim_{n\to\mathcal{U}} 2\varepsilon_n^{1/2} = 0.\end{aligned}$$

This means that b is a nonzero μ-harmonic 1-cocycle. □

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