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QUASI-PROJECTIVITY OF THE MODULI SPACE OF SMOOTH KÄHLER-EINSTEIN FANO MANIFOLDS

BY CHI LI, XIAOWEI WANG AND CHENYANG XU

ABSTRACT. – In this paper, we prove that there is a canonical continuous Hermitian metric on the CM line bundle over the proper moduli space $\overline{\mathcal{M}}$ of smoothable Kähler-Einstein Fano varieties. The Chern curvature of this Hermitian metric is the Weil-Petersson current, which exists as a closed positive (1,1)-current on $\overline{\mathcal{M}}$ and extends the canonical Weil-Petersson current on the moduli space $\mathcal{M}$ of smooth Kähler-Einstein Fano manifolds. As a consequence, we show that the CM line bundle is nef and big on $\overline{\mathcal{M}}$ and its restriction on $\mathcal{M}$ is ample.

RÉSUMÉ. – Dans cet article, nous montrons qu'il existe une métrique hermitienne continue et canonique sur le fibré en droites CM au-dessus de l'espace de modules $\overline{\mathcal{M}}$ des variétés de Kähler-Einstein régularisables. La courbure de Chern de cette métrique hermitienne est le courant de Weil-Petersson, qui existe en tant que (1,1)-courant fermé positif sur $\overline{\mathcal{M}}$, et étend le courant canonique de Weil-Petersson défini sur l'espace de modules $\mathcal{M}$ des variétés de Kähler-Einstein Fano régulières. Nous montrons aussi, en guise d'application de notre résultat, que le fibré des lignes CM est nef et big sur $\overline{\mathcal{M}}$, et que sa restriction à $\mathcal{M}$ est ample.

1. Introduction

The study of moduli spaces of polarized varieties is a fundamental topic in algebraic geometry. The most classical case is the moduli space of Riemann surfaces of genus ≥ 2, whose compactification is a Deligne-Mumford stack admitting a projective coarse moduli space. People have been trying to generalize this picture to higher dimensions, leading to the development of KSBA compactification of moduli space of canonically polarized varieties (see [37]). In [66], Viehweg proved a deep result that the moduli space of polarized manifolds with nef canonical line bundles is quasi-projective. Building on the fundamental work of [35] and the development of Minimal Model Program, it is proved in [29] that the KSBA compactification is projective.

On the other hand, there are negative results concerning the projectivity of moduli spaces. Kollár [36] showed that the moduli space of polarized manifolds may not be quasi-projective. In particular, he proved that any toric variety can be a moduli space of polarized uniruled manifolds. The quasi-projectivity of moduli is still open for polarized manifolds that are not uniruled and whose canonical line bundles are not nef.

Differential geometric methods have also played important roles in studying moduli spaces of complex manifolds. For examples, the moduli space $\mathcal{M}_g$ has been studied using Teichmüller spaces equipped with Weil-Petersson metrics; the moduli spaces of Calabi-Yau manifolds and its Weil-Petersson metrics were studied by Tian and Todorov. Moreover, Tian's results in [58] imply that there is a Hermitian line bundle on the moduli space of Calabi-Yau manifolds whose curvature form is the Weil-Petersson metric. Fujiki-Schumacher later [28] considered the more general case of moduli space of Kähler manifolds admitting constant scalar curvature Kähler (cscK) metrics. They proved that the natural Weil-Petersson metric is always Kähler by interpreting it as the Chern curvature of a determinant line bundle equipped with a Quillen metric. To achieve this, they applied the study of determinant line bundles and the Quillen metrics by Bismut-Gillet-Soulé [11]. As a consequence, it was proved in [28] that any compact subvariety in the moduli space of cscK manifolds with discrete automorphisms is projective. However, all the cases considered above require the fibration to be smooth, which is not the case in general. In [60] Tian studied similar determinant line bundles in a singular setting and introduced the notion of CM ($\mathbb{Q}$-)line bundle (see Definition 4.2 and Remark 4.3), which will be denoted by λ_{CM} from now on in this paper.

It follows from Kollár's negative result that, for uniruled manifolds extra constraints must be imposed in order for the moduli space to be quasi-projective/projective. However, Tian's study of CM line bundle and Fujiki-Schumacher's results suggest that the moduli space of manifolds admitting canonical metrics could be quasi-projective. In this paper, we confirm this speculation for the moduli space of Fano Kähler-Einstein (KE) manifolds, which was first conjectured by Tian in [60].

Fano manifolds in dimension 2 are called del Pezzo surfaces. It was proved in [59] that a smooth del Pezzo surface admits a Kähler-Einstein metric if and only if its automorphism is reductive. Recently, based on the study of degenerations of smooth Kähler-Einstein del Pezzo surfaces in [59], proper moduli spaces of smoothable Kähler-Einstein del Pezzo varieties were constructed in [47]. Moreover, it was shown in [47] that these proper moduli spaces are actually projective except possibly for the case of del Pezzo surfaces of degree 1.

The higher dimensional generalization of the results in [59] and [47] was made possible thanks to the celebrated solutions to the Yau-Tian-Donaldson conjecture ([14], [15], [16], [64]). The moduli space of higher dimensional smooth Kähler-Einstein Fano manifolds, denoted by $\mathcal{M}$ from now on, was studied in [62], [24], [46]. More recently, a proper algebraic compactification $\overline{\mathcal{M}}$ of $\mathcal{M}$ was constructed in [41] (see also [45]). It is further believed that $\overline{\mathcal{M}}$ should be projective (see [47], [41], [45]). This paper is a step towards establishing this. The main technical result of this paper is the following descent and extension result.

THEOREM 1.1. – *The* CM *line bundle λ_{CM} descends to a $\mathbb{Q}$-line bundle Λ_{CM} on the proper moduli space $\overline{\mathcal{M}}$. There is a canonically defined continuous Hermitian metric h_{DP} on Λ_{CM} whose curvature form is a positive current ω_{WP} on $\overline{\mathcal{M}}$ which extends the canonical Weil-Petersson current $\omega^{\circ}_{\mathrm{WP}}$ on $\mathcal{M}$.*

We remark that the descending of CM line bundle has been expected once the existence of a *good moduli* in the sense of [2] is verified, see e.g., [47, Section 6.2]. In this paper we will give a detailed account of this fact based on our construction of $\overline{\mathscr{M}}$ in [41] and Kempf's descending criterion (see [2, Theorem 10.3] and [26, Theorem 2.3]). The main challenge remaining is proving its positivity.

Note that in Theorem 1.1 although we use the notion of current on singular complex spaces defined in Definition 2.6, $\omega_{\mathrm{WP}}^{\circ}$ is actually a smooth Kähler-metric on a dense open set $\mathscr{M}'$ of $\mathscr{M}$ (see Section 4.1.1 and the proof of Theorem 1.2 in Section 6). So equivalently, we can say that ω_{WP} extends the canonical smooth Kähler metric $\omega_{\mathrm{WP}}^{\circ}|_{\mathscr{M}'}$ on $\mathscr{M}'$.

With $(\Lambda_{\mathrm{CM}}, h_{\mathrm{DP}})$ at hand (or equivalently, Weil-Petersson current ω_{WP} with a controlled behavior), we can apply a quasi-projective criterion as Theorem 6.1 to get the following result.

THEOREM 1.2. – Λ_{CM} *is nef and big over* $\overline{\mathscr{M}}$. *Moreover, for the normalization morphism* $n : \overline{\mathscr{M}}^{\mathrm{n}} \to \overline{\mathscr{M}}$ *which induces an isomorphism over* $\mathscr{M}$, *the rational map* $\Phi_{|n^*(m\Lambda_{\mathrm{CM}})|}$ *associated to the complete linear system* $|n^*(m\Lambda_{\mathrm{CM}})|$ *embeds* $\mathscr{M}$ *into* $\mathbb{P}^{N_m-1}$ *for* $m \gg 1$ *with* $N_m = \dim H^0(\overline{\mathscr{M}}^{\mathrm{n}}, n^*(m\Lambda_{\mathrm{CM}}))$. *In particular,* $\mathscr{M}$ *is quasi-projective.*

In some sense, the quasi-projectivity of $\mathscr{M}$ in Theorem 1.2 could be seen as a consequence of Tian's partial C^0-estimates recently established in the fundamental works of Donaldson-Sun [25] and Tian [63]. Actually such kind of implication was stated without proof in [60, end of Section 8] which used the notion of CM stability (introduced in [60]). However, because of the subtlety pointed out by Kollár [36], it is still not clear to us how to deduce the quasi-projectivity directly using CM stability. Nevertheless, if we only look at the open locus of $\mathscr{M}$ which parametrizes Kähler-Einstein Fano manifolds with finite automorphism groups, then [20] and [46] have already shown that it is quasi-projective as we know the Fano manifolds it parametrizes are all asymptotically Chow stable by [20]. On the other hand, if we drop the finite automorphism assumption, there exists a Kähler-Einstein Fano manifold which is asymptotically Chow unstable (see [48]).

Our proof of Theorem 1.1, which heavily depends on the recent development in the theory of Kähler-Einstein metrics on Fano varieties, is also inspired by the work of Schumacher-Tsuji [55] and Schumacher [53]. In [53], Schumacher re-proved the quasi-projectivity of $\mathscr{M}^-$ which is the moduli space of canonically polarized manifolds by using some compactification of $\mathscr{M}^-$ and the extension of Weil-Petersson metric. Our argument uses a similar approach. First, by applying the theory of Deligne pairings, for any smooth variey S together with a flat family of Kähler-Einstein Fano varieties $\mathscr{X} \to S$ containing an open dense $S^\circ \subset S$ such that the fibers of $\mathscr{X}|_{S^\circ} \to S^\circ$ are all Kähler-Einstein Fano manifolds, we can construct a Hermitian metric h_{DP} on the CM line bundle $\lambda_{\mathrm{CM}} \to S$ whose restriction to S° is the classical Weil-Petersson metric. Second, the partial-C^0 estimate established in [25, 63] together with an extension of continuity results in [39] allow us to show that this metric is indeed *continuous* whose curvature form can be extended to a positive current on S. Third, by using the local GIT description of the canonical compactification $\overline{\mathscr{M}}$ constructed in [41] and the fact that $\overline{\mathscr{M}}$ is a *good moduli* in the sense of [2] (see Section 5), CM line bundle λ_{CM} together with the metric h_{DP} can be descended to an Hermitian line bundle $(\Lambda_{\mathrm{CM}}, h_{\mathrm{DP}})$ on $\overline{\mathscr{M}}$, whose curvature form is exactly the Weil-Petersson current we want. The descending construction is partly inspired by [28, Section 11] and based on Kempf's descent lemma proved in [26]. Finally, to obtain the quasi-projectivity, we establish the quasi-projectivity criterion Theorem 6.1 for *normal* algebraic spaces, which can be regarded as an algebro-geometric version of the analytic criterion in [55, Section 6].

The paper is organized in the following way: In the next section, we derive some estimate which will play the key role in proving the extension of Weil-Petersson metric. In Section 4, assuming the existence of a universal family over a parameter space we obtain a canonical *continuous* Hermitian metric on the CM line bundle with curvature form being a positive current over the base via the formalism of Deligne pairing . In Section 5, we descend the metrized CM line bundle to $\overline{\mathscr{M}}$ and prove Theorem 1.1 based on a crucial uniform convergence lemma established in Section 7. In Section 6.1, we finish the proof of Theorem 1.2 by applying the quasi-projective criterion of Theorem 6.1.

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2. Plurisubharmonic functions on complex spaces

For later reference we recall the following known facts. In this section X is a possibly singular complex space. We will denote the open unit disk by $\Delta = \{z \in \mathbb{C}; |z| < 1\}$.

DEFINITION 2.1 ([34, Def. 1, Section 4.1]). – A function $\psi(x)$ on X is called plurisubharmonic on X if the following conditions are satisfied:

1. The value of $\psi(x)$ is a real number or $-\infty$.
2. $\psi(x)$ is upper semi-continuous at any point $x_0 \in X$: $\overline{\lim_{x \to x_0}} \psi(x) \leq \psi(x)$.
3. For any holomorphic map $\tau : \Delta \to X$, the function $\psi \circ \tau$ is subharmonic on Δ.

When X is smooth, the above definition recovers the ordinary definition of plurisubharmonic functions on smooth complex manifolds.

REMARK 2.2. – Note that in the literature, the plurisubharmonic functions in Definition 2.1 are sometimes called weakly plurisubharmonic functions. The plurisubharmonic functions are then defined as local restrictions of plurisubharmonic functions on $\mathbb{C}^N$ under local embeddings of X into $\mathbb{C}^N$. However, by a basic result by Fornaess-Narasimhan [27, Theorem 5.3.1] we know that weakly plurisubharmonic functions are the same as plurisubharmonic functions.

We have the following important Riemann extension theorem for plurisubharmonic functions (see also [18, Theorem 5.24]):

THEOREM 2.3 ([34, Satz 3, Section 1.7]). – *Let X be a normal complex space and D be a proper subvariety of X with* $\operatorname{codim}_X D \geq 1$. *Assume that ψ° is a plurisubharmonic function on $X \setminus D$ and that for each point $x \in D$ there exists a neighborhood U such that ψ° is bounded from above on $U \setminus (U \cap D)$. Then ψ° extends uniquely to a plurisubharmonic function over X.*

This theorem generalizes the following result which is useful for us too:

THEOREM 2.4 (Brelot, Grauert-Remmert [34, Satz 5, Section 2.1]).
Assume ψ° is a subharmonic function on $\Delta \setminus \{0\}$ such that ψ is bounded from above in a neighborhood of 0, then the following function is the unique subharmonic extension of ψ on Δ:

$$\psi(z) = \begin{cases} \psi^\circ(z) & \text{for } z \neq 0, \\ \overline{\lim_{z \to 0}}\, \psi^\circ(z) & \text{for } z = 0. \end{cases} \tag{1}$$

The following proposition is formulated in such a way that it can be applied to our setting of moduli spaces later, and we believe that it should hold under more general assumptions. See [33, Chapter IV] for the definition of seminormal complex spaces.

PROPOSITION 2.5. – *Let X be a seminormal complex space and $D \subset X$ be a proper subvariety such that* $\operatorname{codim}_X D \geq 1$ *and $X \setminus D$ is normal. Let ψ be a continuous function on X such that $\psi^\circ := \psi|_{X \setminus D}$ is plurisubharmonic. Then ψ is plurisubharmonic on X.*

Proof. – By Definition 2.1, we just need to show that for any holomorphic map $\tau : \Delta \to X$, the composition $\psi \circ \tau$ is subharmonic. Let $\nu : \tilde{X} \to X$ be the normalization and $\tilde{D} = \nu^{-1}(D)$. Then ν is an isomorphism outside $\tilde{D}$ and $\operatorname{codim}_{\tilde{X}} \tilde{D} \geq 1$. Moreover $\tilde{\psi} := \psi \circ \nu$ is a continuous function on $\tilde{X}$ such that $\tilde{\psi}$ is plurisubharmonic on $\tilde{X} \setminus \tilde{D} \cong X \setminus D$. Since $\tilde{X}$ is normal, it's easy to see that Theorem 2.3 implies that $\tilde{\psi}$ is plurisubharmonic on $\tilde{X}$. Let $\tilde{\tau} : \Delta \to \tilde{X}$ be a holomorphic lifting of τ. Then $\psi \circ \tau = \tilde{\psi} \circ \tilde{\tau}$ is indeed subharmonic. □

As in [32, Section 3.3], it's natural to make the following definition.

DEFINITION 2.6. – A closed positive (1,1)-current ω on X is by definition a closed positive (1,1)-current ω on X^{reg} such that for any $x \in X$, there exists an open neighborhood U of $x \in X$ and a plurisubharmonic function ψ on (the complex space) U such that $\omega|_{U \cap X^{\text{reg}}} = \sqrt{-1}\partial\bar{\partial}\left(\psi|_{U \cap X^{\text{reg}}}\right)$.

To globalize the above definitions and results, we consider a Hermitian line bundle (L, h) over X. We fix an open covering of $\{U_\alpha\}$ of X and choose generator l_α of $\mathcal{O}_X(U_\alpha)$. Then the Hermitian metric h is represented by a family of real valued functions $\{\psi_\alpha\}$ with $\|l_\alpha\|_h^2 = e^{-\psi_\alpha}$. We say that h is continuous if ψ_α is continuous for every α. We say h is smooth if $\psi_\alpha = \Psi_\alpha|_{U_\alpha}$ for some local embedding $U_\alpha \to \mathcal{U}_\alpha \subset \mathbb{C}^N$ and a smooth function Ψ_α on $\mathcal{U}_\alpha$. We say that (L, h) has a positive curvature current if Ψ_α is plurisubharmonic on U_α for every α. In this case, we define the Chern curvature current of (L, h) by

$$c_1(L, h) := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi_\alpha|_{X^{\text{reg}}}.$$

It is easy to verify this is a well-defined positive current in the sense of Definition 2.6.

3. Consequence of partial C^0-estimate

Let $\pi : \mathcal{X} \to S$ be a family of smoothable K-polystable Fano variety over a complex space S. The consideration here is local in S and so we will assume S is an affine variety in this section. For any $t \in S$, denote by $\mathcal{X}_t = \pi^{-1}\{t\}$ the fiber above t and by $K_{\mathcal{X}_t}$ its canonical $\mathbb{Q}$-line bundle. Let $(\omega_t, h_t) := (\omega_{\mathrm{KE}}(t), h_{\mathrm{KE}}(t))$ be a Kähler-Einstein metrics on $(\mathcal{X}_t, K^{-1}_{\mathcal{X}_t})$. Notice that, over $t \in S^\circ$ we could assume that ω_t varies smoothly with respect to t by using the slice theorem as shown in [22, 57]. But in fact our following argument will depend only on the Gromov-Hausdorff continuity of $(\mathcal{X}_t, \omega_t)$, and the continuous slice of Kähler-Einstein metrics and associated embeddings constructed in [41] can be applied. For any integer $m > 0$, we choose $\{\widetilde{s}_i\}_{i=1}^{N_m}$ to be a fixed basis of the locally free $\mathcal{O}_S$-module $\pi_*\mathcal{O}_{\mathcal{X}}(-mK_{\mathcal{X}})$, and denote $\widetilde{s}_i(t) = \widetilde{s}_i|_{\mathcal{X}_t}$. By [25] and [63] (see also [41, Lemma 8.3]), there exists $m_0 = m_0(n) > 0$, such that for any $m \geq m_0$, we can embed $\widetilde{\iota}_t : \mathcal{X}_t \hookrightarrow \mathbb{P}^{N_m-1}$ using $\{\widetilde{s}_i(t)\}$ such that $\iota_t^* H_i = \widetilde{s}_i$ where H_i's are coordinate hyperplane sections of $\mathbb{P}^{N_m-1}$. We will fix this identification of $\mathbb{P}\left(H^0(\mathcal{X}_t, K^{-m}_{\mathcal{X}_t})^*\right)$ and $\mathbb{P}^{N_m-1}$ from now on. We denote the pull back of the Fubini-Study metric by:

$$\widetilde{\omega}_t := \frac{1}{m}\widetilde{\iota}_t^*\omega_{\mathrm{FS}} = \frac{1}{m}\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\sum_{i=1}^{N_m}|\widetilde{s}_i(t)|^2. \tag{2}$$

Here the right hand side means that if we choose $e \in \mathcal{O}(K^{-m}_{\mathcal{X}/S})$ to be any local generator and denote $e_t = e|_{\mathcal{X}_t}$, then the following is well defined:

$$\frac{1}{m}\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\sum_{i=1}^{N_m}|\widetilde{s}_i(t)|^2 = -\frac{1}{m}\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\frac{|e_t|^2}{\sum|\widetilde{s}_i(t)|^2}.$$

We now recall the definition of Bergman kernels and Bergman metrics of $(\mathcal{X}_t, \omega_t)$. The metrics (ω_t, h_t) induce an L^2-inner product on $H^0(\mathcal{X}_t, K^{-m}_{\mathcal{X}_t})$ as follows:

$$\langle s, s'\rangle_{L^2} = \int_{\mathcal{X}_t}\langle s, s'\rangle_{h_t^{\otimes m}}\omega_t^n.$$

We choose an orthonormal basis $\{s_i(t)\}$ of $(H^0(\mathcal{X}_t, K^{-m}_{\mathcal{X}_t}), \langle\cdot,\cdot\rangle_{L^2})$ and define the m-th *Bergman kernel* as follows:

$$\rho_m(t) := \rho^{(m)}_{\mathrm{KE}}(x,t) = \sum_{i=1}^{N_m}|s_i(t)|^2_{h_t^{\otimes m}}. \tag{3}$$

It is independent of the choice of the orthonormal basis. For the basis $\{\widetilde{s}_i\}_{i=1}^{N_m}$ we fixed at the beginning, let $A_{ij}(t) := \langle\widetilde{s}_i(t), \widetilde{s}_j(t)\rangle_{L^2}$. Then $\{s_i(t)\} := A^{-1/2}\{\widetilde{s}_i(t)\}$ is an orthonormal basis. Now we write

$$\iota_t := A(t)^{-1/2}\circ\widetilde{\iota}_t$$

with $\widetilde{\iota}_t : \mathscr{X}_t \hookrightarrow \mathbb{P}^{N_m-1}$ being the embedding given by $\{\widetilde{s}_i(t)\}$. Define the *Bergman metric* $\check{\omega}_t$ as the following:

$$\check{\omega}_t := \frac{1}{m}\iota_t^*\omega_{\mathrm{FS}} = \frac{1}{m}\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\sum_{i=1}^{N_m}|s_i(t)|^2 = -\frac{1}{m}\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\frac{|e_t|^2}{\sum|s_i(t)|^2}. \tag{4}$$

Then by (3) and (4), we see that ω_t and $\check{\omega}_t$ are related to each other via:

$$\check{\omega}_t = \frac{1}{m}\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\rho_m(t) - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log|e|^2_{h_t} = \omega_t + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(\frac{1}{m}\log\rho_m(t)\right).$$

In particular, the Kähler-Einstein metrics ω_t satisfies the following complex Monge-Ampère equation on $\mathscr{X}_t$:

$$\omega_t^n = \mathrm{V}\frac{\rho_m^{1/m}\Omega_{\mathscr{X}_t}}{\int_{\mathscr{X}_t}\rho_m^{1/m}\Omega_{\mathscr{X}_t}} := e^{-\mathfrak{u}}\Omega_{\mathscr{X}_t}, \tag{5}$$

where the volume form $\Omega_{\mathscr{X}_t}$ and the potential $\mathfrak{u}$ in (5) are given by:

$$\Omega_{\mathscr{X}_t} = \left(\sum_{i=1}^{N_m}|s_i(t)|^2\right)^{-1/m}, \quad \mathfrak{u} = -\log\left(\frac{\rho_m^{1/m}}{\int_{\mathscr{X}_t}\rho_m^{1/m}\Omega_{\mathscr{X}_t}}\right) - \log\mathrm{V}.$$

The right-hand-side of (5) has that form because we have that $\int_{\mathscr{X}_t}\omega_t^n = \mathrm{V} := (c_1(x))^n$ is a fixed constant. For the purpose of later estimates, we rewrite the right hand side of (5) into a form using the data from the original (holomorphic) data $\{\widetilde{s}_i\}$: $e^{-\mathfrak{u}}\Omega_{\mathscr{X}_t} = e^{-\widetilde{\mathfrak{u}}}\widetilde{\Omega}_{\mathscr{X}_t}$, such that

$$\omega_t^n = e^{-\widetilde{\mathfrak{u}}}\widetilde{\Omega}_{\mathscr{X}_t}, \tag{6}$$

where we have denoted:

$$\widetilde{\Omega}_{\mathscr{X}_t} = \left(\sum_{i=1}^{N_m}|\widetilde{s}_i(t)|^2\right)^{-1/m}, \quad \widetilde{\mathfrak{u}} = \mathfrak{u} - \log\frac{\Omega_{\mathscr{X}_t}}{\widetilde{\Omega}_{\mathscr{X}_t}} = -\log\left(\frac{\rho_m^{1/m}\frac{\Omega_{\mathscr{X}_t}}{\widetilde{\Omega}_{\mathscr{X}_t}}}{\int_{\mathscr{X}_t}\rho_m^{1/m}\frac{\Omega_{\mathscr{X}_t}}{\widetilde{\Omega}_{\mathscr{X}_t}}\widetilde{\Omega}_{\mathscr{X}_t}}\right) - \log\mathrm{V}.$$

Now denote by $\widetilde{\Omega} = \{\widetilde{\Omega}_{\mathscr{X}_t}\}$ (resp. $\Omega = \{\Omega_{\mathscr{X}_t}\}$) the family of volume forms on $\mathscr{X}_t$. Then $\widetilde{\Omega}$ (resp. Ω) defines a Hermitian metric on the relative anti-canonical line bundle $K_{\mathscr{X}/S}$. We denote its Chern curvature on the total space $\mathscr{X}$ by:

$$-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\widetilde{\Omega} = \widetilde{\omega} \quad \left(\text{resp. } -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\Omega = \check{\omega}\right). \tag{7}$$

Then clearly we have: $\widetilde{\omega}|_{\mathscr{X}_t} = \widetilde{\omega}_t$ (resp. $\check{\omega}|_{\mathscr{X}_t} = \check{\omega}_t$).

By the uniform convergence of Kähler-Einstein potential functions under Gromov-Hausdorff convergence (see [25, 63]), we know that for a fixed $t \in S$, the function $\mathfrak{u}$ and $\widetilde{\mathfrak{u}}$ are continuous on $\mathscr{X}_t$. The main result in this section is the following proposition.

PROPOSITION 3.1. – *We have that* $\mathfrak{u}$ *and* $\widetilde{\mathfrak{u}}$ *are continuous and uniformly bounded on* $\mathscr{X}$.

Proof. – By using Moser iteration, we know that $\log\rho_m$ is uniformly bounded from above by [59, (5.2)]. Tian's partial C^0-estimate states that $\log\rho_m$ is uniformly bounded from below. The partial C^0-estimate is now known to be true by the fundamental works of [25] and [63]. Moreover, by their proofs, we know that if $(\mathscr{X}_{t_i}, \omega_{t_i})$ Gromov-Hausdorff converges to $(\mathcal{X}_0, \omega_0)$, then in the ambient space $\mathbb{P}^{N_m-1}$, we have $\iota_{t_i}(\mathcal{X}_{t_i}) \to \iota_0(\mathcal{X}_0)$ and $\rho_m \circ \iota_{t_i} \to \rho_m \circ \iota_0$ uniformly. Now in [41] (see Lemma 2.2), we proved that $(\mathcal{X}_0, \omega_0)$ is indeed the unique Gromov-Hausdorff limit as $t \to 0$ independent of the chosen sequence $\{t_i\}$.

For any $(x,t) \in \mathcal{X}_t \subset \mathcal{X}$, denote $f(x,t) = \log \frac{\Omega_{\mathcal{X}_t}}{\widetilde{\Omega}_{\mathcal{X}_t}}$.

We now argue that $f(t) := f(x,t)$ is continuous. Indeed, note that

$$f(t) = \frac{1}{m} \log \frac{\sum_i |\widetilde{s_i} \circ \widetilde{\iota_t}|^2}{\sum_i |s_i \circ \iota_t|^2} = \frac{1}{m} \log \frac{\sum_i |\widetilde{s_i}(t)|^2}{\sum_{i,j} |A(t)_{ij}^{-1/2} \widetilde{s_j}(t)|^2}.$$

So we just need to show that $t \mapsto A(t)^{-1/2}$ is a continuous map from the base S to $GL^+(N_m, \mathbb{C})$. By [25] and [63] we can assume that $A(t)^{-1/2}\{\widetilde{s_i}(t)\} = \{s_i(t)\} \to \{s_i(0)\} = A(0)^{-1/2}\{\widetilde{s_i}(0)\}$ if $(\mathcal{X}_{t_i}, \omega_{t_i}) \to (\mathcal{X}_0, \omega_0)$ in the Gromov-Hausdorff topology. So $A(t_i)^{-1/2} \to A(0)^{-1/2}$ for such a sequence $\{t_i\}$. Again by Lemma 3.3 (see also [41]), this holds for any sequence $t_i \to 0$.

So we have proved the continuity of the function:

$$F(t) := F(x,t) = \frac{1}{m} \log \rho_m + \log \frac{\Omega_{\mathcal{X}_t}}{\widetilde{\Omega}_{\mathcal{X}_t}},$$

which enters into the expression of $\widetilde{\mathfrak{u}}(t)$:

$$\widetilde{\mathfrak{u}}(t) = -\log \left(\frac{e^{F(t)}}{\int_{\mathcal{X}_t} e^{F(t)} \widetilde{\Omega}_{\mathcal{X}_t}} \right) - \log \mathrm{V}.$$

The continuity of $\widetilde{\mathfrak{u}}(t)$ will then follow by similar arguments as in the proof of [39, Lemma 1] except that we need to use a more general convergence Lemma 7.1 in the Appendix, which may have independent interest. Indeed, as in [39, (28)] we can estimate:

$$\begin{aligned} \left| \int_{\mathcal{X}_t} e^{F(t)} \widetilde{\Omega}_{\mathcal{X}_t} - \int_{\mathcal{X}_0} e^{F(0)} \widetilde{\Omega}_{\mathcal{X}_0} \right| &\le \left| \int_{\mathcal{X}_t \setminus \mathcal{W}(\delta)} e^{F(t)} \widetilde{\Omega}_{\mathcal{X}_t} - \int_{\mathcal{X}_0 \setminus \mathcal{W}(\delta)} e^{F(0)} \widetilde{\Omega}_{\mathcal{X}_0} \right| \\ &\quad + \left| \int_{\mathcal{X}_t \cap \mathcal{W}(\delta)} e^{F(t)} \widetilde{\Omega}_{\mathcal{X}_t} - \int_{\mathcal{X}_0 \cap \mathcal{W}(\delta)} e^{F(0)} \widetilde{\Omega}_{\mathcal{X}_0} \right| \\ &\le \left| \int_{\mathcal{X}_t \setminus \mathcal{W}(\delta)} e^{F(t)} \widetilde{\Omega}_{\mathcal{X}_t} - \int_{\mathcal{X}_0 \setminus \mathcal{W}(\delta)} e^{F(0)} \widetilde{\Omega}_{\mathcal{X}_0} \right| \\ &\quad + e^{\|F(t)\|_{L^\infty}} \left(\int_{\mathcal{X}_t \cap \mathcal{W}(\delta)} \widetilde{\Omega}_{\mathcal{X}_t} + \int_{\mathcal{X}_0 \cap \mathcal{W}(\delta)} \widetilde{\Omega}_{\mathcal{X}_0} \right). \end{aligned} \tag{8}$$

Here $\mathcal{W}(\delta)$ denotes a small neighborhood of $\mathcal{X}^{\text{sing}}$ in the analytic topology such that $\lim_{\delta \to 0} \mathcal{W}(\delta) = \mathcal{X}^{\text{sing}}$ in Hausdorff topology of subsets of $\mathbb{P}^{N_m-1} \times S$. By the continuity of $F(x,t)$ and $\widetilde{\mathfrak{u}}(x,t)$ away from $\mathcal{X}^{\text{sing}}$, the first term on the right hand side of (8) can be arbitrarily small if t is sufficiently close to 0 for a fixed $\delta > 0$. On the other hand, by Lemma 7.1, we have

$$\lim_{t \to 0} \int_{\mathcal{X}_t \cap \mathcal{W}(\delta)} \widetilde{\Omega}_{\mathcal{X}_t} = \int_{\mathcal{X}_0 \cap \mathcal{W}(\delta)} \widetilde{\Omega}_{\mathcal{X}_0}. \tag{9}$$

Now by choosing δ sufficiently small, we can make the right hand side of (9) sufficiently small, and hence the left hand side of (9) can also be made sufficiently small as long as t is sufficiently close to 0. Combining the above estimates, we indeed see that $\widetilde{\mathfrak{u}}(t)$ is continuous as $t \to 0$. □

REMARK 3.2. – Constrast to the canonically polarized case studied in [53], in which the Aubin-Yau's C^0-estimate fails to be uniform near the boundary and hence there is no uniform lower bound for the Kähler potential of Weil-Petersson metric (see [53, 9]), in our case we will see that, as a consequence of Proposition 3.1, there are two sided bounds for the potential of the Weil-Petersson metric for the Fano case.

For the reader's convenience, we record the following uniqueness result from [41] and sketch its proof. When the automorphism groups are discrete, this is also proved in [56] using a different method.

LEMMA 3.3 ([41]). – *In Gromov-Hausdorff topology $\mathcal{X}_t \to \mathcal{X}_0$, equivalently $\mathcal{X}_{t_i} \to \mathcal{X}_0$ for any sequence $t_i \to 0$.*

Proof. – Assume $\mathcal{X}_{t_i} \to \mathcal{X}_0$ and $\mathcal{X}_{t_i'} \to \mathcal{X}_0'$ such that $\mathcal{X}_0 \neq \mathcal{X}_0'$. Without loss of generality, we can assume $|t_i| < |t_i'|$ where $|\cdot|$ is any continuous distance function to $0 \in S$. Then by the *Intermediate Value Type result* in [41, Lemma 6.9.(2)], there exists t_i'' such that $|t_i| < |t_i''| < |t_i'|$ such that $(\mathcal{X}_{t_i''}, \omega_{t_i''})$ converges in Gromov-Hausdorff topology to a Kähler-Einstein Fano variety (Y, ω_Y) as $t_i'' \to 0$, which satisfies:

$$\mathrm{Hilb}(\mathcal{X}_{t_i''}, \omega_{t_i''}) \to \mathrm{Hilb}(Y, \omega_Y) \in \left[\overline{O_{\mathrm{Hilb}(\mathcal{X}_0,\omega_0)}} \bigcup \left(\mathrm{GL}(N_m)\cdot(\mathcal{U}\cap\overline{O})\right)\right] \setminus O_{\mathrm{Hilb}(\mathcal{X}_0,\omega_0)},$$

where $\mathrm{Hilb}(\mathcal{X}_t, \omega_t)$ denotes the Hilbert point of $\iota_t(\mathcal{X}_t)$ as a subvariety of $\mathbb{P}^{N_m-1}$, and

$$O_{\mathrm{Hilb}(\mathcal{X}_0,\omega_0)} = \mathrm{GL}(N_m,\mathbb{C})\cdot \mathrm{Hilb}(\mathcal{X}_0,\omega_0), \quad \overline{O} = \lim_{t\to 0} \overline{O_{\mathrm{Hilb}(\mathcal{X}_t,\omega_t)}},$$

and $\mathcal{U}$ is an open set of $\mathrm{Hilb}(\mathcal{X}_0, \omega_0)$ constructed in [41, Lemma 3.1] using local Luna slice theorem. If $Y \in \mathrm{GL}(N_m,\mathbb{C})\cdot\left(\mathcal{U}\cap\overline{O}\right)$ then by [41, Lemma 3.1] there exists a special test configuration of Y to $\mathcal{X}_0$. This contradicts to Y being K-polystable (see [7]). So we must have $Y \in \overline{O_{\mathrm{Hilb}(\mathcal{X}_0,\omega_0)}} \setminus O_{\mathrm{Hilb}(\mathcal{X}_0,\omega_0)}$. However, this implies again that there is a special test configuration of $\mathcal{X}_0$ to Y by [23], which contradicts the fact that $\mathcal{X}_0$ is K-polystable. □

4. Canonical metric on the CM line bundles

In this section, we assume that $\pi : \mathcal{X} \to S$ is a family of Kähler-Einstein Fano varieties over a *smooth complex manifold* (see Remark 4.16) S such that the generic fiber is smooth. For each $t \in S$, denote by ω_t the Kähler-Einstein metrics on $\mathcal{X}_t = \pi^{-1}(t)$. Let $A \subset S$ be the analytic set parametrizing singular $\mathbb{Q}$-Fano varieties (which are smoothable and K-polystable). Denote $S^\circ = S \setminus A$ and $\mathcal{X}^\circ = \pi^{-1}(S^\circ)$. By [22, Section 5.3] and [57], we can assume ω_t varies smoothly on $\mathcal{X}^\circ$.

4.1. Preliminaries

4.1.1. *Weil-Petersson metric and CM line bundle on the smooth locus.* – We know that on the open sub-space $S^\circ = S \setminus A$ there is a well defined Weil-Petersson metric $\omega_{\mathrm{WP}}^\circ$. Let's briefly recall its definition and refer to [28] for detailed discussions. Fix any $t \in S^\circ$. We denote by $\mathcal{T}\mathcal{X}_t$ the holomorphic tangent sheaf/bundle of $\mathcal{X}_t$ and by $\mu_t : T_t S \to H^1(\mathcal{X}_t, \mathcal{T}\mathcal{X}_t)$ the Kodaira-Spencer map associated to the family $\mathcal{X}^\circ \to S^\circ$.

We denote by $A^{0,k}(X, \mathcal{T}\mathcal{X}_t)(k \geq 0)$ the space of smooth $\mathcal{T}\mathcal{X}_t$-valued $(0,k)$-forms on $\mathcal{X}_t$. Then the Kähler-Einstein metric ω_t induces L^2 inner products:

$$(\theta_1, \theta_2)_{L^2_t} = \int_{\mathcal{X}_t} \langle \theta_1, \theta_2 \rangle_{\omega_t} \omega_t^n, \text{ for any } \theta_1, \theta_2 \in A^{0,k}(\mathcal{X}_t, \mathcal{T}\mathcal{X}_t)$$

where $\langle \cdot, \cdot \rangle_{\omega_t}$ is the induced inner product on $\mathcal{T}\mathcal{X}_t \otimes T^{*(0,k)}\mathcal{X}_t$. Then we can define the L^2_t-adjoint $\bar{\partial}^*$ of the operator $\bar{\partial} : A^{0,1}(X, \mathcal{T}\mathcal{X}_t) \to A^{0,2}(X, \mathcal{T}\mathcal{X}_t)$ and then the Laplacian operator $\Box_t = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$ on $A^{0,1}(\mathcal{X}_t, \mathcal{T}\mathcal{X}_t)$. For each $[\theta] \in H^1(\mathcal{X}_t, \mathcal{T}\mathcal{X}_t)$ we denote by θ_{H} the unique harmonic representative of $[\theta]$. For any $v, v' \in T_t S^\circ$, we then define the Weil-Petersson metric by the following formula:

$$\omega^\circ_{\mathrm{WP}}(v, v') = (\mu_t(v)_{\mathrm{H}}, \mu_t(v')_{\mathrm{H}})_{L^2_t}.$$

We say that a tangent direction $v \in T_t S$ is *effective* if $\mu_t(v) \neq 0$. Then by its definition, ω_{WP} is a semipositive smooth form (possibly degenerate metric) over S° and is positive definite along directions of effective infinitesimal deformations. We have the following formula (see also Section 6.2):

Theorem 4.1 ([28, Theorem 7.9]). – *The metric $\omega^\circ_{\mathrm{WP}}$ has the following representation using the fiber integral:*

$$\omega^\circ_{\mathrm{WP}} = -\int_{\mathcal{X}^\circ/S^\circ} \omega^{n+1}_{\mathcal{X}^\circ}. \tag{10}$$

Here we have denoted

$$\omega_{\mathcal{X}^\circ} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\{\omega_t^n\} \tag{11}$$

where the family of volume forms $\{\omega_t^n\}$ is regarded as an Hermitian metric on $-K_{\mathcal{X}^\circ/S^\circ}$.

As mentioned in the introduction, there is a determinant line bundle equipped with a Hermitian metric whose curvature is equal to the Weil-Petersson metric.

Definition 4.2 ([60]). – Let $\pi : \mathcal{X} \to S$ be a flat family of $\mathbb{Q}$-Fano varieties such that $mK_{\mathcal{X}/S}$ is Cartier for some integer m. We define the CM $\mathbb{Q}$-line bundle $\lambda_{\mathrm{CM}} = \lambda_{\mathrm{CM}}(S)$ on S as the determinant line bundle associated to the push-forward of a virtual $\mathbb{Q}$-line bundle (in the sense of Grothendieck):

$$\frac{1}{2^{n+1}m^{n+1}} \det\left[\pi_!\left(-(K^{-m}_{\mathcal{X}/S} - K^{m}_{\mathcal{X}/S})^{n+1}\right)\right]. \tag{12}$$

Remark 4.3. – In the following if it's clear from the context we will just write line bundle instead of $\mathbb{Q}$-line bundle for convenience. Equivalently, we can define the CM-line bundle using Knudsen-Mumford expansion (see [50], [49]):

$$\det\left(\pi_*\left(K^{-mr}_{\mathcal{X}/S}\right)\right) = -\lambda_{\mathrm{CM}}\frac{(mr)^{n+1}}{(n+1)!} + O(r^n).$$

By Grothendieck-Riemann-Roch theorem, the first Chern class of $\lambda_{\mathrm{CM}}(S)$ is given by the formula:

$$
\begin{aligned}
c_1(\lambda_{\mathrm{CM}}) &= \frac{1}{2^{n+1}m^{n+1}}\pi_*\left[Ch\left(-(K^{-m}_{\mathcal{X}/S}-K^{m}_{\mathcal{X}/S})^{n+1}\right)Td(\mathcal{X}/S)\right]_{(2)}\\
&= \pi_*\left(-c_1(K^{-1}_{\mathcal{X}/S})^{n+1}\right). \qquad (13)
\end{aligned}
$$

By applying the fundamental works of Bismut-Gillet-Soulé in [11], Fujiki-Schumacher showed that the above identity also holds at the curvature level, at least for the family over S°:

THEOREM 4.4 ([11, Theorem 0.1], [28, Section 10]). – *There is a Quillen metric h°_{QM} on $\lambda_{\mathrm{CM}}|_{S^\circ}$ such that*

$$
-\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log h^\circ_{\mathrm{QM}}=\omega^\circ_{\mathrm{WP}}.
$$

It's natural to expect that h°_{QM} in Theorem 4.4 extends to h_{QM} on λ_{CM} over S. For this purpose, one needs to study the behavior of the Quillen metric near $A=S\setminus S^\circ$ which a priori is difficult (see e.g., [67]). In [17] Deligne proposed a program to calculate the Quillen metric (or equivalently the analytic torsion) for general determinant line bundle of cohomology. The (metrized) Deligne pairing in the next sub-section is an example of his approach.

4.1.2. *Deligne pairing with Hermitian metrics.* – Let's first recall the definition of Deligne pairings following [17, 68]. Let $\pi:\mathcal{X}\to S$ be a flat and projective morphism of integral schemes of pure relative dimension n. For any $(n+1)$-tuples of line bundles $\{\mathcal{L}_0,\dots,\mathcal{L}_n\}$ on $\mathcal{X}$, Deligne [17] defined a line bundle on S, which is denoted by $\langle\mathcal{L}_0,\dots,\mathcal{L}_n\rangle$ or $\langle\mathcal{L}_0,\dots,\mathcal{L}_n\rangle(\mathcal{X}/S)$. If S is just one point and $\mathcal{X}=X$, then $\langle\mathcal{L}_0,\dots,\mathcal{L}_n\rangle$ is a one-dimensional complex vector space generated by the symbol $\langle l_0,\dots,l_n\rangle$ (also denote as $\langle l_0,\dots,l_n\rangle(X)$) where l_i are meromorphic sections whose divisors have empty intersections, with the following relations satisfied. For some $0\le i\le N$ and a meromorphic function f on $\mathcal{X}$, if the intersection $\bigcap_{j\neq i}\mathrm{div}(l_j)=\sum_\alpha n_\alpha P_\alpha$ is a 0-cycle and has empty intersection with $\mathrm{div}(f)$, then

$$
\langle l_0,\dots,fl_i,\dots\dots,\dots,l_n\rangle=\prod_\alpha f(P_\alpha)^{n_\alpha}\cdot\langle l_0,\dots,l_n\rangle.
$$

Now assume each $\mathcal{L}_i$ has a smooth Hermitian metric h_i. Then one can define a metric on $\langle\mathcal{L}_0,\dots,\mathcal{L}_n\rangle$ as follows. For each $0\le i\le n$, let $c_1'(\mathcal{L}_i)=\frac{1}{2\pi\sqrt{-1}}\partial\bar\partial\log h_i$ denote the Chern curvature of $(\mathcal{L}_i,h_i)$. Then we define inductively (see [17, 8.3.2]):

$$
\log\|\langle l_0,\dots,l_n\rangle(X)\|^2=\log\|\langle l_0,\dots,l_{n-1}\rangle(\mathrm{div}\, l_n)\|^2+\int_X\log\|l_n\|^2\bigwedge_{i=0}^{n-1}c_1'(\mathcal{L}_j). \qquad (14)
$$

The above construction can then be generalized to the case of a flat family. In that case, the local generator of $\langle\mathcal{L}_0,\dots,\mathcal{L}_n\rangle(\mathcal{X}/S)$ over any Zariski open set U of S are symbols of the form $\langle l_0,\dots,l_n\rangle$ where $l_i's$ are meromorphic sections of $\mathcal{L}_i$ over $\pi^{-1}(U)$ such that

$\bigcap_{i=0}^n \mathrm{div}(l_i) = \emptyset$. (14) becomes the following induction formula for metrized Deligne pairing ([68, (1.2.1)]):
(15)

$$\langle \mathcal{L}_0, \dots, \mathcal{L}_n \rangle(\mathcal{X}/S) = \langle \mathcal{L}_0, \dots, \mathcal{L}_{n-1} \rangle(\mathrm{div}(l_n)/S) \otimes \mathcal{O}\left(-\int_{\mathcal{X}/S} \log \|l_n\|^2 \bigwedge_{i=0}^{n-1} c_1'(\mathcal{L}_i) \right),$$

where we assume each component of $\mathrm{div}(l_n)$ is flat over S (which can be achieved by choosing l_i to be general sections), and $\mathcal{O}(\phi)$ denotes the trivial line bundle over S with metric $\|1\|^2 = \exp(-\phi)$. We will need the following regularity result.

Theorem 4.5 ([68], [43]). – *Suppose h_i are smooth metrics on $\mathcal{L}_i (1 \leq i \leq n+1)$. Then Deligne's metric is continuous on $\langle \mathcal{L}_1, \dots, \mathcal{L}_{n+1} \rangle$.*

Deligne's metric is important for us because its curvature is given by the appropriate fiber integral.

Theorem 4.6 ([17, Proposition 8.5]). – *The following curvature formula holds for Deligne's metric:*

$$c_1'\left(\langle \mathcal{L}_0, \dots, \mathcal{L}_n \rangle\right) = \int_{\mathcal{X}/S} c_1'(\mathcal{L}_0) \wedge \cdots \wedge c_1'(\mathcal{L}_n). \tag{16}$$

Remark 4.7. – Notice that the right-hand-side is well defined. See for example [65, Section 3.4]. The regularity result in Theorem 4.5 is also related to the results in [65]. See Proposition 3.4.1, Theorem 2 and Theorem 3 in [65].

Using the inductive formula in (15), we immediately get:

$$\langle \mathcal{L}_0, \dots, \mathcal{L}_n \otimes \mathcal{O}(\phi) \rangle = \langle \mathcal{L}_0, \dots, \mathcal{L}_n \rangle \otimes \mathcal{O}\left(\int_{\mathcal{X}/S} \phi \bigwedge_{i=0}^{n-1} c_1'(\mathcal{L}_i) \right).$$

Since the Deligne pairing is symmetric, we get the following *change of metric formula* (see [51, (2.8)], [61]):

$$\begin{aligned}&\langle \mathcal{L}_0 \otimes \mathcal{O}(\phi_0), \dots, \mathcal{L}_n \otimes \mathcal{O}(\phi_k) \rangle \\ &\qquad = \langle \mathcal{L}_0, \dots, \mathcal{L}_n \rangle \otimes \mathcal{O}\left(\int_{\mathcal{X}/S} \sum_{j=0}^n \phi_j \bigwedge_{k<j} c_1'(\mathcal{L}_k \otimes \mathcal{O}(\phi_k)) \bigwedge_{l>j} c_1'(\mathcal{L}_l) \right).\end{aligned}$$

In particular, if $\mathcal{L}_i = \mathcal{L}$ and $\phi_i = \phi$ are the same, then we have:

$$\left(\mathcal{L} \otimes \mathcal{O}(\phi)\right)^{\langle n+1 \rangle} = \mathcal{L}^{\langle n+1 \rangle} \otimes \mathcal{O}\left(\sum_{j=0}^n \int_{\mathcal{X}/S} \phi\, c_1'(\mathcal{L} \otimes \mathcal{O}(\phi))^{n-j} \wedge c_1'(\mathcal{L})^j \right). \tag{17}$$

Here we have denoted:

$$\mathcal{L}^{\langle n+1 \rangle} = \langle \overbrace{\mathcal{L}, \dots, \mathcal{L}}^{(n+1)\text{ times}} \rangle.$$

More generally if $\mathcal{L}$ is a $\mathbb{Q}$-line bundle and $m\mathcal{L}$ is a genuine line bundle for $m > 0 \in \mathbb{Z}$, we will denote by $\mathcal{L}^{\langle n+1 \rangle}$ the following $\mathbb{Q}$-line bundle:

$$\mathcal{L}^{\langle n+1 \rangle} := \left((m\mathcal{L})^{\langle n+1 \rangle} \right)^{\frac{1}{m^{n+1}}}. \tag{18}$$

We need the following simple but useful lemma.

LEMMA 4.8. – *For any base change $g : S' \to S$, denote the pull-back family by $(g^*\mathcal{X}, g^*\mathcal{L}) := (\mathcal{X}, \mathcal{L}) \times_{g,S} S'$ where $g^*\mathcal{L}$ is endowed with the pull-back metric. Then we have an isometric isomorphism: $\left(g^*\mathcal{L}\right)^{\langle n+1\rangle} \cong g^*\left(\mathcal{L}^{\langle n+1\rangle}\right)$. In particular, if the group G acts equivariantly on $(\mathcal{X}/S, \mathcal{L})$, then for all $\sigma \in G$, we have an isometry: $\sigma^*\mathcal{L}^{\langle n+1\rangle} \cong (\sigma^*\mathcal{L})^{\langle n+1\rangle}$.*

Proof. – This follows from the above functorial construction of metrized Deligne pairings (see [17, 8]). For any Zariski open set $U \subset S$, let $l_i, i = 0, \dots, n$ be meromorphic sections of $\mathcal{L}$ over $\pi^{-1}(U)$ satisfying $\bigcap_{i=0}^n \mathrm{div}(l_i) = \emptyset$ so that $\langle l_0, \dots, l_n\rangle$ is a local generator of $\mathcal{L}^{\langle n+1\rangle}$ over U. Then g^*l_i are meromorphic sections of $g^*\mathcal{L}$ over $\pi'^{-1}(g^{-1}(U))$ where $\pi' : g^*\mathcal{X} = \mathcal{X} \times_{g,S} S' \to S'$ is the induced projection. Then $g^*\langle l_0, \dots, l_n\rangle := \langle g^*l_0, \dots, g^*l_n\rangle$ is a local generator of $(g^*\mathcal{L})^{\langle n+1\rangle}$ over $g^{-1}(U)$. Furthermore, the Deligne metrics on $\mathcal{L}^{\langle n+1\rangle}$ and $(g^*\mathcal{L})^{\langle n+1\rangle}$ are compatible under g^* by using the inductive formula (14). □

REMARK 4.9. – One could also get the statement in the above lemma by using the functoriality of the constructions of determinant line bundles and Quillen metrics, at least in the case that $\mathcal{X} \to S$ is a smooth fibration.

4.2. Canonical continuous hermitian metric on the Deligne pairing

We can apply the Deligne pairing to study the CM line bundle, because it's known that:

THEOREM 4.10 ([17, 50, 49, 7]). – *Using the definition in* (18)*, we have the identity:*

$$\lambda_{\mathrm{CM}} = -\left(K_{\mathcal{X}/S}^{-1}\right)^{\langle n+1\rangle}.$$

Notice that by (13) and (16) the curvatures on both sides are the same. In this section, we will use the formalism of Deligne pairings to prove the following result, which is motivated by [53, Theorem 4].

THEOREM 4.11. – *There exists a continuous metric h_{DP} on λ_{CM} over S such that*

1. $\omega_{\mathrm{WP}} := -\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log h_{\mathrm{DP}}$ *is a positive current.*
2. $\omega_{\mathrm{WP}}|_{S^\circ} = \omega^\circ_{\mathrm{WP}}$.

Proof of Theorem 4.11. – Recall that $\widetilde\Omega = \{\widetilde\Omega_{\mathcal{X}_t}\}$ defines a smooth metric on $K_{\mathcal{X}/S}^{-1}$ which is nothing but the pull back of the Fubini-Study metric by a holomorphic family of embeddings (see Section 3).

By Theorem 4.5, the associated metric $\widetilde h_{\mathrm{DP}}$ on $\lambda_{\mathrm{CM}} = -\left(K_{\mathcal{X}/S}^{-1}\right)^{\langle n+1\rangle}$ is continuous. From now on we choose an open covering $\{\mathcal{U}_\alpha\}$ of S and generators l_α of $\mathcal{O}_S(\lambda_{\mathrm{CM}})(\mathcal{U}_\alpha)$, such that

$$\|l_\alpha\|^2_{\widetilde h_{\mathrm{DP}}} = e^{-\widetilde\Psi_\alpha},$$

where $\widetilde\Psi_\alpha$ is a continuous function on $\mathcal{U}_\alpha$. Moreover, by curvature formula in (16) we have:

$$\frac{\sqrt{-1}}{2\pi}\partial\bar\partial\widetilde\Psi_\alpha = -\int_{\mathcal{X}/S}\widetilde\omega^{n+1} =: \widetilde\omega_S. \tag{19}$$

Notice that $\widetilde{\omega}_S$ is however not known to be a positive Kähler form. Next we bring in the Kähler-Einstein metrics $\{\omega_t\}$. By (6) we have:

$$\omega_t^n = e^{-\widetilde{\mathfrak{u}}}\widetilde{\Omega}_{\mathcal{X}_t}. \tag{20}$$

$\{\omega_t^n\}$ (resp. $\widetilde{\Omega}$) defines a continuous (resp. smooth) metric on $K^{-1}_{\mathcal{X}/S}$. So by the *change of metric formula* in (17), we define:

$$h_{\mathrm{DP}} = \widetilde{h}_{\mathrm{DP}} \cdot e^{-\mathfrak{U}} \tag{21}$$

where

$$\mathfrak{U} = -\sum_{j=1}^{n} \int_{\mathcal{X}/S} \widetilde{\mathfrak{u}}\, \widetilde{\omega}_t^j \wedge \omega_t^{n-j}. \tag{22}$$

Over each fiber $\mathcal{X}_t$, $\widetilde{\omega}_t$ is a smooth Kähler metric (pull back of Fubini-Study) and $\omega_t = \widetilde{\omega}_t + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\widetilde{\mathfrak{u}}$ is positive current with continuous potentials. So by pluripotential theory $\mathfrak{U}$ is a well defined function on S. Moreover, by Lemma 3.1, we know that $\widetilde{\mathfrak{u}}$ is continuous and uniformly bounded on $\mathcal{X}$. Then we can show that $\mathfrak{U}$ in (22) is continuous and uniformly bounded by the following Lemma 4.13 (see [39, 56]). So we conclude that h_{DP} is a continuous Hermitian metric on λ_{CM}.

Next we look at these data at the curvature level. By taking the curvature on both sides of (20), we have:

$$\omega_{\mathcal{X}} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\{\omega_t^n\} = \widetilde{\omega} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\widetilde{\mathfrak{u}}. \tag{23}$$

From (23) we see that on $\mathcal{X}^\circ$ the following identity holds:

$$\omega_{\mathcal{X}^\circ}^{n+1} = \widetilde{\omega}^{n+1} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(\widetilde{\mathfrak{u}}\sum_{j=0}^{n}\widetilde{\omega}^j \wedge \left(\widetilde{\omega} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\widetilde{\mathfrak{u}}\right)^{n-j}\right).$$

Since over $\mathcal{X}^\circ$ both $\omega_{\mathcal{X}^\circ}$ and $\widetilde{\omega}$ are smooth (1,1)-forms and $\widetilde{\mathfrak{u}}$ is a smooth function, we can do fiber integrals to get:

$$-\int_{\mathcal{X}^\circ/S^\circ} \omega_{\mathcal{X}^\circ}^{n+1} = -\int_{\mathcal{X}^\circ/S^\circ} \widetilde{\omega}^{n+1} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\mathfrak{U}, \tag{24}$$

where $\mathfrak{U}$ was defined in (22). By (10) and (19), we see that the above identity is equivalent to:

$$\omega_{\mathrm{WP}}^\circ = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\widetilde{h}_{\mathrm{DP}} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\mathfrak{U} = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h_{\mathrm{DP}}|_{S^\circ}.$$

Note that $\omega_{\mathrm{WP}}^\circ$ is a positive form over S°. Now the conclusion of the theorem follows from Lemma 4.14. ☐

REMARK 4.12. – One may like to try to extend $\omega_{\mathrm{WP}}^\circ$ to S by defining

$$\omega_{\mathrm{WP}} = -\int_{\mathcal{X}/S} \omega_{\mathcal{X}}^{n+1}. \tag{25}$$

The problem is that we need to verify that this ω_{WP} in (25) is well-defined. There are several technical difficulties. For one thing, $\omega_{\mathcal{X}}$ can not be a positive current because we expect ω_{WP} to be positive. So this would prevent us to define the "Monge-Ampère measure" $\omega_{\mathcal{X}}^{n+1}$. Even if we could define this wedge product, we still need to make sense of the fiber integral, for which the work in [65] may be helpful. To get around these difficulties, we proved the above result by combining the continuity and the uniqueness of plurisubharmonic extension.

LEMMA 4.13. – *The function $\mathfrak{U}$ is continuous and uniformly bounded on S.*

Proof. – The argument to prove this was known by [39] (see also [7] and [56]). For the reader's convenience, we briefly sketch the proof and refer to [39] and [56] for more details. We can estimate in the way similar to (8),

$$\left|\int_{\mathcal{X}_t} \widetilde{\mathfrak{u}}\widetilde{\omega}_t^j \wedge \omega_t^{n-j} - \int_{\mathcal{X}_0} \widetilde{\mathfrak{u}}\widetilde{\omega}_0^j \wedge \omega_0^{n-j}\right| \leq \left|\int_{\mathcal{X}_t\setminus\mathcal{W}(\delta)} \widetilde{\mathfrak{u}}\widetilde{\omega}_t^j \wedge \omega_t^{n-j} - \int_{\mathcal{X}_0\setminus\mathcal{W}(\delta)} \widetilde{\mathfrak{u}}\widetilde{\omega}_0^j \wedge \omega_0^{n-j}\right| + \|\widetilde{\mathfrak{u}}\|_{L^\infty}\left(\int_{\mathcal{X}_t\cap\mathcal{W}(\delta)} \widetilde{\omega}_t^j \wedge \omega_t^{n-j} + \int_{\mathcal{X}_0\cap\mathcal{W}(\delta)} \widetilde{\omega}_0^j \wedge \omega_0^{n-j}\right).$$

As in (8), $\mathcal{W}(\delta)$ is a sufficiently small neighborhood of $\mathcal{X}^{\mathrm{sing}}$. $\|\widetilde{\mathfrak{u}}\|_{L^\infty}$ is finite because of Proposition 3.1.

Now to estimate the first term, we can choose a partition of unity $\{\mathcal{V}_\alpha, \rho_\alpha\}$ of $\mathcal{X} \setminus \mathcal{W}(\delta)$ such that $\pi : \mathcal{V}_\alpha \to S$ is a local fibration. Note that both $\widetilde{\omega}_t$ and $\omega_t = \widetilde{\omega}_t + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\widetilde{\mathfrak{u}}|_{\mathcal{X}_t}$ are positive currents with *continuous* potentials, and $\widetilde{\mathfrak{u}}$ is continuous by Lemma 3.1. So by convergence of Monge-Ampère measures (see [18, Corollary 3.6, Chapter 3]), we can show that

$$\lim_{t\to 0}\int_{\mathcal{V}_\alpha\cap\mathcal{X}_t} \rho_\alpha\widetilde{\mathfrak{u}}\,\widetilde{\omega}_t^j \wedge \omega_t^{n-j} = \int_{\mathcal{V}_\alpha\cap\mathcal{X}_0} \rho_\alpha\widetilde{\mathfrak{u}}\,\widetilde{\omega}_0^j \wedge \omega_0^{n-j}. \tag{26}$$

By patching together the convergence (26) on all $\mathcal{V}_\alpha$, we see that the first term approaches 0 as $t \to 0$ for any fixed δ. Note that when we write $t \to 0$, we mean the limit holds for any sequence $t_i \to 0$. Because $\mathcal{X}^{\mathrm{sing}}$ is a pluripolar set, it's immediate that for any $\epsilon > 0$, there exists $0 < \delta \ll 1$ such that

$$\int_{\mathcal{X}_0\cap\mathcal{W}(\delta)} \widetilde{\omega}_0^j \wedge \omega_0^{n-j} \leq \epsilon. \tag{27}$$

Lastly, to estimate the first term in the bracket, we can use the following trick (see [56] and [39]):

$$\int_{\mathcal{X}_t\cap\mathcal{W}(\delta)} = \left(\int_{\mathcal{X}_t} - \int_{\mathcal{X}_t\setminus\mathcal{W}(\delta)}\right). \tag{28}$$

Now $\int_{\mathcal{X}_t} \widetilde{\omega}_t^j \wedge \omega_t^{n-j} = K_{\mathcal{X}_t}^{-n} = K_{\mathcal{X}_0}^{-n} = \int_{\mathcal{X}_0} \widetilde{\omega}_0^j \wedge \omega_0^{n-j}$ is a constant independent of t and

$$\lim_{t\to 0}\int_{\mathcal{X}_t\setminus\mathcal{W}(\delta)} \widetilde{\omega}_t^j \wedge \omega_t^{n-j} = \int_{\mathcal{X}_0\setminus\mathcal{W}(\delta)} \widetilde{\omega}_0^j \wedge \omega_0^{n-j} \tag{29}$$

using the similar reasoning in (26). Then by combining (26)-(29), we show that the first term in the bracket can indeed be made arbitrarily small as long as t and δ are sufficiently small. So we are done. □

LEMMA 4.14. – *Let S be a smooth complex manifold, and $(L, h) \to S$ be a line bundle with a continuous Hermitian metric h. Let $A \subset S$ be a proper subvariety. Denote by $c_1(L, h)$ the curvature current of h. If $c_1(L, h)^\circ := c_1(L, h)|_{S^\circ}$ is a positive current over $S^\circ = S \setminus A$, then $c_1(L, h)$ is a positive current over the whole S.*

Proof. – Choose a covering $\{\mathscr{U}_\alpha\}$ of S. The metric h is locally represented by a positive function $\exp(-\Psi_\alpha)$ over $\mathscr{U}_\alpha$. From the assumption, we know that Ψ_α is continuous (in particular locally uniformly bounded) on $\mathscr{U}_\alpha$ and plurisubharmonic on $\mathscr{U}_\alpha \setminus S$. By Theorem 2.3 (see also [18, Theorem 5.23, 5.24, Chapter 1]) we know that $\Psi_\alpha^\circ := \Psi_\alpha|_{\mathscr{U}_\alpha \setminus A}$ extends *uniquely* across A to become a plurisubharmonic function on $\mathscr{U}_\alpha$. Because Ψ_α is continuous, this extension must coincide with Ψ_α itself, as can be seen by restricting to any analytic curve and using the expression (1) in Theorem 2.4. So we see that Ψ_α is plurisubharmonic on $\mathscr{U}_\alpha$. Using the Definition 2.6 in Section 2, we immediately transform this into the statement of the lemma. □

PROPOSITION 4.15. – *CM-line bundle λ_{CM} is nef over S. Moreover, if S is proper such that $\mathscr{X}^\circ \to S^\circ$ is a* generically effective *family, then $\int_S c_1(\Lambda_{\mathrm{CM}})^{\dim S} > 0$.*

Proof. – We have $\lambda_{\mathrm{CM}} \cdot C = \int_C \omega_{\mathrm{WP}} \geq 0$ for any curve $C \subset S$. Because ω_{WP} has continuous bounded potentials, by [4] the Monge-Ampère measure and hence the integral is well defined. The last statement is true because $\omega_{\mathrm{WP}}|_{S^\circ} = \omega_{\mathrm{WP}}^\circ$ is positive definite along effective directions. □

We conclude this section with the following remark.

REMARK 4.16. – The results in this section still hold true when we just assume S is a normal complex space. To see this, we choose a resolution of singularities $f : \widetilde{S} \to S$ and consider the induced family of K-polystable Fano varieties $\widetilde{\mathscr{X}} := \mathscr{X} \times_S \widetilde{S}$. Then we can carry out the above constructions for the new family $\widetilde{\mathscr{X}} \to \widetilde{S}$. Because CM line bundle is functorial, we have $f^*\lambda_{\mathrm{CM}}(S) = \lambda_{\mathrm{CM}}(\widetilde{S})$. For the data of positively curved metrics, because the fiber of f is connected by Zariski's main theorem, it's easy to see that the data obtained over $\widetilde{S}$ naturally descends to S.

5. Descending CM line bundle and Deligne metrics to the Moduli space

In this section, we will construct the descendent of CM line bundle on $\overline{\mathscr{M}}$, the proper moduli space constructed in [41]. Following [41, Section 8], let us introduce the following parameter space with one minor difference, that is, we will work with Hilbert schemes instead of Chow variety.

DEFINITION 5.1. – We have

$$Z := \Big\{ \mathrm{Hilb}(Y) \mid Y \subset \mathbb{P}^{N_m-1} \text{ is a smooth Fano manifold with } \dim H^0(K_Y^{\otimes t}) = \chi(t), \forall t \gg 1 \text{ and } \mathscr{O}_{\mathbb{P}^{N_m-1}}(1)|_{Y \cong K_Y^{-\otimes m}}. \Big\} \subset \mathrm{Hilb}(\mathbb{P}^{N_m-1}, \chi)\,. \tag{30}$$

By [41, Lemma 8.3] all smoothable K-semistable $\mathbb{Q}$-Fano varieties with a fixed dimension form a bounded family. This bounded family includes the Gromov-Hausdorff limits of Kähler-Einstein Fano manifolds of a fixed dimension. This essentially follows from Donaldson-Sun's work in [25] (compare the boundedness of smooth Fano manifolds with fixed dimension by [38]). As a consequence, by re-embedding in a bigger projective space, we may choose $m \gg 1$ such that $\chi(m) = N_m$ and the closure of Z includes all such $\mathbb{Q}$-Fano varieties. Now following [41, Section 8], let $\overline{Z} \subset \mathrm{Hilb}(\mathbb{P}^{N_m-1},\chi)$ be the closure of Z inside $\mathrm{Hilb}(\mathbb{P}^{N_m-1},\chi)$, Z^{kss} (resp. $Z^{\mathrm{kps}} \subset Z^{\mathrm{kss}}$) be the *open* (resp. *constructible*) subset of $\overline{Z}$ parametrizing the *K-semistable* (resp. *K-polystable*) $\mathbb{Q}$-Fano subvarieties, and $(Z^{\mathrm{kps}})^{\circ}$(resp. $(Z^{\mathrm{kss}})^{\circ}$) $\subset Z^{\mathrm{kps}}$ be the subset parametrizing smooth K-polystable *Fano manifolds*. Let Z^* denote the seminormalization of $Z^{\mathrm{kss}}_{\mathrm{red}}$, the reduction of Z^{kss} and $(Z^{\mathrm{kps}})^* = Z^{\mathrm{kps}} \times_{Z^{\mathrm{kss}}} Z^*$ denote the pull back of Z^{kps}. Here let us explain the reason why we take the semi-normalization for the reader's convenience: A priori, the good quotient moduli space of $Z^{\mathrm{kss}}/\mathrm{SL}(N_m+1)$ may have a different scheme structure when we vary m. However, since it pointwisely parametrizes K-polystable smoothable $\mathbb{Q}$-Fano varieties, its seminormalization, which is the same as $Z^*/\mathrm{SL}(N_m+1)$, does not depend on m. Nevertheless, if we are able to construct a moduli space which corepresents the functor of families of all K-semistable $\mathbb{Q}$-Fano varieties (not just those smoothable ones), then its scheme structure is uniquely determined, and the good moduli quotient space $\overline{\mathcal{M}}$ of $Z^*/\mathrm{SL}(N_m+1)$ we consider above will be identical to the seminormalization of the reduced closure of those components whose generic fiber parametrizes smooth objects.

Let $\mathcal{X} \to Z$ be the universal family over Z and by abusing of notation we will still let $\mathcal{X} \to Z^*$ denote the pull back. Now for each $z \in Z^{\mathrm{kps}}$, the corresponding $\mathcal{X}_z$ is equipped with a weak Kähler-Einstein metric ω_z. Its volume form $\omega_z^n = e^{-\widetilde{u}}\widetilde{\Omega}$ defines a continuous Hermitian metric on $K^{-1}_{\mathcal{X}_z}$, hence a Hermitian metric $h_{\mathrm{DP}}(z)$ on $\lambda_{\mathrm{CM}}|_z$ by (21). Now our main result of this section is the following equivalent version of Theorem 1.1.

Theorem 5.2. – *Let $\overline{\mathcal{M}}$ be the proper good moduli space (cf.* [41, Theorem 1.3]*) for the quotient stack $[Z^*/SL(N+1)]$. Then there is a $k=k(r,\chi)$ such that the bundle $\lambda_{\mathrm{CM}}^{\otimes k} \to Z^{\mathrm{kss}}$ descends to a $\mathbb{Q}$-line bundle Λ_{CM} on $\overline{\mathcal{M}}$ with a well defined continuous metric h_{DP}, whose curvature is a positive current.*

Before we descend $\lambda_{\mathrm{CM}} \to Z^*$ to $\overline{\mathcal{M}}$, let us recall the theory developed [2] and [3].

Definition 5.3. – Let $\mathcal{Z}$ be an algebraic stack of finite type over $\mathbb{C}$, and let $z \in \mathcal{Z}(\mathbb{C})$ be a closed point with reductive stabilizer G_z. We say $f_z : \mathcal{V}_z \to \mathcal{Z}$ is a *local quotient presentation around z* if

1. $\mathcal{V}_z = [\mathrm{Spec}\, A/G_z]$, with A being a finite type $\mathbb{C}$-algebra.
2. f_z is étale and affine.
3. There exists a point $v \in \mathcal{V}_z$ such that $f_z(v) = z$ and f_z induces isomorphism $G_v \cong G_z$.

We say $\mathcal{Z}$ *admits a local quotient presentation* if there exists a local quotient presentation around every closed point $z \in \mathcal{Z}$.

Then we have the following

Theorem 5.4 (Theorem 10.3 in [2] and Theorem 4.1 in [3]). – *Let $\mathcal{Z}$ be an algebraic stack of finite type over $\mathbb{C}$,*

1. *For every closed point $z \in \mathcal{Z}$, there is a local quotient presentation $f_z : \mathcal{V}_z \to \mathcal{Z}$ around z such that*

a) *f_z is stabilizer preserving at closed points of $\mathcal{V}_z$, i.e., for any $v \in \mathcal{V}_z(\mathbb{C})$, $\mathrm{Aut}_{\mathcal{V}_z(\mathbb{C})}(v) \to \mathrm{Aut}_{\mathcal{Z}(\mathbb{C})}(f(v))$ is an isomorphism;*
b) *f_z sends closed points to closed points.*

2. *For any $\mathbb{C}$-point $z \in \mathcal{Z}$, the closed substack $\overline{\{z\}}$ admits a good moduli space.*

Then $\mathcal{Z}$ admits a good moduli space M. Furthermore, if $\mathcal{Z}$ admits a line bundle $\mathcal{L}$ such that for any closed point $z \in \mathcal{Z}(\mathbb{C})$, the stabilizer G_z acts on $\mathcal{L}|_z$ trivially, then $\mathcal{L}$ descends to a line bundle L on M.

Remark 5.5. – The general local condition for descending the line bundle to a good quotient, e.g., (2) in Theorem 5.4, already appeared in [26, Theorem 2.3]. See also [28, Lemma 11.7] and the discussion in [47, Section 6.2]. In our case of Kähler-Einstein smoothable $\mathbb{Q}$-Fano varieties, this was established in [41, Section 8].

To apply the theorem above, we first verify that some multiple of the line bundle $\lambda_{\mathrm{CM}}^{\otimes k}$ can be descent to $\overline{\mathcal{M}}$. For the reader's convenience, let us briefly recall how we show $\overline{\mathcal{M}}$ as the good moduli space of $[Z^*/\mathrm{SL}(N_m)]$. More precisely, in [41, Section 8], we give a local GIT description which provides the local quotient presentation needed in the first part of Theorem 5.4: Fix any closed point $z \in Z^{\mathrm{kps}}$ parametrizing a K-polystable smoothable $\mathbb{Q}$-Fano variety. If we let $\mathrm{Hilb}(\mathbb{P}^{N_m-1}, \chi) \subset \mathbb{P}^K$ be the Plücker embedding, then for any fixed representative $z \in O_z$ there is an $\mathrm{Aut}(\mathcal{X}_z)$-invariant linear subspace $z \in \mathbb{P}W \subset \mathbb{P}^K$ and an $\mathrm{Aut}(\mathcal{X}_z)$-invariant open neighborhood $z \in \mathcal{U}_z \subset \mathbb{P}W \cap Z^{\mathrm{kss}}$ such that

$$[\mathcal{U}_z \,//\, \mathrm{Aut}(\mathcal{X}_z)] \to [Z^*/\mathrm{SL}(N_m)]$$

where $\mathrm{Aut}(\mathcal{X}_z)$ is the stabilizer of z, gives the local quotient presentation of the quotient stack $[Z^*/\mathrm{SL}(N_m)]$. Passing to the quotient moduli space, the GIT quotient

$$\mathcal{U}_z \,//\, \mathrm{Aut}(\mathcal{X}_z) \to \overline{\mathcal{M}}$$

gives rise to a local étale chart $[z] \in \mathcal{V}_{[z]} \subset \overline{\mathcal{M}}$ around $[z]$. By abusing the notation, we will still let $\mathcal{U}_z$ to denote $\mathcal{U}_z \times_{Z^{\mathrm{kss}}_{\mathrm{red}}} Z^*$. Knowing $\mathcal{U}_z \,//\, \mathrm{Aut}(\mathcal{X}_z) \to \overline{\mathcal{M}}$ being a good moduli space, in order to descend $\lambda_{\mathrm{CM}}^{\otimes k}$, it suffices to show the stabilizer $\mathrm{Aut}(\mathcal{X}_z)$ of any *closed* point $z \in Z^*(\mathbb{C})$ acts trivially on $\lambda_{\mathrm{CM}}^{\otimes k}|_z$ (see [2, Theorem 10.3]). By the Futaki invariant vanishes, we know the Lie algebra $\mathfrak{aut}_z$ acts trivially on $\lambda_{\mathrm{CM}}|_z$. Indeed, it is now well known that the Futaki invariant ([30], [31], [19]) is the same as the action of $\mathfrak{aut}_z$ on $\lambda_{\mathrm{CM}}|_z$ (see [60], [21], [49]). In order to trivialize the action of $\mathrm{Aut}(\mathcal{X}_z)$, let us introduce $k_z := |\mathrm{Aut}(\mathcal{X}_z)/\mathrm{Aut}(\mathcal{X}_z)_0|$. Then $\mathrm{Aut}(\mathcal{X}_z)$ acts trivially on $\lambda_{\mathrm{CM}}^{\otimes k_z}|_z$.

Lemma 5.6. – *k_z is uniformly bounded for $z \in Z^*$, that is $k_z < k = k(m, \chi)$ with m, χ being fixed in* (30).

Proof. – Let us consider the universal family $\mathcal{X} \to Z^*$, which is a bounded family. Then $\mathrm{Aut}_{Z^*}\mathcal{X} := \mathrm{Isom}_{Z^*}(\mathcal{X}, \mathcal{X})$ is a group scheme over Z^*. In particular, this implies that number of component over each Zariski open set of Z^* is uniformly bounded. □

As a consequence, the action of $\mathrm{Aut}(\mathscr{X}_z)$ on $\lambda_{\mathrm{CM}}^{\otimes k}$ is trivial for all closed point $z \in Z(\mathbb{C})$. This together implies all the assumptions of Theorem 5.4 are met, and hence $\lambda_{\mathrm{CM}}^{\otimes k}$ descends to a line bundle $\Lambda_{\mathrm{CM}}^{\otimes k}$ over $\overline{\mathscr{M}}$ as we desired, where we consider Λ_{CM} as a $\mathbb{Q}$-line bundle.

With Λ_{CM} in hand, we may proceed the proof of the main result of this section.

Proof of Theorem 5.2 (=Theorem 1.1). – To finish the proof, we need to descend h_{DP} to a metric on Λ_{CM}. To do that, let us fix $[z] \in \overline{\mathscr{M}}$. We choose $z \in Z^{\mathrm{kps}}$ and a $\mathrm{Aut}(\mathscr{X}_z)$-equivariant slice $\mathscr{U}_z$ such that $\mathscr{U}_z /\!/ \mathrm{Aut}(\mathscr{X}_z) = \mathscr{V}_{[z]}$ is an étale neighborhood of $[z]$ as in [41, Theorem 8.5]. Fix a generator $\mathfrak{l}_z = \langle l_0, \ldots, l_n\rangle(\mathscr{X}_z)$ of $\lambda_{\mathrm{CM}}(\mathscr{X}_z)$. By the proof of [26, Theorem 2.3] or [2, Theorem 10.3], we see that $\mathfrak{l}_z$ can be extended to an $\mathrm{Aut}(\mathscr{X}_z)$-invariant section $\mathfrak{l} \in H^0\big(\mathscr{U}_z, \lambda_{\mathrm{CM}}^{\otimes k}|_{\mathscr{U}_z}\big)^{\mathrm{Aut}(\mathscr{X}_z)}$, which descends to a local section $[\mathfrak{l}] \in H^0\big(\mathscr{V}_{[z]}, \Lambda_{\mathrm{CM}}^{\otimes k}|_{\mathscr{V}_{[z]}}\big)$. Then we define a Hermitian metric on Λ_{CM} by

$$\|[\mathfrak{l}]\|_{h_{\mathrm{DP}}}([z]) = \|\mathfrak{l}\|_{h_{\mathrm{DP}}}(z).$$

Because $\mathrm{Aut}(\mathscr{X}_z)$ acts trivially on the fiber of λ_{CM}^k at z and the metrized Deligne pairing is $\mathrm{Aut}(\mathscr{X}_z)$-equivariant by Lemma 4.8, the Hermitian metric h_{DP} on Λ_{CM} is indeed well defined. Now we claim that h_{DP} is continuous on Λ_{CM}. For that, let $[z_i] \overset{i\to\infty}{\longrightarrow} [z]$ in $\overline{\mathscr{M}}$ be a sequence and $z_i \to z \in \mathscr{U}_z \cap Z^{\mathrm{kps}}$ be the lifting; we need to show that $\|\mathfrak{l}\|_{h_{\mathrm{DP}}}(z_i) \to \|\mathfrak{l}\|_{h_{\mathrm{DP}}}(z)$. Recall that, by the change of metric Formula (17), we have Formula (21):

$$\|\mathfrak{l}\|^2_{h_{\mathrm{DP}}}(z_i) = \|\mathfrak{l}\|^2_{\widetilde{h}_{\mathrm{DP}}} e^{-\mathfrak{U}_i}(z_i),$$

where

$$\mathfrak{U}_i = -\sum_{j=0}^{n} \int_{\mathscr{X}_{z_i}} \widetilde{\mathfrak{u}}_{z_i} \omega_{z_i}^j \wedge \widetilde{\omega}_{z_i}^{n-j}$$

and $\widetilde{h}_{\mathrm{DP}}$ is the Deligne metric on λ_{CM} defined using the volume form $\widetilde{\Omega}$ on $K^{-1}_{\mathscr{X}/Z^*}$. By Theorem 4.5, we know that $\|\mathfrak{l}\|^2_{\widetilde{h}_{\mathrm{DP}}}(z_i) \to \|\mathfrak{l}\|^2_{\widetilde{h}_{\mathrm{DP}}}(z)$. By the proof of Lemma 4.13, all we need is that

$$\widetilde{\mathfrak{u}}(z_i) \overset{i\to\infty}{\longrightarrow} \widetilde{\mathfrak{u}}(z). \tag{31}$$

Now by our construction $\mathscr{X}|_{\mathscr{U}_z} \to \mathscr{U}_z$ is a family of klt Fano varieties. So by the proof of Proposition 3.1, (31) is a consequence of

$$\lim_{z'\to z} \int_{\mathscr{X}_{z'}} \widetilde{\Omega}_{\mathscr{X}_{z'}} = \int_{\mathscr{X}_z} \widetilde{\Omega}_{\mathscr{X}_z}, \text{ for } z' \in \mathscr{U}_z.$$

which will be proved in Lemma 7.1.

Finally we show that $(\Lambda_{\mathrm{CM}}, h_{\mathrm{DP}})$ has positive curvature in the sense of Definition 2.6 in Section 2. To apply Proposition 2.5, we first notice that $\mathscr{M}$ is normal. Indeed, since the deformation of any smooth Fano manifold is unobstructed, the Artin stack classifying n-dimensional smooth Fano manifolds is *smooth*. Now we consider the open substack parametrizing K-semistable Fano manifolds and its *good moduli* space in the sense of [2] is hence a *normal* algebraic space. Now because $\mathrm{codim}_{\overline{\mathscr{M}}}(\overline{\mathscr{M}} \setminus \mathscr{M}) \geq 1$, by the continuity of h_{DP} and Proposition 2.5, we just need to verify the positivity of the curvature of h_{DP} over $\mathscr{M}$. By Definition 2.1, we need to verify the positivity along any analytical curve. So letting $\tau : \Delta \to \mathscr{M}$ be any holomorphic map, we need to verify the positivity for $(\tau^*\Lambda_{\mathrm{CM}}, \tau^* h_{\mathrm{DP}})$. After possibly finite base change $p_1 : \Delta \to \Delta$, we can lift τ to a holomorphic map: $\tau_1 : \Delta \to (Z^{\mathrm{kss}})^\circ$, such that $\tau_1(\Delta^\circ)$ is contained in a component of $(Z^{\mathrm{kps}})^\circ$ where $\Delta^\circ = \Delta \setminus \{\text{finite points}\}$. However, by [41, 3.1], we know that after shrinking Δ and replacing τ, we can always assume that for every point $t \in \Delta$, $\tau_1(t) \in (Z^{\mathrm{kps}})^\circ$.

Let $\tilde{\tau} = p_2 \circ \tau_1$ with $p_2 : (Z^{\mathrm{kss}})^\circ \to \mathscr{M}$ be the quotient morphism. Since p_1 is generically smooth and h_{DP} is continuous, by Theorem 2.4 we just need to verify the positivity of $(\tilde{\tau}^*\lambda_{\mathrm{CM}}, \tilde{\tau}^* h_{\mathrm{DP}})$. Now since $\mathscr{X} \times_{\tilde{\tau},(Z^{\mathrm{kss}})^\circ} \Delta$ is a flat family Kähler-Einstein Fano manifolds over Δ, we get the positivity by the positivity of $\omega^\circ_{\mathrm{WP}}$ explained in Section 4.1.1 (see also the proof of Theorem 1.2 in the next Section). □

REMARK 5.7. – In [53], in order to prove the quasi-projectivity of the moduli space $\mathscr{M}^-$ of canonically polarized manifolds, Schumacher proved that there exist some compactification $\overline{\mathscr{M}^-}$ together with a holomorphic line bundle $\overline{\lambda}$ equipped with a singular hermitian metric $\overline{h}^Q$ such that $(\overline{\lambda}, \overline{h}^Q)$ extends (λ, h^Q) where λ is a canonically defined determined line bundle equipped with a positively curved Quillen metric h^Q over $\mathscr{M}^-$ (see [53, Theorem 5]). For this, he proved an extension theorem for Hermitian line bundles whose curvature forms extend as positive currents (see [53, Theorem II] and [54, Theorem II']). In our Fano case, we have constructed a distinguished compactification $\overline{\mathscr{M}}$ that arises from K-polystable $\mathbb{Q}$-Fano varieties of the corresponding Hilbert scheme. Moreover, we showed that there is a canonically defined extension $(\Lambda_{\mathrm{CM}}, h_{\mathrm{DP}})$ by exploiting the local GIT nature of our moduli space and the continuity of the canonically defined Deligne metric.

REMARK 5.8. – It may be possible to verify the continuity of h_{DP} using directly the Hermitian metric on $K^{-1}_{\mathscr{X}/S}$ by $\{\omega_t^n\}$. By [13], we know that the volume measure is continuous under the GH convergence. So we indeed expect that the Hermitian metric $\{\omega_t^n\}$ changes continuously with respect to t so that the metric on the Deligne pairing changes continuously. However, since the volume measure is not exactly the same as the volume form ω_t^n, extra arguments are needed.

6. Quasi-projectivity of $\mathscr{M}$

6.1. Proof of Theorem 1.2

In this section, we verify the criterion for quasi-projectivity embedding in Theorem 6.1, which generalizes the classical Nakai-Moishezon criterion to the normal non-complete algebraic space U with a compactification M. Theorem 6.1 follows from [44, 10] when the underlying space M is known to be projective. We reduce the case of normal algebraic space to this known case. We do not know whether this holds for general proper algebraic space.

THEOREM 6.1. – *Let M be a* normal *proper algebraic space that is of finite type over $\mathbb{C}$. Let L be a line bundle on M and $M^\circ \subset M$ an open subspace. We assume $L^m \cdot Z \geq 0$ for any m-dimensional irreducible subspace and the strictly inequality holds for any Z meets M°. Then for sufficiently large power k, $|L^k|$ induces a rational map which is an embedding restricting on M°.*

Proof. – We first show that it suffices to prove that $L^{\otimes k}$ separates any two points in M° for sufficiently large k. In fact, if this is true, then we can blow up the indeterminacy ideal I of the rational map induced by $|L^{\otimes k}|$ and then take a normalization to get $\mu : M' \to M$. Then $\mu^*(L^{\otimes k}) = L_1 + E$ where $L_1 \geq 0$ and E is base point on M', with the induced morphism separate any two points on $\mu^{-1}(M^\circ) \cong M^\circ$. Then we know that for sufficiently large k_1,

$$|L^{\otimes kk_1}| = \mu^*|L^{\otimes kk_1}| \supset |L_1^{\otimes k_1}| + k_1 E$$

embeds M° as M is normal.

Since there is always a Galois finite surjective morphism $f : M_1 \to M$ from a normal scheme M_1 (cf. [35, Lemma 2.8]), using the Norm map, one easily sees that f^*L separate any two points on $f^{-1}(M^\circ)$ implies

$$\mathrm{Nm}|f^*(L^{\otimes k})| \subset |L^{\otimes k \cdot \deg f}|$$

separate two points on M° for $k \gg 0$. So we can assume M is a normal proper (possibly non-projective) scheme.

For any point $x \in M^\circ$, there is quasi-projective neighborhood $U_x \subset M^\circ$ with U_x being an open set of a projective scheme M_x. Considering the rational map $M_x \dashrightarrow M$ and applying [52, 5.7.11] to the morphism $\Gamma_x \to M$ from its graph Γ_x to M, we see that the indeterminacy locus of $M_x \dashrightarrow M$ can be resolved by a sequence of blow ups. We know that there exists a normal variety M' which admits morphisms $p : M' \to M$ and $q : M' \to M_x$ such that q is relative projective over M_x. In particular, M' is projective.

Consider p^*L and the open set $U'_x \subset M'$ which is isomorphic to U_x. Then the triple (M', U'_x, p^*L) satisfies the same assumption as (M, M°, L) in the theorem. Since M' is projective, by our assumption, we know that U'_x does not meet the exceptional locus $\mathbb{E}(p^*L)$, which is the union of subvarieties on which p^*L is not big. Then it follows from [10, Theorem 1.3] that for sufficiently large k, $p^*L^{\otimes k}$ does not have base points along U'_x. Since M is normal, this implies $L^{\otimes k}$ does not have base points along U_x. Therefore, we know that for sufficiently large k, $|L^{\otimes k}|$ contains no base point in M° and does not contract any curve on M°. Thus after replacing k by its multiple, we know $|L^{\otimes k}|$ separate any two points in M°. □

REMARK 6.2. – In [55, Theorem 6], a similar quasi-projectivity criterion was given in analytic setting.

Proof of Theorem 1.2. – As mentioned above, $\mathscr{M}$ is a normal algebraic space. We can apply Theorem 6.1 to the pair $(M, L) := (\overline{\mathscr{M}}^{\mathrm{n}}, n^*\Lambda_{\mathrm{CM}}^{\otimes k})$ with k being fixed in Lemma 5.6 and $M^\circ = \mathscr{M}$. For that, we need to show that for any irreducible subspace $Y \subset \overline{\mathscr{M}}$ satisfying $Y \cap \mathscr{M} \neq \emptyset$ we have $L^m \cdot Y > 0$.

To achieve that, without loss of generality, we may assume that Y is reduced. We claim that there is a point $[z] \in V_{[z]} \subset Y \cap \mathscr{M}$ together with an open neighborhood $V_{[z]}$, on which $(V_{[z]}, \omega_{\mathrm{WP}}|_{V_{[z]}})$ is a smooth Kähler manifold, from which we deduce $L^m \cdot Y = \int_Y \omega_{\mathrm{WP}}^m > 0$ (cf. Proposition 4.15) and hence finish our proof.

To do that, let us take a smooth point $[z'] \in Y$ and

$$Z^* \supset \mathscr{U}_{z'} \xrightarrow{f_{z'}} \mathscr{V}_{[z']} \subset \mathscr{M}$$

be the local quotient presentation as in Section 5.2. Let $Z^*_{Y,\mathcal{U}_{z'}}$ be a component of $f_{z'}^{-1}(\mathcal{V}_{[z']}\cap Y)$ which dominates $(\mathcal{V}_{[z']}\cap Y)$. Then by our construction

$$f_{z'}|_{Z^*_{Y,\mathcal{U}_{z'}}} : Z^*_{Y,\mathcal{U}_{z'}} \longrightarrow \mathcal{V}_{[z']}\cap Y$$

is surjective, hence there is smooth point $z \in Z^*_{Y,\mathcal{U}_{z'}}$ such that $df_{z'}(z)$ is surjective by Bertini-Sard's theorem. In particular, we are able to find a local slice $z \in S$ of equal dimension locally isomorphic to an open neighborhood $V_{[z]} \subset S$ of $[z] = f_{z'}(z) \in V_{[z]} \subset \mathcal{V}_{[z']}$. So the slice S must be transversal to $\mathrm{Aut}(\mathcal{X}_{z'})$-orbits near z; this implies that the restriction of the universal family $\mathcal{X}|_S \to S$ is *generically effective* of $(f_{z'}|_S)^{-1}(V_{[z]})$ by Kuranshi's local completeness Theorem. By Section 4.1.1, we conclude that the restriction of ω_{WP} to a dense open subset of $V_{[z]}$ is a *smooth Kähler form.* □

6.2. Remarks on the projectivity of $\overline{\mathcal{M}}$

We expect that the proper moduli space $\overline{\mathcal{M}}$ constructed in [41] is actually projective. Indeed, the CM line bundle Λ_{CM} can very well be ample (not only nef and big). Using Nakai-Moishezon's criterion for proper algebraic spaces [35, Theorem 3.11], we just need to verify the positivity of intersection number $\Lambda_{\mathrm{CM}}^{\dim Z}\cdot Z$ for any subvariety Z contained in $\overline{\mathcal{M}}\setminus\mathcal{M}$. Using the notation as before, we just need to verify that the curvature of h_{DP} is strictly positive over an open set of the base Z. Here we verify that this holds if Z parametrizes Fano varieties with orbifold singularities.

By the above functorial construction of the moduli space $\overline{\mathcal{M}}$, we consider the following set-up. Let $\pi : \mathcal{X} \to S$ be a flat family of Kähler-Einstein $\mathbb{Q}$-Fano varieties such that there is a dense open subset $S^\circ \subset S$ with the following conditions satisfied:

1. for each $t \in S^\circ$, the fiber $\mathcal{X}_t$ is a smooth Kähler-Einstein Fano variety;
2. for each $t \in S\setminus S^\circ$, the fiber $\mathcal{X}_t$ is a Kähler-Einstein $\mathbb{Q}$-Fano variety with orbifold singularities such that orbifold locus has complex codimension ≥ 2.

As in Section 3, we fix an embedding $\mathcal{X} \to S\times\mathbb{P}^{N_m}$ by using a fixed basis $\{\widetilde{s}_i\}$ of locally free $\mathcal{O}_S$ module $\pi_*\mathcal{O}_{\mathcal{X}}(-mK_{\mathcal{X}})$ and get a (smooth) family of reference metrics $\widetilde{\omega}_t$ by pulling back the Fubini-Study metric on $\mathbb{P}^{N_m}$. Then we can write $\omega_t = \widetilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\widetilde{\mathfrak{u}}$ where ω_t is a Kähler-Einstein metric on $\mathcal{X}_t$ for each $t \in S$. By the discussion in Section 3, we can assume that $\widetilde{\mathfrak{u}}$ is a continuous function on $\mathcal{X}$ so that for each $t \in Z$ the metric $\omega_t = \widetilde{\omega}_t + \sqrt{-1}\partial\bar{\partial}\widetilde{\mathfrak{u}}(t)$ is a priori only a weak Kähler-Einstein metric on the $\mathbb{Q}$-Fano variety $\mathcal{X}_t$ (with orbifold singularities) which is also the Gromov-Hausdorff limit of any sequence of smooth Kähler-Einstein manifolds $\mathcal{X}_{t_k}$ for $t_k \in S^\circ$ with $t_k \to t$. However it is now well known that any weak Kähler-Einstein metric on a Fano variety with orbifold singularities is automatically a smooth orbifold Kähler-Einstein metric (see [40] and the reference therein). In other words, $\widetilde{\mathfrak{u}}$ is *orbifold smooth* on $\mathcal{X}_t$ for any $t \in S\setminus S^\circ$. Recall that we need to show that for each subvariety $Z \subset S\setminus S^\circ$ that parametrizes an *effective* flat family of Fano varieties (with orbifold singularities), $\Lambda_{\mathrm{CM}}^{\dim Z}\cdot Z$ is strictly positive. By restricting to an open subset of Z, we can further assume that the effective flat family of Fano orbifolds, denoted again by $\pi : \mathcal{X} \to Z$, is a fibration with diffeomorphic orbifold fibers, i.e., $\mathcal{X}_t$ is diffeomorphic to $\mathcal{X}_{t'}$ as smooth orbifolds for any $t, t' \in Z$. In particular, the Kodaira-Spencer class comes from $H^1(\mathcal{X}_t, \mathcal{T}^{\mathrm{orb}})$. Here for any point x, there is an open neighborhood U_x and a uniformization covering $\Pi_x : \widetilde{U}_x \to U_x$ such that $U_x = \widetilde{U}_x/G_x$ for a finite group G_x. Then $\mathscr{T}^{\mathrm{orb}}(U_x)$ is defined to be $\mathscr{T}(\widetilde{U}_x)^{G_x}$. We can assume that ω_t is a smooth family of orbifold Kähler-Einstein metrics by a straight-forward generalization of the slice theorem in [22, Section 5.3] and [57] to the orbifold setting. On the other hand, by pulling back to local uniformization covering $\widetilde{U}_x$ it's easy to see that both $\widetilde{\omega}$ and $\widetilde{\Omega}$ in Section 3 are orbifold smooth. From above discussion and the Equation (6), we know that $\widetilde{\mathfrak{u}}$ is an orbifold smooth function on $\mathcal{X}$. Now we can do the following calculations. For simplicity, let us assume Z is of complex dimension 1 with a local coordinate function t.

1. Using the Stokes formula for fiber integrals along *orbifold smooth* fibers, $\partial\bar{\partial}$ and $\int_{\mathcal{X}/Z}$ can be interchanged.

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\mathfrak{U} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(-\sum_{j=0}^{n}\int_{\mathcal{X}/Z}\widetilde{\mathfrak{u}}\left(\widetilde{\omega}+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\widetilde{\mathfrak{u}}\right)^{j}\wedge\widetilde{\omega}^{n-j}\right)$$
$$= -\sum_{j=0}^{n}\int_{\mathcal{X}/Z}\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\widetilde{\mathfrak{u}}\wedge\left(\widetilde{\omega}+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\widetilde{\mathfrak{u}}\right)^{j}\wedge\widetilde{\omega}^{n-j}.$$

By the proof of Theorem 4.11, we know that locally $h_{\mathrm{DP}} = e^{-\Psi_\alpha} = e^{-\widetilde{\Psi}_\alpha}e^{-\mathfrak{U}}$. So using (19) we have:

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\Psi_\alpha = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\widetilde{\Psi}_\alpha + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\mathfrak{U}$$
$$= -\int_{\mathcal{X}/Z}\widetilde{\omega}^{n+1} - \sum_{j=0}^{n}\int_{\mathcal{X}/Z}\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\widetilde{\mathfrak{u}}\wedge\left(\widetilde{\omega}+\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\widetilde{\mathfrak{u}}\right)^{j}\wedge\widetilde{\omega}^{n-j} = -\int_{\mathcal{X}/Z}\omega^{n+1}.$$

2. ω is orbifold smooth and locally is equal to $\sqrt{-1}\partial\bar{\partial}\psi$. We can write:

$$\omega^{n+1} = c(\psi)\omega_t^n \wedge \frac{\sqrt{-1}}{2\pi}dt\wedge d\bar{t},$$

where $c(\psi)$ essentially measures the negativity of ω in the horizontal direction. By the same calculation as in [53, Proposition 3] (see also [5]), we see that $c(\psi)$ satisfies an elliptic equation:

$$-\Delta_t\, c(\psi) - c(\psi) = |A|^2_{\omega_t}.$$

Here Δ_t is the Laplace operator associated to the orbifold Kähler-Einstein metric ω_t on $\mathcal{X}_t$ and $A \in A^{0,1}(\mathscr{T}^{\mathrm{orb}}_{\mathcal{X}_t})$ represents the Kodaira-Spencer class of the deformation, which is obtained as follows. We choose local orbifold holomorphic coordinate $\{z^i, t\}$ on $\mathcal{X}$. By the non-degeneracy of ω along the fiber $\mathcal{X}_t$, there is a unique horizontal lifting V of ∂_t satisfying:

$$d\pi(V) = \partial_t, \quad \omega(V, \partial_{\bar{z}^j}) = 0, \forall 1 \le j \le n.$$

Then $A = A^i_{\bar{j}} d\bar{z}^j \otimes \partial_{z^i}$ is given by $(\bar{\partial}V)|_{\mathcal{X}_t}$. For details, see [53].

3. So we have:

(32)

$$-\int_{\mathcal{X}/Z}\omega^{n+1} = -\int_{\mathcal{X}/Z} c(\psi)\omega_t^n \wedge \frac{\sqrt{-1}}{2\pi}dt\wedge d\bar{t}$$
$$= -\int_{\mathcal{X}/Z}(\Delta_t c(\psi)+c(\psi))\,\omega_t^n \wedge \frac{\sqrt{-1}}{2\pi}dt\wedge d\bar{t} = \left(\int_{\mathcal{X}/Z}|A|^2_{\omega_t}\omega_t^n\right)\frac{\sqrt{-1}}{2\pi}dt\wedge d\bar{t}.$$

Since $\mathcal{X}\to Z$ is generically effective, we know that $[A]\in H^1(\mathcal{X}_t,\mathcal{T}^{\mathrm{orb}})$ is generically nonzero over Z. So for generic $t\in Z$, A is non-vanishing over $\mathcal{X}_t$ and the right-hand-side of (32) is strictly positive.

REMARK 6.3. – It's clear that the above argument and calculations depend on the extension process in Section 4.2 and the orbifold regularity of weak Kähler-Einstein metrics. The extension process in turn relies on the fact that metricized line bundle $(\Lambda_{\mathrm{CM}}, h_{\mathrm{DP}})$ is constructed by using the formalism of Deligne pairing, which allows the fiber of the flat family to be singular. As pointed out by the referee, if one would like to use the formalism of Bismut-Gillet-Soulé as in [28, 53], one needs to show that the curvature of the Quillen metric of the corresponding determinant line bundle is still given by the formula $-\int_{\mathcal{X}/Z}\omega^{n+1}$ for the orbifold smooth form ω on $\mathcal{X}$. This should be possible via generalizing the work of Bismut-Gillet-Soulé in [11] to the orbifold setting (see [42]).

Using this strict positivity and similar arguments as in the proof of the quasi-projectivity of $\mathcal{M}$, we get the following result:

PROPOSITION 6.4. – *Let $\overline{\mathcal{M}}^{\mathrm{orb}}$ be the locus parametrizing smoothable K-polystable Fano varieties with at worst orbifold singularities. Then the normalization of $\overline{\mathcal{M}}^{\mathrm{orb}}$ is quasi-projective.*

As a direct consequence, if we consider the case of del Pezzo surfaces, in which we know that $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{\mathrm{orb}}$ by [59]. By applying the above strict positivity and Nakai-Moishezon's criterion for proper algebraic spaces ([35]), we immediately obtain:

COROLLARY 6.5. – *The proper moduli spaces of smoothable K-polystable del Pezzo surfaces are projective.*

As mentioned before, this was known by [47] except for del Pezzo surfaces of degree 1.

7. Appendix: A uniform convergence lemma

Let $\pi:\mathcal{X}\to S$ be a flat family of klt Fano variety, which is *holomorphically* embedded into $\mathbb{P}^N\times S$. Assume that m is chosen in such a way that $-mK_{\mathcal{X}/S}$ is relatively base-point-free. Denote by $\{\tilde{s}_i, 1\le i\le N_m\}$ the (holomophic) basis of the $\mathcal{O}_S$ module $\pi_*\mathcal{O}_{\mathcal{X}}(-mK_{\mathcal{X}/S})$. For any $t\in S$, denote $\tilde{s}_i(t)=\tilde{s}_i|_{\mathcal{X}_t}$. We can define a volume form on $\mathcal{X}_t$ by

$$\widetilde{\Omega}_t = \left(\sum_{i=1}^{N_m}|\tilde{s}_i(t)|^2\right)^{-1/m}.$$

The main technical lemma is

LEMMA 7.1. – *In the above setting, we have the following uniform convergence:*

$$\lim_{t\to 0}\int_{\mathcal{X}_t}\widetilde{\Omega}_t=\int_{\mathcal{X}_0}\widetilde{\Omega}_0.$$

We make some remarks before proving this convergence. In [39] this lemma was proved under the assumption that the generic fiber is smooth and $\dim S = 1$. Here we generalize the calculations there to the general situation. In [39] the simpler case of Lemma 7.1 was proved by lifting the integrals on both sides to a log-resolution of singularities $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ and calculating carefully under the normal crossing coordinates. There it was proved that the limit of the left hand side of (9) as $t \to 0$ concentrates on the strict transform of $\mathcal{X}_0$ under π which coincides with the right hand side of (9). This concentration phenomenon essentially only depends on a fundamental result in birational algebraic geometry: *inversion of adjunction*, which says that, in the $\dim_{\mathbb{C}} S = 1$ case, if $\mathcal{X}_0$ is Kawamata-log-terminal (klt), then the pair $(\mathcal{X}, \mathcal{X}_0)$ is purely-log-terminal (plt) in a neighborhood of $\mathcal{X}_0$. The klt property holds in our situation by [8, 25] because each $(\mathcal{X}_t, \omega_t)$ is a Kähler-Einstein Fano variety. The plt property is expressed in terms of $a(\mathcal{X}, \mathcal{X}_0; E) > -1$ for any exceptional divisor E of π which does not have center on $\mathcal{X}_0$. It's well known that this lower bound of discrepancy (or complex exponent) implies an integrability condition, which turns out to be enough for us to apply dominant convergence theorem on the log-resolution $\widetilde{\mathcal{X}}$ to get uniform integrability and confirm the convergence in (9).

REMARK 7.2. – The calculation of a similar kind was first carried out in [7] and was then sharpened in [39] (see also [12, Corollary 8.3]). Indeed, a related continuity of Ding energy was speculated in [7] and its importance was pointed out to the first author by Berman [6].

Here we use the similar arguments to deal with the higher codimensional case. We need to use a form of inversion of adjunction for higher codimensional klt subvariety (35)-(36). Moreover, we need to use the existence of toroidal reduction of family $\mathcal{X} \to S$ proved by Abramovich-Karu ([1]) to replace the role of log resolution in $\dim_{\mathbb{C}} S = 1$ case.

Let's start by applying the toroidal reduction constructed in [1] to obtain the following commutative diagram:

$$\begin{array}{ccccc}\mathcal{Y} & \xrightarrow{\mu} & \mathcal{X}\times_S T & \xrightarrow{m_{\mathcal{X}}} & \mathcal{X}\\ & \searrow^{\pi_{\mathcal{Y}}} & \downarrow \pi & & \downarrow \pi\\ & & T & \xrightarrow{m_S} & S,\end{array} \tag{33}$$

such that $\mathcal{Y}$ and T admit toroidal structures, μ and m_S are birational morphisms and $\pi_{\mathcal{Y}}$ is a flat *toroidal* map. It's clear that we just need to verify (7.1) for $\mathcal{Y} \to T$, since $\mathcal{X}\times_S T \to T$ is still a flat family of klt Fano varieties. So without loss of generality we assume $T = S$ and fix a point $0 \in S$ from now on. We will lift the calculation of integrals and limits to the space $\mathcal{Y} \to T = S$.

Assume $\dim T = d$ such that $\dim \mathcal{Y} = n + d$. Choose general hyperplane divisors L_k on T and H_k its pull back on $\mathcal{X}$ so that $\mathcal{X}_0 = \bigcap_{k=1}^{d} H_k$. Since $\mathcal{Y}_0$ and $\mathcal{X}_0$ have the same dimension and $\mathcal{Y}_0 \to \mathcal{X}_0$ has connected fibers, we know $\mathcal{Y}_0$ has a component $\mathcal{X}_0'$ which is the strict transform of $\mathcal{X}_0$ under μ. Furthermore, the components of $\bigcap_{k=1}^d H'_k$ are normal and yield the log canonical centers of the sub-lc pair $(\mathcal{Y}, -K_{\mathcal{Y}/\mathcal{X}}+\mu^*(\sum H_{k=1}^d))$ where H'_k is the strict transform of H_k, then we indeed know that $\bigcap_{k=1}^d H'_k$ is irreducible because $\mathcal{X}_0$ is a minimal log canonical center of $(\mathcal{X}, \sum_{k=1}^d H_k)$.

We denote by $\{E_i, i \in I\}$ the vertical exceptional divisors of μ on $\mathcal{Y}$ which are not dominant under $\pi_{\mathcal{Y}}$. The following identities define multiplicities a_{ki} which are nonnegative integers:

$$\mu^* H_k = \pi_{\mathcal{Y}}^* L_k = H'_k + \sum_{i=1}^{I} a_{ki} E_i. \tag{34}$$

Now comes the key ingredient. We write down the identity defining the discrepancies:

$$K_{\mathcal{Y}/S} + \sum_{k=1}^{d} H'_k = \mu^*(K_{\mathcal{X}/S} + \sum_{k=1}^{d} H_k) - \sum_{i=1}^{I} b_i E_i - \sum_{j=1}^{J} c_j F_j, \tag{35}$$

where E_i are vertical exceptional divisors and F_j are horizontal exceptional divisors. Because $\mathcal{X}_0$ is klt, by *inversion of adjunction*, we have:

$$b_i < 1, \text{ for } 1 \le i \le I; \quad c_j < 1, \text{ for } 1 \le j \le J. \tag{36}$$

Combining (35) and (34) we also get:

$$K_{\mathcal{Y}/S} = \mu^* K_{\mathcal{X}/S} - \sum_{i=1}^{I} \left(b_i - \sum_{k=1}^{d} a_{ki}\right) E_i - \sum_{j=1}^{J} c_j F_j. \tag{37}$$

In the following, we will denote:

$$a_i = \sum_{k=1}^{d} a_{ki}, \quad 1 \le i \le I.$$

As explained in [39], using the partition of unity argument, it's enough to show the following local convergence properties:

$$\lim_{t\to 0} \int_{\mathcal{Y}\cap\mathcal{U}(p,\delta)} \mu^*(v \wedge \bar{v})^{1/m} = \int_{\mathcal{Y}\cap\mathcal{U}(p,\delta)} \mu|_{\mathcal{Y}}^*(v \wedge \bar{v})^{1/m}, \tag{38}$$

where p is any point in $\mathcal{Y}_0$, $\mathcal{U}(p,\delta)$ is a small neighborhood of p inside $\mathcal{Y}$ and v is a local generator of $\mathcal{O}_{\mathcal{X}}(-mK_{\mathcal{X}/S})(\mathcal{U}(p,\delta))$.

We will generalize the calculations as in [39, Section 4] to verify (38). For any point $p \in \mathcal{Y}_0$, there are 2 possibilities:

1. $p \in \mathcal{X}'_0 \cap \bigcap_{i=1}^{N_v} E_i \cap \bigcap_{j=1}^{N_h} F_j$;
2. $p \in \left(\bigcap_{i=1}^{N_v} E_i \cap \bigcap_{j=1}^{N_h} F_j\right) \setminus \mathcal{X}'_0$.

In the above, $N_v = N_v(p)$ (resp. $N_h = N_h(p)$) is the number of vertical (resp. horizontal) exceptional divisors passing through p. So if $N_v = 0$ (resp. $N_h = 0$), then there are no vertical (resp. horizontal) exceptional divisor passing through p and the corresponding intersection does not appear.

1. Case 1: Using the toroidal property of the map $\pi_{\mathcal{Y}}$, we can choose local coordinates $\{(x_1, \ldots, x_d; y_1, \ldots, y_n)\} =: \{x, y\}$ which are regular functions on $\mathcal{Y}$ near p and $\{t_1, \ldots, t_d\}$ near $0 \in S$ such that
   - $x(p) = y(p) = 0$,
   - locally $L_k = \{t_k = 0\}$, $H'_k = \{x_k = 0\}$ $(1 \le k \le d)$, $E_j = \{y_j = 0\}$ $(1 \le i \le N_v)$ and $F_j = \{y_j = 0\}$ $(N_v + 1 \le j \le N_v + N_h)$.
   - Since the pull back of L_k is the sum of the reduced divisor $H'_k = 0$ and other components, by (34) the map $\pi_{\mathcal{Y}}$ is locally given as

$$
\begin{aligned}
t_1 &= g_1(x, y) \cdot x_1 \textstyle\prod_{i=1}^{N_v} y_i^{a_{1i}} \\
&\cdots \\
t_d &= g_d(x, y) \cdot x_d \textstyle\prod_{i=1}^{N_v} y_i^{a_{di}}
\end{aligned}
\tag{39}
$$

where $g_i(x, y)(1 \le i \le d)$ are non vanishing holomorphic functions. For the simplicity of notations, we will assume $g_i(x, y) = 1$ since it will be easy to modify the calculation for general non vanishing $g_i(x, y)$.

Now notice that the space $\mathcal{Y}$ in general has toric singularities which are good enough for us to carry out the local calculations by locally lifting to finite covers. So possibly by passing to finite covers, let's consider the polydisk region:

$$\mathcal{U}(p, \delta) = \{|x_i| \le \delta, |y_j| \le \delta;\ 1 \le i \le d, 1 \le j \le n\}.$$

When $t_i \neq 0$ $(1 \le i \le d)$, we can choose $\{y_1, \ldots, y_n\}$ as the local coordinate system on the local fiber $\mathcal{V}_t(p, \delta) = \mathcal{U}(p, \delta) \cap \mathcal{Y}_t$:

$$x_i = x_i(t, y_1, \ldots, y_n) = \frac{t_i}{\prod_{j=1}^{N_v} y_j^{a_{ij}}}, \quad 1 \le i \le d. \tag{40}$$

So when $t_i \neq 0$ $(1 \le i \le d)$, $\mathcal{V}_t(p, \delta)$ is biholomorphic to the following region in the y-space via the projection:

$$\mathcal{V}_t(\delta) := \left\{ y = (y_1, \ldots, y_n);\ |y_j| \le \delta, 1 \le j \le n, \prod_{j=1}^{N_v} |y_j|^{a_{ij}} \ge |t_i|\delta^{-1}, 1 \le i \le d \right\}. \tag{41}$$

Note that $\{\mathcal{V}_t(\delta)\}$ is an increasing sequence of sets on the y-space with respect to the variable t. The limit is:

$$\lim_{t \to 0} \mathcal{V}_t(\delta) = \{y = (y_1, \ldots, y_n) \in \mathbb{C}^n;\ |y_j| \le \delta, j = 1, \ldots, n\} =: \mathcal{V}_0(\delta).$$

Now choose a local generator $v = \{v_t\}$ of $mK_{\mathcal{X}/\mathbb{S}}$ near $q = \mu(p)$. We have

$$\mu^* \left(v^{1/m}\right) = g(x, y) \prod_{i=1}^{N_v} y_i^{a_i - b_i} \prod_{j=N_v}^{N_v+N_h} y_j^{-c_j} (dx \wedge dy \otimes \partial_t). \tag{42}$$

Taking adjunction's m-times, we get:

$$K_{\mathcal{X}'_0} = \mu|^*_{\mathcal{X}'_0} K_{\mathcal{X}_0} - \sum_{i=1}^{I} b_i E_i|_{\mathcal{X}'_0} - \sum_{j=1}^{J} c_j F_j|_{\mathcal{X}'_0}. \tag{43}$$

It will be useful for us to see this adjunction analytically. We will denote

$$\mathcal{Z}_k = \bigcap_{1 \le l \le k} H'_l = \{x_l = 0\} \text{ for } k = 1, \dots, d.$$

Then

$$\mathcal{X}'_0 = \mathcal{Z}_d \subset \mathcal{Z}_{d-1} \subset \cdots \subset \mathcal{Z}_1.$$

By (39) and (42), we get:

$$\begin{aligned}
\mu|^*_{\mathcal{Z}_1}(v^{1/m}) &= g(x,y) \prod_{i=1}^{N_v} y_i^{a_i - b_i} \prod_{j=N_v+1}^{N_v+N_h} y_j^{-c_j} \frac{dt_1}{\prod_{i=1}^{N_v} y_i^{a_{1i}}} \bigwedge_{k=2}^{d} dx_k \wedge dy \otimes \partial_{t_1} \bigwedge_{k=2}^{d} \partial_{t_k}|_{\mathcal{Z}_1} \\
&= \pm\, g(0, x_2, \dots, x_d, y) \prod_{i=1}^{N_v} y_i^{(a_i - a_{1i}) - b_i} \prod_{j=N_v+1}^{N_v+N_h} y_j^{-c_j} \left(\bigwedge_{k=2}^{d} dx_k \otimes \partial_{t_k} \right) \wedge dy.
\end{aligned}$$

Inductively, we indeed get the analytic formula corresponding to (43):

$$\mu|_{\mathcal{X}'_0}(v^{1/m}) = \pm\, g(\mathbf{0}, y) \prod_{i=1}^{N_v} y_i^{-b_i} \prod_{j=N_v+1}^{N_v+N_h} y_j^{-c_j}\, dy.$$

For the same reasons, the local volume form along the fiber in (42) restricted $\mathcal{U}_t$ becomes:

$$\mu^* \left(v^{1/m}\right)|_{\mathcal{U}_t} = \pm\, g(x(t,y), y) \prod_{i=1}^{N_v} y_i^{-b_i} \prod_{j=N_v+1}^{N_v+N_h} y_j^{-c_j}\, dy.$$

So

$$\mu^*(v \wedge \bar{v})^{1/m} = \pm |g(x(t,y), y)|^2 \prod_{i=1}^{N_v} |y_i|^{-2b_i} \prod_{j=N_v+1}^{N_v+N_h} |y_j|^{-2c_j} \cdot dy \wedge d\bar{y}.$$

By (40), $\lim_{t \to 0} x(t,y) = \mathbf{0}$. So we see that for any $y \in \mathcal{V}_t(\delta)$, we have:

$$\begin{aligned}
\lim_{t \to 0} \mu^*(v \wedge \bar{v})^{1/m} &= \pm |g(\mathbf{0}, y)|^2 \prod_{i=1}^{N_v} |y_i|^{-2b_i} \prod_{j=N_v+1}^{N_v+N_h} |y_j|^{-2c_j} \cdot dy \wedge d\bar{y} \\
&= \mu|^*_{\mathcal{X}'_0}(v_0 \wedge \bar{v}_0)^{1/m}.
\end{aligned}$$

Now it's straightforward to use the dominant convergence theorem to verify that (see [39, (43)]):

$$\begin{aligned}
\lim_{t \to 0} \int_{\mathcal{U}(p,\delta) \cap \widetilde{\mathcal{X}}_t} \mu^*(v \wedge \bar{v})^{1/m} &= \pm \lim_{t \to 0} \int_{\mathcal{V}_t(\delta)} \frac{|g(x(t,y), y)|^2}{\prod_{i=1}^{N_v} |y_i|^{2b_i} \prod_{j=N_v+1}^{N_v+N_h} |y_j|^{2c_j}} dy \wedge d\bar{y} \\
&= \pm \int_{\mathcal{V}_0(\delta)} \frac{|g(\mathbf{0}, y)|^2}{\prod_{i=1}^{N_v} |y_i|^{2b_i} \prod_{j=N_v+1}^{N_v+N_h} |y_j|^{2c_j}} dy \wedge d\bar{y} \\
&= \int_{\widetilde{\mathcal{X}}_0 \cap \mathcal{U}(p,\delta)} \mu|^*_{\mathcal{X}'_0}(v \wedge \bar{v})^{1/m}.
\end{aligned}$$

Notice that here we need to use the crucial fact from (36) that $b_i < 1$ and $c_j < 1$, which follows from the *inversion of adjunction*.

2. Case 2: There are subcases: The number of $H_i's$ containing p is equal to l for some $0 \le l \le d-1$. In each subcase, we can choose coordinates $(x_1,\ldots,x_l;y_1,\ldots,y_{n+d-l})$ denoted (x,y) on $\mathcal{Y}$ and $(t_1,\ldots,t_d)$ denoted t on S such that the toroidal map $\pi_{\mathcal{Y}}$ is defined by:

$$
\begin{aligned}
t_1 &= x_1\prod_{i=1}^{N_v} y_i^{a_{1i}}\cdot g_1(x,y);\\
&\vdots\\
t_l &= x_l\prod_{i=1}^{N_v} y_i^{a_{li}}\cdot g_l(x,y);\\
t_{l+1} &= \prod_{i=1}^{N_v} y_i^{a_{(l+1)i}}\cdot g_{l+1}(x,y);\\
&\vdots\\
t_d &= \prod_{i=1}^{N_v} y_i^{a_{di}}\cdot g_d(x,y).
\end{aligned}
\tag{44}
$$

Here $g_i(x,y)$ are non vanishing holomorphic functions. As before, we only deal with the case when $g_i(x,y)\equiv 1$ since the modification to the general case will be straightforward. Also we will only consider the extremal case: $l=0$, because it will be clear that the other cases are mixture of Case (1) and this extremal case. So in the following we assume the following equalities hold:

$$
\begin{aligned}
t_1 &= \textstyle\prod_{i=1}^{N_v} y_i^{a_{1i}};\\
&\cdots\\
t_d &= \textstyle\prod_{i=1}^{N_v} y_i^{a_{di}}.
\end{aligned}
\tag{45}
$$

Since the map $\pi_{\mathcal{Y}}$ is dominant, we know that the matrix (a_{ki}) is of rank d. Consider again the following polydisk region by passing to local toric covers:

$$\mathcal{U}(p,\delta)=\{|y_i|\le\delta, 1\le i\le n+d\}.$$

We will show that the integral over $\mathcal{U}_t=\mathcal{U}\cap\mathcal{Y}_t$ converges to 0 as $t\to 0$. Similar as in [39], it will be convenient to use the logarithmic coordinates. So we denote $t_k=e^{s_k}e^{\sqrt{-1}\phi_k}=e^{\tau_k}$, $y_i=e^{u_j}e^{\sqrt{-1}\theta_j}=e^{w_i}$, and (45) becomes

$$s_k=\sum_{i=1}^{N_v}a_{ki}u_i,\quad 1\le k\le d. \tag{46}$$

Then it is easy to see that we have

$$\mathcal{U}_t(p,\delta)\cong\underline{\mathcal{V}}_t(\delta)\times(S^1)^{N_v-d}\times\left\{|y_j|\le\delta, N_v+1\le j\le n+d\right\},$$

where the first factor on the right is a *bounded* polytope:

$$\underline{\mathcal{V}}_s:=\underline{\mathcal{V}}_s(p,\delta)=\left\{u_i<\log\delta,\sum_{i=1}^{N_v}a_{ki}u_i=s_k, 1\le k\le d\right\}\subset\mathbb{R}^{N_v}.$$

For the pull-back of holomorphic form, we have the similar formula as in (42) which follows from (37):

$$\mu^*(v^{1/m}) = g(y)\prod_{i=1}^{N_v} y_i^{a_i-b_i}\prod_{j=N_v+1}^{N_v+N_h} y_j^{-c_j}\,(dy\otimes\partial_t).$$

To transform into logarithmic coordinates, we use:

$$y_i^{a_i-b_i}dy_i = e^{(1+a_i-b_i)w_i}dw_i, \quad \partial_{t_k} = \frac{\partial_{\tau_k}}{t_k} = \frac{\partial_{\tau_k}}{\prod_{j=1}^{N_v} y_j^{a_{kj}}} = \frac{\partial_{\tau_k}}{\prod_{j=1}^{N_v} e^{a_{kj}w_j}}.$$

So we can get:

$$\mu^*(v^{1/m}) = \pm g(y)\left(\prod_{i=1}^{N_v} e^{(1-b_i)w_i}\bigwedge_{i=1}^{N_v} dw_i\otimes\partial_\tau\right)\bigwedge_{j=N_v+1}^{N_v+N_h} y_j^{-c_j}dy_j\wedge dy',$$

where $dy' = \bigwedge_{j=N_v+N_h+1}^{n+d} dy_j$. So we have:

$$\mu^*(v\wedge\bar v)^{1/m} = |g(y)|^2\left(\bigwedge_{i=1}^{N_v} e^{2(1-b_i)u_i}du_i\otimes\bigwedge_{k=1}^{d}\partial_{s_k}\right)\left(\bigwedge_{i=1}^{N_v} d\theta_i\otimes\bigwedge_{k=1}^{d}\partial_{\phi_k}\right)$$
$$\wedge\left(\bigwedge_{j=N_v+1}^{N_v+N_h}|y_j|^{-2c_j}dy_j\wedge d\bar y_j\right)\wedge dy'\wedge d\overline{y'}.$$

Notice that since $c_j < 1$, $|y_j|^{-2c_j}dy_j\wedge d\overline{y}_j$ is integrable. So we just need to estimate:

$$\int_{\underline{\mathcal{V}}_s}\bigwedge_{i=1}^{N_v} e^{2(1-b_i)u_i}du_i\otimes\bigwedge_{k=1}^{d}\partial_{s_k}. \tag{47}$$

Note that $\underline{\mathcal{V}}_s$ is a (N_v-d)-dimensional polytope in $\mathbb{R}^{N_v}$ defined by linear functions. By co-area formula, we know that

$$\bigwedge_{i=1}^{N_v} du_i = \frac{1}{A}\mathrm{dvol}\otimes\bigwedge_{k=1}^{d} ds_k.$$

Here we have denoted by dvol the Euclidean volume form on $\underline{\mathcal{V}}_s$, and $A = \det(\langle\mathbf{a}_k,\mathbf{a}_l\rangle)^{1/2}$, where

$$\mathbf{a}_k = \{a_{ki}\} = \nabla s_k, \quad \langle\mathbf{a}_k,\mathbf{a}_l\rangle = \sum_{i=1}^{N_v} a_{ki}a_{li}.$$

So we see that the integral in (47) is equal to:

$$\frac{1}{A}\int_{\underline{\mathcal{V}}_s}\prod_{i=1}^{N_v} e^{2(1-b_i)u_i}\,\mathrm{dvol}. \tag{48}$$

Using $b_i < 1$ for $1\le i\le N_v$, it's now an easy exercise to verify that the integral in (48) converges *uniformly* to 0 as $s\to-\infty$ (meaning $s_k\to-\infty$ uniformly with respect to k).

As mentioned before, for the general sub cases in (44), we can first use calculus of adjunction in Case (1) l-times to kill the variables $x_1,\dots,x_l$ and reduce to the extremal sub case in Case (2). So we know that the contribution in all sub cases of Case (2) are indeed 0 as $t \to 0$.

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