

quatrième série - tome 51 fascicule 4 juillet-août 2018

*ANNALES
SCIENTIFIQUES
de
L'ÉCOLE
NORMALE
SUPÉRIEURE*

Hung Yean LOKE & Jia-Jun MA

Local theta correspondences between supercuspidal representations

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Responsable du comité de rédaction / *Editor-in-chief*

Patrick BERNARD

Publication fondée en 1864 par Louis Pasteur

Continuée de 1872 à 1882 par H. SAINTE-CLAIRE DEVILLE

de 1883 à 1888 par H. DEBRAY

de 1889 à 1900 par C. HERMITE

de 1901 à 1917 par G. DARBOUX

de 1918 à 1941 par É. PICARD

de 1942 à 1967 par P. MONTEL

Comité de rédaction au 1^{er} mars 2018

P. BERNARD

A. NEVES

S. BOUCKSOM

J. SZEFTEL

R. CERF

S. VŨ NGỌC

G. CHENEVIER

A. WIENHARD

Y. DE CORNULIER G. WILLIAMSON

E. KOWALSKI

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,

45, rue d'Ulm, 75230 Paris Cedex 05, France.

Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.

annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France

Case 916 - Luminy

13288 Marseille Cedex 09

Tél. : (33) 04 91 26 74 64

Fax : (33) 04 91 41 17 51

email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 420 euros.

Abonnement avec supplément papier :

Europe : 540 €. Hors Europe : 595 € (\$ 863). Vente au numéro : 77 €.

© 2018 Société Mathématique de France, Paris

En application de la loi du 1^{er} juillet 1992, il est interdit de reproduire, même partiellement, la présente publication sans l'autorisation de l'éditeur ou du Centre français d'exploitation du droit de copie (20, rue des Grands-Augustins, 75006 Paris).

All rights reserved. No part of this publication may be translated, reproduced, stored in a retrieval system or transmitted in any form or by any other means, electronic, mechanical, photocopying, recording or otherwise, without prior permission of the publisher.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret

Périodicité : 6 n^{os} / an

LOCAL THETA CORRESPONDENCES BETWEEN SUPERCUSPIDAL REPRESENTATIONS

BY HUNG YEAN LOKE AND JIA-JUN MA

ABSTRACT. – By the works of Yu, Kim and Hakim-Murnaghan, we have a parameterization and construction of all supercuspidal representations of a reductive p -adic group in terms of supercuspidal data, when p is sufficiently large. In this paper, we will define a correspondence of supercuspidal data via moment maps and theta correspondences over finite fields. Then we will show that local theta correspondences between supercuspidal representations are completely described by this notion. In Appendix B, we give a short proof of a result of Pan on “depth preservation”.

RÉSUMÉ. – Par les travaux de Yu, Kim et Hakim-Murnaghan, on a une paramétrisation et une construction de toutes les représentations supercuspidales d’un groupe réductif p -adique en termes de données supercuspidales, quand p est suffisamment grand. Dans cet article, nous définirons une correspondance entre les données supercuspidales par l’intermédiaire d’applications moments et de correspondances thêta sur des corps finis. Ensuite, nous montrerons que les correspondances thêta locales entre les représentations supercuspidales sont complètement décrites par cette notion. Dans l’Appendice B, nous fournissons une courte démonstration d’un résultat de Pan sur la « préservation de la profondeur ».

1. Introduction

In this paper, we give an explicit description of the local theta correspondences between tamely ramified supercuspidal representations in terms of the supercuspidal data developed in [15, 37, 17, 11].

1.1. Notation

Throughout this paper, we fix a non-Archimedean local field F of characteristic zero with ring of integers \mathfrak{o} , and finite residual field \mathfrak{f} . Let “val” denote the normalized valuation map such that $\text{val}(F) = \mathbb{Z}$. Suppose E is a finite extension of F or the central simple quaternion division algebra over F , let \mathfrak{o}_E denote its ring of integers, let \mathfrak{p}_E denote the maximal ideal in \mathfrak{o}_E and let $\mathfrak{f}_E := \mathfrak{o}_E/\mathfrak{p}_E$ denote the residue field. We continue to let “val” denote the natural extension of valuations to E . When $E = F$, we sometimes omit the subscript. We

fix a non-trivial additive character $\psi: F \rightarrow \mathbb{C}^\times$ with conductor \mathfrak{p} (i.e., $\psi|_{\mathfrak{p}}$ is trivial but $\psi|_{\mathfrak{o}}$ is non-trivial). Let $\bar{\psi}$ denote the additive character on \mathfrak{f} induced by ψ . For a vector space \mathfrak{W} with an endomorphism \star , we let $\mathfrak{W}^{\star, \varepsilon}$ denote the ε -eigenspace of \star in \mathfrak{W} .

1.2. The set of data

Let (D, τ) denote one of the division algebras over F given in Section 2.1 with an F -linear involution τ . Let $\epsilon \in \{\pm 1\}$ and $\epsilon' = -\epsilon$. Let $(V, \langle \cdot, \cdot \rangle_V)$ (respectively $(V', \langle \cdot, \cdot \rangle_{V'})$) denote a right D -module equipped with an ϵ -Hermitian form $\langle \cdot, \cdot \rangle_V$ (respectively ϵ' -Hermitian form $\langle \cdot, \cdot \rangle_{V'}$). Then $W := V \otimes_D V'$ is naturally a symplectic space. Let $(G, G') = (U(V), U(V'))$ be an irreducible type I reductive dual pair in the symplectic group $\underline{\mathrm{Sp}} := \mathrm{Sp}(W)$. For any subset E of Sp let \tilde{E} be its inverse image in the metaplectic \mathbb{C}^\times -cover $\widetilde{\mathrm{Sp}}(W)$ of $\mathrm{Sp}(W)$. See Section 2 for more details of the notation.

We assume that p is large enough compared to the sizes of G and G' since we need the hypotheses in [17, §3.5] to hold. We will give a lower bound for p in Corollary 3.2. We will review the construction of supercuspidal representations for \tilde{G} following [37, 17] in Section 3. Let $\Sigma := (x, \Gamma, \phi, \rho)$ be a supercuspidal datum as in [17]. We briefly explain the entries in Σ : (i) Γ is a semisimple element in \mathfrak{g} and $G^0 := Z_G(\Gamma)$; (ii) x is a point in the building $\mathcal{B}(G^0)$ of G^0 ; (iii) ϕ and ρ are certain representations of G_x^0 . See Definition 3.4 for details. Then Σ will determine an open compact subgroup $K \subseteq G$ and an irreducible K -module η_Σ and, $\pi_\Sigma := \mathrm{c}\text{-Ind}_K^G \eta_\Sigma$ is a supercuspidal representation of G . By [17], under the assumption that p is large enough, this construction gives all supercuspidal representations of G . Let \mathcal{D}_V be the set of all supercuspidal data and let \hat{G}_{sc} be the equivalence classes of irreducible supercuspidal G -modules. In [11] an equivalence relation \sim on \mathcal{D}_V is defined so that $\overline{\mathcal{D}}_V := \mathcal{D}_V / \sim \rightarrow \hat{G}_{\mathrm{sc}}$ given by $[\Sigma] \mapsto [\pi_\Sigma]$ is a bijection. In other words, $\overline{\mathcal{D}}_V$ parametrizes \hat{G}_{sc} . In fact, the equivalence relation is just G -conjugacy in our situation (cf. Definition 3.6).

Now we consider the covering group \tilde{G} . It is well known that the cover $\tilde{K} \rightarrow K$ splits. Given a certain splitting $\xi: K \rightarrow \tilde{K}$, we identify \tilde{K} with $K \times \mathbb{C}^\times$. We call $\tilde{\Sigma} := (\Sigma, \xi) = (x, \Gamma, \phi, \rho, \xi)$ a supercuspidal datum of \tilde{G} . Define $\tilde{\eta}_{\tilde{\Sigma}} := \eta_\Sigma \boxtimes \mathrm{id}_{\mathbb{C}^\times}$ which is an irreducible \tilde{K} -module. Then $\tilde{\pi}_{\tilde{\Sigma}} := \mathrm{c}\text{-Ind}_{\tilde{K}}^{\tilde{G}} \tilde{\eta}_{\tilde{\Sigma}}$ is an irreducible supercuspidal representation of \tilde{G} . We will see in Section 3.5.4 that under the assumption that p is large enough, the construction of $\tilde{\pi}_{\tilde{\Sigma}}$ exhausts all the irreducible supercuspidal genuine⁽¹⁾ representations of \tilde{G} . The equivalence relation on the set of data of \tilde{G} could also be deduced from that of G easily (cf. Section 3.5).

1.3. Statement of the main theorem

We retain the notation in Section 1.2. Fix a Witt tower \mathcal{F}' of ϵ' -Hermitian spaces. The covering group \tilde{G} in the dual pair $(G, G') = (U(V), U(V'))$ for all $V' \in \mathcal{F}'$ are canonically isomorphic to one another. Let ω be the Weil representation of $\widetilde{\mathrm{Sp}}(W)$ with respect to the character ψ and let

$$(1.1) \quad \mathcal{R}(\tilde{G}, \omega) := \{ \tilde{\pi} \in \mathrm{Irr}_{\mathrm{gen}}(\tilde{G}) \mid \mathrm{Hom}_{\tilde{G}}(\omega, \tilde{\pi}) \neq 0 \}$$

⁽¹⁾ Here genuine means $\mathbb{C}^\times \subseteq \tilde{G}$ acts by multiplication.

be the equivalence classes of irreducible smooth genuine \tilde{G} -modules which could be realized as a quotient of ω . Let $\theta_{V,V'}: \mathcal{R}(\tilde{G}, \omega) \rightarrow \mathcal{R}(\tilde{G}', \omega)$ denote the theta correspondence map.

Let $\tilde{\pi}$ be an irreducible supercuspidal genuine \tilde{G} -module. Note that the $\tilde{\pi}$ -isotypic component $\omega[\tilde{\pi}]$ of ω is naturally a $\tilde{G} \times \tilde{G}'$ module, say $\omega[\tilde{\pi}] \cong \tilde{\pi} \boxtimes \Theta_{V,V'}(\tilde{\pi})$ where $\Theta_{V,V'}(\tilde{\pi})$ is a genuine \tilde{G}' -module. Let

$$m_{\mathcal{F}'}(\tilde{\pi}) = \min \{ \dim_D(V'') \mid \Theta_{V,V''}(\tilde{\pi}) \neq 0 \text{ where } V'' \in \mathcal{F}' \}$$

which is called the *first occurrence index* of π with respect to the Witt tower \mathcal{F}' .

It is well known that (cf. [23, Section 3.IV.4, théorème principal]):

- (i) $\Theta_{V,V'}(\tilde{\pi})$ is either zero or irreducible.
- (ii) $m_{\mathcal{F}'}(\tilde{\pi}) \leq 2 \dim V + a_{\mathcal{F}'}$ where $a_{\mathcal{F}'} = \min \{ \dim_D V'' \mid V'' \in \mathcal{F}' \}$ is the dimension of the anisotropic kernel in \mathcal{F}' (cf. [20]).
- (iii) $\Theta_{V,V'}(\tilde{\pi}) \neq 0$ if and only if $\dim_D(V') \geq m_{\mathcal{F}'}(\tilde{\pi})$ in which case $\theta_{V,V'}(\tilde{\pi}) = \Theta_{V,V'}(\tilde{\pi})$.
- (iv) $\theta_{V,V'}(\tilde{\pi})$ is supercuspidal if and only if $\dim(V') = m_{\mathcal{F}'}(\tilde{\pi})$. In this case, we say that the first occurrence of $\tilde{\pi}$ is at V' .

The aim of this paper is to describe the first occurrences of theta lifts of supercuspidal representations in terms of the supercuspidal data.

Let

$$(1.2) \quad \overline{\mathcal{D}}_{\mathcal{F}'} = \bigsqcup_{V' \in \mathcal{F}'} \overline{\mathcal{D}}_{V'}.$$

Using the moment maps and theta correspondences over finite fields, we will define theta lifts of equivalence classes of supercuspidal data in Section 5, i.e., we will define a map

$$(1.3) \quad \vartheta_{V,\mathcal{F}'}: \overline{\mathcal{D}}_V \hookrightarrow \overline{\mathcal{D}}_{\mathcal{F}'}.$$

Fix a pair of data $(\Sigma, \Sigma') \in \mathcal{D}_V \times \mathcal{D}_{V'}$. There is a canonical splitting

$$\xi_{x,x'}: K \times K' \longrightarrow \tilde{K} \times \tilde{K}'$$

constructed from the generalized lattice model (cf. (2.4)). We always set $\tilde{\Sigma} = (\Sigma, \xi_{x,x'}|_K)$ and $\tilde{\Sigma}' = (\Sigma', \xi_{x,x'}|_{K'})$.

MAIN THEOREM. – (i) Suppose $\Sigma \in \mathcal{D}_V$ and $[\Sigma'] := \vartheta_{V,\mathcal{F}'}([\Sigma]) \in \overline{\mathcal{D}}_{V'}$ for certain $V' \in \mathcal{F}'$. Then $\theta_{V,V'}(\tilde{\pi}_{\tilde{\Sigma}}) = \tilde{\pi}'_{\tilde{\Sigma}'}$. (ii) Conversely, suppose $\theta_{V,V'}(\tilde{\pi}) = \tilde{\pi}'$, such that $\tilde{\pi}$ and $\tilde{\pi}'$ are supercuspidal representations. Then there exists $\Sigma \in \mathcal{D}_V$ such that $\tilde{\pi} = \tilde{\pi}_{\tilde{\Sigma}}$ and $\tilde{\pi}' = \tilde{\pi}'_{\tilde{\Sigma}'}$, where $[\Sigma'] = \vartheta_{V,\mathcal{F}'}([\Sigma])$ and \mathcal{F}' is the Witt class of V' .

REMARKS. – 1. If $\tilde{\pi}$ is a depth zero supercuspidal representation, then $\vartheta_{V,\mathcal{F}'}(\tilde{\pi})$ is essentially constructed in [28].

2. After the completion of the first draft of this paper, we received a preprint [30] from Pan which describes the theta lifts of certain positive depth supercuspidal representations.

3. The main theorem generalizes our earlier results with Savin for epipelagic representations [22].

4. The construction of $\vartheta_{V,\mathcal{F}'}$ provides a criterion on the occurrence of supercuspidal representations by conditions on the isomorphism classes of the Hermitian spaces modulo the theta correspondences over finite fields. On the other hand, for some supercuspidal

representations, theta correspondences over finite fields do not show up in the descriptions of their first occurrences. See Section 5.4 for details.

5. In the proof of Main Theorem 1.3 (ii), we need a generalization of [27, Proposition 6.3] which is proved in Appendix B. This also leads to a simpler proof of Pan's theorem on "depth preservation" [2, 27].

6. A similar result in terms of the parametrization developed by Bushnell-Kutzko [5] and Stevens [34] should also be established. We hope to take on this problem in a future project.

1.4. Organization of the paper

In Section 2 we recall some basic definitions and notations of local theta correspondences and generalized lattice models. In Section 3, we review the definition of supercuspidal data and the constructions of supercuspidal representations for both linear and covering groups.

In Section 4 we define the block decompositions of supercuspidal data in terms of valuations of eigenvalues. In Section 5, we first review the correspondence for depth zero representations and then define the lift of a single block supercuspidal datum using the moment maps. By taking direct sum, the lift in the general case is defined in the end.

We begin the proof of the main theorem with the single block case in Section 6 and Section 7. The geometric structures of moment maps are studied in Section 6 and refined K -types are constructed in Section 7 using these structures. These two sections are the most technical parts of the paper.

By induction on the number of blocks, we prove part 1.3 (i) of the Main Theorem in Section 8. Using Main Theorem and a similar induction, part 1.3 (ii) of the Main Theorem is finally proved in Section 9.

In Appendix A, we review the Heisenberg-Weil representations over a finite field and the special isomorphisms of Yu. These are used freely in Sections 6, 8 and 9. In Appendix B, we first prove the generalization of an identity of Pan needed in Section 9 and then finish the paper by giving a quick proof of the "depth preservation".

Acknowledgments

We would like to thank Gordan Savin and Jiu-Kang Yu for inspiring discussions. We would like to thank Wee Teck Gan and Eitan Sayag for their valuable comments and encouragement. We also would like to thank the referees for their careful reading and comments, in particular for their call to discuss supercuspidal representations of Metaplectic groups in greater detail. Hung Yean Loke is supported by a MOE-NUS AcRF Tier 1 grant R-146-000-208-112. Jia-Jun Ma is partially supported by HKRGC Grant CUHK 405213 during his postdoctoral fellowship in IMS of CUHK and by MOE Tier 1 Grant R-146-000-189-112 and Shanghai Jiao Tong University Startup Fund WF220407110 during the revision of this paper.

2. Preliminaries: Local theta correspondence

In this section, we set up some notations and review some facts about the generalized lattice model of the oscillator representation.

2.1. Type I dual pairs and moment maps

Let (D, τ) denote a division algebra D over F with an F -linear involution τ in one of the following situations:

- (a) $D = F$ and τ is the identity map;
- (b) D is a quadratic field extension of F and τ is the nontrivial element in $\text{Gal}(D/F)$;
- (c) D is the central division quaternion algebra over F and τ is the main involution.

2.1.1. – Let $\epsilon \in \{\pm 1\}$. Let $(V, \langle \cdot, \cdot \rangle_V)$ or simply V denote a right D -module equipped with an ϵ -Hermitian form $\langle \cdot, \cdot \rangle_V$. Let $\mathfrak{gl}(V) := \text{End}_D(V)$ be the Lie algebra of $\text{GL}_D(V)$. For $X \in \mathfrak{gl}(V)$, let $X^* \in \mathfrak{gl}(V)$ denote the adjoint of X which is defined by

$$\langle Xv_1, v_2 \rangle_V = \langle v_1, X^*v_2 \rangle_V \quad \forall v_1, v_2 \in V.$$

Then the isometry group of V and its Lie algebra are given by

$$\begin{aligned} \text{U}(V) &= \{ g \in \mathfrak{gl}(V) \mid gg^* = \text{id} \} \quad \text{and} \\ \mathfrak{u}(V) &= \{ X \in \mathfrak{gl}(V) \mid X + X^* = 0 \} = \mathfrak{gl}(V)^{*, -1} \end{aligned}$$

respectively. We will always view $\text{U}(V)$ and $\mathfrak{u}(V)$ as subsets of $\text{End}_D(V)$.

Let $\text{tr}_{D/F}: D \rightarrow F$ be the reduced trace on D . We set $\text{tr}_F := \text{tr}_{D/F} \circ \text{tr}: \mathfrak{gl}(V) \rightarrow F$. Clearly $\text{tr}_F(X) = \text{tr}_F(X^*)$. Let

$$\mathbb{B}(X, Y) := \frac{1}{2} \text{tr}_F(XY).$$

It is the invariant non-degenerate bilinear form on $\mathfrak{gl}(V)$ and $\mathfrak{u}(V)$ which we fix throughout this paper.

2.1.2. – Let $\epsilon' = -\epsilon$ and $(V', \langle \cdot, \cdot \rangle_{V'})$ be a right D -module equipped with an ϵ' -Hermitian sesquilinear form $\langle \cdot, \cdot \rangle_{V'}$. We view V' as a left D -module by $av = va^\tau$ for all $a \in D$ and $v \in V'$. Let $W = V \otimes_D V'$. We always identify W with $\text{Hom}_D(V, V')$ by $v \otimes v' \mapsto (v_1 \mapsto v' \langle v, v_1 \rangle_V)$. For any $w \in \text{Hom}_D(V, V')$, let $w^\star \in \text{Hom}_D(V', V)$ denote its adjoint which is defined by

$$\langle wv, v' \rangle_{V'} = \langle v, w^\star v' \rangle_V \quad \forall v \in V, v' \in V'.$$

The F -vector space W will be equipped with a symplectic form $\langle \cdot, \cdot \rangle_W$ given by

$$\langle v_1 \otimes v'_1, v_2 \otimes v'_2 \rangle_W = \text{tr}_{D/F}(\langle v_1, v_2 \rangle_V \langle v'_1, v'_2 \rangle_{V'}^\tau).$$

Let $G = \text{U}(V, \langle \cdot, \cdot \rangle_V)$ and $G' = \text{U}(V', \langle \cdot, \cdot \rangle_{V'})$. The pair (G, G') is called an irreducible reductive dual pair of type I in $\text{Sp}(W)$ following Howe. The above construction gives all such pairs when F varies (cf. [13, §5] or [21, Lecture 5]).

Let $\mathfrak{g} := \mathfrak{u}(V)$ and $\mathfrak{g}' := \mathfrak{u}(V')$ denote the Lie algebras of G and G' respectively. For $w \in W$, it is not hard to see that $w^\star w \in \mathfrak{g} \subseteq \text{End}_D(V)$ and $ww^\star \in \mathfrak{g}' \subseteq \text{End}_D(V')$.

DEFINITION 2.1. – We define the *moment maps* $M: W \rightarrow \mathfrak{g}$ and $M': W \rightarrow \mathfrak{g}'$ for the dual pair (G, G') by

$$M(w) = w^\star w \quad \text{and} \quad M'(w) = ww^\star \quad \forall w \in W.$$

The Lie algebras \mathfrak{g} and \mathfrak{g}' act on $W = \text{Hom}_D(V, V')$ by

$$(X \cdot w)(v) = w(-Xv) \text{ and } (X' \cdot w)(v) = X'(w(v))$$

for all $w \in W$, $v \in V$, $X \in \mathfrak{g}$ and $X' \in \mathfrak{g}'$. We leave the proof of the following simple formulas to the reader.

LEMMA 2.2. – *Let $w, w_1, w_2 \in W$, $X \in \mathfrak{g}$ and $X' \in \mathfrak{g}'$. Then*

- (i) $\langle w_1, w_2 \rangle_W = \text{tr}_F(w_1^{\star} w_2)$,
- (ii) $\langle X \cdot w, w \rangle_W = 2\mathbb{B}(X, M(w))$ and
- (iii) $\langle X' \cdot w, w \rangle_W = 2\mathbb{B}(X', -M'(w))$.

2.2. Lattice functions and Bruhat-Tits Buildings

We recall some well known facts about self-dual lattice functions. We refer to [22, § 4] for more details.

DEFINITION 2.3. – *A lattice function \mathcal{L} in V is a function which maps $s \in \mathbb{R} \sqcup \mathbb{R}^+$ to an \mathfrak{o}_D -lattice \mathcal{L}_s in V such that (i) $\mathcal{L}_s \supseteq \mathcal{L}_t$ if $s < t$, (ii) $\mathcal{L}_{s+\text{val}(a)} = \mathcal{L}_s a$ for all $a \in D^\times$, (iii) $\mathcal{L}_s = \bigcap_{t < s} \mathcal{L}_t$ and, (iv) $\mathcal{L}_{s+} = \bigcup_{t > s} \mathcal{L}_t$. For a lattice function \mathcal{L} , we set*

$$\text{Jump}(\mathcal{L}) := \{r \in \mathbb{R} \mid \mathcal{L}_r \not\subseteq \mathcal{L}_{r+}\}.$$

For an \mathfrak{o}_D -lattice L in V , we denote its dual lattice

$$L^* := \{v \in V \mid \langle v, L \rangle_V \subseteq \mathfrak{p}_D\}.$$

A lattice function \mathcal{L} in V is called *self-dual* if $(\mathcal{L}_t)^* = \mathcal{L}_{-t+}$.

We always let $\mathcal{B}(G)$ denote the (extended) Bruhat-Tits building of G . Then $\mathcal{B}(G)$ is naturally identified with the set of self-dual lattice functions (cf. [3, 4] and [22, § 4]). For any $x \in \mathcal{B}(G)$, we let \mathcal{L}_x denote the corresponding lattice function. Let G_x denote the stabilizer of x in G . For $r \in \mathbb{R} \sqcup \mathbb{R}^+$ and $r \geq 0$ (respectively $r \in \mathbb{R} \sqcup \mathbb{R}^+$), we let $G_{x,r}$ denote the corresponding Moy-Prasad subgroup of G (respectively Lie subalgebra of \mathfrak{g}) [24, 25]. For $r < t$, we set

$$\mathfrak{g}_{x,r:t} := \mathfrak{g}_{x,r} / \mathfrak{g}_{x,t}.$$

Let \mathcal{L}_x and $\mathcal{L}'_{x'}$ be two self-dual lattice functions in V and V' respectively. We define a lattice function $\mathcal{B}_{x,x'}$ on $W = V \otimes_D V'$ by

$$(2.1) \quad \mathcal{B}_{x,x',t} := (\mathcal{L}_x \otimes_D \mathcal{L}'_{x'})_t := \sum_{t=t_1+t_2} \mathcal{L}_{x,t_1} \otimes_{\mathfrak{o}_D} \mathcal{L}'_{x',t_2}.$$

Then $\mathcal{B}_{x,x'}$ is a self-dual lattice function on W . We view $\text{Hom}_{\mathfrak{o}_D}(\mathcal{L}_{x,t}, \mathcal{L}'_{x',t'})$ as a lattice in $\text{Hom}_D(V, V') = W$. Then $\mathcal{B}_{x,x',t} = \bigcap_r \text{Hom}_{\mathfrak{o}_D}(\mathcal{L}_{x,r}, \mathcal{L}'_{x',t+r})$.

Now $(x, x') \mapsto \mathcal{B}_{x,x'}$ gives a natural $G \times G'$ -equivariant map⁽²⁾

$$\mathcal{B}(G) \times \mathcal{B}(G') \longrightarrow \mathcal{B}(\text{Sp}(W)).$$

⁽²⁾ One can show that it is an embedding. In fact, the map is a restriction of the natural embedding $\mathcal{B}_{\text{red}}(\text{GL}(V)) \times \mathcal{B}_{\text{red}}(\text{GL}(V')) \rightarrow \mathcal{B}_{\text{red}}(\text{GL}(V \otimes_D V'))$ between reduced buildings.

If it is clear what x and x' are, then we will suppress x, x' and simply write $\mathcal{L} = \mathcal{L}_x$, $\mathcal{L}' = \mathcal{L}'_{x'}$, and $\mathcal{B} = \mathcal{B}_{x,x'}$. For $s < t$, we denote

$$\mathcal{L}_{s:t} := \mathcal{L}_s / \mathcal{L}_t, \quad \mathcal{L}'_{s:t} := \mathcal{L}'_s / \mathcal{L}'_t \quad \text{and} \quad \mathcal{B}_{s:t} := \mathcal{B}_s / \mathcal{B}_t.$$

2.3. Generalized lattice model

Let W be a symplectic space. Let $H(W) = W \times F$ denote the corresponding Heisenberg group and let $\widetilde{\text{Sp}}(W)$ denote the metaplectic \mathbb{C}^\times -covering of $\text{Sp}(W)$.

Let (ω, \mathcal{S}) or simply ω denote the oscillator representation of $\widetilde{\text{Sp}}(W) \times H(W)$ with central character ψ . We recall below the definition of the generalized lattice model of the oscillator representation. See [36] or [22, § 3] for more details.

2.3.1. – Fix a self-dual lattice function \mathcal{B} in W . Let $\mathfrak{b} := \mathcal{B}_0 / \mathcal{B}_{0+}$. The symplectic form $\langle \cdot, \cdot \rangle_W$ induces a non-degenerate symplectic form on the \mathfrak{f} -vector space \mathfrak{b} . Let $\mathbf{H}(\mathfrak{b}) = \mathfrak{b} \times \mathfrak{f}$ be the Heisenberg group defined by \mathfrak{b} . Let $(\overline{\omega}_{\mathfrak{b}}, \mathbb{S}(\mathfrak{b}))$ be the oscillator representation of $\mathbf{SH}(\mathfrak{b}) := \mathbf{Sp}(\mathfrak{b}) \times \mathbf{H}(\mathfrak{b})$ with central character $\overline{\psi}$ (cf. Section 1.1). See Appendix A.1. Let $H(\mathcal{B}_0) := \mathcal{B}_0 \times \mathfrak{o} \subseteq H(W)$, $\text{Sp}_{\mathcal{B}} := \{g \in \text{Sp}(W) \mid g\mathcal{B}_0 = \mathcal{B}_0\}$ and $\text{Sp}_{\mathcal{B},0+} := \{g \in \text{Sp}(W) \mid (g-1)\mathcal{B}_0 \subseteq \mathcal{B}_{0+}\}$. By an abuse of notation, we also let $\overline{\omega}_{\mathfrak{b}}$ denote its inflation to $\text{Sp}_{\mathcal{B}} \times H(\mathcal{B}_0)$ via the natural quotient map.

A generalized lattice model with respect to \mathcal{B}_0 of the oscillator representation (ω, \mathcal{S}) is realized on the following space of functions

$$\mathcal{S}(\mathcal{B}_0) := \left\{ f: W \rightarrow \mathbb{S}(\mathfrak{b}) \mid \begin{array}{l} f \text{ is locally constant and compactly supported,} \\ f(a+w) = \psi(\frac{1}{2} \langle w, a \rangle_W) \overline{\omega}_{\mathfrak{b}}(a) f(w) \quad \forall a \in \mathcal{B}_0 \end{array} \right\}.$$

Via the generalized lattice model $\mathcal{S}(\mathcal{B}_0)$, we get a splitting $\xi_{\mathcal{B}}: \text{Sp}_{\mathcal{B}} \hookrightarrow \widetilde{\text{Sp}}_{\mathcal{B}} \subseteq \widetilde{\text{Sp}}(W)$ given by

$$(2.2) \quad ((\omega \circ \xi_{\mathcal{B}}(k))f)(w) = \overline{\omega}_{\mathfrak{b}}(k) f(k^{-1} \cdot w) \quad \forall k \in \text{Sp}_{\mathcal{B}}, w \in W, f \in \mathcal{S}(\mathcal{B}_0).$$

The splitting $\xi_{\mathcal{B}}: \text{Sp}_{\mathcal{B}} \rightarrow \widetilde{\text{Sp}}_{\mathcal{B}}$ extends to an isomorphism

$$\widetilde{\xi}_{\mathcal{B}}: \text{Sp}_{\mathcal{B}} \times \mathbb{C}^\times \xrightarrow{\sim} \widetilde{\text{Sp}}_{\mathcal{B}}$$

given by $(k, c) \mapsto \xi_{\mathcal{B}}(k)c$.

If there is no fear of confusion, we will write $\omega \circ \xi_{\mathcal{B}}(k)$ as $\omega(k)$ for $k \in \text{Sp}_{\mathcal{B}}$. By [22, Appendix C], the splitting restricted on $\text{Sp}_{\mathcal{B},0+}$ is independent of the choices of \mathcal{B} and agrees with Kudla’s splitting.⁽³⁾ In particular we have following canonical splitting on the pro-unipotent part of Sp :

$$(2.3) \quad \Xi: \bigcup_{\mathcal{B} \in \mathcal{B}(\text{Sp})} \text{Sp}_{\mathcal{B},0+} \longrightarrow \bigcup_{\mathcal{B} \in \mathcal{B}(\text{Sp})} \widetilde{\text{Sp}}_{\mathcal{B},0+}.$$

⁽³⁾ We only checked the compatibility of splittings for lattice model in [22, Appendix C]. We still need to check the compatibility between generalized lattice model and lattice model. However this is clear by testing on the unique (up to scalar) fixed vector of a certain self-dual lattice.

Alternatively, one can prove this using the fact that the first and second cohomologies of a pro- p group taking values in a 2-group is trivial when $p \neq 2$. See [12, Proposition 2.3].

We warn that the canonical splitting does not extend to the union $\bigcup_{\mathcal{B} \in \mathcal{B}(\text{Sp})} \text{Sp}_{\mathcal{B}}$

For any subset $\Omega \subseteq W$ and any element $w \in W$, we set

$$\mathcal{S}(\mathcal{B}_0)_\Omega := \{ f \in \mathcal{S}(\mathcal{B}_0) \mid \text{supp}(f) \subseteq \Omega + \mathcal{B}_0 \}$$

and $\mathcal{S}(\mathcal{B}_0)_w := \mathcal{S}(\mathcal{B}_0)_{\{w\}}$.

Suppose $\mathcal{B} = \mathcal{B}_{x,x'}$ where (x, x') is a pair of points in $\mathcal{B}(G) \times \mathcal{B}(G')$. Then $G_x \times G'_{x'} \subseteq \text{Sp}_\mathcal{B}$. The restriction of $\xi_\mathcal{B}$ gives a splitting

$$(2.4) \quad \xi_{x,x'} := \xi_\mathcal{B}|_{G_x \times G'_{x'}} : G_x \times G'_{x'} \longrightarrow \tilde{G}_x \times \tilde{G}'_{x'}$$

of the covering $\tilde{G}_x \times \tilde{G}'_{x'} \twoheadrightarrow G_x \times G'_{x'}$. The restriction of $\xi_{x,x'}$ to the subgroup $K \times K' \subseteq G_x \times G'_{x'}$ (still called $\xi_{x,x'}$) is the canonical splitting we referred to in Section 1.3.

2.3.2. – We now study a subspace of \mathcal{S} as an induced representation which plays a key role later in this paper.

Fix an element $w \in W$ and let

$$S_w := \text{Stab}_{\text{Sp}_\mathcal{B}}(w + \mathcal{B}_0) = \{ h \in \text{Sp}_\mathcal{B} \mid h \cdot w - w \in \mathcal{B}_0 \}.$$

The evaluation at w given by $f \mapsto f(w)$ induces an isomorphism $\mathcal{S}(\mathcal{B}_0)_w \xrightarrow{\sim} \mathbb{S}(\mathbf{b})$. Clearly S_w acts on $\mathcal{S}(\mathcal{B}_0)_w$ which translates to an action on $\mathbb{S}(\mathbf{b})$. We will denote the resulting S_w -action on $\mathbb{S}(\mathbf{b})$ by $\bar{\omega}_w$.

LEMMA 2.4. – *The group S_w acts on $\mathbb{S}(\mathbf{b})$ by*

$$(2.5) \quad \begin{aligned} \bar{\omega}_w(h) &:= \bar{\omega}_\mathbf{b}(h) \bar{\omega}_\mathbf{b}(h^{-1} \cdot w - w) \psi\left(\frac{1}{2} \langle w, h^{-1}w - w \rangle_W\right) \\ &= \bar{\omega}_\mathbf{b}(h) \bar{\omega}_\mathbf{b}(h^{-1} \cdot w - w) \psi\left(\frac{1}{2} \langle h \cdot w - w, w \rangle_W\right) \end{aligned}$$

for all $h \in S_w$.

Let H be a subgroup of $\text{Sp}_\mathcal{B}$ and $S := \text{Stab}_H(w + \mathcal{B}_0) = H \cap S_w$. We have an isomorphism of H -modules

$$\mathfrak{T}: \mathcal{S}(\mathcal{B}_0)_{H \cdot w + \mathcal{B}_0} \xrightarrow{\sim} \text{c-Ind}_S^H \mathcal{S}(\mathcal{B}_0)_w \cong \text{c-Ind}_S^H \bar{\omega}_w$$

given by $(\mathfrak{T}(f))(k) = (\omega(k)f)(w)$ for all $k \in H$.

Proof. – Let $h \in S_w$. Then $h^{-1} \cdot w - w \in \mathcal{B}_0$. Hence, for any $f \in \mathcal{S}(\mathcal{B}_0)$,

$$\begin{aligned} (\omega(h)f)(w) &= \bar{\omega}_\mathbf{b}(h) f(h^{-1} \cdot w) = \bar{\omega}_\mathbf{b}(h) f(w + (h^{-1} \cdot w - w)) \\ &= \bar{\omega}_\mathbf{b}(h) \bar{\omega}_\mathbf{b}(h^{-1} \cdot w - w) \psi\left(\frac{1}{2} \langle w, h^{-1} \cdot w - w \rangle_W\right) f(w) \\ &= \bar{\omega}_w(h) f(w). \end{aligned}$$

Observe that $\langle w, h^{-1} \cdot w - w \rangle_W = \langle w, h^{-1} \cdot w \rangle_W = \langle h \cdot w, w \rangle_W = \langle h \cdot w - w, w \rangle_W$. The second equality in (2.5) follows.

Note that $h \mapsto h \cdot w + \mathcal{B}_0$ defines a bijection $H/S \cong (H \cdot w + \mathcal{B}_0)/\mathcal{B}_0 \subseteq W/\mathcal{B}_0$. Hence $\mathcal{S}(\mathcal{B}_0)_{H \cdot w + \mathcal{B}_0} = \text{Span} \{ \omega(H) \mathcal{S}(\mathcal{B}_0)_w \}$ and \mathfrak{T} is an isomorphism. \square

REMARKS. – It is easy to see that (2.5) could be simplified greatly in some cases.

1. Suppose that $w \in \mathcal{B}_{-s}$ for certain $s > 0$. Let $h := (g, g') = (\exp(X), \exp(X')) \in G_{x,s} \times G'_{x',s}$ where $(X, X') \in \mathfrak{g}_{x,s} \oplus \mathfrak{g}'_{x',s}$. Then

$$\begin{aligned}
 (2.6) \quad \frac{1}{2} \langle w, h^{-1} \cdot w - w \rangle_W &= \frac{1}{2} \langle h \cdot w - w, w \rangle_W \\
 &\equiv \frac{1}{2} \left\langle (X, X') \cdot w + \frac{1}{2} (X, X') \cdot (X, X') \cdot w, w \right\rangle_W \pmod{\mathfrak{p}} \\
 &= \frac{1}{2} \langle (X, X') \cdot w, w \rangle_W - \frac{1}{4} \langle (X, X') \cdot w, (X, X') \cdot w \rangle_W \\
 &= \frac{1}{2} \langle X \cdot w, w \rangle_W + \frac{1}{2} \langle X' \cdot w, w \rangle_W \\
 &= \mathbb{B}(X, M(w)) + \mathbb{B}(X', -M'(w)) \quad (\text{by Lemma 2.2}).
 \end{aligned}$$

This immediately implies $\psi(\frac{1}{2} \langle h \cdot w - w, w \rangle_W) = \psi_{M(w)}(g) \psi_{-M'(w)}(g')$ (see (3.2) for the definition of $\psi_{M(w)}$) and

$$(2.7) \quad \bar{\omega}_w(h) = \bar{\omega}_b(-(X, X') \cdot w) \psi_{M(w)}(g) \psi_{-M'(w)}(g').$$

2. Suppose $h = (g, g') = \exp(X, X') \in G_{x,s+} \times G'_{x',s+}$. Then $(X, X') \cdot w \in \mathcal{B}_{0+}$ and (2.7) could be further simplified into

$$(2.8) \quad \bar{\omega}_w(h) = \psi_{M(w)}(g) \psi_{-M'(w)}(g').$$

3. Preliminaries: Supercuspidal representations

In this section, we will first review the parametrization of tamely ramified supercuspidal representations for classical groups G when p is sufficiently large. Then we will extend the notion to the covering groups \tilde{G} . We follow closely the notations and formulation in [17].

3.1. Residue characteristics

We assume that the residue characteristics p is large enough compared to the size of G so that all the hypotheses in [17, §3.4] hold. In this subsection, we will find a lower bound for p . Let

- (i) e_D be the absolute ramification index of D/\mathbb{Q}_p if D is a field, or
- (ii) $e_D = 2e_F$ if D is the quaternion division algebra over F .

PROPOSITION 3.1. – *Suppose V is an ϵ -Hermitian space over D such that $n := \dim_D(V)$. Kim's hypotheses [17, § 3.4] are satisfied for $U(V)$ if*

$$(3.1) \quad p \geq \max \{ 2n + 1, e_D n + 2 \}.$$

Proof. – We check each of Kim's hypotheses (Hk), (HB), (HT) and (HN):

1. (Hk.1) requires the exponential map to be well-defined on $\mathfrak{g}_{x,0+}$, which is ensured by $p \geq e_D n + 2$ [7, Section B.1].
2. (Hk.2) translates to $p \geq e_D n + 2$ for $G \subset \mathrm{GL}_n(D)$.
3. (HB) holds for classical groups when $p \neq 2$ since it holds for GL and classical group is the fixed point of an involution. We would like to thank J. Adler for the discussion.
4. (HT) holds by the Howe factorization (cf. Proposition 4.3).

5. (HN) consists of the set of hypotheses in [6, § 4.2]. Hypothesis 4.2.3 holds for $p \geq 2n + 1$. Hypothesis 4.2.1 holds by Hypothesis 4.2.3 in characteristic zero case (see [6, Appendix A]). Hypothesis 4.2.4 and Hypothesis 4.2.5 hold since F is characteristic zero. Hypothesis 4.2.7 holds for the exponential map by (Hk).

This proves the proposition. \square

COROLLARY 3.2. – *Let $(G, G') := (U(V), U(V'))$ be a type I dual pair with $n := \dim_D V$. Let $\tilde{\pi}$ and $\tilde{\pi}'$ be irreducible supercuspidal genuine representations of \tilde{G} and \tilde{G}' respectively such that $\tilde{\pi}' = \theta_{V, V'}(\tilde{\pi})$. Then Kim's hypotheses in [17, § 3.4] are satisfied for $U(V)$ and $U(V')$ if*

$$p \geq \max \{ 4n + 9, e_D(2n + 4) + 2 \}.$$

Proof. – By Section 1.3 (ii) $\dim_D V' \leq 2 \dim_D V + a_{\mathcal{G}'} \leq 2n + 4$. Then p satisfying the inequality in the corollary will satisfy (3.1) for both $U(V)$ and $U(V')$. \square

3.2. Good factorization

Let $\Gamma \in \mathfrak{g}$ be a semisimple element. We say that Γ is *tamely ramified* if Γ lies in a Cartan subalgebra \mathfrak{t} which splits over certain tamely ramified extension E of F . Let $\text{depth}: \mathfrak{g} \rightarrow \mathbb{Q}$ denote the depth function given by

$$\text{depth}(X) = \sup_{x \in \mathcal{B}(G)} \{ r \mid X \in \mathfrak{g}_{x,r} \setminus \mathfrak{g}_{x,r+} \} \quad \forall X \in \mathfrak{g}.$$

We say that Γ is *good* or *G-good* if for every root α of $\mathfrak{g}(E) := \mathfrak{g} \otimes_F E$ with respect to $\mathfrak{t}(E)$, $d\alpha(\Gamma)$ is either zero or has valuation $\text{depth}(\Gamma)$. See [1] and [18, § 2].

DEFINITION 3.3. – Suppose Γ is a tamely ramified semisimple element in \mathfrak{g} with depth $-r < 0$. A decomposition of $\Gamma = \sum_{i=-1}^d \Gamma_i$ in \mathfrak{g} is called a *good factorization* if the following hold:

- (a) $\{ \Gamma, \Gamma_d, \dots, \Gamma_{-1} \}$ is a set of commuting semisimple elements in \mathfrak{g} ;
- (b) $\text{depth}(\Gamma_{-1}) \geq 0$ and we set $r_{-1} = 0$;
- (c) If $0 \leq i < d$, then Γ_i is a good element and $-r_i := \text{depth}(\Gamma_i) < 0$;
- (d) $\Gamma_d \in \mathbf{Z}(\mathfrak{g})$;
- (e) If $\Gamma_d = 0$ (called Case I), then $-r_{d-1} < \dots < -r_0 < 0$ and we set $r_d := r_{d-1} = r$;
- (f) If $\Gamma_d \neq 0$ (called Case II), then $-r_d < -r_{d-1} < \dots < -r_0 < 0$ where $r_d := r = -\text{depth}(\Gamma_d)$.

Fix a good factorization of Γ as above. We define $G^d = G$ and $G^i = \mathbf{Z}_{G^{i+1}}(\Gamma_i)$ for $0 \leq i \leq d - 1$.

REMARKS. – 1. Good factorization of Γ exists. It is not unique but the set $\{ G^i : 0 \leq i \leq d \}$ are independent of the choice of the good factorization (cf. [17, Prop. 4.7]).

2. By [17, Remarks 5.10], Γ_{-1} plays no role in the construction of supercuspidal data. In general, we always assume $G^0 = \mathbf{Z}_G(\Gamma)$. For example, this could be achieved via replacing Γ by $\Gamma - \Gamma_{-1}$. By the argument in Section 4.1, the condition $G^0 = \mathbf{Z}_G(\Gamma)$ is equivalent to $\Gamma_{-1} \in \mathbf{Z}(\mathfrak{g}^0) = F'[\Gamma_d, \dots, \Gamma_0]$ for one (and so for any) good factorization $\Gamma = \sum_{i=-1}^d \Gamma_i$ where $F' := \mathbf{Z}(D)$ is the center of D .

3.3. Tamely ramified supercuspidal representations for classical groups

We now quickly review the notion of supercuspidal data and the constructions of supercuspidal representations.

We only study classical groups that appear in Type I dual pairs. Let G be such a classical group. In this case, the center $Z(G)$ is anisotropic so the reduced building and the extended building of G are the same. Therefore we will use x instead of its image $[x]$ in the reduced building (cf. [37]).

Under our assumption that p is big enough, the exponential map \exp is well defined on $\bigcup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,0^+}$. Let \log denote the inverse map whenever it makes sense.

3.3.1. – For an element $\Gamma \in \mathfrak{g}$, we define a function ψ_Γ on the domain of \log by

$$(3.2) \quad \psi_\Gamma(g) := \psi(\mathbb{B}(\log(g), \Gamma)).$$

DEFINITION 3.4. – A *supercuspidal datum* for G is a tuple $\Sigma = (x, \Gamma, \phi, \rho)$ satisfying the following conditions:

- (a) Γ is a tamely ramified semisimple element in \mathfrak{g} which admits a good factorization $\Gamma = \sum_{i=-1}^d \Gamma_i$ such that $\Gamma_{-1} \in F'[\Gamma_d, \dots, \Gamma_0]$;
- (b) The center $Z({}^\circ G^0)$ of the connected component ${}^\circ G^0$ of $G^0 := Z_\Gamma(G)$ is anisotropic ⁽⁴⁾;
- (c) The point x is a vertex in $\mathcal{B}(G^0)$, i.e., the connected component of $G_x^0 := \text{Stab}_{G^0}(x)$ is a maximal parahoric subgroup in G^0 ;
- (d) $\phi: G_x^0 \rightarrow \mathbb{C}^\times$ is a character such that $\phi|_{G_{x,0^+}^0} = \psi_\Gamma|_{G_{x,0^+}^0}$. Note that $G_{x,0^+}^0$ is the pro- p unipotent radical of G_x^0 ;
- (e) ρ is an irreducible cuspidal representation of the finite group $G_x^0 := G_x^0/G_{x,0^+}^0$.

We define the *depth* of the datum Σ , denoted by $\text{depth}(\Sigma)$, to be $\max\{-\text{depth}(\Gamma), 0\}$. Note that if Σ is a depth zero data, then $\Gamma \in Z(\mathfrak{g})$ by definition.

REMARK. – If we only require that the G_x^0 -module ρ in (e) is irreducible but not necessarily cuspidal, then we call the tuple (x, Γ, ϕ, ρ) a (*refined*) *K-type datum*. We will use such *K-type data* in Definition 5.17 and Section 9.

3.3.2. – Let $\Sigma = (x, \Gamma, \phi, \rho)$ be a supercuspidal datum. We fix a good factorization of $\Gamma = \sum_{i=0}^d \Gamma_i$. Since $Z({}^\circ G^0)$ is anisotropic, there are canonical embeddings of buildings

$$\mathcal{B}(G^0) \hookrightarrow \mathcal{B}(G^1) \hookrightarrow \dots \hookrightarrow \mathcal{B}(G^{d-1}) \hookrightarrow \mathcal{B}(G^d).$$

We now define some notations and review the construction of supercuspidal G -module π_Σ attached to Σ . These notations will be used freely in the rest of the paper.

DEFINITION 3.5. – Let Σ be a supercuspidal data. We set

- (a) $s_i := r_i/2$,
- (b) $K^i := G_{x,s_0}^0 G_{x,s_0}^1 \cdots G_{x,s_{i-1}}^i$,
- (c) $K_{0^+}^i := G_{x,0^+}^0 G_{x,s_0}^1 G_{x,s_1}^2 \cdots G_{x,s_{i-1}}^i = K^i \cap G_{x,0^+}$,
- (d) $K_+^i := G_{x,0^+}^0 G_{x,s_0^+}^1 G_{x,s_1^+}^2 \cdots G_{x,s_{i-1}^+}^i$,

⁽⁴⁾ For a general connected reductive group G , this condition would be that $Z(G^0)/Z(G)$ is anisotropic. However this is equivalent to (b) in our case.

- (e) $K := K^d$, $K_{0+} := K_{0+}^d$ and $K_+ := K_+^d$.
- (f) The character ϕ extends to a character of $G_x^0 K_+$ by setting $\phi|_{K_+} = \psi_\Gamma$. By an abuse of notation, we still denote it by ϕ .
- (g) Let κ^i be the canonically constructed irreducible K^i -module such that $\kappa^i|_{K_+^i}$ is $\psi_\Gamma|_{K_+^i}$ -isotypic. See Appendix A.2.2 for the precise definition. Let $\kappa := \kappa^d$.
- (h) Let $\eta_\Sigma := \rho \otimes \kappa$, which is an irreducible K -module. Here ρ is identified with its inflation to G_x^0 .
- (i) Let $\pi_\Sigma := \text{c-Ind}_K^G \eta_\Sigma$.

Suppose G is a connected reductive group. Yu proves that π_Σ is an irreducible supercuspidal representations of G [37]. If the residue characteristic of F is big enough (see (3.1)), Kim proves that the set of π_Σ exhausts all the supercuspidal representations of connected G [17].

Note that every odd orthogonal group is a direct product of a special orthogonal group with $\{\pm 1\}$. Hence, the above results of Yu and Kim, as well as those of Hakim-Murnaghan in Section 3.3.3 below, extend to odd orthogonal groups. We will show in Section 3.4 that they also extend to even orthogonal groups.

We call π_Σ the supercuspidal representation of G constructed from the datum Σ .

3.3.3. – We now describe the equivalence relation on supercuspidal data.

DEFINITION 3.6. – Let $\Sigma = (x, \Gamma, \phi, \rho)$ and $\dot{\Sigma} = (\dot{x}, \dot{\Gamma}, \dot{\phi}, \dot{\rho})$ be two supercuspidal data. We say that Σ and $\dot{\Sigma}$ are *equivalent* with each other if there exists an element $g \in G$ such that

- (a) $x = g \cdot \dot{x}$,
- (b) $\text{Ad}_g(\dot{\Gamma}) \in \Gamma + \mathfrak{g}_{x,0}$ and,
- (c) $\dot{\rho} \otimes \dot{\phi} \cong (\rho \otimes \phi) \circ \text{Ad}_g$ as G_x^0 -modules⁽⁵⁾.

REMARK. – Since we assume Definition 3.4 (a) in the definition of supercuspidal data, we may further assume $\text{Ad}_g(\dot{\Gamma}) \in \Gamma + (\mathbb{Z}(\mathfrak{g}^0) \cap \mathfrak{g}_{x,0}^0)$ in Definition 3.6 (b) thanks to [19, Lemma 5.1.3 (3)]. On the other hand, a depth zero data (x, Γ, ϕ, ρ) is always equivalent to $(x, 0, \mathbf{1}, \rho \otimes \phi)$ which is considered as a typical representative of the equivalence class.

Let Σ and $\dot{\Sigma}$ be two supercuspidal data. Hakim and Murnaghan show that π_Σ and $\pi_{\dot{\Sigma}}$ are isomorphic if and only if Σ and $\dot{\Sigma}$ differ by an elementary transform, conjugation and refactorization. Definition 3.6 could be read off from [11, Lemma 6.4, Theorem 6.7] by observing that, in our situation, (a) $Z(G)$ is anisotropic so there is no elementary transform; (b) the refactorization corresponds to a refactorization of the semisimple element Γ in terms of G^i -good elements, so the notion of “refactorization” also could be suppressed.

We now record their theorem as follows.

THEOREM 3.7 (Hakim-Murnaghan). – *Suppose G is a connected classical group, a special orthogonal group or an odd orthogonal group. Let Σ and $\dot{\Sigma}$ be two supercuspidal data for G . Then $\pi_\Sigma \simeq \pi_{\dot{\Sigma}}$ if and only if Σ and $\dot{\Sigma}$ are equivalent with each other. \square*

We record following easy consequence of the equivalence of data.

⁽⁵⁾ Note that (a) and (b) imply $G_x^0 \cong \text{Ad}_g(G_{\dot{x}}^0)$

LEMMA 3.8. – Suppose G be as in Theorem 3.7. Let $\Sigma = (x, \Gamma, \phi, \rho)$ be a supercuspidal data for G and let $\kappa = \kappa_\Sigma$. Then the multiplicity space

$$(3.3) \quad \rho' := \text{Hom}_{K_{0+}}(\kappa, \pi_\Sigma)$$

is isomorphic to ρ as G_x^0 -modules.

Proof. – Let $\dot{\rho}$ be an irreducible component of ρ' . By [17, Proposition 17.2], $\dot{\rho}$ is a cuspidal G_x^0 -module. By Frobenius reciprocity and Yu's construction, $\pi_\Sigma \cong \pi_{\dot{\Sigma}}$ where $\dot{\Sigma} = (x, \Gamma, \phi, \dot{\rho})$. Now by Theorem 3.7, $\rho \cong \dot{\rho}$. Hence, ρ' is ρ -isotypic with multiplicity

$$m_\rho = \text{Hom}_K(\rho, \text{Hom}_{K_{0+}}(\kappa, \pi_\Sigma)) = \text{Hom}_K(\eta_\Sigma, \pi_\Sigma) = \text{Hom}_G(\pi_\Sigma, \pi_\Sigma) = 1. \quad \square$$

3.4. Even orthogonal groups

We now show that the results of Yu, Kim and Hakim-Murnaghan extend to the even orthogonal groups. The contents in this subsection is well known to the experts. We include the proofs for completeness.

Let V be an even dimensional quadratic space. Let $G = \text{O}(V)$ and ${}^\circ G = \text{SO}(V)$.

Suppose V is anisotropic, then there is nothing to prove. Suppose V is a two dimensional hyperbolic space, then $\text{O}(V) = \text{O}(1, 1)$. The subgroup $\text{SO}(1, 1)$ in $\text{O}(1, 1)$ is defined to be a parabolic subgroup since it is the stabilizer of an isotropic subspace. This implies that all representations of $\text{O}(1, 1)$ are non-supercuspidal. Therefore it suffices to consider $\dim V > 2$.

For any subgroup H of G , we denote ${}^\circ H := {}^\circ G \cap H$. For a subgroup H of G and a H -module τ , we will let ${}^c \tau$ denote the $\text{Ad}_c(H)$ -module defined by ${}^c \tau(h) = \tau(\text{Ad}_{c^{-1}} h)$.

3.4.1. – In this section, we only assume that G is a group and ${}^\circ G$ is an index two normal subgroup of G . We first review some simple relationships between irreducible representations of group G and ${}^\circ G$. Let $c \in G \setminus {}^\circ G$. Suppose ${}^\circ \pi$ is an irreducible representation of ${}^\circ G$. Then $\text{Ind}_{{}^\circ G}^G {}^\circ \pi|_{{}^\circ G} \cong {}^\circ \pi \oplus {}^c({}^\circ \pi)$. The induced representation $\text{Ind}_{{}^\circ G}^G {}^\circ \pi$ is either (I) an irreducible representation of G , which happens if and only if ${}^\circ \pi$ and ${}^c({}^\circ \pi)$ are non-isomorphic as ${}^\circ G$ -modules or (II) it is a direct sum of two irreducible G -modules.

Conversely, the restriction of an irreducible representation π of G to ${}^\circ G$ is either (I) a direct sum of two non-isomorphic irreducible ${}^\circ G$ -modules or (II) an irreducible ${}^\circ G$ -module.

3.4.2. – As the basic step, we first show that the theory of depth zero supercuspidal representation of connected group extends to G . Note that $G_x \cap (G \setminus {}^\circ G) \neq \emptyset$ for each vertex $x \in \mathcal{B}(G)$.

Let π be a depth zero supercuspidal G -module. We consider its restriction to ${}^\circ G$ and relate to the two cases (I) and (II) in Section 3.4.1 above.

- (I) Suppose $\pi|_{{}^\circ G} = {}^\circ \pi_1 \oplus {}^\circ \pi_2$. Then there is a depth zero minimal K -type $(x, {}^\circ \rho)$ of ${}^\circ \pi_1$ where x is a vertex in $\mathcal{B}(G)$ and ${}^\circ \rho$ is a cuspidal ${}^\circ G_x$ -module. We fix $c \in G_x \cap (G \setminus {}^\circ G)$. By [25], ${}^c {}^\circ \rho \not\cong {}^\circ \rho$ since ${}^c {}^\circ \pi_1 \cong {}^\circ \pi_2 \not\cong {}^\circ \pi_1$. Hence, $\pi = \text{c-Ind}_{G_x}^G \rho$ where $\rho := \text{Ind}_{G_x}^{G_x} {}^\circ \rho$ is an irreducible cuspidal G_x -module.
- (II) Suppose $\pi|_{{}^\circ G}$ is an irreducible supercuspidal. Then it has a minimal K -type $(x, {}^\circ \rho)$ where $x \in \mathcal{B}(G)$ is a vertex. Let ρ be the natural representation of G_x on the $G_{x,0+}$ -invariant subspace of π . Clearly $\rho|_{{}^\circ G_x} = {}^\circ \rho$. Hence $\pi = \text{c-Ind}_{G_x}^G \rho$ since $\text{c-Ind}_{G_x}^G \rho|_{{}^\circ G} = \text{c-Ind}_{G_x}^{{}^\circ G_x} {}^\circ \rho = {}^\circ \pi$ is irreducible.

In summary, Condition **D4** and related claims in [37, p. 590] hold for even orthogonal groups.

3.4.3. – The centralizer $Z_G(\gamma)$ of a semisimple element $\gamma \in \mathfrak{g}$ is called a *twisted Levi subgroup* of G if $Z_{\circ G}(\gamma)$ is a twisted Levi subgroup of $\circ G$ [37, p. 586]. Therefore a twisted Levi subgroup is a product of general linear groups, unitary groups and at most one even orthogonal group. Combining with Section 3.4.2 above, Yu's definition of generic G -datum in [37, p. 615] extends to the orthogonal groups without any change.

Our formulation of supercuspidal data follows Kim's simplification [17, Section 5]. To translate between Kim and Yu's formulations, we see that the following variation of [17, Lemma 5.5] holds for all twisted Levi subgroups appearing in the construction of supercuspidal representations:

LEMMA 3.9. – *Suppose \check{G} is a twisted Levi subgroup of G such that $Z(\circ\check{G})/Z(G)$ is anisotropic, γ is a negative depth element in the center of the Lie algebra of \check{G} and $x \in \mathcal{B}(\check{G})$, then there exists a character ϕ of \check{G} such that $\phi|_{\check{G}_{x,0^+}}(g) = \psi(\mathbb{B}(\gamma, \log(g)))$ for every $g \in \check{G}_{x,0^+}$.*

Proof. – Since $Z(\circ\check{G})/Z(G)$ is anisotropic, \check{G} cannot have any $O(1, 1)$ factor or general linear group factor. Therefore, the center of the Lie algebra of \check{G} is contained in a product of unitary Lie algebra factors. Now the lemma follows immediately from its connected group version [17, Lemma 5.5]. \square

3.4.4. – In this subsection, we refer to Conditions **GE1** and **GE2** and the notation in [37, § 8]. Let X be a good element in G and let $\check{G} := Z_G(X)$ be the corresponding twisted Levi subgroup. This is **GE1** under our settings. The following modification of **GE2** is clearly implied by **GE1** for orthogonal groups:

CLAIM (**GE2'**). – *Let $T \subset \check{G} \subset G$ be maximal torus of \check{G} and $X \in \text{Lie}(T)$. Let \bar{F} be the algebraic closure of F . Let \check{X}^* be as in [37, p. 596]. Let $W := N_{G(\bar{F})}(T(\bar{F}))/T(\bar{F})$ and $\check{W} := N_{\check{G}(\bar{F})}(T(\bar{F}))/T(\bar{F})$ be the absolute Weyl groups of G and \check{G} respectively. Then $Z_W(\check{X}^*) = \check{W}$.* \square

3.4.5. – Let $\Sigma = (x, \Gamma, \phi, \rho)$ be a supercuspidal datum of G as in Definition 3.4. Argue as in [17, Remarks 5.10], there is a datum $(\vec{G}, x, \vec{r}, \vec{\phi}, \vec{\rho})$ such that ϕ_i is represented by the good element Γ_i and $\rho \otimes \phi = \vec{\rho} \otimes \prod_{i=0}^d \phi_i$. In particular, η_Σ constructed in Definition 3.5 (h) is the same as the K -type constructed following Yu's recipe.

3.4.6. – We now explain how to extend the proofs in [37] to G .

THEOREM 3.10. – *The representation $\pi_\Sigma := \text{c-Ind}_K^G \eta_\Sigma$ constructed in Definition 3.5 (i) is an irreducible supercuspidal representation of $G = O(V)$.*

Proof. – In [37, § 4], Yu defines conditions **SC1_i**, **SC2_i** and **SC3_i** which do not assume that the group is connected so they are applicable to G . We now verify these conditions and then [37, Proposition 4.6] will imply that π_Σ is an irreducible supercuspidal representation of G .

First we consider **SC1_i**. Its proof in [37, Theorem 9.4] relies on [37, Lemma 8.3] which still applies. In the proof of [37, Lemma 8.3], Yu uses conditions **GE1** and **GE2** which are

satisfied by the discussion in Section 3.4.4. In addition, one also needs the existence of certain integral model of the Moy-Prasad groups. This is clear by viewing the orthogonal group as a symmetric subgroup of the general linear group, see [3].

The condition **SC2_i** is about the existence of Heisenberg-Weil representation. This is taken care of by Appendix A.2.1.

The proof of **SC3_i** takes up [37, § 12-13]. Though it is long but the proof extends without change to our case. □

3.4.7. – Next we extend the exhaustion result of [17] to G .

THEOREM 3.11. – *Given (3.1), the set of π_Σ exhausts all the supercuspidal representations of $G = \mathbf{O}(V)$.*

Proof. – Let π be an irreducible supercuspidal representation of G . Then π contains an irreducible supercuspidal representation ${}^\circ\pi$ of ${}^\circ G$. By [17], ${}^\circ\pi = {}^\circ\pi_{\circ\Sigma}$ for some supercuspidal datum ${}^\circ\Sigma = (x, \Gamma, {}^\circ\phi, {}^\circ\rho)$. Using Γ and x we define G_x^0 and K etc. By Lemma 3.9, we can assume ${}^\circ\phi$ extends to a character ϕ of G^0 . Let κ be the K -module defined by the procedure in Appendix A.2.2.

We note that $K_{0+} = {}^\circ K_{0+}$. Define $\rho' = \text{Hom}_{K_{0+}}(\kappa, \pi)$ to be the multiplicity space of $\kappa|_{K_{0+}}$. It is a natural G_x^0 -module and the ${}^\circ\rho$ -isotypic subspace $\rho'[\circ\rho] \neq 0$. Pick any irreducible G_x^0 -submodule ρ in $\rho'[\circ\rho]$ and define $\Sigma = (x, \Gamma, \phi, \rho)$. Then π is a submodule of $\pi_\Sigma = \text{c-Ind}_K^G \eta_\Sigma$. By Theorem 3.10, π_Σ is irreducible so $\pi = \pi_\Sigma$. This completes the proof. □

3.4.8. – Finally we extend Theorem 3.7 to G .

THEOREM 3.12. – *Let Σ and $\dot{\Sigma}$ be two supercuspidal data for $G = \mathbf{O}(V)$. Then $\pi_\Sigma \simeq \pi_{\dot{\Sigma}}$ if and only if Σ and $\dot{\Sigma}$ are equivalent with each other in the sense of Definition 3.6.*

Proof. – Suppose $\Sigma = (x, \Gamma, \phi, \rho)$ and $\dot{\Sigma} = (\dot{x}, \dot{\Gamma}, \dot{\phi}, \dot{\rho})$. We argue case by case.

Case A. First we suppose $G_x^0 \subset {}^\circ G$. Then $K := K_\Sigma \subset {}^\circ G$ and Σ is also a supercuspidal datum for ${}^\circ G$. Fixing $c \in G \setminus {}^\circ G$, then $\pi_\Sigma|_{{}^\circ G} = \text{c-Ind}_K^{{}^\circ G} \eta_\Sigma \oplus \text{c-Ind}_{\text{Ad}_c({}^\circ K)}^{{}^\circ G} \eta_{c\Sigma}$, where ${}^c\Sigma := (c \cdot x, \text{Ad}_c \Gamma, {}^c\phi, {}^c\rho)$. On the other hand, let ${}^\circ\dot{\rho}_1$ be an irreducible ${}^\circ\dot{K}$ -submodule in $\dot{\rho}$ and ${}^\circ\dot{\Sigma}_1 = (\dot{x}, \dot{\Gamma}, \dot{\phi}|_{{}^\circ G_x^0}, {}^\circ\dot{\rho}_1)$. Then $\pi_{\dot{\Sigma}}|_{{}^\circ G} \cong \pi_{\dot{\Sigma}_1}|_{{}^\circ G}$ contains $\text{c-Ind}_K^{{}^\circ G} \eta_{{}^\circ\dot{\Sigma}_1}$. Now by [11], there is a $g \in {}^\circ G$ such that ${}^\circ\dot{\Sigma}_1$ is equivalent to either Σ or ${}^c\Sigma$. In particular, $\dot{K} \subset {}^\circ G$ so that $\dot{\rho} = {}^\circ\dot{\rho}_1$ is an irreducible ${}^\circ G_x^0 = G_x^0$ -module. Hence, Σ and $\dot{\Sigma}$ are equivalent.

Case B. Next, we suppose that $G_x^0 \cap (G \setminus {}^\circ G) \neq \emptyset$. Fix an element c in $G_x^0 \cap (G \setminus {}^\circ G)$.

(1) Suppose $\rho|_{{}^\circ G_x^0} \cong {}^\circ\rho \oplus {}^c({}^\circ\rho)$ where ${}^\circ\rho$ is an irreducible cuspidal ${}^\circ G_x^0$ -module. Using similar proof in Case A, we see that, up to G -conjugacy, $\dot{x} = x, \dot{\Gamma} \in \Gamma + \mathfrak{g}_{x,0}$ and $(\dot{\rho} \otimes \dot{\phi})|_{{}^\circ G_x^0}$ contains ${}^\circ\rho \otimes (\phi|_{{}^\circ G_x^0})$. Hence, Σ and $\dot{\Sigma}$ are equivalent by applying the discussion in Section 3.4.1 to $K > {}^\circ K$.

(2) Suppose ${}^\circ\rho := \rho|_{{}^\circ G_x^0}$ is already irreducible. Let ${}^\circ\Sigma = (x, \Gamma, \phi|_{{}^\circ G_x^0}, {}^\circ\rho)$ be the corresponding supercuspidal datum for ${}^\circ G$. Then $\eta_{\circ\Sigma} = \eta_\Sigma|_{{}^\circ K}$ is the supercuspidal type of ${}^\circ G$ determined by ${}^\circ\Sigma$ and $\pi_\Sigma|_{{}^\circ G} = \text{Ind}_K^{{}^\circ G} \eta_{\circ\Sigma}$. Again by the argument in Case A, up to ${}^\circ G$ -conjugacy, we could assume $\dot{x} = x, \dot{\Gamma} \in \Gamma + \mathfrak{g}_{x,0}, \dot{\phi} = \phi$ and $\rho|_{{}^\circ G_x^0} \cong \dot{\rho}|_{{}^\circ G_x^0}$.

One observes that both ρ and $\dot{\rho}$ must be G_x^0 -submodules in the multiplicity space ρ' in (3.3). However $\rho'|_{\circ G_x^0} \simeq \rho|_{\circ G_x^0}$ is irreducible. Hence, $\rho = \dot{\rho}$. \square

REMARK. – As a consequence of Theorem 3.12, Lemma 3.8 also extends to even orthogonal groups. We leave the details to the reader.

3.5. Tamely ramified supercuspidal representation for covering groups

We now state and review some results on tamely ramified supercuspidal representations of \tilde{G} . We will supply some proofs although they follow almost immediately from those in the algebraic group case. ⁽⁶⁾ Depth zero representations of non-linear covers of p -adic groups were studied by Howard and Weissman [12].

3.5.1. – The supercuspidal data for \tilde{G} is an extension of the supercuspidal data for G by a splitting of the covering:

DEFINITION 3.13. – A *supercuspidal datum* for \tilde{G} is a tuple $\tilde{\Sigma} = (x, \Gamma, \phi, \rho, \xi)$ such that $\Sigma = (x, \Gamma, \phi, \rho)$ is a supercuspidal datum for G as in Definition 3.4 and

- (f) $\xi: K \rightarrow \tilde{K}$ is a splitting of the \mathbb{C}^\times -covering $\tilde{K} \twoheadrightarrow K$ such that $\xi|_{K_{0+}} = \Xi|_{K_{0+}}$ where Ξ is the canonical splitting defined in (2.3) and, K and K_{0+} are defined in Definition 3.5 with respect to Σ . The splitting ξ induces an identification of \tilde{K} with $K \times \mathbb{C}^\times$. Let

$$(3.4) \quad \tilde{\xi}: K \times \mathbb{C}^\times \xrightarrow{\sim} \tilde{K}.$$

denote the corresponding isomorphism.

REMARK. – Suppose ξ_1 is another splitting of K , then ξ and ξ_1 differ by a character. More precisely we have a character

$$(3.5) \quad \mu_{\xi_1, \xi}: K \rightarrow \mathbb{C}^\times \quad \text{given by} \quad \mu_{\xi_1, \xi}(k)\xi_1(k) = \xi(k) \quad \text{for all } k \in K.$$

3.5.2. – Let $\tilde{\Sigma} = (x, \Gamma, \phi, \rho, \xi)$ be a supercuspidal datum for \tilde{G} . We assume all the notations in Section 3.3.2. We let

$$(3.6) \quad \tilde{\eta}_{\tilde{\Sigma}} := (\eta_\Sigma \boxtimes \text{id}_{\mathbb{C}^\times}) \circ \tilde{\xi}^{-1}$$

which is an irreducible \tilde{K} -module and let

$$\tilde{\pi}_{\tilde{\Sigma}} := \text{c-Ind}_{\tilde{K}}^{\tilde{G}} \tilde{\eta}_{\tilde{\Sigma}}.$$

THEOREM 3.14. – *The representation $\tilde{\pi}_{\tilde{\Sigma}}$ is an irreducible supercuspidal representation.*

⁽⁶⁾ The essences of most proofs for the algebraic groups are related to the positive depth parts. Hence these proofs translate to our case by identifying the positive depth parts via the canonical splitting (2.3).

Proof. – Let $I(\eta_\Sigma) := \{g \in G \mid \text{Hom}_{gK \cap K}(\eta_\Sigma, {}^g \eta_\Sigma) \neq 0\}$ be the set of intertwiners of η_Σ . We will follow the proof of [37, Proposition 4.6] in which Yu proves that

$$(3.7) \quad I(\eta_\Sigma) = K.$$

(Also see the proof of Theorem 3.10 for even orthogonal groups.) In order to adapt Yu’s proof to our theorem, it suffices to show that the set

$$I(\widetilde{\eta}_\Sigma) := \left\{ \tilde{g} \in \widetilde{G} \mid \text{Hom}_{\tilde{g}\widetilde{K} \cap \widetilde{K}}(\widetilde{\eta}_\Sigma, {}^{\tilde{g}} \widetilde{\eta}_\Sigma) \neq 0 \right\}$$

of intertwiners of $\widetilde{\eta}_\Sigma$ is exactly \widetilde{K} . Fix $\tilde{g} \in \widetilde{G}$ and let g be its image in G . Note that the adjoint action of \widetilde{G} factors through G . Then

- (a) ${}^{\tilde{g}} \widetilde{K} \cap \widetilde{K} = {}^g K \cap K$ which we identified with $({}^g K \cap K) \times \mathbb{C}^\times$ using ξ in (3.4), and
- (b) ${}^{\tilde{g}} \widetilde{\eta}_\Sigma|_{\tilde{g}\widetilde{K} \cap \widetilde{K}} = ({}^g \eta_\Sigma \boxtimes \text{id}_{\mathbb{C}}) \circ \tilde{\xi}^{-1}|_{\tilde{g}\widetilde{K} \cap \widetilde{K}}$.

Therefore, we have $I(\widetilde{\eta}_\Sigma) = \widetilde{K}$ which is \widetilde{K} by (3.7). □

DEFINITION 3.15. – We call $\tilde{\pi}_\Sigma$ the supercuspidal representation of \widetilde{G} attached to the datum $\widetilde{\Sigma}$.

3.5.3. – Now we describe the equivalence of supercuspidal data for covering groups.

DEFINITION 3.16. – Let $\widetilde{\Sigma} = (x, \Gamma, \phi, \rho, \xi)$ and $\check{\Sigma} = (\dot{x}, \dot{\Gamma}, \dot{\phi}, \dot{\rho}, \dot{\xi})$ be two supercuspidal data for \widetilde{G} . We say that $\widetilde{\Sigma}$ and $\check{\Sigma}$ are *equivalent* data if there exists an element $g \in G$ such that

- (a) $x = g \cdot \dot{x}$,
- (b) $\text{Ad}_g(\dot{\Gamma}) \in \Gamma + \mathfrak{g}_{x,0}$ and
- (c) $((\dot{\rho} \otimes \dot{\phi}) \boxtimes \text{id}_{\mathbb{C}^\times}) \circ \check{\xi}^{-1} \cong ((\rho \otimes \phi) \boxtimes \text{id}_{\mathbb{C}^\times}) \circ \xi^{-1} \circ \text{Ad}_g$ as \widetilde{G}_x^0 -module ⁽⁷⁾.

We remark that Condition (c) is equivalent to

- (c') $\dot{\rho} \otimes \dot{\phi} \otimes \mu_{\dot{\xi}, \dot{\xi}^g} \cong (\rho \otimes \phi) \circ \text{Ad}_g$ as G_x^0 -module where $\mu_{\dot{\xi}, \dot{\xi}^g}$ is defined by (3.5) and $\xi^g := \text{Ad}_{g^{-1}} \circ \xi \circ \text{Ad}_g$.

Since our choices of splittings agree on the pro-unipotent part, $\mu_{\dot{\xi}, \dot{\xi}^g}$ is a character that is trivial on $G_{\dot{x},0^+}^0$, i.e., a character of $G_x^0 := G_{\dot{x},0}^0 / G_{\dot{x},0^+}^0$. Condition (c') is simpler and seems easier to check because $\mu_{\dot{\xi}, \dot{\xi}^g}$ is trivial in most of the cases (cf. [26]).

The following theorem is a variation of a result in [11]. The reader may consult [11] for notations when reading the proof and should note that the notations in the proof may not agree with other parts of our paper.

THEOREM 3.17. – *Let $\widetilde{\Sigma} = (x, \Gamma, \rho, \phi, \xi)$ and $\check{\Sigma} = (\dot{x}, \dot{\Gamma}, \dot{\rho}, \dot{\phi}, \dot{\xi})$ be two supercuspidal data. Then $\tilde{\pi}_\Sigma \cong \tilde{\pi}_{\check{\Sigma}}$ if and only if $\widetilde{\Sigma}$ and $\check{\Sigma}$ are equivalent with each other.*

⁽⁷⁾ Here Ad_g acts on \widetilde{G} since the adjoint action factors through the center.

Proof. – Suppose $\widetilde{\Sigma}$ and $\check{\Sigma}$ are equivalent. By definition $\widetilde{\eta}_{\widetilde{\Sigma}}$ and $\widetilde{\eta}_{\check{\Sigma}}$ are isomorphic up to G -conjugacy so $\widetilde{\pi}_{\widetilde{\Sigma}} \cong \widetilde{\pi}_{\check{\Sigma}}$.

We now assume $\widetilde{\pi}_{\widetilde{\Sigma}} \cong \widetilde{\pi}_{\check{\Sigma}}$. Let K and \check{K} be the compact subgroups determined by (x, Γ) and $(\dot{x}, \dot{\Gamma})$ respectively. Let $\bullet \mapsto (\bullet)^\vee$ denote the operation of taking contragredient.

In order to apply the result in [11, § 5], we make following definitions. We let $\mathcal{G} := G \times G$ equipped with an involution θ sending (g_1, g_2) to (g_2, g_1) and identify G with the diagonal subgroup of \mathcal{G} . Let $\Sigma = (x, \Gamma, \rho, \phi)$ and $\check{\Sigma}^\vee = (\dot{x}, -\dot{\Gamma}, \dot{\rho}^\vee, \dot{\phi}^\vee)$. We view $\Psi := \Sigma \times \check{\Sigma}^\vee$ as a supercuspidal datum for \mathcal{G} .

The \widetilde{G} -module $\widetilde{\pi}_{\widetilde{\Sigma}} \otimes (\widetilde{\pi}_{\check{\Sigma}})^\vee$ factors to a G -module and we have

$$\begin{aligned} \mathbb{C} &= \text{Hom}_G(\widetilde{\pi}_{\widetilde{\Sigma}} \otimes \widetilde{\pi}_{\check{\Sigma}}^\vee, \mathbf{1}) = \text{Hom}_G(\text{c-Ind}_{\widetilde{K} \times \check{K}}^{\widetilde{G} \times \check{G}}(\widetilde{\eta}_{\widetilde{\Sigma}} \otimes \widetilde{\eta}_{\check{\Sigma}}^\vee), \mathbf{1}) \\ &= \sum_{g \in K \backslash G / \check{K}} \text{Hom}_{\widetilde{K} \cap g \check{K}}(\widetilde{\eta}_{\widetilde{\Sigma}} \otimes {}^g \widetilde{\eta}_{\check{\Sigma}}^\vee, \mathbf{1}). \end{aligned}$$

Therefore, exactly one term of the above summation is non-vanishing and of dimension 1. By replacing $\widetilde{\Sigma}$ by its G -conjugate, we may assume that

$$(3.8) \quad \text{Hom}_{\widetilde{K} \cap \check{K}}(\widetilde{\eta}_{\widetilde{\Sigma}} \otimes \widetilde{\eta}_{\check{\Sigma}}^\vee, \mathbf{1}) = \mathbb{C}.$$

By Section 2.3.1 $\xi|_{K_+ \cap \check{K}_+} = \dot{\xi}|_{K_+ \cap \check{K}_+}$. Hence (3.8) implies

$$\text{Hom}_{K_+ \cap \check{K}_+}(\psi_\Gamma \otimes \psi_{-\dot{\Gamma}}, \mathbf{1}) \neq 0.$$

This means Ψ is *weakly compatible* with the involution θ in the sense of [11, Definition 5.6]. Now [11, Proposition 5.7] implies that Ψ is *weakly θ -symmetric* up to a conjugation of $K \times \check{K}$. By the definition of weakly θ -symmetric in [11, Definition 3.13], we may assume $\Gamma = \dot{\Gamma}$ and so $G^0 = Z_G(\Gamma) = Z_G(\dot{\Gamma}) = \dot{G}^0$.

Since we are in the “group case”, the theorem could be proven by reducing to the depth zero case: Thanks to [17, Lemma 5.5], we can fix a character ϕ_0 of G^0 extending $\psi_\Gamma|_{G_{0+}^0}$. Consider the depth zero supercuspidal data $\widetilde{\Sigma}_0 := (x, 0, \mathbf{1}, \rho \otimes \phi \otimes \phi_0^{-1}, \xi)$ and $\check{\Sigma}_0 := (\dot{x}, 0, \mathbf{1}, \dot{\rho} \otimes \dot{\phi} \otimes \phi_0^{-1}, \dot{\xi})$. Let $\widetilde{\eta}_{\widetilde{\Sigma}_0}$ and $\widetilde{\eta}_{\check{\Sigma}_0}$ be depth zero supercuspidal K -types of \widetilde{G}^0 defined by data $\widetilde{\Sigma}_0$ and $\check{\Sigma}_0$ respectively. Restricting (3.8) to $\widetilde{G}_x^0 \cap \widetilde{G}_{\dot{x}}^0$ gives

$$0 \neq \text{Hom}_{\widetilde{G}_x^0 \cap \widetilde{G}_{\dot{x}}^0}(\widetilde{\eta}_{\widetilde{\Sigma}} \otimes \widetilde{\eta}_{\check{\Sigma}}^\vee, \mathbf{1}) = \text{Hom}_{\widetilde{G}_x^0 \cap \widetilde{G}_{\dot{x}}^0}(\widetilde{\eta}_{\widetilde{\Sigma}_0} \otimes \widetilde{\eta}_{\check{\Sigma}_0}^\vee, \mathbf{1})^{\oplus m}$$

where m is certain multiplicity. On the other hand, $\text{Hom}_{\widetilde{G}_x^0 \cap \widetilde{G}_{\dot{x}}^0}(\widetilde{\eta}_{\widetilde{\Sigma}_0} \otimes \widetilde{\eta}_{\check{\Sigma}_0}^\vee, \mathbf{1}) \neq 0$ implies the depth zero supercuspidal \widetilde{G}^0 -modules $\widetilde{\pi}_{\widetilde{\Sigma}_0} = \text{c-Ind}_{\widetilde{G}_x^0}^{\widetilde{G}^0} \widetilde{\eta}_{\widetilde{\Sigma}_0}$ and $\widetilde{\pi}_{\check{\Sigma}_0} = \text{c-Ind}_{\widetilde{G}_{\dot{x}}^0}^{\widetilde{G}^0} \widetilde{\eta}_{\check{\Sigma}_0}$ are isomorphic to each other. Since all depth zero unrefined minimal K -types are associates (see [12, Proposition 3.6]), there is an element $g \in G^0$ such that $x = g \cdot \dot{x} \in \mathcal{B}(G^0) = \mathcal{B}_{\text{red}}(G^0)$ and $((\rho \otimes \phi \otimes \phi_0^{-1}) \boxtimes \text{id}_{\mathbb{C}^\times}) \circ \widetilde{\xi}^{-1} \circ \text{Ad}_g \cong ((\dot{\rho} \otimes \dot{\phi} \otimes \phi_0^{-1}) \boxtimes \text{id}_{\mathbb{C}^\times}) \circ \dot{\xi}^{-1}$ as \widetilde{G}_x^0 -modules. This finishes the proof of the theorem. \square

3.5.4. – We will extend the results of [17] to every covering group \widetilde{G} appearing in a type I dual pair. More precisely we will show that the set of $\widetilde{\pi}_{\widetilde{\Sigma}}$ exhausts all the genuine ⁽⁸⁾ supercuspidal representations of \widetilde{G} under the assumption that p is big enough. Gan and Kim are currently preparing a manuscript on such kinds of non-linear covering groups [9].

Since all covering groups in this paper occur in a type I dual pair, these are split central covering groups except the odd orthogonal-metaplectic dual pairs. Therefore Kim's exhaustion result applies except for metaplectic groups. We now show that the exhaustion for metaplectic groups could be obtained from that of odd-orthogonal groups:

COROLLARY 3.18. – *Let V' be a symplectic space over F of even dimension n . Let \mathbf{Mp} be the metaplectic \mathbb{C}^\times -cover of $\mathrm{Sp}(V')$. Suppose $p \geq \max\{2n + 3, e_F(n + 1) + 2\}$ where e_F is the ramification index of F/\mathbb{Q}_p . Then every genuine supercuspidal representation of \mathbf{Mp} is of the form $\widetilde{\pi}_{\widetilde{\Sigma}}$ where $\widetilde{\Sigma}$ is a supercuspidal data of \mathbf{Mp} .*

Proof. – Let $\widetilde{\pi}'$ be an irreducible supercuspidal genuine \mathbf{Mp} -module. By the conservation relation [35], there is an odd dimensional quadratic space V such that (i) $\dim_F V \leq n + 1$, (ii) $(G, G') = (\mathrm{U}(V), \mathrm{U}(V'))$ form a type I dual pair so that $\widetilde{G}' = \mathbf{Mp}$ and (iii) $\widetilde{\pi}' = \theta_{V, V'}(\widetilde{\pi})$ for an irreducible supercuspidal genuine \widetilde{G} -module.

By Proposition 3.1 both $\mathrm{U}(V)$ and $\mathrm{U}(V')$ satisfy Kim's hypotheses [17, § 3.4]. Hence there is a supercuspidal data $\widetilde{\mathcal{D}}$ such that $\widetilde{\pi} := \widetilde{\pi}_{\widetilde{\mathcal{D}}}$ by [17]. This is the starting point of the proof of 1.3 (ii) of the main theorem in Section 9.1. Using Proposition 9.1, the proof gives supercuspidal data Σ and Σ' such that $\widetilde{\mathcal{D}}$ and $\widetilde{\Sigma}$ are equivalent (i.e., $\widetilde{\pi} = \widetilde{\pi}_{\widetilde{\Sigma}}$) and $\widetilde{\pi}' = \widetilde{\pi}'_{\widetilde{\Sigma}'}$. In particular, $\widetilde{\pi}'$ is realized as a supercuspidal representation of \widetilde{G}' attached to the supercuspidal datum $\widetilde{\Sigma}'$. This finishes the proof. \square

REMARK. – The proof of Proposition 9.1 does not depend on Kim's work on exhaustion except a variation of [17, Proposition 17.2]. Hence there is no circular reasoning.

4. Good factorizations and block decompositions

In this section, we first construct GL-good factorizations which will be used in Section 6. Then we will define a notion of block decompositions for supercuspidal data. The theta lifting map of supercuspidal data in Section 5 is defined based on this notion. We remark that parts of the treatment resemble those of [33]*§ 3.

4.1. GL-good factorization

We now construct a good factorization of a tamely ramified semisimple element $\Gamma \in \mathfrak{g}$ following Howe [15]. Let $F' := Z(D)$ be the center of D which is also identified with the center of $\mathrm{End}_D(V)$.

Let $A := F'[\Gamma] \subseteq \mathrm{End}_D(V)$. Then A is isomorphic to a product $\prod_{j \in \mathfrak{J}} F_j$ of (tamely ramified) finite extensions of F' where \mathfrak{J} is a finite index set. Furthermore, we have factorization of the A -module $V = \bigoplus_j V_j$ where V_j is an (F_j, D) -bimodule. Since $\Gamma^* = -\Gamma$, $*$ induces an involution on A and on the set \mathfrak{J} respectively. An orbit of the $*$ -action on \mathfrak{J} has at most 2 elements. Therefore we have a decomposition of \mathfrak{J} and A such that

⁽⁸⁾ Here “genuine” means the restriction of the representation on \mathbb{C}^\times is the scalar multiplication $\mathrm{id}_{\mathbb{C}^\times}$.

- (A1) $\mathfrak{J} = \mathfrak{J}_0 \sqcup \mathfrak{J}_1 \sqcup *(\mathfrak{J}_1)$ and $A = F'[\Gamma] = \prod_{j \in \mathfrak{J}_0} F_j \times \prod_{j \in \mathfrak{J}_1} (F_j^+ \times F_j^-)$ where $F_j^+ := F_j$ and $F_j^- := F_{*(j)}$ for $j \in \mathfrak{J}_1$;
 (A2) the $*$ -action is an involution or the identity⁽⁹⁾ on the field F_j when $j \in \mathfrak{J}_0$.
 (A3) the $*$ -action permutes F_j^+ and F_j^- when $j \in \mathfrak{J}_1$.

LEMMA 4.1. – Let ${}^\circ G^0$ be the connected component of $G^0 := Z_G(\Gamma)$. Let \mathfrak{g}^0 be its Lie algebra and let $\mathfrak{gl}^0 := Z_{\mathfrak{gl}_D(V)}(\Gamma)$. Then

- (i) $Z(\mathfrak{gl}^0) = F'[\Gamma]$;
 (ii) the center $Z({}^\circ G^0)$ is anisotropic if and only if (a) $\mathfrak{J}_1 = \emptyset$ in the decomposition ((A1)) and (b) there is no $\mathrm{SO}(1, 1)(F)$ factor in ${}^\circ G^0$;
 (iii) $Z(\mathfrak{g}^0) = F'[\Gamma] \cap \mathfrak{g}$ when the equivalent conditions in part (ii) hold.

Proof. – Let $D_j := F_j \otimes_{F'} D$. Then D_j is a central simple algebra over F_j and $D[\Gamma] = \prod_j D_j$. Hence $\mathfrak{gl}^0 = Z_{\mathfrak{gl}_D(V)}(\Gamma) = \prod_i \mathrm{End}_{D_i} V_i$ where each factor $\mathfrak{gl}_i := \mathrm{End}_{D_i} V_i$ is a central simple algebra over F_i . This proves part (i).

The $*$ -action permutes \mathfrak{gl}_j and $\mathfrak{gl}_{*(j)}$. If $j \in \mathfrak{J}_0$, then the form $\langle \cdot, \cdot \rangle_V$ restricted on V_j is non-degenerate. If $j \in \mathfrak{J}_0$ and $F_j \neq F'$, then the $*$ -action on F_j is nontrivial since $(\Gamma|_{V_j})^* = -\Gamma|_{V_j}$. In this case, $U_j := \{g \in \mathrm{End}_{D_j}(V_j) \mid g^*g = \mathrm{id}_{V_j}\}$ is a unitary group defined over the $*$ -fixed point sub-field F_j^* of F_j . In summary, we have

$$G^0 = \prod_{j \in \mathfrak{J}_1} \mathrm{GL}_{D_j}(V_j) \times \prod_{\substack{j \in \mathfrak{J}_0 \\ F_j \neq F'}} U_j \times \prod_{\substack{j \in \mathfrak{J}_0 \\ F_j = F'}} \mathrm{U}(V_j).$$

Now parts (ii) and (iii) follow. □

By Lemma 4.1 and Definition 3.4 (b), we may and will assume that $\mathfrak{J}_1 = \emptyset$ and (A3) will not happen since we only study those Γ which are contained in supercuspidal data. Lemma 4.2 and Proposition 4.3 below also apply to $\mathfrak{J}_1 \neq \emptyset$. We will leave the details to the reader.

LEMMA 4.2. – Let $\gamma \in \mathfrak{g}$ be a tamely ramified semisimple element. If it is $\mathrm{GL}_D(V)$ -good, then it is G -good.

Proof. – It is enough to prove this lemma after a base change to a tamely ramified extension of F such that Γ is contained in a split Cartan subalgebra of \mathfrak{g} . Since the set of roots of \mathfrak{g} is the restrictions of a subset of roots of $\mathfrak{gl}_D(V)$, the lemma follows. □

PROPOSITION 4.3. – Let Γ be a tamely ramified semisimple element in \mathfrak{g} . Then there is a G -good factorization $\Gamma = \sum_{i=-1}^d \Gamma_i$ such that Γ_i is $\mathrm{GL}_D(V)$ -good for $0 \leq i \leq d$.

The construction of the factorization is essentially the *Howe factorization*.

⁽⁹⁾ This is because F_j could be equal to F in certain cases. For example, if $D = F$ and Γ is not a full rank matrix, then $F[\Gamma] \cong F \oplus (F[x]/P(x))$ where P is the minimal polynomial of Γ .

Proof. – We fix a uniformizer ϖ_F of F . We recall that $A = F'[\Gamma] = \prod_{j \in \mathfrak{J}} F_j$ is a product of fields.

First we assume that the product has only one factor, i.e., $\Gamma \neq 0$ and $A = F'[\Gamma]$ is a field. Let e_A be the ramification index of A/F . Suppose Γ has valuation $\frac{k}{e}$ such that k and e are coprime. Consider $b := \varpi_F^{-k} \Gamma^e$. Then $b \in \mathfrak{o}_A \setminus \mathfrak{p}_A$. Let C be the set of roots of unity in $F'[\Gamma]$. Then $(b + \mathfrak{p}_A) \cap C$ has a unique element, say \hat{b} .

CLAIM 1. – *The equation $\varpi_F^{-k} \gamma^e = \hat{b}$ has a unique solution $\gamma = \hat{\gamma}$ in A such that $\Gamma \equiv \hat{\gamma} \pmod{\mathfrak{p}_A^{\frac{k}{e} e_A + 1}}$. Moreover $\hat{\gamma}^* = -\hat{\gamma}$, i.e., $\hat{\gamma} \in \mathfrak{g}$.*

Proof. – Note that $p \nmid e$. Hence the map $1 + \mathfrak{p}_A \rightarrow b + \mathfrak{p}_A = b(1 + \mathfrak{p}_A)$ given by $1 + x \mapsto b(1 + x)^e$ is a surjection (in fact it is a bijection) for any $b \in \mathfrak{o}_A \setminus \mathfrak{p}_A$. Therefore $\varpi_F^{-k} \gamma^e = \hat{b}$ has a solution $\hat{\gamma} = \Gamma(1 + x)$ for certain $x \in \mathfrak{p}_A$. On the other hand, all solutions of $\varpi_F^{-k} \gamma^e = \hat{b}$ are of the form $\gamma := c\hat{\gamma}$ such that $c^e = 1$. However among them only $\hat{\gamma}$ satisfies $\Gamma \equiv \gamma \pmod{\mathfrak{p}_A^{\frac{k}{e} e_A + 1}}$. This proves the first assertion of the claim.

Both $\hat{\gamma}^*$ and $-\hat{\gamma}$ are solutions of $\varpi_F^{-k} \gamma^e = \hat{b}^* = (-1)^e \hat{b}$ and $\hat{\gamma}^* \equiv \Gamma^* \equiv -\Gamma \equiv -\hat{\gamma} \pmod{\mathfrak{p}_A^{\frac{k}{e} e_A + 1}}$. Hence $\hat{\gamma}^* = -\hat{\gamma}$ by the uniqueness of the first part. This finishes the proof of the claim \square

Next we consider the general case when $F'[\Gamma] = A = \prod_{j \in \mathfrak{J}} F_j$ is a product of fields. Let Γ^j denote the F_j component of Γ to F_j . Let $-r_j$ denote the valuation of Γ^j and let $-r = \frac{k}{e}$ denote the depth of Γ where k and e are coprime integers. Let $b_j = \varpi_F^{-k} \Gamma_j^e \in \mathfrak{o}_{F_j} \setminus \mathfrak{p}_{F_j}$ when $r_j = r$. Define \hat{b}_j and $\hat{\gamma}_j$ as before. Let $\beta_1 = \sum_{r_j=r} \hat{\gamma}_j$.

CLAIM 2. – *The element β_1 is $\mathrm{GL}_D(V)$ -good.*

Proof. – We could base change to a tamely ramified extension E of F which at least contains a e -th root ϖ_E of ϖ_F and sufficiently many roots of unity so that β_1 splits over E .

Each factor $\hat{\gamma}_j$ satisfies $\varpi_F^{-k} \hat{\gamma}_j^e = \hat{b}_j$ and \hat{b}_j is a certain root of unity. The eigenvalue of β_1 is of the form $\zeta_l \varpi_E^k$ where ζ_l is a root of unity. The evaluation of β_1 on any root (with respect to any split maximal torus in $\mathfrak{gl}(V) \otimes_F E$ containing β_1) is a difference of two eigenvalues. It is either 0 or has valuation $\frac{k}{e} = -r$. This proves the claim. \square

Let $\Gamma' = \Gamma - \beta_1$. Apply the same construction to Γ' and stop if $\mathrm{depth}(\Gamma') \geq 0$. Note that $\mathrm{depth}(\Gamma') > \mathrm{depth}(\Gamma)$. Moreover the denominator of $\mathrm{depth}(\Gamma')$ is bounded by the maximum of the ramification index of F_j/F , since $\Gamma' \subseteq F'[\Gamma]$. Therefore the procedure stops after finitely many steps. We get $\beta_1, \beta_2, \dots, \beta_{d'}$ for certain $d' \geq 1$. If $\beta_1 \notin F'$, then we define $d = d'$, $\Gamma_d = 0$ and $\Gamma_{d-i} = \beta_i$ for $1 \leq i \leq d$. If $\beta_1 \in F'$, then we define $d = d' - 1$, $\Gamma_d = \beta_1$ and $\Gamma_{d-i} = \beta_{i+1}$ for $1 \leq i \leq d$. Finally we set $\Gamma_{-1} = \Gamma - \sum_{i=0}^d \Gamma_i$.

Now $\Gamma = \sum_{i=-1}^d \Gamma_i$ is the required factorization. \square

4.2. Block decomposition

Let $\Sigma = (x, \Gamma, \phi, \rho)$ be a supercuspidal data of $G = \mathbf{U}(V)$.

Let Γ^j be the F_j component of Γ in $\prod_{j \in \mathfrak{J}} F_j$ in Section 4.1 (A1). Let

$$(4.1) \quad \{ {}^b r > \dots > {}^1 r \} = \{ -\text{val}(\Gamma^j) > 0 \mid j \in \mathfrak{J} \}$$

be a set of positive numbers arranged in decreasing order where b is the size of the set. We set

$${}^l A := \prod_{\text{val}(\Gamma^j) = -{}^l r} F_j \quad \forall 1 \leq l \leq b.$$

For $l = 0$, we define ${}^0 A := \prod_j F_j$ where product is taken over those $j \in \mathfrak{J} = \mathfrak{J}_0$ such that $\Gamma^j = 0$ or $\text{val}(\Gamma^j) \geq 0$.

Let $\mathbf{1}$ and ${}^l \mathbf{1}$ be the multiplicative identity elements of A and ${}^l A$ respectively. Then we set ${}^l \Gamma = {}^l \mathbf{1} \cdot \Gamma \cdot {}^l \mathbf{1}$ and ${}^l V = {}^l \mathbf{1} \cdot V$. These give

$$(4.2) \quad A = F'[\Gamma] = \prod_{l=0}^b {}^l A, \quad \mathbf{1} = \bigoplus_{l=0}^b {}^l \mathbf{1}, \quad \Gamma = \bigoplus_{l=0}^b {}^l \Gamma \quad \text{and} \quad V = \bigoplus_{l=0}^b {}^l V.$$

It is easy to see that $\langle \cdot, \cdot \rangle_V$ restricted to ${}^l V$ is non-degenerate and $V = \bigoplus_{l=0}^b {}^l V$ is an orthogonal decomposition⁽¹⁰⁾. We call (4.2) the *block decompositions* of A , $\mathbf{1}$, Γ and V respectively.

DEFINITION 4.4. – (a) We say that a supercuspidal datum $\Sigma = (x, \Gamma, \phi, \rho)$ is a *single block of positive depth r* if Γ^j has the same valuation $-r < 0$ for all $j \in \mathfrak{J}$ in Section 4.1 (A1). Equivalently this means $\text{depth}(\Gamma) = -r$, $\Gamma = {}^1 \Gamma$ and $V = {}^1 V$ in (4.2).

(b) A depth zero supercuspidal datum $\Sigma = (x, \Gamma, \mathbf{1}, \rho)$ is called a *single block of zero depth*.

PROPOSITION 4.5. – *Let $\Sigma = (x, \Gamma, \phi, \rho)$ be a supercuspidal datum of G . Then there exists an orthogonal decomposition of $V = \bigoplus_{l=0}^b {}^l V$ such that*

- (i) $\Gamma = \bigoplus_{l=0}^b {}^l \Gamma$ where ${}^l \Gamma \in \text{End}_D({}^l V) \cap \mathfrak{g}$;
- (ii) ${}^b r > \dots > {}^1 r > 0 = {}^0 r$ where ${}^l r = \max \{ -\text{depth}({}^l \Gamma), 0 \}$;
- (iii) $G^0 = \prod_{l=0}^b {}^l G^0$;
- (iv) $x = ({}^0 x, \dots, {}^b x) \in \mathcal{B}({}^0 G^0) \times \dots \times \mathcal{B}({}^b G^0) \hookrightarrow \mathcal{B}(G)$;
- (v) $\phi = {}^0 \phi \boxtimes \dots \boxtimes {}^b \phi$ and $\rho = {}^0 \rho \boxtimes \dots \boxtimes {}^b \rho$ as $G_x^0 = \prod_{l=0}^b {}^l G_x^0$ -modules;
- (vi) ${}^l \Sigma = ({}^l x, {}^l \Gamma, {}^l \phi, {}^l \rho)$ is a *single block supercuspidal datum of positive depth ${}^l r$* for $1 \leq l \leq b$; and
- (vii) ${}^0 \Sigma = ({}^0 x, {}^0 \Gamma, {}^0 \phi, {}^0 \rho)$ is a *single block supercuspidal datum of zero depth*.

Here $G^0 := Z_G(\Gamma)$, ${}^l G := G \cap \text{End}_D({}^l V) = \mathbf{U}({}^l V)$ and ${}^l G^0 = Z_{{}^l G}({}^l \Gamma)$ for $0 \leq l \leq b$ (cf. Section 4.1).

Proof. – The proposition is straightforward probably except (iii). Part (iii) follows easily from the fact that ${}^l \mathbf{1} \in F'[\Gamma]$ for all $1 \leq l \leq b$ (cf. (4.2)). \square

Motivated by the above proposition, we make the following definition.

⁽¹⁰⁾ If we set $\Gamma_{-1} = 0$, then ${}^0 V$ is the kernel of Γ .

DEFINITION 4.6. – Retaining the notation in Proposition 4.5, we write $\Sigma = \bigoplus_{l=0}^b {}^l\Sigma$ and we call it *the block decomposition* of Σ (cf. next section). In this situation, we say that Σ has b blocks (by counting from 0).

In fact, the block decomposition is unique. See [33, Remark 3.3 (iii)] for an elementary argument.

4.3. Direct sum of supercuspidal data

We now define the direct sum of single block supercuspidal data with different depths.

LEMMA 4.7. – *Suppose b is a positive integer and $\{ {}^l\Sigma : 0 \leq l \leq b \}$ is a set of supercuspidal data for $U({}^lV)$ such that*

- (a) lV is an ϵ -Hermitian space;
- (b) ${}^0\Sigma$ has zero depth;
- (c) ${}^l\Sigma$ is single block of positive depth ${}^l r$ for $1 \leq l \leq b$;
- (d) ${}^b r > \dots > {}^1 r > 0$.

Let $V := \bigoplus_{l=0}^b {}^lV$ be the orthogonal direct sum of ϵ -Hermitian spaces. Define x, Γ, ϕ and ρ by the formula in Proposition 4.5 (i), (iv) and (v) respectively.

Then $\Sigma = (x, \Gamma, \phi, \rho)$ is a supercuspidal datum with depth ${}^b r$ for $U(V)$ called the direct sum of $\{ {}^l\Sigma \}$ and we write $\Sigma = \bigoplus_l {}^l\Sigma$.

REMARK. – This construction also induces a notion of direct sum on the set of equivalence classes of data.

Proof. – We recall $G^0 = Z_{U(V)}(\Gamma)$ and ${}^lG = U({}^lV)$ as in Proposition 4.5. We claim that $G^0 = \prod_l {}^lG^0$. Indeed this follows from the observation that, after a certain base change, lV is exactly the direct sum of eigenspaces of Γ whose eigenvalues have valuation $-{}^l r$ if $1 \leq l \leq b$. The lemma now follows from this claim and we will leave the details to the reader. \square

5. Theta Lifts of supercuspidal data

The purpose of this section is to define the notion of theta lifts of supercuspidal data. We first define the lift for a single block of zero depth or positive depth. The general case is defined using direct sum (cf. Section 4.3).

Recall that \mathfrak{f} is the residual field of F and \mathfrak{f}_D is the residual field of D which is at most a quadratic extension of \mathfrak{f} . We fix an uniformizer $\varpi_D \in \mathfrak{p}_D$ such that $\tau(\varpi_D) = \epsilon_D \varpi_D$ and $\epsilon_D \in \{ \pm 1 \}$. We retain the notation of Section 2.1.

5.1. Theta lifts of depth zero representations

Local theta lifts between depth zero supercuspidal representations were studied by Pan in [28]. It is reduced to theta correspondences over finite fields. We summarize these results below.

5.1.1. – We recall the definition of the dual lattice in Definition 2.3. A lattice L in V is called a *good lattice* if $L^* \mathfrak{p}_D \subseteq L \subseteq L^*$.

The set of vertices in $\mathcal{B}(G)$ naturally corresponds to a subset of good lattices.⁽¹¹⁾

Let L be a good lattice in V corresponding to a point or a good lattice $x \in \mathcal{B}(G)$. Then

- (a) $G_x = \{g \in G \mid gL = L\}$.
- (b) $G_{x,0+} = \{g \in G \mid (g-1)L^* \subseteq L, (g-1)L \subseteq L^* \mathfrak{p}_D\}$.
- (c) The \mathfrak{f}_D -modules $\ell := L/L^* \mathfrak{p}_D$ and $\ell^* := L^*/L$ are equipped with \mathfrak{f}_D -sesquilinear forms induced by $\varpi_D^{-1} \langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_V$ respectively. Clearly ℓ is $\epsilon_D \epsilon$ -Hermitian and ℓ^* is ϵ -Hermitian.
- (d) We have $G_x/G_{x,0+} \cong U(\ell) \times U(\ell^*)$.

The Witt classes of ℓ and ℓ^* are completely determined by the Witt class of V but independent of the choices of L in V . Indeed, the anisotropic kernel of the Witt class of ℓ (resp. ℓ^*) is equal to $L_{\min}/L_{\min}^* \mathfrak{p}_D$ (resp. L_{\max}^*/L_{\max}) where L_{\min} (resp. L_{\max}) is a minimal (resp. maximal) good lattice.

Let \mathcal{T} be a Witt class of ϵ -Hermitian D -modules. Let \mathcal{T} and \mathcal{T}^* be the corresponding Witt classes of ℓ and ℓ^* determined by \mathcal{T} . Then it is clear that there is a map

$$(5.1) \quad \begin{array}{ccc} \Upsilon: \mathcal{T} \times \mathcal{T}^* & \longrightarrow & \mathcal{T} \\ & & (\ell, \ell^*) \longmapsto V \end{array}$$

such that $\dim_{\mathfrak{f}_D} \ell + \dim_{\mathfrak{f}_D} \ell^* = \dim_D V$, and ℓ and ℓ^* are constructed from a vertex $x \in \mathcal{B}(U(V))$.

DEFINITION 5.1. – (a) We say that a pair (x, ρ) is a *depth zero K -type* for $G := U(V)$ when x is a vertex in $\mathcal{B}(G)$ and ρ is an irreducible $G_x := G_x/G_{x,0+}$ -module.

(b) Suppose \tilde{G} is a certain central \mathbb{C}^\times -covering of G in (a). We say that a pair $(x, \tilde{\rho})$ is a *depth zero K -type* for \tilde{G} when x is a vertex in $\mathcal{B}(G)$ and $\tilde{\rho}$ is an irreducible $\tilde{G}_x := \tilde{G}_x/G_{x,0+}$ -module.

(c) We equip an equivalence relation on the set of depth zero K -types by G -conjugacy.

(d) A \tilde{G} -module $\tilde{\pi}$ is said to have a depth zero K -type $(x, \tilde{\rho})$ if $\tilde{\rho}$ occurs in $\tilde{\pi}|_{\tilde{G}_x}$.

We warn that the depth zero K -type in (5.1)(a) and (5.1)(d) above is more general than the depth zero minimal K -type in [24, 25], where ρ must be cuspidal.

Clearly, a depth zero supercuspidal datum $\Sigma = (x, 0, \mathbf{1}, \rho)$ corresponds to the depth zero K -type (x, ρ) . This gives an embedding of the set of equivalence classes of depth zero supercuspidal datum to the set of equivalence classes of depth zero K -types. The image precisely consists of the equivalence classes $[(x, \rho)]$ where ρ is cuspidal.

⁽¹¹⁾ Suppose x is a vertex in $\mathcal{B}(G)$ and \mathcal{L} is the corresponding lattice function, then $\mathcal{L} \mapsto \mathcal{L}_{0+}$ gives the correspondence. The subset of good lattices could be proper, see [38, Example 2.2.3.1].

5.1.2. – We review some basic results of theta correspondences over a finite field. Let $(U(\ell), U(\ell'))$ be a type I reductive dual pair in a symplectic \mathfrak{f} -space \mathbf{W} and let $\overline{\omega}_{\mathbf{W}}$ be the oscillator representation of $\mathbf{Sp}(\mathbf{W})$ with respect to the additive character $\overline{\psi}$ (cf. Section 1.1).

DEFINITION 5.2. – Let ρ and ρ' be irreducible representations of $U(\ell)$ and $U(\ell')$ respectively. Then ρ and ρ' are said to correspond with each other if $\rho \otimes \rho'$ is a submodule of $\overline{\omega}_{\mathbf{W}}|_{U(\ell) \times U(\ell')}$. Such correspondence is not one-to-one in general, so we can only say that ρ' is “ a ” theta lift of ρ .

If ρ is cuspidal, then there is at most one ρ' which corresponds to ρ . In particular, when restricted to cuspidal representations, theta lift still provide an one-to-one correspondence (cf. [23, Section 3.IV.4]). In this case, we write $\rho' = \theta_{\ell, \ell'}(\rho)$.

5.1.3. – In the definition of lift of data, a zero dual pair (i.e., $\ell = 0$ or $\ell' = 0$) may occur as the degenerate case⁽¹²⁾.

DEFINITION 5.3. – A type I reductive dual pair $(U(\ell), U(\ell'))$ defined over the field \mathfrak{f} is called a zero dual pair if ℓ or ℓ' is the zero vector space.

In this case, $\mathbf{Sp}(\ell \otimes_{\mathfrak{f}_D} \ell')$ degenerates to the trivial group and the corresponding oscillator representation is the trivial representation. Since the roles of ℓ and ℓ' are symmetric, we will assume that $\ell = 0$. Then $U(\ell)$ is the trivial group and it has only one representation, namely, the trivial representation $\pi := \mathbf{1}_{U(\ell)}$. Now $\theta_{\ell, \ell'}(\pi)$ is the trivial representation of $U(\ell')$. Note that, the trivial representation of $U(\ell')$ is cuspidal if and only if ℓ' is anisotropic.

5.1.4. – Fix a Witt tower \mathcal{F}' of ϵ' -Hermitian D -modules. The discussion in Section 5.1.1 also applies to \mathcal{F}' and we add primes $'$ to extend the corresponding notations.

We fix $V' \in \mathcal{F}'$ and a vertex $x' \in \mathcal{B}(G')$. Then $(U(\ell), U(\ell'^*))$ and $(U(\ell^*), U(\ell'))$ form two reductive dual pairs over the finite field \mathfrak{f}_D . Here the zero dual pair may appear.

DEFINITION 5.4. – Let (x, ρ) and (x', ρ') be two depth zero K -types of G and G' respectively. We write $\rho = \rho_{\ell} \boxtimes \rho_{\ell^*}$ and $\rho' = \rho_{\ell'} \boxtimes \rho_{\ell'^*}$ where ρ_{\diamond} is an irreducible $U(\diamond)$ -module with $\diamond \in \{\ell, \ell^*, \ell', \ell'^*\}$. We say (x', ρ') is a (theta) lift of (x, ρ) if $\rho_{\ell'^*}$ is a theta lift of ρ_{ℓ} and $\rho_{\ell'}$ is a theta lift of ρ_{ℓ^*} .

Now we are ready to state a theorem of Pan.

THEOREM 5.5 ([28, Theorem 5.6]). – Let $(G, G') = (U(V), U(V'))$ be a type I dual pair over F and $\tilde{\pi}' = \theta_{V, V'}(\tilde{\pi})$. Suppose $\tilde{\pi}$ has a depth zero K -type $(x, \tilde{\rho})$. Then there exists a depth zero K -type (x, ρ) and a theta lift (x', ρ') of it such that (see (2.4) for the definition of $\tilde{\xi}_{x, x'}$)

- (a) $\tilde{\rho} = (\rho \boxtimes \text{id}_{\mathbb{C}^\times}) \circ \tilde{\xi}_{x, x'}^{-1}$, and
- (b) $\tilde{\pi}'$ has depth zero K -type $(x', \tilde{\rho}')$ where $\tilde{\rho}' := (\rho' \boxtimes \text{id}_{\mathbb{C}^\times}) \circ \tilde{\xi}_{x, x'}^{-1}$. □

On the other hand, suppose (x', ρ') is a theta lift of (x, ρ) . Let $\tilde{\rho}$ and $\tilde{\rho}'$ be defined as in Theorem 5.5 (a) and (b). Then the $\tilde{G}_x \times \tilde{G}'_{x'}$ -module $\tilde{\rho} \boxtimes \tilde{\rho}'$ occurs in $\mathcal{S}(\mathcal{B}_0)_{\mathcal{B}_0}$ where $\mathcal{B} = \mathcal{B}_{x, x'}$.

⁽¹²⁾ Similar notion also applies to dual pairs over other fields.

5.1.5. – Now let $\Sigma = (x, 0, \mathbf{1}, \rho)$ be a depth zero supercuspidal datum of $U(V)$ and let \mathcal{F}' be a fixed Witt tower of ϵ' -Hermitian spaces. Then we have $\ell, \ell^*, \mathcal{F}', \mathcal{F}'^*$ defined in Section 5.1.1. Moreover, $\rho = \rho_\ell \boxtimes \rho_{\ell^*}$ with ρ_ℓ and ρ_{ℓ^*} cuspidal.

Let $\rho_{\ell'^*} := \theta_{\ell, \mathcal{F}'^*}(\rho_\ell)$ be the theta lift of ρ_ℓ such that it is at the first occurrence, say $\ell'^* \in \mathcal{F}'^*$, with respect to the Witt tower \mathcal{F}'^* . Likewise define ℓ' and the $U(\ell')$ -module $\rho_{\ell'} := \theta_{\ell^*, \mathcal{F}'}(\rho_{\ell^*})$. Note that $\rho_{\ell'} \otimes \rho_{\ell'^*}$ is a cuspidal representation. By (5.1), let $V' := \Upsilon'(\ell', \ell'^*) \in \mathcal{F}'$ and let x' be the corresponding vertex in $\mathcal{B}(U(V'))$.

DEFINITION 5.6. – Define $\Sigma' := (x', 0, \mathbf{1}, \rho_{\ell'} \boxtimes \rho_{\ell'^*}) \in \mathcal{D}_{V'}$ and call it the *theta lift* of Σ with respect to the Witt tower \mathcal{F}' . Furthermore, the theta lift of data map $\vartheta_{V, \mathcal{F}'}$ is defined by

$$\vartheta_{V, \mathcal{F}'}([\Sigma]) = [\Sigma']$$

when restricted on the set of depth zero supercuspidal data. By an abuse of notation, we also write $\Sigma'' = \vartheta_{V, V'}(\Sigma) = \vartheta_{V, \mathcal{F}'}(\Sigma)$ where Σ'' is any element in the equivalence class $[\Sigma']$.

The next corollary follows immediately from the above discussion (cf. [28, § 9]).

COROLLARY 5.7. – *The Main Theorem holds when restricted on the set of depth zero supercuspidal representations.* \square

5.2. Theta lift of a single block of positive depth

Throughout this subsection, we assume that $\Sigma = (x, \Gamma, \phi, \rho)$ is a single block datum of positive depth r for $G = U(V)$. Let $s := r/2$ and $\mathcal{L} = \mathcal{L}_x$. Since x is a point in $\mathcal{B}(G^0)$, we have

($\Gamma 1$) $\mathcal{L}_{t-r} = \Gamma \mathcal{L}_t$ for all $t \in \mathbb{R} \sqcup \mathbb{R}^+$, and

($\Gamma 2$) each element in $\Gamma + \mathfrak{g}_{x, -r+}$ is invertible with depth $-r$.

DEFINITION 5.8. – For $\Gamma \in \mathfrak{g}$ which is invertible in $\mathfrak{gl}(V)$, we define V_Γ to be the $(-\epsilon)$ -Hermitian D -module with the same underlying D -module as V and equipped with the form $\langle v_1, v_2 \rangle_{V_\Gamma} := \langle v_1, \Gamma v_2 \rangle_V$ for $v_1, v_2 \in V_\Gamma$.

REMARK. – In fact, there is an element $w \in W := \text{Hom}_D(V, V_\Gamma)$ such that $M(w) = \Gamma$. Let $\iota \in \text{Hom}_D(V, V_\Gamma)$ be the identity map with respect to the underlying D -modules of V_Γ and V . Let $w := \iota$. Then $w^\star = \Gamma \circ \iota^{-1}$ and $M(w) = w^\star w = \Gamma$.

5.2.1. – In this subsection, we let V' be an ϵ' -Hermitian D -module such that

$$\dim_D V' = \dim_D V \quad \text{and} \quad \epsilon' = -\epsilon.$$

LEMMA 5.9. – *Suppose that there is a $w \in \text{Hom}_D(V, V')$ such that $M(w) \in \Gamma + \mathfrak{g}_{x, -r+}$. By ($\Gamma 2$), w is an isomorphism of D -modules. We define a lattice function in V' by*

$$(5.2) \quad \mathcal{L}'_t := w \mathcal{L}_{t+s}.$$

Then

- (i) $\text{Jump}(\mathcal{L}') = \text{Jump}(\mathcal{L}) - s$,
- (ii) the lattice function \mathcal{L}' is self-dual and
- (iii) \mathcal{L}' is the unique self-dual lattice function on V' such that $w \in (\mathcal{L} \otimes_D \mathcal{L}')_{-s}$.

Proof. – (i) This is clear from the definition of \mathcal{L}' .

(ii) For any $v \in V$, $\langle wv, \mathcal{L}'_t \rangle_{V'} = \langle wv, w\mathcal{L}_{t+s} \rangle_V = \langle v, w^\star w\mathcal{L}_{t+s} \rangle_V = \langle v, \Gamma\mathcal{L}_{t+s} \rangle_V = \langle v, \mathcal{L}_{t-s} \rangle_V$. Therefore $(\mathcal{L}'_t)^\star = (w\mathcal{L}_{-(t-s)}) = \mathcal{L}'_{-t+}$, i.e., \mathcal{L}' is self-dual.

(iii) Clearly, $w \in (\mathcal{L} \otimes_D \mathcal{L}')_{-s}$ by the definition of \mathcal{L}' . Suppose $\check{\mathcal{L}}'$ is another self-dual lattice function such that $w \in (\mathcal{L} \otimes \check{\mathcal{L}}')_{-s}$. Then $\mathcal{L}'_t = w\mathcal{L}_{t+s} \subseteq \check{\mathcal{L}}'_t$ for all $t \in \mathbb{R}$. Taking the dual lattices gives $\check{\mathcal{L}}'_{-t+} = (\check{\mathcal{L}}'_t)^\star \subseteq (\mathcal{L}'_t)^\star = \mathcal{L}'_{-t+}$ for all $t \in \mathbb{R}$. Hence $\mathcal{L}'_t = \check{\mathcal{L}}'_t$. \square

5.2.2. – The following proposition shows the surjectivity of moment maps on certain cosets. This is a key proposition in the single block of positive depth case.

PROPOSITION 5.10. – *Let \mathcal{L}' be a self-dual lattice in V' and $\mathcal{B} := \mathcal{L} \otimes_D \mathcal{L}'$. Let $w \in \mathcal{B}_{-s}$. Suppose that $M(w) \in \Gamma + \mathfrak{g}_{x,-r+}$. Then*

$$M(w + \mathcal{B}_t) = M(w) + \mathfrak{g}_{x,-s+t} \quad \forall t > -s.$$

Proof. – We first prove the following claim.

CLAIM 1. – *The map*

$$w + \mathcal{B}_t \rightarrow (M(w) + \mathfrak{g}_{x,-s+t}) / \mathfrak{g}_{x,-s+t+} \subsetneq \mathfrak{g} / \mathfrak{g}_{x,-s+t+}$$

given by $w' \mapsto M(w') + \mathfrak{g}_{x,-s+t+}$ is a surjection.

Proof. – Let $b \in \mathcal{B}_t$. Since $t > -s$, we have

$$\begin{aligned} (5.3) \quad M(w + b) &= (w + b)^\star (w + b) = w^\star w + w^\star b + b^\star w + b^\star b \\ &\equiv M(w) + w^\star b + b^\star w \pmod{\mathfrak{g}_{x,-s+t+}}. \end{aligned}$$

On the other hand, by Lemma 5.9, $X \mapsto wX$ gives an isomorphism $\mathfrak{gl}(V)_{x,s+t} \xrightarrow{\sim} \mathcal{B}_t$. Hence we assume that $b = wX$ for some $X \in \mathfrak{gl}(V)_{x,s+t}$.

Pick a good element $\check{\Gamma} \in M(w) + \mathfrak{g}_{x,-r+} = \Gamma + \mathfrak{g}_{x,-r+}$. We have

$$\begin{aligned} (5.4) \quad w^\star b + b^\star w &= w^\star wX + (wX)^\star w = M(w)X + X^\star M(w) \\ &\equiv \check{\Gamma}X + X^\star \check{\Gamma} \pmod{\mathfrak{g}_{x,-s+t+}}. \end{aligned}$$

We set $\mathfrak{gl}(V)_{x,t_1:t_2} := \mathfrak{gl}(V)_{x,t_1} / \mathfrak{gl}(V)_{x,t_2}$ for $t_1 < t_2$. Now Claim 1 reduces to the following claim.

CLAIM 2. – *The map $\beta: \mathfrak{gl}(V)_{x,s+t:s+t+} \rightarrow \mathfrak{g}_{x,-s+t:-s+t+}$ defined by $X \mapsto \check{\Gamma}X + X^\star \check{\Gamma}$ is surjective.*

Under the \ast -action, $\mathfrak{gl}(V)_{x,-s+t} = \mathfrak{g}_{x,-s+t} \oplus \mathfrak{gl}(V)_{x,-s+t}^{\ast,+1}$ where $\mathfrak{gl}(V)_{x,-s+t}^{\ast,+1}$ is the \ast -invariant subspace. Under the $\check{\Gamma}$ -action, we have decomposition $\mathfrak{g} = \check{\mathfrak{g}} \oplus \check{\mathfrak{g}}^\perp$ where $\check{\mathfrak{g}} = Z_{\mathfrak{g}}(\check{\Gamma})$ and $\check{\mathfrak{g}}^\perp$ is the orthogonal complement of $\check{\mathfrak{g}}$ in \mathfrak{g} under the G -invariant bilinear form \mathbb{B} . We also have a similar decomposition of $\mathfrak{gl}(V)$. Since $\check{\Gamma}$ is \ast -skew invariant, these decompositions are compatible with each other.

First assume that $X \in \mathfrak{g}_{x,-s+t}$, i.e., $X^\star = -X$. Then $\beta(X) = \check{\Gamma}X - X\check{\Gamma} = [\check{\Gamma}, X]$.

Now [18, Lemma 2.3.4] states that $X \mapsto [\check{\Gamma}, X]$ induces an isomorphism

$$(5.5) \quad \beta: \check{\mathfrak{g}}_{x,s+t:s+t+}^\perp \xrightarrow{\sim} \check{\mathfrak{g}}_{x,-s+t:-s+t+}^\perp.$$

Since $\mathfrak{g}_{x,-s+t:-s+t+} = \check{\mathfrak{g}}_{x,-s+t:-s+t+} \oplus \check{\mathfrak{g}}_{x,-s+t:-s+t+}^\perp$ it remains to show that $\check{\mathfrak{g}}_{x,-s+t:-s+t+}$ is in the image of β . Let $\check{\mathfrak{gl}} = Z_{\mathfrak{gl}(V)}(\check{\Gamma})$. Suppose $X \in \check{\mathfrak{gl}}_{x,s+t}^{*,+1} \subseteq \mathfrak{gl}_{x,s+t}^{*,+1}$. Then

$$(5.6) \quad \beta(X) = \check{\Gamma}X + X^*\check{\Gamma} = \check{\Gamma}X + X\check{\Gamma} = 2\check{\Gamma}X.$$

Therefore β restrict to an isomorphism $\check{\mathfrak{gl}}_{x,s+t}^{*,+1} \xrightarrow{\beta} \check{\mathfrak{gl}}_{x,-s+t}^{*,+1} = \check{\mathfrak{g}}_{x,-s+t}$. This proves Claim 2 and also Claim 1. \square

We now prove Proposition 5.10. By (5.3) we have $M(w + \mathcal{B}_t) \subseteq M(w) + \mathfrak{g}_{x,-s+t}$. Fix an element $\gamma \in M(w) + \mathfrak{g}_{x,-s+t}$. Clearly $M(w) \in \gamma + \mathfrak{g}_{x,-s+t}$. Let $w_1 := w$ and $t_1 := t$. We construct sequences $\{w_i\}$ and $\{t_i\}$ inductively. Suppose we have $M(w_i) \in \gamma + \mathfrak{g}_{x,-s+t_i}$ for some $w_i \in w + \mathcal{B}_t$. Apply the above claim with $w = w_i$ and $t = t_i$, we get a certain $b_i \in \mathcal{B}_{x,t_i}$ such that $M(w_i + b_i) \in \gamma + \mathfrak{g}_{x,-s+t_i+}$. Let $w_{i+1} := w_i + b_i$ and $t_{i+1} = \max\{t \mid t > t_i, \mathfrak{g}_{x,-s+t} = \mathfrak{g}_{x,-s+t_i+}\} \in \text{Jump}(\mathfrak{g}_x)$ where \mathfrak{g}_x denote the lattice function on \mathfrak{g} corresponding to the Moy-Prasad filtration. Clearly $M(w_{i+1}) \in \gamma + \mathfrak{g}_{x,-s+t_i+} = \gamma + \mathfrak{g}_{x,-s+t_{i+1}}$. Since $\text{Jump}(\mathfrak{g}_x)$ is a discrete set in \mathbb{R} , $t_i \rightarrow \infty$ and w_i converges to some $w_\infty \in w + \mathcal{B}_t$. Since the moment map is continuous, we have $M(w_\infty) = \lim_{i \rightarrow \infty} M(w_i) = \gamma$. This proves the proposition. \square

Now we present some corollaries of Proposition 5.10.

COROLLARY 5.11. – *The set of self-dual lattice functions \mathcal{L}' in V' such that*

$$(\Gamma + \mathfrak{g}_{x,r+}) \cap M((\mathcal{L} \otimes_D \mathcal{L}')_{-s}) \neq \emptyset$$

is a G' -orbit.

Proof. – Let \mathcal{L}' and \mathcal{L}'' be two self-dual lattice functions in the set. Let $w \in (\mathcal{L} \otimes \mathcal{L}')_{-s}$ such that $M(w) \in \Gamma + \mathfrak{g}_{x,r+}$. By Proposition 5.10, we may assume that $M(w) = \Gamma$. By Lemma 5.9 $\mathcal{L}'_t = w\mathcal{L}_{t+s}$ for all t . Similarly we pick a ${}^1w \in (\mathcal{L} \otimes \mathcal{L}'')_{-s}$ such that $M({}^1w) = \Gamma$ and $\mathcal{L}''_t = {}^1w\mathcal{L}_{t+s}$ for all t . Note that Γ is invertible. By Witt's theorem (see [8, Section 1.11] and [14, Thm 3.7.1]) there is $g' \in G'$ such that ${}^1w = g'w$. Hence $\mathcal{L}''_t = {}^1w\mathcal{L}_{t+s} = g'w\mathcal{L}_{t+s} = g'\mathcal{L}'_t$ for all t . \square

We recall the definition of V_Γ in Definition 5.8.

COROLLARY 5.12. – *The set of ϵ' -Hermitian D -modules*

$$\mathfrak{V}'_\Gamma = \{V' \mid \dim_D V' = \dim_D V \text{ and } M^{-1}(\Gamma + \mathfrak{g}_{x,-r+}) \neq \emptyset\}$$

is the isometry class of V_Γ .

Proof. – By the remark after Definition 5.8, we see that \mathfrak{V}'_Γ contains V_Γ . Let $V' \in \mathfrak{V}'_\Gamma$. By Proposition 5.10 there exists a $w \in \text{Hom}(V, V')$ such that $M(w) = \Gamma$. Now $\langle wv_1, wv_2 \rangle_{V'} = \langle v_1, \Gamma v_2 \rangle_V = \langle v_1, v_2 \rangle_{V_\Gamma}$ for all $v_1, v_2 \in V$. In other words w gives an isometry from V_Γ to V' . This proves the corollary. \square

Corollary 5.12 shows that V_Γ is the unique ϵ' -Hermitian D -module V' up to isometry such that $\dim_D V' = \dim_D V$ and there exists a $w \in W$ such that $M(w) \in \Gamma + \mathfrak{g}_{x,r+}$.

5.2.3. – Recall the notation in the beginning of Section 5.2, where $\Sigma = (x, \Gamma, \phi, \rho)$ is a single block supercuspidal datum of positive depth r . We assume that V' is isomorphic to V_Γ . Let $W = V \otimes_D V'$ and fix a $w \in W$ such that $M(w) = \Gamma$. Let $G' = U(V')$ and let $x' \in \mathcal{B}(G')$ be the point corresponding to the lattice function \mathcal{L}' defined by (5.2). We define

$$\Gamma' := M'(w) = ww^\star = w\Gamma w^{-1} \in \mathfrak{g}'.$$

PROPOSITION 5.13. – Let $\Gamma = \sum_{i=-1}^d \Gamma_i$ be a $GL_D(V)$ -good factorization of Γ in \mathfrak{g} (which always exists by Proposition 4.3). Let

$$\Gamma'_j := w\Gamma_j w^{-1}.$$

Then $\Gamma'_j \in \mathfrak{g}'$ and $\Gamma' = \sum_{j=-1}^d \Gamma'_j$ is a $GL_D(V')$ -good factorization of Γ' in \mathfrak{g}' .

Proof. – Since Γ commutes with Γ_j , we have

$$\begin{aligned} (\Gamma'_j)^* &= (w^{-1})^\star \Gamma_j^* w^\star = -(w^\star)^{-1} \Gamma_j w^\star = -(w^\star)^{-1} \Gamma_j \Gamma w^{-1} \\ &= -(w^\star)^{-1} \Gamma \Gamma_j w^{-1} = -w\Gamma_j w^{-1} = -\Gamma'_j. \end{aligned}$$

This shows that $\Gamma'_j \in \mathfrak{g}'$. Note that $w: V \rightarrow V'$ is an isomorphism of D -modules. Hence $\Gamma' = \sum_{j=-1}^d \Gamma'_j$ is a $GL(V')$ -good factorization. □

REMARKS. – We collect some easy consequences of Proposition 5.13.

1. By Lemma 4.2, $\Gamma' = \sum_{j=-1}^d \Gamma'_j$ is also a G' -good factorization.
2. Γ' satisfies Definition 3.4 (a) with respect to G' and therefore

$$G^0 := Z_{G'}(\Gamma'_d, \dots, \Gamma'_0) = Z_{G'}(\Gamma').$$

3. We have an isomorphism

$$(5.7) \quad \alpha: G^0 \xrightarrow{\sim} G'^0 \quad \text{defined by} \quad g \mapsto wgw^{-1}.$$

Thanks to Lemma 5.9, α restricted to an isomorphism $\alpha|_{G_x^0}: G_x^0 \xrightarrow{\sim} G_{x'}^0$.

4. The point $x' \in \mathcal{B}(G'^0)$ is also a vertex.

Let $\phi' := \phi^* \circ \alpha^{-1}$ and $\rho' := \rho^* \circ \alpha^{-1}$ viewed as a character and a cuspidal representation of $G_{x'}^0/G_{x',0+}^0$ respectively. Clearly

$$(5.8) \quad \Sigma'_{\Sigma,w} := (x', -\Gamma', \phi', \rho')$$

is a single block supercuspidal datum of positive depth r for $G' = U(V')$.

The following lemma shows that (5.8) is well-defined up to equivalence classes.

LEMMA 5.14. – Let $\Sigma := (x, \Gamma, \phi, \rho)$ be a single block datum of positive depth r . Let $w \in M^{-1}(\Gamma)$ and define $\Sigma'_{\Sigma,w}$ as in (5.8). Then the equivalence class $[\Sigma'_{\Sigma,w}]$ is independent of the choice of the element in the equivalence class $[\Sigma]$ and w .

Proof. – First we fix $\Sigma = (x, \Gamma, \phi, \rho)$ in $[\Sigma]$. For any $w \in M^{-1}(\Gamma)$, let $\Sigma'_{\Sigma,w}$ denote the datum defined via (5.8). Since $M^{-1}(\Gamma)$ is a single G' -orbit, we see that elements in $\{\Sigma'_{\Sigma,w} \mid w \in M^{-1}(\Gamma)\}$ are G' -conjugates of one another.

Suppose $\dot{\Sigma} = (\dot{x}, \dot{\Gamma}, \dot{\phi}, \dot{\rho}) \in [\Sigma]$. We will show that $\Sigma'_{\Sigma,w}$ and $\Sigma'_{\dot{\Sigma},\dot{w}}$ are equivalent. By G -conjugacy, we could assume $\dot{x} = x$ and $\dot{\Gamma} = \Gamma + \gamma$ such that $\gamma \in Z(\mathfrak{g}^0) \cap \mathfrak{g}_{x,0} \subseteq F'[\Gamma]$ (see remarks after Definition 3.6 and Lemma 4.1).

CLAIM. – Let $F'[\Gamma]_r^* := F'[\Gamma] \cap \mathfrak{gl}(V)_{x,r}^{*,+1}$ be the set of elements in $F'[\Gamma]$ which are $*$ -invariant and whose depth is not smaller than r . Then there is an element $c \in F'[\Gamma]_r^*$ such that $M(w(1+c)) = \dot{\Gamma}$.

Proof. – By a similar calculation as in (5.3), (5.4), we have $M(w(1+c)) = \Gamma + \Gamma(2c+c^2)$. Then $M(w(1+c)) = \dot{\Gamma}$ if and only if $2c+c^2 = \Gamma^{-1}\gamma$. Observe that $F'[\Gamma] \cap \mathfrak{gl}(V)_{x,r}^{*,+1}$ is a product of non-Archimedean local field(s) and $F'[\Gamma]_r^*$ is an ideal in its integral ring. Since $p \neq 2$, the map $F'[\Gamma]_r^* \mapsto F'[\Gamma]_r^*$ defined by $c \mapsto 2c+c^2$ is a bijection by Hensel’s lemma. Now the claim follows because $\Gamma^{-1}\gamma \in F'[\Gamma]_r^*$. \square

Let c be the element given by the above claim and $\dot{w} := w(1+c)$. It is straightforward to check that $\dot{\Gamma}' = \Gamma' + \gamma'$ with $\gamma' := w(2c+c^2)w^\star \in \mathbf{Z}_{\mathfrak{g}'(\Gamma')} \cap \mathfrak{g}'_{x',0}$. Moreover $G'^0, x', \alpha, \phi', \rho'$ are exactly the same objects for w and \dot{w} . In other words, $\Sigma'_{\Sigma,w}$ and $\Sigma'_{\dot{\Sigma},\dot{w}}$ are equivalent. This completes the proof of the lemma. \square

DEFINITION 5.15. – We retain the notation in Lemma 5.14. The isomorphism class of V' is independent of the choice of the element in $[\Sigma]$ by Corollary 5.12. We define $\vartheta^+([\Sigma])$ to be the equivalence class $[\Sigma'_{\Sigma,w}] \in \overline{\mathcal{D}}_{V'}$. By an abuse of notation, we will also write $\Sigma' = \vartheta^+(\Sigma)$ where $\Sigma' := \Sigma'_{\Sigma,w}$ and w is implicitly fixed.

5.3. The general case

Let $\Sigma = (x, \Gamma, \phi, \rho)$ be a supercuspidal datum of $G := \mathbf{U}(V)$. By Definition 4.6, let $\Sigma = \bigoplus_{l=0}^b {}^l\Sigma$ be the block decomposition of Σ into b positive depth blocks $\{ {}^l\Sigma \mid 1 \leq l \leq b \}$ and a depth zero block ${}^0\Sigma$. In addition, we have $\Gamma = \bigoplus {}^l\Gamma$ and $V = \bigoplus {}^lV$.

For any ϵ' -Hermitian D -module V' , let $[V']$ represent its Witt class in the Witt group.

DEFINITION 5.16. – Let \mathcal{F}' be a fixed Witt class of ϵ' -Hermitian D -modules. We recall $\overline{\mathcal{D}}_{\mathcal{F}'}$ in (1.2). We set

- (a) ${}^l\Sigma' := \vartheta^+({}^l\Sigma) \in \mathcal{D}_{{}^lV_{{}^l\Gamma}}$ for $1 \leq l \leq b$;
- (b) ${}^0\mathcal{F}' := \mathcal{F}' - \sum_{l=1}^b [{}^lV_{{}^l\Gamma}]$ and
- (c) ${}^0\Sigma' := \vartheta_{0V,0\mathcal{F}'}({}^0\Sigma)$ (cf. Definition 5.6).

Then we define

$$\vartheta_{V,\mathcal{F}'}([\Sigma]) := \left[\bigoplus_{i=0}^b {}^i\Sigma' \right] \in \overline{\mathcal{D}}_{\mathcal{F}'}$$

By an abuse of notation again, we also write $\Sigma' = \vartheta_{V,\mathcal{F}'}(\Sigma)$ where $\Sigma' := \bigoplus_{i=0}^b {}^i\Sigma'$.

REMARKS. – 1. Note that the ${}^l\Sigma'$ s have different depths. It follows from Lemma 4.7 that $\Sigma' := \vartheta_{V,\mathcal{F}'}(\Sigma) = (x', -\Gamma', \phi', \rho')$ is a supercuspidal datum of $\mathbf{U}(V')$ for a well-defined $[V'] \in \mathcal{F}'$.

2. In the construction, we get an element ${}^l w \in {}^lV \otimes_D {}^lV'$ such that ${}^l\Gamma = M({}^l w)$ and ${}^l\Gamma' = M'({}^l w)$ for each $0 < l \leq b$. Therefore we get an element

$$(5.9) \quad w := \bigoplus_{0 < l \leq b} {}^l w \in \bigoplus_{0 < l \leq b} {}^lV \otimes_D {}^lV' \subseteq V \otimes_D V'$$

so that $\Gamma \equiv M(w) \pmod{\mathfrak{g}_{x,0}}$ and $\Gamma' \equiv M'(w) \pmod{\mathfrak{g}'_{x',0}}$.

3. In the above definition of $\vartheta_{V, \mathcal{G}'}$, the key is the correspondence of semisimple elements via the moment maps. We expect an explicit description of the correspondences between cuspidal representations of dual pairs over finite fields using similar construction. Indeed there are some partial results in this aspect by Pan [31, 30].

4. The discussions in Sections 3 to 5 extend to K -type data defined in the remark to Definition 3.4. More precisely, the notion of K -type data extends to the covering group and the notions of equivalence relation, block decomposition, direct sum etc. extend under exactly the same definition as well.

5. Suppose Σ (resp. $\widetilde{\Sigma}$) is a K -type data for G (resp. \widetilde{G}). Then η_Σ and $\widetilde{\eta}_{\widetilde{\Sigma}}$ are also defined in the same way.

DEFINITION 5.17. – A K -type datum Σ' is a *theta lift* of a supercuspidal datum Σ for the dual pair $(U(V), U(V'))$, if

- (a) $\Sigma = \bigoplus_{l=0}^b {}^l\Sigma$ is a block decomposition of a K -type datum;
- (b) ${}^l\Sigma' = \vartheta^+({}^l\Sigma)$ for $1 \leq l \leq b$;
- (c) ${}^0\Sigma'$ is a (not necessary supercuspidal) depth zero data which is a theta lift of ${}^0\Sigma$ (cf. Definition 5.4);
- (d) $V' = \bigoplus_{l=0}^b {}^lV'$ and $\Sigma' := \bigoplus_{l=0}^b {}^l\Sigma'$.

5.4. An example

To illustrate the content of the definitions made above, we provide the following example which could be considered as a generic case:

EXAMPLE. – Let $\widetilde{\Sigma}$ be a supercuspidal datum of G such that 0V is the zero space under the block decomposition (cf. Section 4.2). Equivalently, this means every eigenvalue of Γ over \bar{F} has negative valuation when we view V as an F -vector space and Γ as an F -linear map on V . Since Γ is invertible, we let V_Γ denote the ϵ' -Hermitian space in Definition 5.8. Let $V_{\mathcal{G}'-[V_\Gamma]}^\circ$ be the anisotropic ϵ' -Hermitian space in the Witt tower $\mathcal{G}' - [V_\Gamma]$. Then the first occurrence of $\widetilde{\pi}_{\widetilde{\Sigma}}$ is at $V' := V_\Gamma \oplus V_{\mathcal{G}'-[V_\Gamma]}^\circ$ in the Witt tower \mathcal{G}' . Using this explicit formula, one may check the conservation relation [35] of the first occurrence indices directly in this case. If $\mathcal{G}' = [V_\Gamma]$, then $\vartheta_{V, \mathcal{G}'}(\Sigma)$ is essentially the “contragredient” of Σ (cf. (5.8)). Otherwise, $\vartheta_{V, \mathcal{G}'}(\Sigma)$ is the direct sum of $\vartheta_{V, [V_\Gamma]}(\Sigma)$ and the datum attached to the trivial representation of $U(V_{\mathcal{G}'-[V_\Gamma]}^\circ)$.

6. One positive depth block case I: orbit structure

6.1. Assumptions

Throughout this section, we retain the notation in Section 5.2 and make the following assumptions.

- (I) Let $\Sigma = (x, \Gamma, \phi, \rho)$ be a single block datum with positive depth $r = 2s$. In particular Γ is an invertible element in $\text{End}(V)$. We fix a $\text{GL}(V)$ -good factorization $\Gamma = \sum_{i=-1}^d \Gamma_i$.
- (II) The space $(V', \langle \cdot, \cdot \rangle_{V'})$ is isomorphic to $(V_\Gamma, \langle \cdot, \cdot \rangle_{V_\Gamma})$ and $w \in \text{Hom}_D(V, V')$ is a fixed element such that $M(w) = \Gamma$. In particular $\dim_D V = \dim_D V'$.

We make following definitions:

- DEFINITION 6.1. – (a) We refer to Cases I and II in Definition 3.3. If we are in Case I, i.e., $\Gamma_d = 0$, then we set $\mathfrak{J}_\Gamma := \{0, \dots, d - 1\}$ and $\check{\Gamma} = \Gamma_{d-1}$. Otherwise if we are in Case II, i.e., $\Gamma_d \neq 0$, then we set $\mathfrak{J}_\Gamma := \{0, \dots, d\}$ and $\check{\Gamma} = \Gamma_d$. Under this definition $\check{\Gamma}$ is a nonzero good element in $\Gamma + \mathfrak{g}_{x, -r+}$. Let $\check{G} := Z_G(\check{\Gamma})$.
- (b) Define G^i with its Lie algebra \mathfrak{g}^i as in Definition 3.3. In particular $G^0 = Z_G(\Gamma)$.
- (c) Let $\mathfrak{g}^{i-1\perp}$ be the orthogonal complement of \mathfrak{g}^{i-1} in \mathfrak{g}^i with respect to the invariant bilinear form \mathbb{B} in Section 2.1.1, i.e., $\mathfrak{g}^i = \mathfrak{g}^{i-1} \oplus \mathfrak{g}^{i-1\perp}$. Let $\mathfrak{g}_{x,r}^{i\perp} = \mathfrak{g}^{i\perp} \cap \mathfrak{g}_{x,r}$.
- (d) Let $\mathfrak{gl} := \mathfrak{gl}^d := \mathfrak{gl}(V)$ and $\mathfrak{gl}^i = Z_{\mathfrak{g}^{i+1}}(\Gamma_i)$ for $0 \leq i < d$.
- (e) Let \mathcal{L} be the self-dual lattice function on V corresponding to x . Let \mathcal{L}' be the self dual lattice function in V' defined by $\mathcal{L}'_t = w\mathcal{L}_{t+s}$ as in Lemma 5.9 and let x' be the corresponding point in $\mathcal{B}(G^0)$.
- (f) Let $\Gamma' = M'(w)$ and $\Gamma' = \sum_{i=-1}^d \Gamma'_i$ be the good factorization of Γ' given by Proposition 5.13.
- (g) We define similar notation for G' as in (b) to (d).
- (h) Let $\mathcal{B} := \mathcal{L} \otimes_D \mathcal{L}'$. Then $w \in \mathcal{B}_{-s}$ by Lemma 5.9 (iii).
- (i) Let $\alpha: G^0 \xrightarrow{\sim} G'^0$ be the group isomorphism given by $\alpha(g) = wgw^{-1}$ (cf. (5.7)).
- (j) Let $\mathfrak{g}'^\perp := \mathfrak{gl}(V')^{*,+1}$. Then $\mathfrak{gl}' := \mathfrak{gl}(V') = \mathfrak{g}' \oplus \mathfrak{g}'^\perp$ under the $*$ -action.
- (k) For each $X \in \mathfrak{g}$, we define $d\alpha: \mathfrak{g} \rightarrow \mathfrak{g}'$ and $d\alpha^\perp: \mathfrak{g} \rightarrow \mathfrak{g}'^\perp$ by

$$d\alpha(X) = \frac{1}{2}(wXw^{-1} - (wXw^{-1})^*) \quad \text{and} \quad d\alpha^\perp(X) = \frac{1}{2}(wXw^{-1} + (wXw^{-1})^*).$$

Clearly $wXw^{-1} = d\alpha(X) + d\alpha^\perp(X)$.

6.2. Structure of orbits

We apply Definition 3.5 (a)-(e) to data (x, Γ) and (x', Γ') . The purpose of this section is to study the $K \times K'$ -orbit of the coset $w + \mathcal{B}_0$ in W/\mathcal{B}_0 .

6.2.1. – We start by investigating some properties of $d\alpha$ and $d\alpha^\perp$ by elementary linear algebra.

LEMMA 6.2. – For any $t \in \mathbb{R}$, we set $t'_i := r - r_{i-1} + t$. Then the following statements hold:

- (i) $d\alpha(\mathfrak{g}_{x,t}^i) \subseteq \mathfrak{g}_{x',t'}^i$ for $0 \leq i \leq d$.
- (ii) $d\alpha^\perp(\mathfrak{g}_{x,t}^i) \subseteq \mathfrak{g}_{x',t'_i}^i$ for $0 < i \leq d$.
- (iii) $d\alpha: \mathfrak{g}^0 \rightarrow \mathfrak{g}'^0$ is an isomorphism which is the differential of α and $d\alpha^\perp(\mathfrak{g}^0) = 0$.
- (iv) $d\alpha(\mathfrak{g}_{x,t}^i \setminus \mathfrak{g}_{x,t+}^i) \subseteq \mathfrak{g}_{x',t'}^i \setminus \mathfrak{g}_{x',t'+}^i$ for $i \in \mathfrak{J}_\Gamma$.
- (v) The map $\overline{d\alpha}|_{\mathfrak{g}_{x,t,t+}^i} : \mathfrak{g}_{x,t,t+}^i \longrightarrow \mathfrak{g}_{x',t',t'+}^i$ induced by $d\alpha|_{\mathfrak{g}_{x,t}^i}$ is an isomorphism for $i \in \mathfrak{J}_\Gamma$.
Hence, $d\alpha|_{\mathfrak{g}_{x,t}^i} : \mathfrak{g}_{x,t}^i \longrightarrow \mathfrak{g}_{x',t'}^i$ and $d\alpha|_{\mathfrak{g}^i} : \mathfrak{g}^i \longrightarrow \mathfrak{g}'^i$ are also isomorphisms for $i \in \mathfrak{J}_\Gamma$.
- (vi) $d\alpha(\mathfrak{g}^{i\perp}) \subseteq \mathfrak{g}'^{i\perp}$ for $0 \leq i < d$ and so $d\alpha(\mathfrak{g}_{x,t}^{i\perp}) \subseteq \mathfrak{g}_{x',t'}^{i\perp}$ by (ii).

Proof. – We set $\tilde{X} := wXw^{-1} = d\alpha(X) + d\alpha^\perp(X)$. By the definition of \mathcal{L}' , it is clear that $\tilde{X} \in \mathfrak{g}'_{x',t'}$ if and only if $X \in \mathfrak{g}_{x,t}$. Note that $(w^{-1})^\star = (w^\star)^{-1}$ and $\tilde{X}^\star = (wXw^{-1})^\star = (w^\star)^{-1}X^\star w^\star$.

(i) Suppose $X \in \mathfrak{g}^i$. Then $[\tilde{X}, \Gamma'_l] = [wXw^{-1}, w\Gamma_lw^{-1}] = w[X, \Gamma_l]w^{-1} = 0$ for all $i \leq l \leq d$, i.e., $\tilde{X} \in \mathfrak{gl}^i$. Similar argument gives $\tilde{X}^* \in \mathfrak{gl}^i$. Now $d\alpha(X) = (\tilde{X} - \tilde{X}^*)/2 \in \mathfrak{g}^i$. This shows that $d\alpha(\mathfrak{g}^i) \subseteq \mathfrak{g}^i$. Since $\tilde{X}^* \in \mathfrak{gl}_{x',t}$ for $X \in \mathfrak{g}_{x,t}$, $d\alpha(\mathfrak{g}_{x,t}^i) \subseteq \mathfrak{g}_{x',t}^i$.

(ii) Let $X \in \mathfrak{g}_{x,t}^i$. Then

$$\begin{aligned} d\alpha^\perp(X) &= \frac{1}{2}(wXw^{-1} + (wXw^{-1})^*) = \frac{1}{2}(wXw^{-1} - (w^\star)^{-1}Xw^\star) \\ &= \frac{1}{2}(w^\star)^{-1}(w^\star wX - Xw^\star w)w^{-1} = \frac{1}{2}(w^\star)^{-1}[\Gamma, X]w^{-1} \\ &= \frac{1}{2}(w^\star)^{-1}\left(\sum_{l=-1}^d [\Gamma_l, X]\right)w^{-1} = \frac{1}{2}(w^\star)^{-1}\left(\sum_{l=-1}^{i-1} [\Gamma_l, X]\right)w^{-1}. \end{aligned}$$

Note that, for any $\tilde{t} \in \mathbb{R}$, we have $w^{-1}\mathcal{L}'_{\tilde{t}} = \mathcal{L}_{\tilde{t}+s}$, $(w^\star)^{-1}\mathcal{L}_{\tilde{t}} = \mathcal{L}'_{\tilde{t}+s}$, $\Gamma_l\mathcal{L}_{\tilde{t}} \subseteq \mathcal{L}_{\tilde{t}-r_l}$ and $X\mathcal{L}_{\tilde{t}} \subseteq \mathcal{L}_{\tilde{t}+t}$ (cf. Section 5.2.1). Hence $d\alpha^\perp(X) \in \mathfrak{gl}'_{r-r_{i-1}+t}$.

(iii) This is clear from the definition of α , G^0 and G'^0 (cf. Definition 6.1 (b) and (i)).

(iv) If $X \in \mathfrak{g}_{x,t}^i \setminus \mathfrak{g}_{x,t+}^i$, then $\tilde{X} = wXw^{-1} \in \mathfrak{gl}'_{x',t} \setminus \mathfrak{gl}'_{x',t+}$. Since $r > r_{i-1}$, $d\alpha^\perp(X) \in \mathfrak{gl}'_{x',t+}$ by (ii). Therefore, $d\alpha(X) = \tilde{X} - d\alpha^\perp(X) \in \mathfrak{gl}'_{x',t} \setminus \mathfrak{gl}'_{x',t+}$.

(v) The injectivity of $\overline{d\alpha}|_{\mathfrak{g}_{x,t,t+}^i}$ is a restatement of (iv). The surjectivity follows by dimension counting. Note that the roles of G and G' are symmetric. We could define $d\alpha': \mathfrak{g}' \rightarrow \mathfrak{g}$ in the same way. Now

$$\mathfrak{g}_{x,t,t+}^i \xrightarrow{\overline{d\alpha}} \mathfrak{g}_{x',t,t+}^i \xrightarrow{\overline{d\alpha'}} \mathfrak{g}_{x,t,t+}^i$$

where $\overline{d\alpha}$ and $\overline{d\alpha}'$ are injections. Hence $\overline{d\alpha'} \circ \overline{d\alpha}$ is an injection, $\dim_{\mathfrak{f}} \mathfrak{g}_{x,t,t+}^i = \dim_{\mathfrak{f}} \mathfrak{g}_{x',t,t+}^i$ and $\overline{d\alpha}|_{\mathfrak{g}_{x,t,t+}^i}$ is an isomorphism. The claim for $d\alpha|_{\mathfrak{g}_{x,t}^i}$ and $d\alpha|_{\mathfrak{g}^i}$ immediately follows.

(vi) Note that $(w^\star)^{-1}\Gamma_iw^\star = (w^\star)^{-1}w^{-1}\Gamma'_iw^\star = \Gamma'^{-1}\Gamma'_i\Gamma' = \Gamma'_i$. Therefore, for all $X \in \mathfrak{g}$,

$$\begin{aligned} (6.1) \quad d\alpha \circ \text{ad}_{\Gamma_i}(X) &= \frac{1}{2}(w[\Gamma_i, X]w^{-1} - (w^\star)^{-1}[\Gamma_i, X]^*w^\star) \\ &= \frac{1}{2}([\Gamma'_i, \tilde{X}] - (w^\star)^{-1}[\Gamma_i, -X]w^\star) \\ &= \frac{1}{2}([\Gamma'_i, \tilde{X}] - [\Gamma'_i, -(w^\star)^{-1}Xw^\star]) \\ &= \text{ad}_{\Gamma'_i} \circ d\alpha(X). \end{aligned}$$

Now (vi) follows since $\mathfrak{g}^{i\perp}$ and $\mathfrak{g}'^{i\perp}$ are the sums of non-trivial isotypic components in \mathfrak{g}^{i+1} and \mathfrak{g}'^{i+1} under the actions of ad_{Γ_i} and $\text{ad}_{\Gamma'_i}$ respectively. \square

6.2.2. – We define symplectic forms on \mathfrak{g} and \mathfrak{g}' respectively by ⁽¹³⁾

$$(6.2) \quad \begin{aligned} \langle X_1, X_2 \rangle_\Gamma &= \mathbb{B}([X_1, X_2], \Gamma) \quad \forall X_1, X_2 \in \mathfrak{g} \quad \text{and} \\ \langle X'_1, X'_2 \rangle_{-\Gamma'} &= \mathbb{B}([X'_1, X'_2], -\Gamma') \quad \forall X'_1, X'_2 \in \mathfrak{g}'. \end{aligned}$$

We equip $\mathfrak{g} \oplus \mathfrak{g}'$ with the form $\langle \cdot, \cdot \rangle_\Gamma \oplus \langle \cdot, \cdot \rangle_{-\Gamma'}$.

⁽¹³⁾ Do not confuse with $\langle \cdot, \cdot \rangle_{V_\Gamma}$ on V_Γ in Definition 5.8.

LEMMA 6.3. – *The map $\iota: \mathfrak{g} \oplus \mathfrak{g}' \longrightarrow W$ given by*

$$(X, X') \mapsto X \cdot w + X' \cdot w = -wX + X'w$$

is an isometry.

Proof. – Let $X, X_1, X_2 \in \mathfrak{g}$ and $X', X'_1, X'_2 \in \mathfrak{g}'$. Then

$$\begin{aligned} \langle X_1, X_2 \rangle_\Gamma &= \mathbb{B}([X_1, X_2], \Gamma) = \frac{1}{2} \operatorname{tr}_F(X_1 X_2 \Gamma - X_2 X_1 \Gamma) \\ &= \frac{1}{2} \operatorname{tr}_F(X_1 X_2 \Gamma - (X_2 X_1 \Gamma)^*) = \frac{1}{2} \operatorname{tr}_F(X_1 X_2 \Gamma + \Gamma X_1 X_2) \\ &= \operatorname{tr}_F(X_1 X_2 w^\star w) = -\operatorname{tr}_F((-w X_2)^\star (-w X_1)) \\ &= -\langle X_2 \cdot w, X_1 \cdot w \rangle_W = \langle \iota(X_1), \iota(X_2) \rangle_W. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \langle X'_1, X'_2 \rangle_{-\Gamma'} &= -\mathbb{B}([X'_1, X'_2], \Gamma') = -\operatorname{tr}_F(X'_1 X'_2 \Gamma') = -\operatorname{tr}_F(X'_1 X'_2 w w^\star) \\ &= \operatorname{tr}_F((X'_1 w)^\star X'_2 w) = \langle X'_1 \cdot w, X'_2 \cdot w \rangle_W = \langle \iota(X'_1), \iota(X'_2) \rangle_W. \end{aligned}$$

On the other hand

$$\begin{aligned} \langle \iota(X), \iota(X') \rangle_W &= \langle -wX, X'w \rangle_W = \operatorname{tr}_F((-wX)^\star X'w) = \operatorname{tr}_F(Xw^\star X'w) \\ &= \operatorname{tr}_F(w^\star X'wX) = \operatorname{tr}_F((X'w)^\star (-wX)) = \langle \iota(X'), \iota(X) \rangle_W \\ &= -\langle \iota(X), \iota(X') \rangle_W. \end{aligned}$$

Hence $\langle \iota(X), \iota(X') \rangle_W = 0$. Therefore, $\iota(\mathfrak{g})$ and $\iota(\mathfrak{g}')$ are orthogonal to each other. \square

6.2.3. – Let $\overline{\langle \cdot, \cdot \rangle}_\Gamma$ denote the \mathfrak{f} -symplectic form on $\mathfrak{g}_{x,s;s^+}$ induced by $\langle \cdot, \cdot \rangle_\Gamma$. Let $\mathfrak{r} := \operatorname{Rad}(\overline{\langle \cdot, \cdot \rangle}_\Gamma)$ be the radical of $\overline{\langle \cdot, \cdot \rangle}_\Gamma$ in $\mathfrak{g}_{x,s;s^+}$. Likewise we define $\overline{\langle \cdot, \cdot \rangle}_{-\Gamma'}$ and $\mathfrak{r}' := \operatorname{Rad}(\overline{\langle \cdot, \cdot \rangle}_{-\Gamma'})$.

LEMMA 6.4. – (i) *In Case I (i.e., $\Gamma_d = 0$), we have $\mathfrak{r} = \mathfrak{g}_{x,s;s^+}^{d-1}$. In Case II (i.e., $\Gamma_d \neq 0$), we have $\overline{\langle \cdot, \cdot \rangle}_\Gamma \equiv 0$ and $\mathfrak{r} = \mathfrak{g}_{x,s;s^+}$.*
(ii) *Likewise $\mathfrak{r}' = \mathfrak{g}_{x',s';s^+}^{d-1}$ in Case I and $\mathfrak{r}' = \mathfrak{g}'_{x',s';s^+}$ in Case II.*

Proof. – For $X_1, X_2 \in \mathfrak{g}_{x,s}$,

$$\langle X_1, X_2 \rangle_\Gamma = \mathbb{B}([X_1, X_2], \Gamma) = -\mathbb{B}(X_2, [X_1, \Gamma]).$$

Hence $\mathfrak{r} = \{X_1 + \mathfrak{g}_{x,s^+} \mid \operatorname{ad}_\Gamma(X_1) \in \mathfrak{g}_{x,-s^+}\}$. However $\operatorname{ad}_\Gamma|_{\mathfrak{g}_{x,s;s^+}} = \operatorname{ad}_{\check{\Gamma}}|_{\mathfrak{g}_{x,s;s^+}}$. We get the conclusion in Case I by (5.5).

In Case II, we have $\check{\Gamma} = \Gamma_d \in Z(\mathfrak{g})$. So $[X_1, \Gamma] \equiv [X_1, \check{\Gamma}] \equiv 0 \pmod{\mathfrak{g}_{x,-s^+}}$ and $\langle X_1, X_2 \rangle_\Gamma \in \mathfrak{p}$. \square

6.2.4. – We recall that $\mathfrak{b} := \mathcal{B}_{0;0+}$ is an \mathfrak{f} -symplectic space. By Lemma 6.3, ι induces an isometry

$$\bar{\iota} : \mathfrak{g}_{x,s;s+} \oplus \mathfrak{g}'_{x',s;s+} \longrightarrow \mathfrak{b}.$$

Define $\mathbf{\Delta} : \mathfrak{g} \longrightarrow \mathfrak{g} \oplus \mathfrak{g}'$ by $X \mapsto (X, d\alpha(X))$. It induces an injection

$$(6.3) \quad \overline{\mathbf{\Delta}} : \mathfrak{g}_{x,s;s+} \hookrightarrow \mathfrak{g}_{x,s;s+} \oplus \mathfrak{g}'_{x',s;s+}.$$

Let $\mathfrak{b}_+ := \bar{\iota}(\mathfrak{r} \oplus \mathfrak{r}')$. Then \mathfrak{b}_+ is the radical of $\text{Im}(\bar{\iota})$. Let $\mathfrak{b}_0 := \mathfrak{b}_+^\perp / \mathfrak{b}_+$.

LEMMA 6.5. – *The following statements hold:*

- (i) $\overline{d\alpha} : \mathfrak{r} \rightarrow \mathfrak{r}'$ is an isomorphism,
- (ii) $0 \longrightarrow \mathfrak{r} \xrightarrow{\overline{\mathbf{\Delta}}} \mathfrak{g}_{x,s;s+} \oplus \mathfrak{g}'_{x',s;s+} \xrightarrow{\bar{\iota}} \mathfrak{b}$ is exact,
- (iii) $\mathfrak{b}_+ = \bar{\iota}(\mathfrak{r}) = \bar{\iota}(\mathfrak{r}')$,
- (iv) $\dim_{\mathfrak{f}} \mathfrak{g}_{x,s;s+} + \dim_{\mathfrak{f}} \mathfrak{g}'_{x',s;s+} = \dim_{\mathfrak{f}} \mathfrak{b}$,
- (v) $\mathfrak{b}_+^\perp = \text{Im}(\bar{\iota})$,
- (vi) in Case I, the composition of maps

$$\bar{\iota}_{\mathfrak{b}_0} : \mathfrak{g}_{x,s;s+}^{d-1\perp} \oplus \mathfrak{g}'_{x',s;s+}^{d-1\perp} \hookrightarrow \mathfrak{g}_{x,s;s+} \oplus \mathfrak{g}'_{x',s;s+} \xrightarrow{\bar{\iota}} \mathfrak{b}_+^\perp \twoheadrightarrow \mathfrak{b}_0$$

is an isomorphism of non-degenerate symplectic spaces over \mathfrak{f} ,

- (vii) in Case II, $\text{Im}(\bar{\iota}) = \mathfrak{b}_+ = \mathfrak{b}_+^\perp$ is a maximal isotropic subspace of \mathfrak{b} and $\mathfrak{b}_0 = 0$.

Proof. – Throughout this proof, we let $(X, X') \in \mathfrak{g}_{x,s} \oplus \mathfrak{g}'_{x',s}$ and we let (\bar{X}, \bar{X}') denote its image in $\mathfrak{g}_{x,s;s+} \oplus \mathfrak{g}'_{x',s;s+}$.

- (i) This follows from Lemma 6.2 (v) and Lemma 6.4.
- (ii) Let $\bar{X} \in \mathfrak{r} \subseteq \mathfrak{g}_{x,s;s+}$. Then

$$\begin{aligned} \bar{\iota} \circ \overline{\mathbf{\Delta}}(\bar{X}) &\equiv -wX + d\alpha(X)w = (-wXw^{-1} + d\alpha(X))w \\ &= -d\alpha^\perp(X)w \pmod{\mathcal{B}_{0+}}. \end{aligned}$$

Since $r > r_{d-2}$ in Case I (resp. $r > r_{d-1}$ in Case II), Lemma 6.2 (ii) implies $d\alpha^\perp(X) \in \mathfrak{g}'_{x',s+}$. Hence $d\alpha^\perp(X)w \in \mathcal{B}_{0+}$. This proves $\overline{\mathbf{\Delta}}(\mathfrak{r}) \subseteq \text{Ker } \bar{\iota}$.

Now we show the opposite inclusion. Suppose $\bar{\iota}(\bar{X}, \bar{X}') = 0$. Since $\bar{\iota}$ is an isometry, $(\bar{X}, \bar{X}') \in \mathfrak{r} \oplus \mathfrak{r}' = \text{Rad}(\langle \cdot, \cdot \rangle_\Gamma \oplus \langle \cdot, \cdot \rangle_{-\Gamma'})$. Now $\overline{\mathbf{\Delta}}(\bar{X}) = (\bar{X}, \overline{d\alpha}(\bar{X})) \in \text{Ker } \bar{\iota}$ so $\bar{\iota}(0, \bar{X}' - \overline{d\alpha}(\bar{X})) = 0$. Note that $\bar{\iota}|_{\mathfrak{g}'_{x',s;s+}}$ is an injection. This implies $\bar{X}' = \overline{d\alpha}(\bar{X})$

and proves the exactness of (ii).

- (iii) By (ii), $\bar{\iota}(\bar{X}) = -\bar{\iota}(\overline{d\alpha}(\bar{X}))$ for $\bar{X} \in \mathfrak{r}$. Now $\bar{\iota}(\mathfrak{r}) = \bar{\iota}(\mathfrak{r}')$ by part (i).
- (iv) Since $\mathcal{L}'_t = w\mathcal{L}_{t+s}$, we have $\mathfrak{gl}_{x,s;s+} \cong \mathfrak{b}$ by $X \mapsto wX$. Therefore,

$$(6.4) \quad \dim_{\mathfrak{f}} \mathfrak{b} = \dim_{\mathfrak{f}} \mathfrak{gl}_{x,s;s+}.$$

Consider the isomorphism $\eta : \mathfrak{gl} \xrightarrow{\sim} \mathfrak{gl}'$ defined by $X \mapsto w^\star Xw$.

Since $(w^\star)^\star = -w$, we have $\eta(X)^* = (w^\star Xw)^* = -w^\star X^*w = \eta(-X^*)$. By reducing to the residue field, $\bar{\eta} : \mathfrak{gl}_{x,s;s+} \xrightarrow{\sim} \mathfrak{gl}'_{x',-s;-s+}$ induces an isomorphism

$$\bar{\eta} : \mathfrak{gl}_{x,s;s+}^{*,+1} \xrightarrow{\sim} \mathfrak{gl}'_{x',-s;-s+}^{*,-1} = \mathfrak{gl}'_{x',-s;-s+} \cong \text{Hom}_{\mathfrak{f}}(\mathfrak{gl}'_{x',s;s+}, \mathfrak{f}).$$

Therefore

$$(6.5) \quad \dim_{\mathfrak{f}} \mathfrak{g}_{x,s:s^+} = \dim_{\mathfrak{f}} \mathfrak{g}_{x,s:s^+} + \dim_{\mathfrak{f}} \mathfrak{g}_{x,s:s^+}^{*,+1} = \dim_{\mathfrak{f}} \mathfrak{g}_{x,s:s^+} + \dim_{\mathfrak{f}} \mathfrak{g}'_{x',s:s^+}.$$

Combining (6.4) and (6.5) yields ((iv)).

- (v) Note that $\bar{\iota}$ is an isometry. So $\text{Im}(\bar{\iota}) \subseteq \bar{\iota}(\mathfrak{r})^{\perp} = \mathfrak{b}_{+}^{\perp}$ by ((iii)). Since $\bar{\iota}|_{\mathfrak{g}_{x,s:s^+}}$ is an injection and \mathfrak{b} is a non-degenerate symplectic space, $\dim \mathfrak{b}_{+}^{\perp} = \dim \mathfrak{b} - \dim \mathfrak{b}_{+} = \dim \mathfrak{g}_{x,s:s^+} \oplus \mathfrak{g}'_{x',s:s^+} - \dim \mathfrak{r} = \dim \text{Im}(\bar{\iota})$ by ((iv)) and ((ii)).
- (vi) By Lemma 6.4 ((i)), $\mathfrak{g}_{x,s:s^+}^{d-1\perp} \oplus \mathfrak{g}'_{x',s:s^+}^{d-1\perp}$ is a maximal non-degenerate symplectic subspace of $\mathfrak{g}_{x,s:s^+} \oplus \mathfrak{g}'_{x',s:s^+}$. Now the claim follows from ((v)), since $\bar{\iota}$ is an isometry.
- (vii) This follows by a similar argument as that of part (vi) using Lemma 6.4 ((ii)). \square

6.2.5. – We begin with definitions which will be used in the rest of the section.

DEFINITION 6.6. – In order to simplify the notation, let

- (a) $\check{K} := K \times K'$, $\check{K}_{0+} = K_{0+} \times K'_{0+}$ and $\check{K}_{+} := K_{+} \times K'_{+}$;
 (b) $\check{G}_{s_{i-1}}^i := G_{x,s_{i-1}}^i \times G_{x',s_{i-1}}^i$ and $\check{G}_{s_{i-1}^+}^i := G_{x,s_{i-1}^+}^i \times G_{x',s_{i-1}^+}^i$ for $0 < i \leq d$; and
 (c) $\check{G}^0 := G_x^0 \times G_{x'}^0$ and $\check{G}_{0+}^0 := G_{x,0+}^0 \times G_{x',0+}^0$.

We define

- (d) $S := \text{Stab}_{\check{K}}(w + \mathcal{B}_0) = \left\{ (h, h') \in \check{K} \mid (h, h') \cdot w \in w + \mathcal{B}_0 \right\}$,
 (e) $S^i := S \cap \check{G}_{s_{i-1}}^i$ and $S_+^i := S \cap \check{G}_{s_{i-1}^+}^i$ for $i > 0$,
 (f) $S^0 := S \cap \check{G}^0$, $S_+ := S \cap \check{K}_{+}$ and $S_{0+} := S \cap \check{K}_{0+}$.

We extend the group isomorphism $\alpha: G_x^0 \xrightarrow{\sim} G_{x'}^0$ in Definition 6.1 ((i)) to a map

$$(6.6) \quad \alpha: G_x^0 G_{x,0+} \longrightarrow G_{x'}^0 G_{x',0+} \quad \text{by} \quad g \exp(X) \mapsto (w g w^{-1}) \exp(d\alpha(X))$$

for all $g \in G_x^0$ and $\exp(X) \in G_{x,0+}$. This map α is well-defined because $w \exp(X) w^{-1} = \exp(d\alpha(X))$ for $\exp X \in G_x^0 \cap G_{x,0+} = G_{x,0+}^0$. We warn that the map α is not necessarily a group homomorphism. We define

- (g) $\Delta^0 := \{ (g, \alpha(g)) \mid g \in G_x^0 \} \cong G_x^0 \cong G_{x'}^0$,
 (h) $\Delta_+^0 := \{ (g, \alpha(g)) \mid g \in G_{x,0+}^0 \} \cong G_{x,0+}^0 \cong G_{x',0+}^0$,
 (i) $\Delta^i := \{ (g, \alpha(g)) \mid g \in G_{x,s_{i-1}}^i \}$ for $i > 0$ and
 (j) $\Delta_+^i := \{ (g, \alpha(g)) \mid g \in G_{x,s_{i-1}^+}^i \}$ for $i > 0$.

By Lemma 6.2, the map α restricts to a map⁽¹⁴⁾ $\alpha|_{G_{x,s_{i-1}}^i}: G_{x,s_{i-1}}^i \rightarrow G_{x',s_{i-1}}^i$ for $1 \leq i \leq d$ and this restricted map is a bijection if $i \in \mathcal{J}_{\Gamma}$. Therefore $\Delta^i \subseteq \check{G}_{s_{i-1}}^i$ and $\Delta_+^i \subseteq \check{G}_{s_{i-1}^+}^i$ for all $0 \leq i \leq d$.

Obviously $G_{x,s} G_{x',s} \subseteq S$. The following is a key lemma which claims that S is generated by these sets.

LEMMA 6.7. – (i) *If $(g, g') \in \Delta^0$, then $(g, g') \cdot w = w$. Moreover, ι and $\bar{\iota}$ are Δ^0 -equivariant.*

⁽¹⁴⁾ We reiterate that α is not necessarily a group homomorphism.

(ii) Suppose $0 < i \in \mathcal{J}_\Gamma$. Let $g = \exp(X)$ with $X \in \mathfrak{g}_{x,s_{i-1}}^i$. Then $d\alpha^\perp(X)w \in \mathcal{B}_{0+}$, $[d\alpha(X), d\alpha^\perp(X)]w \in \mathcal{B}_s$ and

$$(g, \alpha(g)) \cdot w \subseteq w - d\alpha^\perp(X)w - \frac{1}{2}[d\alpha(X), d\alpha^\perp(X)]w + \mathcal{B}_{s+} \subseteq w + \mathcal{B}_{0+}.$$

In particular, $\Delta^i \subseteq S$.

- (iii) $S \cap G = G_{x,s}$ and $S \cap G' = G'_{x',s}$.
- (iv) For each $i \in \mathcal{J}_\Gamma$, $S^i = \Delta^i G_{x,s}^i = \Delta^i G_{x,s}^i G_{x',s}^i$.
- (v) In Case I, i.e., $\Gamma_d = 0$ and $r = r_d = r_{d-1}$,

$$S = \Delta^0 \Delta^1 \dots \Delta^{d-1} G_{x,s} G'_{x',s} \quad \text{and}$$

$$S_+ = \Delta_+^0 \Delta_+^1 \dots \Delta_+^{d-1} G_{x,s}^{d-1} G_{x',s}^{d-1} G_{x,s+} G'_{x',s+}.$$

In Case II, i.e., $\Gamma_d \neq 0$ and $r = r_d > r_{d-1}$,

$$S = \Delta^0 \Delta^1 \dots \Delta^{d-1} \Delta^d G_{x,s} G'_{x',s} \quad \text{and}$$

$$S_+ = \Delta_+^0 \Delta_+^1 \dots \Delta_+^{d-1} \Delta_+^d G_{x,s} G'_{x',s}.$$

Proof. – (i) Clearly, $(g, g') \cdot w = g'wg^{-1} = \alpha(g)wg^{-1} = wgw^{-1}wg^{-1} = w$. The claim that ι and $\bar{\iota}$ are Δ^0 -equivariant also follows by a straightforward computation which we will leave to the reader.

(ii) Let $X' := d\alpha(X)$, $Y' := d\alpha^\perp(X)$ and $g' := \exp(X')$. Then $(g, g') = (g, \alpha(g)) \in \Delta^i$. Note that $r > r_{i-1} > 0$ and so $r - s_{i-1} > s$. By Lemma 6.2 (i) and (ii), $X' \in \mathfrak{gl}'_{x',s_{i-1}} \subseteq \mathfrak{gl}'_{x',0+}$ and $Y' \in \mathfrak{gl}'_{x',r-s_{i-1}} \subseteq \mathfrak{gl}'_{x',s+} \subseteq \mathfrak{gl}'_{x',0+}$. Therefore $X'Y', Y'X' \in \mathfrak{gl}'_{x',r}$, $Y'w \in \mathcal{B}_{0+}$ and $[X', Y']w \in \mathcal{B}_s$. By Zassenhaus formula,

$$\begin{aligned} ((g, g') \cdot w)w^{-1} &= g'wg^{-1}w^{-1} = \exp(X') \exp(-wXw^{-1}) = \exp(X') \exp(-X' - Y') \\ &= \exp(X') \exp(-X') \exp(-Y') \exp(-\frac{1}{2}[X', Y'])g'_{r+} \\ &= \exp(-Y') \exp(-\frac{1}{2}[X', Y'])g'_{r+}, \end{aligned}$$

where $g'_{r+} \in \text{GL}(V')_{x',r+}$. Hence $(g, g') \cdot w - w \equiv \exp(-Y') \exp(-\frac{1}{2}[X', Y'])w - w \equiv -Y'w - \frac{1}{2}[X', Y']w \pmod{\mathcal{B}_{s+}}$. This finishes the proof of (ii).

(iii) We only prove the identity for G' and the proof for G is similar. Note that $T \mapsto Tw^{-1}$ induces an isomorphism $\mathcal{B}_t \rightarrow \mathfrak{gl}'_{x',t+s}$. Therefore, for $g' \in G'$, $g'w + \mathcal{B}_0 = w + \mathcal{B}_0$ if and only if $g' \in 1 + \mathfrak{gl}'_{x',s}$. Hence $S \cap G' = \text{GL}(V')_{x',s} \cap G' = G'_{x',s}$.

(iv) By ((i)) to ((iii)), $S^i \supseteq \Delta^i G_{x,s}^i G_{x',s}^i \supseteq \Delta^i G_{x',s}^i$. It remains to show $S^i \subseteq \Delta^i G_{x',s}^i$. Indeed suppose $(g, g') \in S^i \subseteq \check{G}_{s_{i-1}}^i$. Then $(1, \alpha(g)^{-1}g') = (g, \alpha(g))^{-1}(g, g') \in S^i \cap G^i \subseteq G_{x',s}^i$ by ((iii)). Hence $(g, g') \in \Delta^i G_{x',s}^i$.

(v) Let $S' = \Delta^0 \dots \Delta^d G_{x,s} G'_{x',s}$. Note that $\Delta^d \subseteq G_{x,s} G'_{x',s}$ in Case I. By (i)-(iv), $S \supseteq S'$. Observe that the projection to the first factor $\text{pr}_1: S' \rightarrow K$ is surjective. Suppose $(g, g') \in S$. Then there is a $g'' \in K'$ such that $(g, g'') \in S'$. Therefore $(1, g''^{-1}g') \in S \cap G' = G'_{x',s}$ by (iii). Hence $S \subseteq S'G'_{x',s} = S'$. This proves that $S = S'$. Intersecting S with \check{K}_+ gives S_+ . \square

6.3. A key identity of ψ

We recall the function ψ_Γ in (3.2) which is clearly well-defined on \check{K}_{0+} .

LEMMA 6.8. – Suppose $1 \leq i \in \mathfrak{I}_\Gamma$ and $(g, g') \in \Delta^i$ or $(g, g') \in G_{x,s} \times G'_{x',s}$. Then

$$(6.7) \quad \psi\left(\frac{1}{2}\langle w, (g, g')^{-1} \cdot w - w \rangle_W\right) = \psi_\Gamma(g)\psi_{-\Gamma'}(g').$$

REMARK. – For $(g, g') \in \Delta^0$, $\psi\left(\frac{1}{2}\langle w, (g, g')^{-1} \cdot w - w \rangle_W\right) = 1$ by Lemma 6.7 ((i)).

Proof. – Suppose $(g, g') = (\exp(X), \exp(X')) \in \Delta^i$ where $X \in \mathfrak{g}_{x,s_{i-1}}^i$ and $X' = d\alpha(X)$. Let $Y' := d\alpha^1(X)$. By Lemma 6.7 ((ii)), $(g, g')^{-1} \cdot w - w \equiv Y'w - \frac{1}{2}[X', Y']w \pmod{\mathcal{B}_{s+}}$.

We claim that $\frac{1}{2}\langle w, [X', Y']w \rangle_W \in \mathfrak{p}$. Indeed by Lemma 2.2,

$$\begin{aligned} \frac{1}{2}\langle w, [X', Y']w \rangle_W &= \mathbb{B}(\Gamma', [Y', X']) = \mathbb{B}([\Gamma', X'], Y') \\ &\equiv \mathbb{B}([\check{\Gamma}', X'], Y') \pmod{\mathfrak{p}} \\ &= 0 \quad (\text{because } X' \in Z_{\mathfrak{g}'}(\check{\Gamma}') \text{ by Lemma 6.2 ((i)) and } i \in \mathfrak{I}_\Gamma). \end{aligned}$$

Note that $wXw^{-1} = X' + Y'$ and so we have

$$\begin{aligned} &\psi_\Gamma(g^{-1})\psi_{-\Gamma'}(g'^{-1})\psi\left(\frac{1}{2}\langle w, (g, g')^{-1} \cdot w - w \rangle_W\right) \\ &= \psi\left(\mathbb{B}(\Gamma, -X) + \mathbb{B}(-\Gamma', -X') + \frac{1}{2}\langle w, Y'w \rangle_W - \frac{1}{4}\langle w, [X', Y']w \rangle_W\right) \\ &= \psi\left(\frac{1}{2}\text{tr}_F(-w^\star wX) + \frac{1}{2}\text{tr}_F(ww^\star X') + \frac{1}{2}\text{tr}_F(w^\star Y'w)\right) \\ &= \psi\left(\frac{1}{2}\text{tr}_F(-ww^\star(wXw^{-1}) + ww^\star(X' + Y'))\right) \\ &= \psi(0) = 1. \end{aligned}$$

This proves the lemma for $(g, g') \in \Delta^i$.

Next suppose $(g, g') := (\exp(X), \exp(X')) \in G_{x,s}G'_{x',s}$. Then

$$\begin{aligned} \psi\left(\frac{1}{2}\langle w, (g, g')^{-1} \cdot w - w \rangle_W\right) &= \psi(\mathbb{B}(X, M(w)) + \psi(\mathbb{B}(X', -M'(w))) \quad (\text{by (2.6)}) \\ &= \psi_{M(w)}(g)\psi_{-M'(w)}(g') = \psi_\Gamma(g)\psi_{-\Gamma'}(g'). \end{aligned}$$

This proves the lemma. □

6.4. A maximal totally isotropic subspace

We refer to Appendix A.2.1 for the notation. Also see [37, p. 591] and [17, Section 12]. Suppose $1 \leq i \in \mathcal{I}_\Gamma$. Let $J^i := (G^{i-1}, G^i)_{x, (r_{i-1}, s_{i-1})}$ and $J_+^i := (G^{i-1}, G^i)_{x, (r_{i-1}, s_{i-1}^+)}$. Likewise we have subgroups J^i and J_+^i in G' . Let $\check{J}^i := J^i \times J^i$ and $\check{J}_+^i := J_+^i \times J_+^i$. Note that \exp induces a group isomorphism $\mathfrak{g}_{x, s_{i-1}: s_{i-1}^+}^{i-1\perp} \xrightarrow{\sim} J^i/J_+^i$ and we identify both sides from now on. We let $\mathbf{W}^i := J^i/J_+^i = \mathfrak{g}_{x, s_{i-1}: s_{i-1}^+}^{i-1\perp}$, $\mathbf{W}^i := J^i/J_+^i = \mathfrak{g}_{x', s_{i-1}: s_{i-1}^+}^{i-1\perp}$ and $\check{\mathbf{W}}^i := \check{J}^i/\check{J}_+^i = \mathbf{W}^i \times \mathbf{W}^i$. Note that $\check{\mathbf{W}}^i$ has a natural non-degenerate symplectic space structure over \mathfrak{f} induced by $\langle \cdot, \cdot \rangle_\Gamma \oplus \langle \cdot, \cdot \rangle_{-\Gamma'}$ (cf. (6.2)). Let \mathbf{D}^i be the image of $\Delta^i \cap \check{J}^i$ under the natural quotient map $\check{J}^i \twoheadrightarrow \check{\mathbf{W}}^i$. By Lemma 6.2, $\Delta^i \cap \check{J}^i = \{(\exp(X), \exp(d\alpha(X)) \mid X \in \mathfrak{g}_{x, r_{i-1}}^{i-1} \oplus \mathfrak{g}_{x, s_{i-1}}^{i-1\perp})\}$. Although $\Delta^i \cap \check{J}^i$ is not a subgroup of $G \times G'$, \mathbf{D}^i is an \mathfrak{f} -subspace in $\check{\mathbf{W}}^i$ isomorphic to $\mathfrak{g}_{x, s_{i-1}: s_{i-1}^+}^{i-1\perp}$.

LEMMA 6.9. – For $1 \leq i \in \mathcal{I}_\Gamma$, \mathbf{D}^i is a maximal totally isotropic subspace of $\check{\mathbf{W}}^i$.

Proof. – By Lemma 6.2 ((v)),

$$\dim_{\mathfrak{f}} \mathfrak{g}_{s_{i-1}: s_{i-1}^+}^{i-1\perp} = \dim_{\mathfrak{f}} \mathfrak{g}_{s_{i-1}: s_{i-1}^+}^i / \mathfrak{g}_{s_{i-1}: s_{i-1}^+}^{i-1} = \dim_{\mathfrak{f}} \mathfrak{g}_{s_{i-1}: s_{i-1}^+}^i / \mathfrak{g}_{s_{i-1}: s_{i-1}^+}^{i-1} = \dim_{\mathfrak{f}} \mathfrak{g}_{s_{i-1}: s_{i-1}^+}^{i-1\perp}.$$

Therefore, by Lemma 6.2 ((vi)), $\overline{d\alpha}$ induces an isomorphism $\mathfrak{g}_{s_{i-1}: s_{i-1}^+}^{i-1\perp} \xrightarrow{\sim} \mathfrak{g}_{s_{i-1}: s_{i-1}^+}^{i-1\perp}$ and $\dim_{\mathfrak{f}} \mathbf{D}^i = \frac{1}{2} \dim_{\mathfrak{f}} \check{\mathbf{W}}^i$. It remains to show that \mathbf{D}^i is isotropic. Let $X_1, X_2 \in \mathfrak{g}_{s_{i-1}}^{i-1\perp}$ and let $Y_1 = d\alpha(X_1), Y_2 = d\alpha(X_2)$. Then the symplectic form is given by

$$\begin{aligned} \langle (X_1, Y_1), (X_2, Y_2) \rangle &\equiv \mathbb{B}([X_1, X_2], \Gamma_{i-1}) + \mathbb{B}([Y_1, Y_2], -\Gamma'_{i-1}) \\ &\equiv \mathbb{B}(X_2, \text{ad}_{\Gamma_{i-1}}(X_1)) - \mathbb{B}(Y_2, \text{ad}_{\Gamma'_{i-1}}(Y_1)) \\ &\equiv \mathbb{B}(X_2, \text{ad}_{\Gamma_{i-1}}(X_1)) - \mathbb{B}(Y_2, d\alpha(\text{ad}_{\Gamma_{i-1}}(X_1))) \quad (\text{by (6.1)}) \\ &\equiv \mathbb{B}(X_2, \text{ad}_{\Gamma_{i-1}}(X_1)) - \mathbb{B}(wX_2w^{-1}, w(\text{ad}_{\Gamma_{i-1}}(X_1))w^{-1}) \\ &\quad (\text{by } r - r_{i-1} > 0, \text{ Lemma 6.2 ((i)) and ((ii))}) \\ &\equiv 0 \quad (\text{mod } \mathfrak{p}). \end{aligned}$$

This finishes the proof. \square

6.5. Triviality of $\chi^{\mathfrak{b}+}$

We recall the \mathfrak{f} -vector space $\mathfrak{b}_+ = \bar{\iota}(\mathfrak{t}) = \bar{\iota}(\mathfrak{t}')$ in Section 6.2.4. The space \mathfrak{b}_+ is an isotropic subspace in \mathfrak{b} and Δ^0 acts on it. Let $\chi^{\mathfrak{b}+}$ be the character of Δ^0 as defined in Appendix A.1. More precisely, $\chi^{\mathfrak{b}+}(g, \alpha(g)) = \det((g, \alpha(g))|_{\mathfrak{b}_+})^{(q-1)/2}$ where $(g, \alpha(g)) \in \Delta^0$ and $q = |\mathfrak{f}|$.

LEMMA 6.10. – We have $\chi^{\mathfrak{b}+}(g, \alpha(g)) = 1$ for all $(g, \alpha(g)) \in \Delta^0$.

The rest of this section is devoted to the proof of the above lemma. The proof does not affect the rest part of the paper, so the reader may skip it without loss of continuity.

First we introduce some notation. Suppose \mathfrak{f}' is an extension of \mathfrak{f} and \mathfrak{V} is an \mathfrak{f}' -module. Let G be a group acting \mathfrak{f}' -linearly on \mathfrak{V} . Let $\det_{\mathfrak{f}}(g|_{\mathfrak{V}})$ denote the determinant of $g \in G$ when we view \mathfrak{V} as an \mathfrak{f} -vector space. Let $\chi_{\mathfrak{f}}^{\mathfrak{V}}$ be the character $g \mapsto \det_{\mathfrak{f}}(g|_{\mathfrak{V}})^{(|\mathfrak{f}|-1)/2}$. More

conceptually, $\chi_f^{\mathfrak{D}}(g)$ is 1 if $\det_f(g|_{\mathfrak{D}})$ is a square in f^\times and is -1 if otherwise, i.e., it is the Legendre symbol of $\det_f(g|_{\mathfrak{D}})$ in f .

Note that $G_x^0 \cong \Delta^0$ via $g \mapsto (g, \alpha(g))$ and $\mathfrak{r} \cong \mathfrak{b}_+$ via $X \mapsto \bar{\iota}(X)$. Clearly for $g \in G_x^0$ and $X \in \mathfrak{r}$,

$$\bar{\iota}(g \cdot X) = -wgXg^{-1} = -\alpha(g)wXg^{-1} = (g, \alpha(g)) \cdot \bar{\iota}(X)$$

so $\chi_f^{\mathfrak{r}}(g) = (\det_f(g|_{\mathfrak{r}}))^{(q-1)/2} = \det_f((g, \alpha(g))|_{\mathfrak{b}_+})^{(q-1)/2} = \chi^{\mathfrak{b}_+}((g, \alpha(g)))$. Then Lemma 6.10 is equivalent to the following statement:

$$(6.8) \quad \chi_f^{\mathfrak{r}}(g) = 1 \quad \forall g \in G_x^0.$$

Recall $\check{\Gamma}$ and \check{G} in Definition 6.1 ((a)). Note that G_x^0 is a subgroup of \check{G}_x and \check{G}_x acts on $\mathfrak{r} = \check{\mathfrak{g}}_{x,s:s^+}$. Hence (6.8) is a consequence of the following:

$$(6.9) \quad \chi_f^{\mathfrak{r}}(g) = 1 \quad \forall g \in \check{G}_x.$$

If we replace Γ by $\check{\Gamma}$, then every object in (6.9) remains unchanged. Therefore, we could assume that $\Gamma = \check{\Gamma}$. In this case $\check{G}_x = G_x^0$.

We recall that $F' = Z(D)$. We consider Case I and Case II separately.

1. In Case I, $F'[\Gamma] = \prod_i F_i$ is a product of fields F_i with involution $*$ and $V = \bigoplus_i V^i$ such that each V^i is a certain Hermitian space over F_i . Let F_i° be the $*$ fixed subfield of F_i . Then $U(V^i)$ is an algebraic group defined over F_i° . Under this decomposition,

$$G^0 = \prod_i U(V^i), x = (x_i) \in \prod_i \mathcal{B}(U(V^i)) \text{ and } \chi_f^{\mathfrak{r}} = \prod_i \chi_f^{u(V^i)_{x_i,s:s^+}}.$$

The residue field f_i° of F_i° could be a finite extension of f . Note $\chi_{f_i^\circ}^{u(V^i)_{x_i,s:s^+}}(g) = 1$ means that $\det_{f_i^\circ}(g) = a^2$ is a square in $(f_i^\circ)^\times$. Hence $\det_f(g) = \text{Norm}_{f_i^\circ/f} \circ \det_{f_i^\circ}(g) =$

$(\text{Norm}_{f_i^\circ/f}(a))^2$ is a square in f^\times . So $\chi_f^{u(V^i)_{x_i,s:s^+}}(g) = 1$. In order to prove (6.9), it

suffices to check that $\chi_{f_i^\circ}^{u(V^i)_{x_i,s:s^+}}|_{U(V^i)_{x_i}}$ is trivial for each i .

2. In Case II, $G^0 = G$ is a unitary group over $D = F' = F[\Gamma]$.

To summarize, we have reduced Lemma 6.10 to the following claim.

CLAIM. – Suppose

- (a) D is a quadratic field extension of a certain p -adic field F ;
- (b) τ is the nontrivial element in $\text{Gal}(D/F)$;
- (c) V is a D -vector space with a Hermitian form $\langle \cdot, \cdot \rangle$;
- (d) $G = U(V)$, $\mathfrak{g} = \mathfrak{u}(V)$ and $x \in \mathcal{B}(G, F)$;
- (e) Γ is an element in $Z(\mathfrak{g}) = D^{\tau,-1}$ with valuation $-r = -2s$.

Then $\chi_f^{\mathfrak{g}_{x,s:s^+}}(g) = 1$ for all $g \in G_x$.

Proof of the Claim. – Without loss of generality, we may assume $\text{val}(D) = \mathbb{Z}$. Let \mathcal{L} be the self-dual lattice function corresponding to x . Define the f_D -space $L_t := \mathcal{L}_t / \mathcal{L}_{t+}$.

1. If D/F is unramified, we let ϖ_D be the fixed uniformizer of both D and F . The residue field f_D is a quadratic extension of f . Moreover L_0 and $L_{\frac{1}{2}}$ (possibly zero spaces) are Hermitian spaces over f_D .

2. If D/F is ramified then $\mathfrak{f}_D = \mathfrak{f}$. We fix a uniformizer ϖ_D of D such that $\varpi_D^r = -\varpi_D$. Then L_0 is an orthogonal space over \mathfrak{f} whose form is induced by $\langle \cdot, \cdot \rangle$ and $L_{\frac{1}{2}}$ is a symplectic space over \mathfrak{f} whose form is induced by $\varpi_D^{-1} \langle \cdot, \cdot \rangle$.

Note that $\chi_{\mathfrak{f}}^{\mathfrak{g}_{x,s;s^+}}$ factors through the group

$$(6.10) \quad G := G_x/G_{x,0^+} = \left(\prod_{t \in \text{Jump}(\mathcal{L}) \cap (0, \frac{1}{2})} \text{GL}_{\mathfrak{f}_D}(L_t) \right) \times U(L_0) \times U(L_{\frac{1}{2}}).$$

Now we consider two separate cases in the next two subsections.

6.5.1. *Case 1: $s \in \text{val}(D) = \mathbb{Z}$.* First we claim that D is an unramified extension of F . Indeed, if D/F is ramified, then $\text{val}(F) = 2\mathbb{Z}$ and $-r = \text{val}(\Gamma)$ is odd because $\Gamma \in D^{\tau,-1}$. This implies $s \notin \mathbb{Z}$, a contradiction.

Now $X \mapsto \varpi_D^s X$ gives a G -equivariant isomorphism $\mathfrak{g}_{x,0;0^+} \xrightarrow{\sim} \mathfrak{g}_{x,s;s^+}$. Therefore $\det_{\mathfrak{f}}(g|_{\mathfrak{g}_{x,s;s^+}}) = \det_{\mathfrak{f}}(g|_{\mathfrak{g}_{x,0;0^+}}) = 1$ since all the simple factors of G in (6.10) are of type A acting on its Lie algebra via adjoint action. Note that $\text{GL}_{\mathfrak{f}_D}(L_t)$ should be viewed as a group defined over \mathfrak{f} by restriction of scalars, but this does not affect the conclusion. Hence we have proved the claim in this case.

6.5.2. *Case 2: $s \notin \text{val}(D) = \mathbb{Z}$.* Then $s = \frac{r}{2} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. We recall that

$$(6.11) \quad \mathfrak{gl}_{x,s;s^+} = \bigoplus_{t \in \mathbb{Q}/\mathbb{Z}} \text{Hom}_{\mathfrak{f}_D}(L_t, L_{t+s}).$$

The adjoint action $*$ (cf. Section 2.1.1) permutes the terms in (6.11) and $\mathfrak{g}_{x,s;s^+}$ is the (-1) -eigenspace of $*$ in $\mathfrak{gl}_{x,s;s^+}$. Let $l_t := \dim_{\mathfrak{f}_D} L_t$.

Now we consider the value of $\chi_{\mathfrak{f}}^{\mathfrak{g}_{x,s;s^+}}$ on each factor of (6.10).

- (i) Suppose $t \not\equiv -t \pmod{\mathbb{Z}}$ and $t \not\equiv -(t+s) \pmod{\mathbb{Z}}$. We consider the action of $\text{GL}(L_t)$. We have

$$\begin{aligned} * : \text{Hom}_{\mathfrak{f}_D}(L_t, L_{t+s}) &\xrightarrow{\sim} \text{Hom}_{\mathfrak{f}_D}(L_{-t-s}, L_{-t}) \quad \text{and} \\ * : \text{Hom}_{\mathfrak{f}_D}(L_{-t}, L_{-t+s}) &\xrightarrow{\sim} \text{Hom}_{\mathfrak{f}_D}(L_{t-s}, L_t). \end{aligned}$$

The two domains and codomains are distinct terms in (6.11). Moreover $l_{t+s} = l_{t-s}$ since $L_{t-s} \cong L_{t+s}$ via multiplication by ϖ_D^{2s} . Therefore

$$\det_{\mathfrak{f}}(\text{Ad}(g)|_{\mathfrak{g}_{x,s;s^+}}) = \text{Norm}_{\mathfrak{f}_D/\mathfrak{f}} \left(\det_{\mathfrak{f}_D}(g|_{L_t})^{-l_{t+s}} \det_{\mathfrak{f}_D}(g|_{L_t})^{l_{t-s}} \right) = 1 \quad \forall g \in \text{GL}(L_t).$$

Hence $\chi_{\mathfrak{f}}^{\mathfrak{g}_{x,s;s^+}}(g) = 1$ for $g \in \text{GL}(L_t)$.

- (ii) Suppose $t \equiv -t$, i.e., $t \equiv 0$ or $\frac{1}{2} \pmod{\mathbb{Z}}$. We consider the actions of $U(L_0)$ and $U(L_{\frac{1}{2}})$. Now $-s + \frac{1}{2}$ is an integer and multiplication by $\varpi_D^{-s+\frac{1}{2}}$ induces isomorphisms

$$\text{Hom}(L_0, L_s) \xrightarrow{\sim} \text{Hom}(L_0, L_{\frac{1}{2}}) \quad \text{and} \quad \text{Hom}(L_{-\frac{1}{2}}, L_{-\frac{1}{2}+s}) \xrightarrow{\sim} \text{Hom}(L_{-\frac{1}{2}}, L_0).$$

Now $* : \text{Hom}(L_0, L_{\frac{1}{2}}) \cong \text{Hom}(L_{-\frac{1}{2}}, L_0)$. Combining with the above gives $\text{Hom}(L_0, L_s) \xrightarrow{\sim} \text{Hom}(L_{-\frac{1}{2}}, L_{-\frac{1}{2}+s})$. As a $U(L_0)$ -module, $\mathfrak{g}_{x,s;s^+}$ is isomorphic

to $\text{Hom}(L_0, L_{\frac{1}{2}})$ direct sum with certain copies of the trivial representation. Suppose

$g_0 \in U(L_0)$. Then $\det_f(g_0|_{\mathfrak{g}_{x,s;s^+}}) = \det_f(g_0|_{L_0})^{-l_{\frac{1}{2}}}$.

- If D/F is unramified, then $U(L_0)$ is a unitary group. Now $\det_{f_D}(g_0|_{L_0}) \in \mathfrak{f}_D$ has norm 1 so $\det_f(g_0|_{L_0}) = 1$. By a similar argument, $\det_f(g_{\frac{1}{2}}|_{\mathfrak{g}_{x,s;s^+}}) = 1$ for $g_{\frac{1}{2}} \in U(L_{\frac{1}{2}})$.

- If D/F is ramified, then L_0 is an orthogonal space and $L_{\frac{1}{2}}$ is a symplectic space.

Hence $\det_f(g_0|_{\mathfrak{g}_{x,s;s^+}}) = \det_f(g_0|_{L_0})^{-l_{\frac{1}{2}}} = 1$ since $l_{\frac{1}{2}}$ is even. Since $L_{\frac{1}{2}}$ is a symplectic space, $\det_f(g_{\frac{1}{2}}|_{\mathfrak{g}_{x,s;s^+}}) = 1$ for $g_{\frac{1}{2}} \in U(L_{\frac{1}{2}})$.

Hence, we have shown that $\chi_f^{\mathfrak{g}_{x,s;s^+}}$ is trivial on $U(L_0) \times U(L_{\frac{1}{2}})$.

- (iii) Suppose $t \equiv -t - s \pmod{\mathbb{Z}}$. Then $t \equiv \pm \frac{1}{4} \pmod{\mathbb{Z}}$. We consider the actions of $GL(L_{\frac{1}{4}})$ in (6.10). Composing with multiplication by $\varpi_D^{-s-\frac{1}{2}}$ induces an isomorphism

$$\text{Hom}(L_{\frac{1}{4}}, L_{\frac{1}{4}+s}) \xrightarrow{\sim} \text{Hom}(L_{\frac{1}{4}}, L_{-\frac{1}{4}})$$

and the $*$ -action on the left hand side commutes with the $(\varepsilon*)$ -action on the right hand side. Here $\varepsilon = 1$ if D/F is unramified and $\varepsilon = (-1)^{s+\frac{1}{2}}$ if D/F is ramified.

Similarly composing with multiplication by $\varpi_D^{-s+\frac{1}{2}}$ induces an isomorphism

$$\text{Hom}(L_{-\frac{1}{4}}, L_{-\frac{1}{4}+s}) \xrightarrow{\sim} \text{Hom}(L_{-\frac{1}{4}}, L_{\frac{1}{4}})$$

and $*$ action on the left hand side commutes with the $(\varepsilon'*)$ -action on the right hand side with $\varepsilon' = 1$ if D/F is unramified and $\varepsilon' = (-1)^{s-\frac{1}{2}}$ if D/F is ramified.

Let $\mathfrak{s} := \text{Hom}(L_{-\frac{1}{4}}, L_{\frac{1}{4}})$ and $\mathfrak{s}' := \text{Hom}(L_{\frac{1}{4}}, L_{-\frac{1}{4}})$. Clearly \mathfrak{s} and \mathfrak{s}' are dual to each other as $GL(L_{\frac{1}{4}})$ -modules over \mathfrak{f}_D via the trace form $(X, Y) \mapsto \text{tr}_{\mathfrak{f}_D}(YX)$. Since the form is $*$ -invariant, $\mathfrak{s}^{*,e}$ and $\mathfrak{s}'^{*,e}$ are dual to each other for $e \in \{\pm 1\}$. As a $GL(L_{\frac{1}{4}})$ -module, $\mathfrak{g}_{x,s;s^+}$ is isomorphic to $\mathfrak{s}^{*,-\varepsilon} \oplus \mathfrak{s}'^{*,-\varepsilon'}$ direct sum with copies of the trivial representation. Let $g_{\frac{1}{4}} \in GL(L_{\frac{1}{4}})$.

- If D/F is unramified, then $\varepsilon = \varepsilon' = 1$. Since $\mathfrak{s}^{*,-1}$ and $\mathfrak{s}'^{*,-1}$ are dual to each other, we have $\det_{f_D}(g_{\frac{1}{4}}|_{\mathfrak{g}_{x,s;s^+}}) = 1$.

- If D/F is ramified, then $\varepsilon = -\varepsilon'$. Since $\det_f(g_{\frac{1}{4}}|_{\mathfrak{s}}) = \det_f(g_{\frac{1}{4}}|_{L_{\frac{1}{4}}})^{2l_{\frac{1}{4}}}$ is a square

and $\mathfrak{s} = \mathfrak{s}^{*,\varepsilon} \oplus \mathfrak{s}^{*,-\varepsilon}$, we have $\chi_f^{\mathfrak{s}^{*,\varepsilon}} = \chi_f^{\mathfrak{s}^{*,-\varepsilon}}$. Hence

$$\chi_f^{\mathfrak{g}_{x,s;s^+}}(g_{\frac{1}{4}}) = \chi_f^{\mathfrak{s}^{*,-\varepsilon}}(g_{\frac{1}{4}})\chi_f^{\mathfrak{s}'^{*,-\varepsilon'}}(g_{\frac{1}{4}}) = \chi_f^{\mathfrak{s}^{*,\varepsilon}}(g_{\frac{1}{4}})\chi_f^{\mathfrak{s}'^{*,\varepsilon}}(g_{\frac{1}{4}}) = 1.$$

We conclude that $\chi_f^{\mathfrak{g}_{x,s;s^+}} = 1$ on $GL(L_{\frac{1}{4}})$.

Combining (i), (ii) and (iii), we have proved the Claim in view of (6.10). □

This concludes the proof of Lemma 6.10. □

7. One positive depth block Case II: the constructions of refined minimal K -types

We retain the notation in Section 6. Recall that $\Sigma = (x, \Gamma, \phi, \rho)$ is a single block datum with positive depth $r = 2s$ as in Section 6.1. We have $\Gamma = M(w)$, $\Gamma' = M'(w)$ and a group isomorphism $\alpha: G_x^0 \xrightarrow{\sim} G_{x'}^0$. In Definition 5.15, we had defined $\Sigma' := \vartheta^+(\Sigma) = (x', -\Gamma', \phi', \rho')$ where $\phi' := \phi^* \circ \alpha^{-1}$ and $\rho' = \rho^* \circ \alpha^{-1}$.

7.1. A key proposition

We always use “ $\check{\sim}$ ” to mark an object in $G \times G'$ having two similar copies in components of G and G' as below. We set

- $\check{K} := K \times K'$, $\check{K}_{0+} = K_{0+} \times K'_{0+}$ and $\check{K}_+ := K_+ \times K'_+$;
- $\check{G}^0 := G_x^0 \times G_{x'}^0 = G_x^0/G_{x,0+}^0 \times G_{x'}^0/G_{x',0+}^0$;
- $\check{\rho} := \rho \boxtimes \rho'$ be the $K \times K'$ -module inflated from the \check{G}^0 -module $\rho \boxtimes \rho'$;
- $\check{\kappa} := \kappa \boxtimes \kappa'$ be the Heisenberg-Weil representation of \check{K} constructed by $(\Gamma, -\Gamma')$ (cf. Appendix A.2.2);
- $\check{\eta} := \eta \boxtimes \eta' = \check{\rho} \otimes \check{\kappa}$ and
- $\check{\Omega} := \check{K} \cdot w + \mathcal{B}_0 \subseteq W$.

REMARK. – The above notations also apply to multiple block Σ and its lift Σ' defined in Remark 5.3 of Definition 5.16.

7.1.1. – If J is a compact group, U is a J -module and χ is an irreducible J -module, then we let $U[\chi]$ denote the χ isotypic component of U . Now we can state a key proposition.

PROPOSITION 7.1. – Under the settings in Section 7.1, we have $\mathcal{S}(\mathcal{B}_0)_\Omega[\eta \boxtimes \eta'] \cong \eta \boxtimes \eta'$.

The proof will be given in Section 7.1.4 based on Lemma 7.4 below.

7.1.2. – We now record an elementary fact which will be used freely in this paper. Suppose H and J are compact groups and J is a subgroup of H . For a J -module τ , we will identify the induced representation with a space of functions:

$$\text{Ind}_J^H \tau = \{ f: H \rightarrow \tau \mid f(jh) = \tau(j)f(h) \ \forall j \in J, h \in H \},$$

where H acts by right translation.

LEMMA 7.2. – Suppose J is a compact normal subgroup of H . Let J_1 be a subgroup of H such that $J < J_1 < H$. Let τ be a J_1 -module and χ be an irreducible J -module. Suppose that H stabilizes χ , i.e., $\chi \circ \text{Ad}_h \cong \chi$ as J -modules for all $h \in H$. Then $\tau[\chi]$ is a J_1 -module and $(\text{Ind}_{J_1}^H \tau)[\chi] = \text{Ind}_{J_1}^H(\tau[\chi])$. □

7.1.3. – Let $\check{\psi} := \psi_\Gamma \boxtimes \psi_{-\Gamma'}$ be the function on $G_{x,0^+} \times G'_{x,0^+}$. Recall that $\check{\psi}$ restricted on \check{K}_+ is a character, \check{K} normalizes \check{K}_+ and stabilizes $\check{\psi}|_{\check{K}_+}$. We could extend $\check{\psi}$ to a function on $\Delta^0 G_{x,0^+} G'_{x,0^+}$ by letting $\check{\psi}(xg) := \check{\psi}(g)$ for all $x \in \Delta^0$ and $g \in G_{x,0^+} G'_{x,0^+}$.

Combining Lemma 2.4, Lemma 6.7 ((v)) and Lemma 6.8 yields the following lemma.

LEMMA 7.3. – *As an S -module realized on $\mathbb{S}(\mathbf{b})$, $\overline{\omega}_w$ is given by*

$$\overline{\omega}_w(h) = \check{\psi}(h) \overline{\omega}_{\mathbf{b}}(h) \overline{\omega}_{\mathbf{b}}(h^{-1} \cdot w - w) \quad \forall h \in S. \quad \square$$

We recall the definitions of \mathbf{b}_+ and $\mathbf{b}_0 := \mathbf{b}_+^\perp / \mathbf{b}_+$ in Section 6.2.4. Let $\mathbb{S}(\mathbf{b})^{\mathbf{b}_+}$ be the $\overline{\omega}_{\mathbf{b}}(\mathbf{b}_+)$ invariant subspace in $\mathbb{S}(\mathbf{b})$. Then $\mathbb{S}(\mathbf{b})^{\mathbf{b}_+} \cong \chi^{\mathbf{b}_+} \otimes \overline{\omega}_{\mathbf{b}_0}$ as $\mathbf{P}(\mathbf{b}_+) \ltimes \mathbf{H}(\mathbf{b}_+^\perp)$ -modules where $\mathbf{P}(\mathbf{b}_+)$ is the parabolic subgroup in $\mathbf{Sp}(\mathbf{b})$ stabilizing \mathbf{b}_+ (see Appendix A.1 for notation).

From Lemma 6.7 ((v)), $S = \left(\prod_{j \in \mathcal{J}_\Gamma} \Delta^j\right) \check{G}_s$ and

$$S \check{K}_+ = \left(\prod_{j \in \mathcal{J}_\Gamma} \Delta^j\right) \check{K}_+ \check{G}_s = \begin{cases} \Delta^0 \Delta^1 \dots \Delta^{d-1} \check{K}_+ \check{G}_s & \text{in Case I,} \\ \Delta^0 \Delta^1 \dots \Delta^d \check{K}_+ \check{G}_s & \text{in Case II.} \end{cases}$$

LEMMA 7.4. – (i) *The evaluation map eva defined by $f \mapsto f(1)$ gives an isomorphism between the vector spaces*

$$(7.1) \quad \text{eva}: \left(\text{Ind}_S^{S \check{K}_+ \overline{\omega}_w}\right) [\check{\psi}|_{\check{K}_+}] \xrightarrow{\sim} \mathbb{S}(\mathbf{b})^{\mathbf{b}_+}.$$

Note that $\mathbb{S}(\mathbf{b})^{\mathbf{b}_+} \cong \mathbb{S}(\mathbf{b}_0)$. The $S \check{K}_+$ -module structure on the left hand side of (7.1) translates to an action ψ^S of $S \check{K}_+$ on $\mathbb{S}(\mathbf{b}_0)$. The action ψ^S is given as following:

- (a) *If $h \in \prod_{1 \leq j \in \mathcal{J}_\Gamma} \Delta^j \check{K}_+$, then $\psi^S(h)$ acts on $\mathbb{S}(\mathbf{b}_0)$ by the scalar $\check{\psi}(h)$.⁽¹⁵⁾*
- (b) *If $h = (\exp(X), \exp(X')) \in G_{x,s} \times G'_{x,s}$, then $\psi^S(g) = \check{\psi}(h) \overline{\omega}_{\mathbf{b}_0}(b)$ where b is the image in \mathbf{b}_0 of $h^{-1} \cdot w - w \equiv wX - X'w \pmod{\mathcal{B}_{0^+}}$.*
- (c) *$\psi^S|_{\Delta^0}$ is the inflation of $\overline{\omega}_{\mathbf{b}_0}$ via $\Delta^0 \rightarrow \mathbf{Sp}(\mathbf{b}_0)$.*

(ii) *We have $\left(\text{Ind}_S^{\Delta^0 \check{K}_+ \overline{\omega}_w}\right) [\check{\psi}|_{\check{K}_+}] \cong \check{\kappa}|_{\Delta^0 \check{K}_+}$ as $\Delta^0 \check{K}_+$ -modules.*

(iii) *Let $\overline{\Delta^0}$ denote the image of Δ^0 in \check{G}^0 . Then we have following \check{K} -module isomorphisms*

$$\left(\text{Ind}_S^{\check{K} \overline{\omega}_w}\right) [\check{\psi}|_{\check{K}_+}] \cong \text{Ind}_{\Delta^0 \check{K}_+}^{\check{K}} \left(\left(\text{Ind}_S^{\Delta^0 \check{K}_+ \overline{\omega}_w}\right) [\check{\psi}|_{\check{K}_+}]\right) \cong \left(\text{Ind}_{\overline{\Delta^0}}^{\check{G}^0} \mathbf{1}\right) \otimes \check{\kappa}.$$

Note that ((iii)) follows immediately from ((ii)). Before we embark on the proofs of ((i)) and ((ii)), we will use the lemma to give a proof of Proposition 7.1.

⁽¹⁵⁾ By definition $\check{\psi}|_{\Delta^0} \equiv 1$. Therefore by ((a)) the function $\check{\psi}$ is a character when restricted on $\prod_{j \in \mathcal{J}_\Gamma} \Delta^j \check{K}_+$.

7.1.4. *Proof of Proposition 7.1.* – We have

$$\begin{aligned}
\mathrm{Hom}_{\check{K}}(\check{\eta}, \mathcal{S}(\mathcal{B}_0)_\Omega) &= \mathrm{Hom}_{\check{K}}(\check{\eta}, \left(\mathrm{Ind}_{\check{S}}^{\check{K}} \bar{\omega}_w\right) [\check{\psi}|_{\check{K}_+}]) \\
&\quad (\text{by Lemma 2.4 and the fact that } \check{\eta}|_{\check{K}_+} \text{ is } \check{\psi}|_{\check{K}_+} \text{-isotypic}) \\
&= \mathrm{Hom}_{\check{K}}(\check{\rho} \otimes \check{\kappa}, \left(\mathrm{Ind}_{\check{\Delta}_0}^{\check{G}_0} \mathbf{1}\right) \otimes \check{\kappa}) \quad (\text{by Lemma 7.4 ((iii))}) \\
&= \mathrm{Hom}_{\check{G}_0}(\check{\rho}, \left(\mathrm{Ind}_{\check{\Delta}_0}^{\check{G}_0} \mathbf{1}\right) \otimes \mathrm{Hom}_{\check{K}_{0+}}(\check{\kappa}, \check{\kappa})) \\
&\quad (\text{since } \check{\rho} \text{ and } \mathrm{Ind}_{\check{\Delta}_0}^{\check{G}_0} \mathbf{1} \text{ are trivial when restricted on } \check{K}_{0+}) \\
&= \mathrm{Hom}_{\check{G}_0}(\check{\rho}, \mathrm{Ind}_{\check{\Delta}_0}^{\check{G}_0} \mathbf{1}) \quad (\text{since } \check{\kappa}|_{\check{K}_{0+}} \text{ is irreducible}) \\
&= \mathrm{Hom}_{\check{\Delta}_0}(\check{\rho}, \mathbf{1}) = \mathbb{C} \quad (\text{since } \rho' = \rho^* \circ \alpha^{-1}).
\end{aligned}$$

This proves the proposition. \square

7.2. Proof of Lemma 7.4

The rest of this section is devoted to proving Lemma 7.4.

7.2.1. *Proof of Lemma 7.4 (i).* – We recall that $S_+ := S \cap \check{K}_+$. Frobenius reciprocity gives the following natural isomorphism of vector spaces:

$$\mathrm{eva}: \left(\mathrm{Ind}_{\check{S}}^{S \check{K}_+} \bar{\omega}_w\right) [\check{\psi}|_{\check{K}_+}] \cong \left(\mathrm{Ind}_{S_+}^{\check{K}_+} \bar{\omega}_w\right) [\check{\psi}|_{\check{K}_+}] \xrightarrow{\sim} \bar{\omega}_w [\check{\psi}|_{S_+}].$$

Now the key is to prove the following claim.

CLAIM. – *We have*

$$(7.2) \quad \bar{\omega}_w [\check{\psi}|_{S_+}] = \mathbb{S}(\mathbf{b})^{\mathbf{b}^+} \subseteq \mathbb{S}(\mathbf{b}).$$

Proof. – We will only prove it for Case I. The proof for Case II is similar and easier, so we leave it to the reader.

We recall Lemma 6.7 ((v)) that

$$(7.3) \quad S_+ = \Delta_+^0 \Delta_+^1 \cdots \Delta_+^{d-1} G_{x,s}^{d-1} G_{x',s}^{d-1} G_{x,s} + G'_{x',s}.$$

Now we consider the $\bar{\omega}_w$ -action (cf. Lemma 7.3) of each factor on the right hand side of (7.3). Note that $\bar{\omega}_w|_{S_+}$ is trivial since $S_+ \subseteq G_{x,0} + G'_{x',0}$.

- (1) Suppose $h = \exp(X) \in G_{x,s}$ where $X \in \mathfrak{g}_{x,s}$. Then $h^{-1} \cdot w - w \in wX + \mathcal{B}_s \subseteq \mathcal{B}_{0+}$. Hence $\bar{\omega}_w(h) = \check{\psi}(h) \bar{\omega}_w(h^{-1} \cdot w - w) = \check{\psi}(h)$. By the same argument, we also have $\bar{\omega}_w(h) = \check{\psi}(h)$ for $h \in G'_{x',s}$.
- (2) Suppose $h = \exp(X) \in G_{x,s}^{d-1}$ with $X \in \mathfrak{g}_{x,s}^{d-1}$. Then $h^{-1} \cdot w - w \in wX + \mathcal{B}_0 \subseteq \mathcal{B}_0$. Hence $\bar{\omega}_w(h) = \check{\psi}(h) \bar{\omega}_w(wX)$. The same argument gives $\bar{\omega}_w(h') = \check{\psi}(h') \bar{\omega}_w(-X'w)$ for $h' = \exp(X') \in G_{x',s}^{d-1}$ where $X' \in \mathfrak{g}_{x',s}^{d-1}$. Hence $\bar{\omega}_w[\check{\psi}|_{S_+}] \subseteq \mathbb{S}(\mathbf{b})^{\mathbf{b}^+}$ since $\mathbf{b}_+ = \bar{\iota}(\mathfrak{g}_{x,s}^{d-1} \oplus \mathfrak{g}_{x',s}^{d-1}) = w\mathfrak{g}_{x,s}^{d-1} + \mathfrak{g}_{x',s}^{d-1}w + \mathcal{B}_0 \subseteq \mathcal{B}_0/\mathcal{B}_{0+}$.
- (3) Suppose $h = (g, g') \in \Delta_+^i$ for $0 \leq i \leq d-1$. By Lemma 6.7 (i) and (ii), $h^{-1}w - w \in \mathcal{B}_{0+}$. Therefore $\bar{\omega}_w(h) = \check{\psi}(h)$.

Combining (1)–(3), we see that the $\check{\psi}|_{S_+}$ isotypic component is exactly the $\bar{\omega}_{\mathbf{b}}(\mathbf{b}_+)$ -invariant subspace in $\mathbb{S}(\mathbf{b})$. This proves the claim. \square

Now we calculate the translated $S\check{K}_+ = \Delta^0 \prod_{0 < i \in \mathfrak{I}_\Gamma} \Delta^i \check{K}_+$ action ψ^S on $\mathbb{S}(\mathbf{b}_0) \cong \mathbb{S}(\mathbf{b})^{\mathbf{b}_+}$.

- (4) Clearly $\psi^S|_{\check{K}_+} = \check{\psi}|_{\check{K}_+}$.
- (5) Suppose $h \in \Delta^i$ for $0 < i \in \mathfrak{I}_\Gamma$. Then $h \in G_{x,0+}G'_{x',0+}$ and $h^{-1} \cdot w - w \in \mathcal{B}_{0+}$ by Lemma 6.7 ((ii)). So $\psi^S(h) = \bar{\omega}_w(h) = \check{\psi}(h)$. Combining this with (4) proves (a).
- (6) Suppose $h \in G_{x,s} \times G'_{x',s}$. By Appendix A.1 (iii),

$$\psi^S(h) = \bar{\omega}_w(h) = \check{\psi}(h)\bar{\omega}_{\mathbf{b}}(h^{-1} \cdot w - w) = \check{\psi}(h)\bar{\omega}_{\mathbf{b}_0}(h).$$

This proves part (b).

- (7) Suppose $h \in \Delta^0$. By Appendix A.1 (iii) and Lemma 6.10,

$$\psi^S(h) = \bar{\omega}_w(h) = \bar{\omega}_{\mathbf{b}}(h) = \chi^{\mathbf{b}_+}(h)\bar{\omega}_{\mathbf{b}_0}(h) = \bar{\omega}_{\mathbf{b}_0}(h).$$

This proves part (c).

These complete the proof of Lemma 7.4 (i).

7.2.2. *Proof of Lemma 7.4 ((ii)).* – By Part (i) and Lemma 7.2, we have

$$(7.4) \quad \left(\text{Ind}_S^{\Delta^0 \check{K}_{0+}} \bar{\omega}_w \right) [\check{\psi}|_{\check{K}_+}] \cong \text{Ind}_{S\check{K}_+}^{\Delta^0 \check{K}_{0+}} \left(\left(\text{Ind}_S^{S\check{K}_+} \bar{\omega}_w \right) [\check{\psi}|_{\check{K}_+}] \right) \cong \text{Ind}_{S\check{K}_+}^{\Delta^0 \check{K}_{0+}} \psi^S$$

as $\Delta^0 \check{K}_{0+}$ -modules.

CLAIM. – *We have*

$$(7.5) \quad \dim \text{Ind}_{S\check{K}_+}^{\Delta^0 \check{K}_{0+}} \psi^S = \dim \check{\kappa}.$$

Proof. – Let $K_{+,s} := K_+ \cap G_{x,s}$ and $K_s := K \cap G_{x,s} = G_{x,s}$. Let $Q := \#(S_{0+}/S_+)$, $N := \#(K_{0+}/K_+)$, $N_s := \#(K_s/K_{+,s})$, $N' := \#(K'_{0+}/K'_+)$ and $N'_s := \#(K'_s/K'_{+,s})$.

We note the following facts.

1. By the definition of $\check{\kappa}$, we have $\dim \check{\kappa} = (\#\check{K}_{0+}/\check{K}_+)^{\frac{1}{2}} = \sqrt{NN'}$.
2. By Lemma 6.5 ((vi)), we have $\dim \mathbb{S}(\mathbf{b}_0) = \sqrt{\#\mathbf{b}_0} = \sqrt{\#(K_s/K_{+,s})\#(K'_s/K'_{+,s})} = \sqrt{N_s N'_s}$.
3. By Lemma 6.7 ((v)), the projection to the first coordinate $S_{0+} \rightarrow K_{0+}$ is surjective and its kernel is K'_s . Hence $K_{0+} \cong S_{0+}/K'_s$. Similarly, we have $K_+ \cong S_+/K'_{+,s}$. Hence

$$(7.6) \quad K_{0+}/K_+ \cong (S_{0+}/K'_s)/(S_+/K'_{+,s}) \cong S_{0+}/K'_s S_+ \cong (S_{0+}/S_+)/(K'_s S_+/S_+).$$

Note that $K'_s/K'_{+,s} = K'_s S_+/S_+$. Counting the elements of the both sides of (7.6), we get $N = Q/N'_s$. A similar argument yields $N' = Q/N_s$.

4. Note that $\check{K}_{0+} \cap S\check{K}_+ = S_{0+}\check{K}_+$ and $S_{0+}\check{K}_+/\check{K}_+ = S_{0+}/(\check{K}_+ \cap S_{0+}) = S_{0+}/S_+$.

Hence

$$\begin{aligned} \dim \operatorname{Ind}_{S\check{K}_+}^{\Delta^0 \check{K}_{0+}} \psi^S &= \dim \mathbb{S}(\mathbf{b}_0) \cdot \#(\check{K}_{0+}/S_{0+}\check{K}_+) \\ &= \dim \mathbb{S}(\mathbf{b}_0) \cdot \#(\check{K}_{0+}/\check{K}_+)/\#(S_{0+}\check{K}_+/\check{K}_+) \\ &= \sqrt{N_s N'_s N N'}/Q = \sqrt{N N'} = \dim \check{\kappa}. \end{aligned}$$

This proves the claim. \square

In Section 7.2.3, we will show that

$$(7.7) \quad \dim \operatorname{Hom}_{\Delta^0 \check{K}_{0+}}(\check{\kappa}, \operatorname{Ind}_{S\check{K}_+}^{\Delta^0 \check{K}_{0+}} \psi^S) = \dim \operatorname{Hom}_{S\check{K}_+}(\check{\kappa}, \psi^S) = 1.$$

Combining (7.4), (7.5) and (7.7) gives Lemma 7.4 (ii). This also completes the proof of the whole Lemma 7.4. \square

7.2.3. *Proof of (7.7).* – The first equality in (7.7) is just Frobenius reciprocity. It remains to prove the second equality. We only give the proof for Case I. The proof for Case II where $\mathbf{b}_0 = 0$ is essentially contained in the proof of Lemma 7.5 (ii) below.

We assume the notation and the construction of κ in Appendix A.2.2. We also retain the notion in Section 6.4. Recall $J^j := (G^{j-1}, G^j)_{x, (r_{j-1}, s_{j-1})}$, $J_+^j := (G^{j-1}, G^j)_{x, (r_{j-1}, s_{j-1}^+)}$, $\check{J}^j := J^j \times J'^j$ and $\check{J}_+^j = J_+^j \times J_+'^j$.

- For $1 \leq j \leq d$, let $Q^j := \Delta^1 \cdots \Delta^j \check{K}_+^j$ and $Q_j := (\Delta^j \cap \check{J}^j) \check{J}_+^j$.
- For $0 \leq j \leq d-1$, let $\check{\kappa}^j := \kappa^j \boxtimes \kappa'^j$ and $P^j := S\check{K}_+ \cap G^j = \Delta^0 \Delta^1 \cdots \Delta^j \check{K}_+^j$. Moreover, let $P := P^{d-1}$.
- For $1 \leq j \leq d$, let $\check{\omega}_{\Gamma_{j-1}}^j := \bar{\omega}_{\Gamma_{j-1}}^j \boxtimes \bar{\omega}_{-\Gamma_{j-1}}^j$ and $\mathbf{1} \times \check{\psi}_{\Gamma^j} := (\mathbf{1} \times \psi_{\Gamma^j}) \boxtimes (\mathbf{1} \times \psi_{-\Gamma^j})$ be $\check{K}^{j-1} \times \check{J}^j = (K^{j-1} \times J^j) \times (K'^{j-1} \times J'^j)$ -modules. Here $\Gamma^j := \sum_{i=j}^d \Gamma_i$ and $\Gamma'^j := \sum_{i=j}^d \Gamma'_i$.

By definition we have

- (a) a surjection $P \times \check{J}^d \rightarrow S\check{K}_+$;
- (b) Q^j and Q_j are groups such that $Q^j = Q^{j-1} Q_j$;
- (c) $P^0 = \Delta^0 K_+^0$ and $P^j = \Delta^0 Q^j = P^{j-1} Q_j$;
- (d) Pulling back via $P^{j-1} \times \check{J}^j \rightarrow \check{K}^j$, $\check{\kappa}^j|_{P^{j-1} \times \check{J}^j} = \check{\kappa}^{j-1} \otimes (\check{\omega}_{\Gamma_{j-1}}^j \otimes (\mathbf{1} \times \check{\psi}_{\Gamma^j}))$, where the \check{K}^{j-1} -module $\check{\kappa}^{j-1}$ is inflated to $P^{j-1} \times \check{J}^j$ via $P^{j-1} \times \check{J}^j \rightarrow P^{j-1} \hookrightarrow \check{K}^{j-1}$ (cf. Appendix A.2.2). In particular, as $P \times J^d$ -module,

$$\check{\kappa} = \check{\kappa}^{d-1} \otimes \check{\omega}_{\Gamma_{d-1}}^d.$$

Now

$$(7.8) \quad \begin{aligned} \operatorname{Hom}_{S\check{K}_+}(\check{\kappa}, \psi^S) &= \operatorname{Hom}_{P \times \check{J}^d}(\check{\kappa}^{d-1}, \operatorname{Hom}_{\mathbb{C}}(\check{\omega}_{\Gamma_{d-1}}^d, \psi^S)) \\ &= \operatorname{Hom}_P(\check{\kappa}^{d-1}, \operatorname{Hom}_{\check{J}^d}(\check{\omega}_{\Gamma_{d-1}}^d, \psi^S)). \end{aligned}$$

The map \bar{t} defined in Section 6.2.4 induces a P -equivariant isomorphism of symplectic spaces:

$$(7.9) \quad \bar{t}_{\mathbf{b}_0}: \check{\mathbf{W}}^d := \mathfrak{g}_{x, s; s^+}^{d-1 \perp} \oplus \mathfrak{g}_{x', s; s^+}^{d-1 \perp} \xrightarrow{\bar{t}} \mathbf{b}_0.$$

In fact, this is just a rephrase of Lemma 6.5 ((vi)) and Lemma 6.7 ((i)) since the P actions on the both sides of $\bar{\tau}_{\mathbf{b}_0}$ factors through Δ^0 .

Consider the map ⁽¹⁶⁾

$$(7.10) \quad \begin{aligned} \xi^{\mathbf{b}_0}: \quad \check{J}^d &\longrightarrow (\mathbf{b}_+^\perp/\mathbf{b}_+) \times \mathfrak{f} = \mathbf{H}(\mathbf{b}_0) \\ h = (\exp(X), \exp(X')) &\longmapsto (-(X, X') \cdot w, \frac{1}{2} \langle w, h^{-1} \cdot w - w \rangle_W). \end{aligned}$$

By the explicit description of special isomorphism in Appendix A.2.1, (2.6), (7.9) and Lemma 7.4 ((i)) part (b), the following diagram commutes:

$$\begin{array}{ccc} & & \mathbf{SH}(\check{\mathbf{W}}^d) \\ \Delta^0 \times \check{J}^d & \begin{array}{c} \nearrow \\ \searrow \end{array} & \downarrow \cong \\ & \xi^{\mathbf{b}_0} \nearrow & \mathbf{SH}(\mathbf{b}_0) \end{array}$$

Hence, as $\Delta^0 \times \check{J}^d$ -module,

$$(7.11) \quad \psi^S \cong \check{\omega}_{\check{\Gamma}_{d-1}}^d.$$

On the other hand, $Q^{d-1} \subseteq \check{K}_{0+}^{d-1}$ acts trivially on $\check{\omega}_{\check{\Gamma}_{d-1}}^d$ and acts as $\check{\psi}$ on ψ^S (cf. Lemma 7.4 (i) (a)). Therefore, as $P = \Delta^0 Q^{d-1}$ -modules, $\text{Hom}_{\check{J}^d}(\check{\omega}_{\check{\Gamma}_{d-1}}^d, \psi^S) \cong \check{\psi}|_P$. Putting this into (7.8), (7.7) becomes

$$(7.12) \quad \dim \text{Hom}_P(\check{\kappa}^{d-1}, \check{\psi}|_P) = 1.$$

This follows from part (ii) of the next lemma.

LEMMA 7.5. – (i) For $1 \leq j \leq d - 1$, we have

$$(7.13) \quad \text{Hom}_{Q_j}(\check{\omega}_{\check{\Gamma}_{j-1}}^j \otimes (\mathbf{1} \times \check{\psi}_{\check{\Gamma}_j}), \check{\psi}) \cong \check{\psi}$$

as P^{j-1} -modules.

(ii) For $0 \leq j \leq d - 1$, we have $\dim \text{Hom}_{P^j}(\check{\kappa}^j, \check{\psi}) = 1$.

Proof. – (i) We first check that the right hand side of (7.13) has dimension one. The image of Q_j under $\check{J}^j \rightarrow \check{J}^j/\check{J}_+^j =: \check{\mathbf{W}}^j$ is \mathbf{D}^j which is a maximal isotropic subspace in $\check{\mathbf{W}}^j$ according to Lemma 6.9. Note that the Q_j -character ⁽¹⁷⁾ $\check{\psi}_{\check{\Gamma}_{j-1}} := \psi_{\Gamma_{j-1}} \boxtimes \psi_{-\Gamma'_{j-1}} \cong \check{\psi}|_{J^j} \otimes (\check{\psi}_{\check{\Gamma}_j}|_{J^j})^{-1}$ factors to a \mathbf{D}^j -character. Therefore

$$\dim \text{Hom}_{Q_j}(\check{\omega}_{\check{\Gamma}_{j-1}}^j \otimes (\mathbf{1} \times \check{\psi}_{\check{\Gamma}_j}), \check{\psi}) = \dim \text{Hom}_{Q_j}(\check{\omega}_{\check{\Gamma}_{j-1}}^j, \check{\psi}_{\check{\Gamma}_{j-1}}) = 1.$$

We now check that the actions of $P^{j-1} = \Delta^0 Q^{j-1}$ on both sides of (7.13) agree.

- The character $\check{\psi}$ is trivial on Δ^0 . By the Appendix A.1, the left hand side of (7.13) is isomorphism to $\chi^{\mathbf{D}^j}$ as Δ^0 -module. We claim that $\chi^{\mathbf{D}^j}|_{\Delta^0}$ is trivial. Indeed $\mathbf{D}^j \cong \mathbf{W}^j$ as $\Delta^0 \cong G_x^0$ -module. Since the right hand side has a symplectic form preserved by G_x^0 -action, $\det_{\mathfrak{f}}(h|_{\mathbf{D}^j}) = 1$ for all $h \in \Delta^0$. This proves the claim.

⁽¹⁶⁾ Note that there is a negative sign before $(X, X') \cdot w$.

⁽¹⁷⁾ One can see that $\check{\psi}_{\check{\Gamma}_{j-1}}$ is a character of Q_j directly from the discussion in Appendix A.2.1.

- The group $Q^{j-1} \subseteq P^{j-1} \cap \check{K}_{0+}^{j-1}$ has trivial action on $\check{\omega}_{\check{\Gamma}^{j-1}} \otimes (\mathbf{1} \times \check{\psi}_{\check{\Gamma}^j})$. Therefore the left hand side of (7.13) is $\check{\psi}$ -isotypic as Q^{j-1} -module.

This proves (i).

(ii) We prove by induction on j .

1. By definition, $\dim \text{Hom}_{P^0}(\check{\kappa}^0, \check{\psi}) = 1$.
2. Now assume $\dim \text{Hom}_{P^{j-1}}(\check{\kappa}^{j-1}, \check{\psi}) = 1$. By (i)

$$\text{Hom}_{P^j}(\check{\kappa}^j, \check{\psi}) = \text{Hom}_{P^{j-1}}(\check{\kappa}^{j-1}, \text{Hom}_{Q^j}(\check{\omega}_{\check{\Gamma}^{j-1}} \otimes (\mathbf{1} \times \check{\psi}_{\check{\Gamma}^j}), \check{\psi})) = \text{Hom}_{P^{j-1}}(\check{\kappa}^{j-1}, \check{\psi}).$$

$$\text{Hence } \dim \text{Hom}_{P^j}(\check{\kappa}^j, \check{\psi}) = 1.$$

This completes the induction process and proves (ii). □

Now (7.12) holds and the proof of (7.7) is complete. □

8. Proof of the main Theorem I: construction of K -types in the general case

In this section, we will prove the part (1.3) of the Main Theorem by reducing the statement into one block cases. The idea is old, already appeared in [16, § 2.4] and [32, Section 3.3] for example. Hence we will omit the proofs of some simple facts.

We retain the notation in the Main Theorem. The Main Theorem of the Main Theorem is a consequence of the following proposition.

PROPOSITION 8.1. – *Suppose $\Sigma' := \vartheta_{V, \mathcal{G}'}(\Sigma) = (x', -\Gamma', \phi', \rho')$. Let $w \in V \otimes_D V'$ be the element defined by (5.9) via the construction of Σ' . We retain the notation in Section 7.1 with respect to w, Σ and Σ' so that $\Omega := \check{K} \cdot w + \mathcal{B}_0$. Then*

$$\dim \text{Hom}_{\check{K}}(\check{\eta}, \mathcal{S}(\mathcal{B}_0)_\Omega) = 1.$$

REMARK. – More generally, if Σ' a theta lift of Σ as in Definition 5.17, then the same proof in this section would show that

$$\text{Hom}_{\check{K}}(\check{\eta}, \mathcal{S}(\mathcal{B}_0)_\Omega) \neq 0.$$

The rest of this section is devoted to proving the proposition by induction on the number of blocks. We first state the induction hypothesis in Section 8.1. Then we prove some lemmas on the block decomposition in Sections 8.2 and 8.3. The proof is completed in Section 8.4.

8.1. Induction hypothesis

Let

- (a) \dot{V} be an ϵ -Hermitian space such that $\dim_D \dot{V} \leq \dim_D V$;
- (b) $\dot{\mathcal{J}}$ be a Witt tower of ϵ' -Hermitian spaces;
- (c) $\dot{\Sigma}$ be a supercuspidal data for $\dot{G} := U(\dot{V})$ such that $\dot{\Sigma}$ has \dot{b} blocks and $\dot{b} < b$;
- (d) $\dot{\Sigma}' := \vartheta_{\dot{V}, \dot{\mathcal{J}}'}(\dot{\Sigma})$ be a supercuspidal data of \dot{V}' where $[\dot{V}'] \in \dot{\mathcal{J}}'$.

We extend all the notations to this dual pair by adding “ $\dot{\cdot}$ ”. We assume that Proposition 8.1 holds for $(\dot{\Sigma}, \dot{\Sigma}')$, i.e.,

$$\dim \text{Hom}_{\dot{K}}(\dot{\eta}, \mathcal{S}(\dot{\mathcal{B}}_0)_{\dot{\Omega}}) = 1.$$

Note that the hypothesis holds when $\dot{b} = 0$, i.e., the depth-zero case (cf. [28] and Section 5.1).

8.2. Block decomposition of vector spaces

We have already treated the depth zero case, so we assume that $b \geq 1$.

8.2.1. – Let $\Sigma = \bigoplus_{l=0}^b {}^l\Sigma$ and $\Gamma = \bigoplus_{l=0}^b {}^l\Gamma$ be the decomposition of datum Σ according to Proposition 4.5 where ${}^l\Gamma$ has depth $-{}^lr$. We denote ${}^a\Sigma := \bigoplus_{i=0}^{b-1} {}^i\Sigma$ so that $\Sigma = {}^b\Sigma \oplus {}^a\Sigma$. In the rest of the section, the index i is reserved specially for $i = b, a$.

DEFINITION 8.2. – We collect the following definitions and facts.

- (i) Let $r = {}^br$ be the depth of Σ and $s = r/2$ as usual.
- (ii) We have $V = {}^bV \oplus {}^aV$ where ${}^aV := \bigoplus_{l=0}^{b-1} {}^lV$.
- (iii) Let ${}^iG := U({}^iV)$. Then ${}^bG \times {}^aG \subseteq \text{End}_D({}^bV) \oplus \text{End}_D({}^aV)$ sitting in $G \subseteq \text{End}_D(V)$ block diagonally.
- (iv) We have $\Gamma = {}^b\Gamma \oplus {}^a\Gamma$ where ${}^a\Gamma := \bigoplus_{l=0}^{b-1} {}^l\Gamma$.
- (v) We have $x = ({}^bx, {}^ax) \in \mathcal{B}({}^bG) \times \mathcal{B}({}^aG)$ and $\mathcal{L} = {}^b\mathcal{L} \oplus {}^a\mathcal{L}$ gives the decomposition in terms of the corresponding lattice functions.
- (vi) We have $G_x^0 = {}^bG_{b_x}^0 \times {}^aG_{a_x}^0$, $\rho = {}^b\rho \boxtimes {}^a\rho$ and $\phi = {}^b\phi \boxtimes {}^a\phi$.
- (vii) Let ${}^iK := K \cap {}^iG$, ${}^iK_{0+} := K_{0+} \cap {}^iG$, ${}^iK_+ := K_+ \cap {}^iG$ and ${}^{\square}K := {}^bK \times {}^aK$.
- (viii) Let ${}^{\square}\mathfrak{g} := \mathfrak{g} \cap {}^{\square}\text{End}$ and ${}^{\square}\mathfrak{g}_{x,s} := {}^{\square}\mathfrak{g} \cap \mathfrak{g}_{x,s}$ where

$${}^{\square}\text{End} := \text{Hom}_D({}^bV, {}^aV) \oplus \text{Hom}_D({}^aV, {}^bV) \subseteq \text{End}_D(V).$$

- (ix) Let ${}^{\square}J := G_{x,r} \exp({}^{\square}\mathfrak{g}_{x,s})$ and ${}^{\square}J_+ := {}^{\square}J \cap K_+ = G_{x,r} \exp({}^{\square}\mathfrak{g}_{x,s+})$. Obviously

$$\exp: {}^{\square}\mathfrak{g}_{x,s;s+} \xrightarrow{\sim} {}^{\square}J / {}^{\square}J_+ =: {}^{\square}\mathbf{W}$$

is a ${}^bG_{b_x} \times {}^aG_{a_x}$ -equivariant isomorphism between abelian groups.

- (x) We have

$$(8.1) \quad K = {}^{\square}K {}^{\square}J = {}^bK {}^aK {}^{\square}J \quad \text{and} \quad K_+ = {}^{\square}K_+ {}^{\square}J_+ = {}^bK_+ {}^aK_+ {}^{\square}J_+.$$

- (xi) Let ${}^{\square}k$ be the pull back of $\bar{\omega}_{\square\mathbf{W}}$ via

$${}^{\square}K \times {}^{\square}J \longrightarrow \mathbf{Sp}({}^{\square}\mathbf{W}) \times \mathbf{H}({}^{\square}\mathbf{W})$$

where ${}^{\square}J \rightarrow \mathbf{H}({}^{\square}\mathbf{W})$ is the restriction of special morphism given by (A.1).

- (xii) Let ${}^{\mathfrak{N}}\kappa$ be the Heisenberg-Weil representation of ${}^{\mathfrak{N}}K = {}^bK \times {}^aK$ determined by $({}^b\Gamma, {}^a\Gamma)$ and let ${}^{\mathfrak{N}}\eta := {}^b\eta \boxtimes {}^a\eta$ where ${}^i\eta$ is the iK -module defined by the datum ${}^i\Sigma$.
- (xiii) We identify ${}^{\mathfrak{N}}\kappa$ and ${}^{\mathfrak{N}}\eta$ with their inflations to ${}^{\mathfrak{N}}K \ltimes {}^{\mathfrak{N}}J$. Directly from the construction of κ and η , we get

$$(8.2) \quad \kappa = {}^{\mathfrak{N}}\kappa \otimes {}^{\mathfrak{N}}\kappa \quad \text{and} \quad \eta = {}^{\mathfrak{N}}\eta \otimes {}^{\mathfrak{N}}\kappa.$$

as ${}^{\mathfrak{N}}K \ltimes {}^{\mathfrak{N}}J$ -modules.

For the data Σ' , we define similar notations by adding “prime” and get corresponding conclusions.

8.2.2. – We now consider the block decomposition in the context of theta correspondence.

DEFINITION 8.3. – We collect the following notations and facts.

- (i) Let ${}^iW := {}^iV \otimes_D {}^iV'$, ${}^{ba}W := {}^bV \otimes_D {}^aV'$, ${}^{ab}W := {}^aV \otimes_D {}^bV'$, ${}^{\mathfrak{N}}W := {}^bW \oplus {}^aW$ and ${}^{\mathfrak{N}}W := {}^{ba}W \oplus {}^{ab}W$. Then we have following orthogonal decompositions of the symplectic spaces:

$$(8.3) \quad W = V \otimes_D V' = {}^{\mathfrak{N}}W \oplus {}^{\mathfrak{N}}W = ({}^bW \oplus {}^aW) \oplus ({}^{ba}W \oplus {}^{ab}W).$$

- (ii) We have irreducible reductive dual pairs $({}^iG, {}^iG')$ in $\mathrm{Sp}({}^iW)$ for $i = b$ and a . They form a reducible reductive dual pair $({}^{\mathfrak{N}}G, {}^{\mathfrak{N}}G') := ({}^bG \times {}^aG, {}^bG' \times {}^aG')$ in $\mathrm{Sp}({}^{\mathfrak{N}}W)$.
- (iii) By the construction of lifting of datum (see Definition 5.16), ${}^b\Sigma' = \vartheta^+({}^b\Sigma)$ and ${}^a\Sigma' = \vartheta_{aV, a\mathcal{G}'}({}^a\Sigma)$ where ${}^a\mathcal{G}' := \mathcal{G}' - [{}^bV']$. In addition, we have $w = {}^bw \oplus {}^aw \in {}^bW \oplus {}^aW$ so that ${}^aw = \bigoplus_{l=0}^{b-1} l w$, $M({}^iw) = {}^i\Gamma$ and $M'({}^iw) = {}^i\Gamma'$.
- (iv) Define lattice functions

$${}^{\mathfrak{N}}\mathcal{B}_t := {}^{\mathfrak{N}}W \cap \mathcal{B}_t \quad \text{and} \quad {}^{\mathfrak{N}}\mathfrak{b} = {}^{\mathfrak{N}}\mathcal{B}_0 / {}^{\mathfrak{N}}\mathcal{B}_{0+} \quad \text{for } \mathfrak{N} = b, a, ba, ab, \mathfrak{N}, \mathfrak{N}.$$

We have ${}^i\mathcal{B} = {}^i\mathcal{L} \otimes {}^i\mathcal{L}'$, ${}^{ba}\mathcal{B} = {}^b\mathcal{L} \otimes {}^a\mathcal{L}'$, ${}^{ab}\mathcal{B} = {}^a\mathcal{L} \otimes {}^b\mathcal{L}'$,

$$\mathcal{B} = {}^{\mathfrak{N}}\mathcal{B} \oplus {}^{\mathfrak{N}}\mathcal{B} = ({}^b\mathcal{B} \oplus {}^a\mathcal{B}) \oplus ({}^{ba}\mathcal{B} \oplus {}^{ab}\mathcal{B}) \quad \text{and}$$

$$\mathfrak{b} = {}^{\mathfrak{N}}\mathfrak{b} \oplus {}^{\mathfrak{N}}\mathfrak{b} = ({}^b\mathfrak{b} \oplus {}^a\mathfrak{b}) \oplus ({}^{ba}\mathfrak{b} \oplus {}^{ab}\mathfrak{b}).$$

- (v) Define doubled objects $\check{K} := K \times K'$, ${}^i\check{K} := {}^iK \times {}^iK'$, ${}^{\mathfrak{N}}\check{J} := {}^{\mathfrak{N}}J \times {}^{\mathfrak{N}}J'$, ${}^{\mathfrak{N}}\check{\eta} := {}^{\mathfrak{N}}\eta \boxtimes {}^{\mathfrak{N}}\eta'$, ${}^{\mathfrak{N}}\check{\kappa} := {}^{\mathfrak{N}}\kappa \boxtimes {}^{\mathfrak{N}}\kappa'$ etc. as usual.
- (vi) Let ${}^i\Omega := {}^i\check{K}{}^i w + {}^i\mathcal{B}_0$ and ${}^iS := \mathrm{Stab}_{{}^i\check{K}}({}^i w + {}^i\mathcal{B}_0)$ for $i = a, b$. We have

$${}^{\mathfrak{N}}S := \mathrm{Stab}_{{}^b\check{K} \times {}^a\check{K}}(w + \mathcal{B}_0) = {}^bS \times {}^aS \quad \text{and} \quad S := \mathrm{Stab}_{\check{K}}(w + \mathcal{B}_0) = {}^{\mathfrak{N}}S \oplus {}^{\mathfrak{N}}\check{J}.$$

8.2.3. – Similar to (7.9), we have following lemma.

LEMMA 8.4. – Consider the map ${}^{\mathfrak{N}}\bar{\iota}: {}^{\mathfrak{N}}\mathfrak{g}_{x,s;s+} \oplus {}^{\mathfrak{N}}\mathfrak{g}'_{x',s;s+} \rightarrow {}^{\mathfrak{N}}\mathcal{B}_{0;0+} = {}^{\mathfrak{N}}\mathfrak{b}$ induced by

$$(X, X') \mapsto (X, X') \cdot w = -wX + X'w \quad \forall X \in {}^{\mathfrak{N}}\mathfrak{g}_{x,s}, X' \in {}^{\mathfrak{N}}\mathfrak{g}'_{x',s}.$$

Both the domain and codomain of ${}^{\mathfrak{N}}\bar{\iota}$ have natural ${}^{\mathfrak{N}}\check{K}$ -module structures and the actions factor through ${}^{\mathfrak{N}}\check{K} / {}^{\mathfrak{N}}\check{K}_{0+}$. Moreover ${}^{\mathfrak{N}}\bar{\iota}$ is an ${}^{\mathfrak{N}}S$ -equivariant isomorphism between \mathfrak{f} -vector spaces.

Proof. – Let $(X, X') \in \mathfrak{g}_{x,s} \oplus \mathfrak{g}'_{x',s}$. In terms of matrices with respect to the decomposition $V = {}^bV \oplus {}^aV$ and $V' = {}^bV' \oplus {}^aV'$, we write

$$X = \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix}, \quad X' = \begin{pmatrix} 0 & A' \\ -A'^* & 0 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} {}^b w & 0 \\ 0 & {}^a w \end{pmatrix}.$$

Here $A \in \text{Hom}({}^a\mathcal{L}, {}^b\mathcal{L})_s \subseteq \text{Hom}_D({}^aV, {}^bV)$ and $*$: $\text{Hom}_D({}^aV, {}^bV) \xrightarrow{\sim} \text{Hom}_D({}^bV, {}^aV)$ is defined by $\langle Av_1, v_2 \rangle_{{}^bV} = \langle v_1, A^*v_2 \rangle_{{}^aV}$ for all $v_1 \in {}^aV, v_2 \in {}^bV$. The notation for V' is defined similarly. Then

$$(X, X') \cdot w = -wX + X'w = \begin{pmatrix} 0 & -{}^b w A + A' {}^a w \\ {}^a w A^* - A'^* {}^b w & 0 \end{pmatrix}.$$

CLAIM 1. – *Then map $\mathfrak{g}_{\bar{t}}$ is injective.*

Proof. – Suppose $\mathfrak{g}_{\bar{t}}(X, X') \in \mathfrak{B}_{0+}$, i.e.,

$$(8.4) \quad -{}^b w A + A' {}^a w \equiv 0 \pmod{{}^{ab}\mathfrak{B}_{0+}} \quad \text{and}$$

$$(8.5) \quad {}^a w A^* - A'^* {}^b w \equiv 0 \pmod{{}^{ba}\mathfrak{B}_{0+}}.$$

Applying \star to (8.5) gives

$$(8.6) \quad A {}^a w^\star - {}^b w^\star A' \equiv 0 \pmod{{}^{ab}\mathfrak{B}_{0+}}.$$

Note that ${}^i w \in \text{Hom}({}^i\mathcal{L}, {}^i\mathcal{L})_{-s}$ for $i = a, b$. Hence ${}^b w^\star(8.4) + (8.6) {}^a w$ yields

$$-{}^b \Gamma A + A {}^a \Gamma \equiv 0 \pmod{\text{Hom}({}^a\mathcal{L}, {}^b\mathcal{L})_{-s+}}.$$

By the definition of block decomposition, ${}^a \Gamma \in {}^a \mathfrak{g}_{a_x, -(b-1)r+} \subseteq {}^a \mathfrak{g}_{a_x, -r+}$ and so $-{}^b \Gamma A \in \text{Hom}({}^a\mathcal{L}, {}^b\mathcal{L})_{-s+}$. On the other hand, the datum ${}^b \Sigma$ is a single positive depth block so multiplying by ${}^b \Gamma$ induces an isomorphism ${}^b \mathcal{L}_{x,t} \xrightarrow{{}^b \Gamma} {}^b \mathcal{L}_{x,t-r}$. Hence $A \in \text{Hom}({}^a\mathcal{L}, {}^b\mathcal{L})_{s+}$, i.e., $X \in \mathfrak{g}_{x,s+}$. A similar argument yields $X' \in \mathfrak{g}'_{x',s+}$. This proves Claim 1. \square

CLAIM 2. – *We have $\dim_{\mathfrak{f}} \mathfrak{g}_{x,s} = \dim_{\mathfrak{f}} {}^{ab}\mathfrak{b}$ and $\dim_{\mathfrak{f}} \mathfrak{g}'_{x',s} = \dim_{\mathfrak{f}} {}^{ba}\mathfrak{b}$.*

Proof. – We recall that ${}^b w {}^b \mathcal{L}_t = {}^b \mathcal{L}'_{t-s}$ by the construction of lift of datum. Hence $A \mapsto {}^b w A$ induces an isomorphism

$$\mathfrak{g}_{x,s;s+} \cong \text{Hom}({}^a\mathcal{L}, {}^b\mathcal{L})_{s;s+} \xrightarrow{\sim} \text{Hom}({}^a\mathcal{L}, {}^b\mathcal{L}')_{0;0+} = {}^{ab}\mathfrak{b}.$$

Similarly, $A' \mapsto A'^* {}^b w$ induces an isomorphism

$$\mathfrak{g}'_{x',s;s+} \cong \text{Hom}({}^a\mathcal{L}', {}^b\mathcal{L}')_{s;s+} \xrightarrow{\sim} \text{Hom}({}^b\mathcal{L}, {}^a\mathcal{L}')_{0;0+} = {}^{ba}\mathfrak{b}. \quad \square$$

Claim 1 and Claim 2 prove that $\mathfrak{g}_{\bar{t}}$ is an isomorphism of \mathfrak{f} -vector spaces.

The group ${}^{\mathfrak{S}}S$ stabilizes the coset $w + {}^{\mathfrak{S}}\mathfrak{B}_0 \in {}^{\mathfrak{S}}W / {}^{\mathfrak{S}}\mathfrak{B}_0$. Using this fact and a direct computation show that $\mathfrak{g}_{\bar{t}}$ is ${}^{\mathfrak{S}}S$ -equivariant. \square

REMARK. – We only use the fact that ${}^a w \in {}^a \mathfrak{B}_{a_x, -s}$ when we prove the injectivity of $\mathfrak{g}_{\bar{t}}|_{\mathfrak{g}_{x,s;s+}}$. Therefore we could and will reuse this proof in Section 9.2.3.

8.3. Block decomposition of representations

As in the one block case, we consider the space $\mathcal{S}(\mathcal{B}_0)_\Omega$. By Lemma 2.4, $\mathcal{S}(\mathcal{B}_0)_\Omega = \text{Ind}_S^{\check{K}} \mathcal{S}(\mathcal{B}_0)_w = \text{Ind}_S^{\check{K}} \bar{\omega}_w$. We now decompose $\bar{\omega}_w$ according to the decomposition of data.

8.3.1. – Using the formula in (7.10), we get a morphism $\zeta^{\mathfrak{b}}: \mathfrak{J} \rightarrow \mathbf{H}(\mathfrak{b})$. Its natural extension to ${}^{\mathfrak{S}}S \times \mathfrak{J} \rightarrow \mathbf{SH}(\mathfrak{b})$ is again denoted by $\zeta^{\mathfrak{b}}$.

Let ${}^{\mathfrak{b}}\bar{\omega}$ denote the pull back of $\bar{\omega}_{\mathfrak{b}}$ via $\zeta^{\mathfrak{b}}$ which is an ${}^{\mathfrak{S}}S \times \mathfrak{J}$ -module realized on $\mathbb{S}(\mathfrak{b})$. More precisely, for $h = (u, (\exp(X), \exp(X'))) \in {}^{\mathfrak{S}}S \times \mathfrak{J}$,

$$(8.7) \quad {}^{\mathfrak{b}}\bar{\omega}(h) := \bar{\omega}_{\mathfrak{b}}(u) \bar{\omega}_{\mathfrak{b}}(-(X, X') \cdot w) \psi\left(\frac{1}{2} \langle w, (\exp(X), \exp(X'))^{-1} \cdot w - w \rangle_W\right).$$

LEMMA 8.5. – *We have ${}^{\mathfrak{b}}\check{k} \cong {}^{\mathfrak{b}}\bar{\omega}$ as ${}^{\mathfrak{S}}S \times \mathfrak{J}$ -modules.*

Proof. – By (2.6), we have $\frac{1}{2} \langle w, h^{-1} \cdot w - w \rangle_W \equiv \mathbb{B}(X, \Gamma) + \mathbb{B}(X', -\Gamma') \pmod{\mathfrak{p}}$ for all $h = (\exp(X), \exp(X')) \in \mathfrak{J} \subseteq G_{x,s} \times G'_{x',s}$. Now the lemma follows immediately from Lemma 8.4 with the same proof of (7.11). \square

Since $\mathfrak{b} = {}^{\mathfrak{b}}\mathfrak{b} \oplus \mathfrak{b}$, we have $\bar{\omega}_{\mathfrak{b}} = \bar{\omega}_{\mathfrak{b}} \boxtimes \bar{\omega}_{\mathfrak{b}}$ as $\mathbf{SH}({}^{\mathfrak{b}}\mathfrak{b}) \times \mathbf{SH}(\mathfrak{b})$ -module, realized on

$$(8.8) \quad \mathbb{S}(\mathfrak{b}) = \mathbb{S}({}^{\mathfrak{b}}\mathfrak{b}) \boxtimes \mathbb{S}(\mathfrak{b})$$

Note that $w = ({}^b w, {}^a w) \in {}^{\mathfrak{S}}W$. Evaluation at w gives an isomorphism of \mathbb{C} -vector spaces

$$(8.9) \quad \mathcal{S}({}^b \mathcal{B}_0)_{b_w} \boxtimes \mathcal{S}({}^a \mathcal{B}_0)_{a_w} = \mathcal{S}({}^{\mathfrak{S}} \mathcal{B}_0)_w \xrightarrow{\sim} \mathbb{S}(\mathfrak{b}) = \mathbb{S}({}^b \mathfrak{b}) \boxtimes \mathbb{S}({}^a \mathfrak{b}).$$

Translating the ${}^{\mathfrak{S}}S$ -module (resp. ${}^i S$ -module for $i = a, b$) structure via (8.9), we let $\bar{\omega}_{\mathfrak{S}\mathfrak{b}}$ (resp. $\bar{\omega}_{i\mathfrak{b}}$) be the resulting module acting on $\mathbb{S}(\mathfrak{b})$ (resp. $\mathbb{S}({}^i \mathfrak{b})$). Clearly, by Lemma 2.4,

$$(8.10) \quad \begin{aligned} \bar{\omega}_{\mathfrak{S}\mathfrak{b}}(h) &= \bar{\omega}_{\mathfrak{b}}(h) \bar{\omega}_{\mathfrak{b}}(h^{-1} \cdot w - w) \psi\left(\frac{1}{2} \langle w, h^{-1} \cdot w - w \rangle_{\mathfrak{S}W}\right) \\ &= \bar{\omega}_{b\mathfrak{S}}({}^b h) \boxtimes \bar{\omega}_{a\mathfrak{S}}({}^a h) \quad \forall h = ({}^b h, {}^a h) \in {}^{\mathfrak{S}}S. \end{aligned}$$

We state a key lemma for the induction process.

LEMMA 8.6. – *By an abuse of notation, let $\bar{\omega}_{\mathfrak{S}\mathfrak{b}}$ also denote its inflation to ${}^{\mathfrak{S}}S \times \mathfrak{J}$. Then*

$$\bar{\omega}_w = \bar{\omega}_{\mathfrak{S}\mathfrak{b}} \otimes {}^{\mathfrak{b}}\bar{\omega}$$

as ${}^{\mathfrak{S}}S \times \mathfrak{J}$ -module under the factorization (8.8).

Proof. – Suppose $h \in {}^{\mathfrak{S}}S$. Then $h^{-1} \cdot w - w \in {}^{\mathfrak{S}}\mathcal{B}_0$, i.e., its component in ${}^{\mathfrak{b}}\mathcal{B}_0$ is zero. Therefore by Lemma 2.4, (8.7) and (8.10),

$$\begin{aligned} \bar{\omega}_w(h) &= \bar{\omega}_{\mathfrak{b}}(h) \bar{\omega}_{\mathfrak{b}}(h^{-1} \cdot w - w) \psi\left(\frac{1}{2} \langle w, h^{-1} \cdot w - w \rangle_W\right) \\ &= (\bar{\omega}_{\mathfrak{b}} \boxtimes \bar{\omega}_{\mathfrak{b}})(h) \bar{\omega}_{\mathfrak{b}}(h^{-1} \cdot w - w) \psi\left(\frac{1}{2} \langle w, h^{-1} \cdot w - w \rangle_{\mathfrak{S}W}\right) \\ &= \bar{\omega}_{\mathfrak{S}\mathfrak{b}}(h) \otimes {}^{\mathfrak{b}}\bar{\omega}(h). \end{aligned}$$

Suppose $h = (\exp(X), \exp(X')) \in \mathfrak{J} \subseteq G_{x,s} \times G_{x',s}$. Then $\bar{\omega}_{\mathfrak{b}}(h) = \text{id}$. Since

$$h^{-1} \cdot w - w \in -(X, X') \cdot w + \mathcal{B}_{0+} \subseteq {}^{\mathfrak{S}}\mathcal{B}_{0+} \oplus {}^{\mathfrak{b}}\mathcal{B}_0$$

we see that $\overline{\omega}_b(h^{-1} \cdot w - w) = \text{id}_{\mathbb{S}(\mathbb{S}_b)} \boxtimes \overline{\omega}_{\mathbb{S}_b}(- (X, X') \cdot w)$. Putting the above into (2.5) gives

$$\overline{\omega}_w(h) = \overline{\omega}_{\mathbb{S}_b}(- (X, X') \cdot w) \psi\left(\frac{1}{2} \langle w, h^{-1} \cdot w - w \rangle_W\right) = \overline{\omega}(h).$$

This completes the proof of the lemma. □

8.4. Proof of Proposition 8.1

Note that ${}^a\Sigma$ has $b - 1$ blocks and the data $({}^a\Sigma, {}^a\Sigma')$ satisfies the induction hypothesis in Section 8.1. Also note that $\overline{\omega}$ is an irreducible $\mathbb{S}\check{J}$ -module since it is a Heisenberg representation. Now

$$\begin{aligned} \text{Hom}_{\check{K}}(\check{\eta}, \mathcal{S}(\mathcal{B}_0)_\Omega) &= \text{Hom}_{\check{K}}(\check{\eta}, \text{Ind}_S^{\check{K}} \overline{\omega}_w) = \text{Hom}_S(\check{\eta}, \overline{\omega}_w) \\ &= \text{Hom}_{\mathbb{S}_S \times \mathbb{S}_J}(\mathbb{S}\check{\eta} \otimes \mathbb{S}\check{\kappa}, \overline{\omega}_{\mathbb{S}_S} \otimes \overline{\omega}) \quad (\text{by (8.2) and Lemma 8.6}) \\ &= \text{Hom}_{\mathbb{S}_S}(\mathbb{S}\check{\eta}, \overline{\omega}_{\mathbb{S}_S} \otimes \text{Hom}_{\mathbb{S}_J}(\mathbb{S}\check{\kappa}, \overline{\omega})) \\ &= \text{Hom}_{\mathbb{S}_S}(\mathbb{S}\check{\eta}, \overline{\omega}_{\mathbb{S}_S}) \quad (\text{by Lemma 8.5}) \\ &= \text{Hom}_{b_S}({}^b\check{\eta}, \overline{\omega}_{b_S}) \otimes \text{Hom}_{a_S}({}^a\check{\eta}, \overline{\omega}_{a_S}). \end{aligned}$$

It has dimension 1 by Proposition 7.1 and the Induction Hypothesis. This completes the induction process and proves the proposition. □

9. Proof of the main Theorem II: Exhaustion

In this section, we prove 1.3 (ii) of the Main Theorem.

9.1. Occurrence of refined K -types

Part 1.3 (ii) of the Main Theorem is a easy consequence of following Proposition 9.1. Its proof consists of the whole Section 9.2 which uses the key identity (9.1).

Recall the notion of K -type data in the remark of Definition 3.4 and its extension to covering groups in Remark 5.3 of Definition 5.16.

PROPOSITION 9.1. – *Let $(G, G') = (U(V), U(V'))$ be a type I reductive dual pair. Let $\widetilde{\mathcal{D}} = (x, \Gamma, \phi, \rho, \xi)$ be a supercuspidal datum of \widetilde{G} . Suppose that $\theta_{V, V'}(\tilde{\pi}_{\widetilde{\mathcal{D}}}) \neq 0$ (or equivalently $\omega[\tilde{\eta}_{\widetilde{\mathcal{D}}}] \neq 0$) with respect to (G, G') . Then there exist a supercuspidal datum Σ for G and a K -type datum Σ' for G' such that*

- (i) $\widetilde{\Sigma}$ is a theta lift of Σ (cf. Definition 5.17), and the pair (Σ, Σ') defines a splitting $\xi_{x, x'}$;
- (ii) $\widetilde{\mathcal{D}}$ is equivalent to $\widetilde{\Sigma} := (\Sigma, \xi_{x, x'})$, i.e., $\pi_{\widetilde{\mathcal{D}}} \simeq \pi_{\widetilde{\Sigma}}$ and
- (iii) $\omega[\eta_\Sigma \boxtimes \eta'_{\Sigma'}] \neq 0$ under the splitting $\xi_{x, x'}$ where η_Σ and $\eta'_{\Sigma'}$ are the refined K and K' -types defined by Σ and Σ' respectively.

Proof of Main Theorem 1.3 (ii). – Let $\widetilde{\mathcal{D}}$ be a supercuspidal datum such that $\tilde{\pi} = \tilde{\pi}_{\widetilde{\mathcal{D}}}$. By Proposition 9.1, we have Σ and Σ' such that $\tilde{\pi} = \tilde{\pi}_{\widetilde{\Sigma}} := \text{c-Ind}_{\check{K}}^{\check{G}} \tilde{\eta}_{\widetilde{\Sigma}}$ and

$$\begin{aligned} 0 \neq \text{Hom}_{\check{K} \times \check{K}'}(\tilde{\eta}_{\widetilde{\Sigma}} \boxtimes \tilde{\eta}'_{\widetilde{\Sigma}'}, \omega) &= \text{Hom}_{\check{G} \times \check{K}'}(\tilde{\pi} \boxtimes \tilde{\eta}'_{\widetilde{\Sigma}'}, \omega) \\ &= \text{Hom}_{\check{G} \times \check{K}'}(\tilde{\pi} \boxtimes \tilde{\eta}'_{\widetilde{\Sigma}'}, \tilde{\pi} \boxtimes \theta_{V, V'}(\tilde{\pi})). \end{aligned}$$

Therefore $\widetilde{\eta}_{\Sigma'}$ occurs in $\theta_{V,V'}(\widetilde{\pi})$. By [17, Proposition 17.2 (2)], we conclude that ρ' in $\widetilde{\Sigma}'$ is cuspidal since $\widetilde{\pi}' = \theta_{V,V'}(\widetilde{\pi})$ is supercuspidal by assumption. Hence $\Sigma' = \vartheta_{V,\mathcal{G}'}(\Sigma)$ by definition, $\widetilde{\Sigma}'$ is a supercuspidal datum and $\widetilde{\pi}' = \widetilde{\pi}'_{\Sigma'}$. \square

REMARK. – In the proof of [17, Proposition 17.2], [17, Lemma 15.4] is used to treat the depth-zero case (see [17, p 315]). The covering group version of this lemma also holds since “The proof of Proposition 6.7 in [25] goes through without changes” as stated in [12, proof of Theorem 3.10]. Meanwhile the other parts of the proof of [17, Proposition 17.2 (2)] only involves subgroups of G which split canonically (because they are either unipotent or pro- p). Thus the proof [17, Proposition 17.2] also adapts “mutatis mutandis”.

9.2. Proof of Proposition 9.1

When $\widetilde{\mathcal{D}}$ has no positive depth block, i.e., $\widetilde{\mathcal{D}}$ is a depth zero data, this is proved by Pan in [28]. See Section 5.1 and in particular Theorem 5.5.

We prove by induction on the number of blocks similar to Section 8. We now assume $\widetilde{\mathcal{D}}$ has b positive depth blocks with $b > 0$.

INDUCTION HYPOTHESIS. – Assume Proposition 9.1 holds for $(\widetilde{\mathcal{D}}, \dot{V}, \dot{V}')$ where $\dim \dot{V} \leq \dim V, \dim \dot{V}' \leq \dim V'$ and $\widetilde{\mathcal{D}}$ has \dot{b} positive depth blocks with $\dot{b} > \dot{b}$.

9.2.1. – Suppose $\widetilde{\mathcal{D}} = (x, \Gamma, \phi, \rho, \xi)$. Let $\mathcal{D} = (x, \Gamma, \phi, \rho)$ and $\mathcal{D} = {}^b\mathcal{D} \oplus {}^a\mathcal{D}$ be the decomposition of \mathcal{D} as that for Σ in Section 8.2 so that $\text{depth}({}^b\mathcal{D}) = r$ and $\text{depth}({}^a\mathcal{D}) < r$. We adopt the notation defined in Definition 8.2 with respect to \mathcal{D} .

Let $\text{pr}_{G_{x,r+}}$ (resp. $\text{pr}_{[\Gamma]}$ and $\text{pr}_{\widetilde{\eta}_{\widetilde{\mathcal{D}}}}$) be the projection operator to the $G_{x,r+}$ -invariant spaces (resp. the $\psi_{\Gamma}|_{G_{x,s+}}$ -isotypic component and the $\widetilde{\eta}_{\widetilde{\mathcal{D}}}$ -isotypic component). Clearly, $\text{pr}_{[\Gamma]} = \text{pr}_{[\Gamma]} \circ \text{pr}_{G_{x,r+}}$ and $\text{pr}_{\widetilde{\eta}_{\widetilde{\mathcal{D}}}} \circ \text{pr}_{[\Gamma]} = \text{pr}_{\widetilde{\eta}_{\widetilde{\mathcal{D}}}}$.

LEMMA 9.2. – *There exist an $x'' \in \mathcal{B}(G')$ and a $w \in \mathcal{B}_{x,x'',-s}$ such that*

- (i) $\text{pr}_{\widetilde{\eta}_{\widetilde{\mathcal{D}}}}(\mathcal{S}(\mathcal{B}_{x,x'',0})_w) \neq 0$ and
- (ii) $M(w) \in \Gamma + {}^b\mathfrak{g}_{b_{x,-s}} \oplus {}^a\mathfrak{g}_{a_{x,-s}} \subseteq {}^b\mathfrak{g} \oplus {}^a\mathfrak{g}$.

Proof. – Since $\text{Jump}(\mathcal{L}_x) \subseteq \mathbb{Q}$, we have

$$(9.1) \quad \mathcal{S}^{G_{x,r+}} = \sum_{y' \in \mathcal{B}(G')} \mathcal{S}_{\mathcal{B}_{x,y',-s}}$$

See Appendix B for the notation and a quick proof of (9.1).

Note that $K \subseteq G_x$ preserves $\mathcal{S}_{\mathcal{B}_{x,y',-s}}$. Therefore, we can find an $x'' \in \mathcal{B}(G')$ such that $\text{pr}_{\widetilde{\eta}_{\widetilde{\mathcal{D}}}}(\mathcal{S}_{\mathcal{B}_{x,x'',-s}}) \neq 0$ since $\text{pr}_{\widetilde{\eta}_{\widetilde{\mathcal{D}}}}(\mathcal{S}) \neq 0$.

We denote $\mathcal{B}_{x,x'',t}$ by \mathcal{B}_t . Since $\mathcal{B}_{-s} = \bigcup_{w \in \mathcal{B}_{-s}} (w + \mathcal{B}_0)$, there is a $w \in \mathcal{B}_{-s}$ such that $\text{pr}_{\widetilde{\eta}_{\widetilde{\mathcal{D}}}}(\mathcal{S}(\mathcal{B}_0)_w) \neq 0$. Clearly $\text{pr}_{[\Gamma]}(\mathcal{S}(\mathcal{B}_0)_w) \neq 0$.

CLAIM. – *If $\text{pr}_{[\Gamma]}(\mathcal{S}(\mathcal{B}_0)_w) \neq 0$, then $M(w) \in \Gamma + \mathfrak{g}_{x,-s}$.*

Proof. – Let $g = \exp(X) \in G_{x,s+}$ with $X \in \mathfrak{g}_{x,s+}$. By (2.8), $\omega(g)f = \psi_{M(w)}(g)f$ for each $f \in \mathcal{S}(\mathcal{B}_0)_{w+\mathcal{B}_0}$. Hence $\text{pr}_{[\Gamma]}(\mathcal{S}(\mathcal{B}_0)_{w+\mathcal{B}_0}) \neq 0$ is equivalent to $\psi_{M(w)}(g) = \psi_\Gamma(g)$ for all $g \in G_{x,s+}$. This is equivalent to

$$\psi(\mathbb{B}(M(w) - \Gamma, X)) = 1 \quad \forall X \in \mathfrak{g}_{x,s+}.$$

Since ψ is a non-trivial character with conductor \mathfrak{p} , the above condition is equivalent to $\mathbb{B}(M(w) - \Gamma, \mathfrak{g}_{x,s+}) \subseteq \mathfrak{p}$, i.e., $M(w) \in \Gamma + \mathfrak{g}_{x,-s}$. \square

By Proposition 4.3, let $\check{\Gamma}$ be a $\text{GL}({}^bV)$ -good element in ${}^b\mathfrak{g}_{b_x,-r}$ representing ${}^b\Gamma + {}^b\mathfrak{g}_{b_x,-r+}$. Then $\check{\Gamma}$ is also a good element in $\mathfrak{g}_{x,r}$ representing $\Gamma + \mathfrak{g}_{x,-r+}$. Clearly $\check{G} := Z_G(\check{\Gamma}) \subseteq {}^bG \times {}^aG$.

We now recall a result of Kim-Murnaghan.

LEMMA ([19, Lemma 5.1.3 (3)]). – *Let $x \in \mathcal{B}(\check{G})$ and $X \in \check{\mathfrak{g}}_{x,-r} \cap \check{\mathfrak{g}}_{-r+}$. Then for $t > -r$ we have*

$$(\text{Ad } G_{x,r+t})(\check{\Gamma} + X + \check{\mathfrak{g}}_{x,t}) = \check{\Gamma} + X + \mathfrak{g}_{x,t}. \quad \square$$

Setting $X = \Gamma - \check{\Gamma}$, and $t = -s$, the above lemma gives $(\text{Ad } G_{x,s})(\Gamma + \check{\mathfrak{g}}_{x,-s}) = \Gamma + \mathfrak{g}_{x,-s}$. In other words, there is an $h \in G_{x,s}$ such that

$$M(h \cdot w) = h \cdot M(w) \in \Gamma + \check{\mathfrak{g}}_{x,-s} \subseteq ({}^b\Gamma + {}^b\mathfrak{g}_{b_x,-s}) \oplus {}^a\mathfrak{g}_{a_x,-s}.$$

Since $G_{x,s}$ normalizes $\tilde{\eta}_{\mathcal{D}}$, $\text{pr}_{\tilde{\eta}_{\mathcal{D}}}(\mathcal{S}(\mathcal{B}_0)_{h \cdot w}) = h \cdot \text{pr}_{\tilde{\eta}_{\mathcal{D}}}(\mathcal{S}(\mathcal{B}_0)_w) \neq 0$. Therefore by replacing w with $h \cdot w$, we may assume that w satisfies Lemma 9.2 (ii).

This completes the proof of Lemma 9.2.

9.2.2. – Let x'' and w satisfy Lemma 9.2. Let $\mathcal{B} := \mathcal{B}_{x,x''}$ and

$$(9.2) \quad M(w) = {}^bX + {}^aX,$$

where ${}^bX \in {}^b\Gamma + {}^b\mathfrak{g}_{b_x,-s}$ and ${}^aX \in {}^a\Gamma + {}^a\mathfrak{g}_{a_x,-s}$.

We define ${}^bV' = w({}^bV)$ and ${}^aV' = {}^bV'^{\perp}$. Let ${}^i w := w|_{{}^iV}$ for $i = a$ and b .

LEMMA 9.3. – *The following statements hold:*

- (i) *Restricting on bV , the map $w|_{{}^bV}: {}^bV \xrightarrow{\sim} {}^bV'$ is an isomorphism.*
- (ii) *The restriction of $\langle \cdot, \cdot \rangle_{V'}$ to ${}^bV'$ is non-degenerate. In particular, $V' = {}^bV' \oplus {}^aV'$.*
- (iii) *The image $w({}^aV) \subseteq {}^aV'$, i.e., ${}^a w \in \text{Hom}({}^aV, {}^aV')$.*

Proof. – (i) All elements in ${}^b\Gamma + {}^b\mathfrak{g}_{b_x,-s}$ are invertible elements in $\text{End}({}^bV)$. In particular bX is invertible. Since $w^{\star}w = {}^bX$ restricted on bV , $w: {}^bV \rightarrow {}^bV'$ is an injection and hence an isomorphism.

(ii) Let $v_1, v_2 \in {}^bV$. Then $\langle wv_1, wv_2 \rangle_{V'} = \langle v_1, M(w)v_2 \rangle_V = \langle v_1, {}^bXv_2 \rangle_{{}^bV}$. Since bX is invertible, the claim follows.

(iii) Suppose $v \in {}^aV$. Then $\langle wu, wv \rangle_{V'} = \langle -{}^bXu, v \rangle_V = 0$ for all $u \in {}^bV$. Hence $wv \in {}^bV'^{\perp} = {}^aV'$. \square

By Lemma 9.3, we see that ${}^i w \in \text{Hom}({}^i V, {}^i V')$ and $w = {}^b w \oplus {}^a w$ is a block diagonal decomposition. Moreover ${}^i M({}^i w) = {}^i X$ for ${}^i X$ in (9.2) where ${}^i M$ is the moment map defined with respect to the dual pair $(U({}^i V), U({}^i V'))$.

Let ${}^i \mathcal{L} = \mathcal{L} \cap {}^i V$ which is the lattice function corresponding to ${}^i x \in \mathcal{B}({}^i G)$ and $\mathcal{L} = {}^b \mathcal{L} \oplus {}^a \mathcal{L}$. The following lemma says that the lattice function \mathcal{L}' corresponding to x'' is split under $V' = {}^b V' \oplus {}^a V'$.

LEMMA 9.4. – Let ${}^i \mathcal{L}'_t = \mathcal{L}'_t \cap {}^i V'$ for $i = b, a$. Then

- (i) ${}^b \mathcal{L}'_t = {}^b w({}^b \mathcal{L}'_{t+s})$ and it is self-dual in ${}^b V'$,
- (ii) $\mathcal{L}' = {}^b \mathcal{L}' \oplus {}^a \mathcal{L}'$ and
- (iii) ${}^a \mathcal{L}'$ is self-dual⁽¹⁸⁾.

Proof. – (i) Let ${}^b \mathcal{L}''_t := {}^b w({}^b \mathcal{L}'_{t+s}) = w({}^b \mathcal{L}'_{t+s})$. By Lemma 5.9, it is a self-dual lattice function in ${}^b V'$. Since $w \in \mathcal{B}_{-s}$, ${}^b \mathcal{L}''_t \subseteq \mathcal{L}'_t \cap {}^b V' = {}^b \mathcal{L}'_t$ for all $t \in \mathbb{R}$. Taking dual lattice in ${}^b V$, we have ${}^b \mathcal{L}''_{-t+} = ({}^b \mathcal{L}''_t)^* \supseteq ({}^b \mathcal{L}'_t)^* \supseteq (\mathcal{L}'_t)^* \cap {}^b V' = {}^b \mathcal{L}'_{-t+}$ for all $t \in \mathbb{R}$. Hence (i) holds.

(ii) Obviously $\mathcal{L}'_t \supseteq {}^b \mathcal{L}'_t \oplus {}^a \mathcal{L}'_t$. Conversely let $v = {}^b v + {}^a v \in \mathcal{L}'_{-t+} = (\mathcal{L}'_t)^*$ with ${}^i v \in {}^i V'$. Then ${}^b v \in {}^b \mathcal{L}'_{-t+} = ({}^b \mathcal{L}'_t)^*$ since $\langle {}^b v, {}^b \mathcal{L}'_t \rangle_{bV} = \langle v, {}^b \mathcal{L}'_t \rangle_V \subseteq \langle v, \mathcal{L}'_t \rangle_V \subseteq \mathfrak{p}_D$. Now ${}^a v = v - {}^b v \in (\mathcal{L}'_{-t+} + {}^b \mathcal{L}'_{-t+}) \cap {}^a V' = {}^a \mathcal{L}'_{-t+}$. Therefore, $\mathcal{L}'_{-t+} \subseteq {}^b \mathcal{L}'_{-t+} \oplus {}^a \mathcal{L}'_{-t+}$ for any $t \in \mathbb{R}$. This proves part (ii).

(iii) By (i) and (ii), ${}^b \mathcal{L}'_{-t+} \oplus {}^a \mathcal{L}'_{-t+} = \mathcal{L}'_{-t+} = (\mathcal{L}'_t)^* = ({}^b \mathcal{L}'_t)^* \oplus ({}^a \mathcal{L}'_t)^* = {}^b \mathcal{L}'_{-t+} \oplus ({}^a \mathcal{L}'_t)^*$. Hence ${}^a \mathcal{L}'_{-t+} = ({}^a \mathcal{L}'_t)^*$, i.e., ${}^a \mathcal{L}'$ is self-dual. \square

Note that ${}^b M({}^b w) \in {}^b \Gamma + {}^b \mathfrak{g}_{x,-s}$. By Proposition 5.10, there is a ${}^b w_0 \in {}^b w + {}^b \mathcal{B}_0$ such that ${}^b M({}^b w_0) = {}^b \Gamma$. Replacing w with ${}^b w_0 \oplus {}^a w \in w + {}^b \mathcal{B}_0 \subseteq w + \mathcal{B}_0$, we assume that ${}^b M({}^b w) = {}^b \Gamma$ from now on.

9.2.3. – We retain all the notations in Definition 8.3 (??). On the other hand, we do not have enough information about Σ' to define K' at the moment. Instead, we replace \check{K} by K in Definition 8.3 ((vi)) and define the following notations.

- (vi') Let $\Omega_K := Kw + \mathcal{B}_0$ and $S_K := \text{Stab}_K(w + \mathcal{B}_0)$.
 Let ${}^i \Omega_K := {}^i K {}^i w + {}^i \mathcal{B}_0$, ${}^i S_K := \text{Stab}_{{}^i K}({}^i w + {}^i \mathcal{B}_0)$ for $i = a, b$.
 Let ${}^\square S_K := {}^b S_K \times {}^a S_K = \text{Stab}_{{}^b K \times {}^a K}(w + \mathcal{B}_0)$ so that $S_K = {}^\square S_K {}^\square J$.

By the remark of Lemma 8.4, we see that the map

$${}^\square \bar{\iota}_K: {}^\square \mathfrak{g}_{x,s;s+} \longrightarrow {}^\square \mathcal{B}_{0;0+} = {}^\square \mathfrak{b}$$

induced by $X \mapsto X \cdot w$ is an ${}^\square S_K$ -equivariant injection between \mathfrak{f} -modules. Moreover ${}^\square \bar{\iota}_K$ is an isometry with respect to natural symplectic forms of the domain and codomain. Let ${}^\square \mathfrak{b}_K^\perp$ be the orthogonal complement of the image of ${}^\square \bar{\iota}_K$ in ${}^\square \mathfrak{b}$.

The next lemma is a variation of Lemma 8.5 and Lemma 8.6 which follows by the same arguments.

⁽¹⁸⁾ We warn that ${}^a \mathcal{L}'_t \neq w_a {}^a \mathcal{L}'_{t+s}$.

LEMMA 9.5. – Let ${}^{\square}K$ be the Heisenberg-Weil representation of ${}^{\square}K \ltimes {}^{\square}J$ defined in Definition 8.2 ((xi)). Let $\bar{\omega}_{{}^{\square}S_K}$ and ${}^{\square}\bar{\omega}$ be the ${}^{\square}S_K \ltimes {}^{\square}J$ -modules realized on $\mathbb{S}({}^{\square}\mathbf{b})$ and $\mathbb{S}({}^{\square}\mathbf{b})$ respectively as in Section 8.3.1 (see (8.7) and (8.10)). Then, as ${}^{\square}S_K \ltimes {}^{\square}J$ -modules,

- (i) ${}^{\square}\bar{\omega} \cong {}^{\square}K \otimes \mathbb{S}({}^{\square}\mathbf{b}_K^{\perp}) \cong ({}^{\square}K)^{\oplus c}$, where $c = \dim \mathbb{S}({}^{\square}\mathbf{b}_K^{\perp}) = (\# {}^{\square}\mathbf{b}_K^{\perp})^{\frac{1}{2}}$,
- (ii) $\bar{\omega}_{{}^{\square}S_K} = \bar{\omega}_{{}^{\square}S_K} \boxtimes \bar{\omega}_{{}^{\square}S_K}$, and
- (iii) $\bar{\omega}_{{}^{\square}S_K} \cong \bar{\omega}_{{}^{\square}S_K} \otimes {}^{\square}\bar{\omega}$. □

9.2.4. – Let $\overset{\circ}{\mathcal{D}} = (x, \Gamma, \phi, \overset{\circ}{\rho})$ where $\overset{\circ}{\rho} := \rho \otimes \mu_{\xi, \xi_{x, x''}}$ and let $\overset{\circ}{\mathcal{D}} = {}^b\overset{\circ}{\mathcal{D}} \oplus {}^a\overset{\circ}{\mathcal{D}}$ be the corresponding block decomposition. Note that $(\overset{\circ}{\mathcal{D}}, \xi_{x, x''}) \sim \widetilde{\overset{\circ}{\mathcal{D}}}$ by definition and $\text{pr}_{\widetilde{\eta_{\overset{\circ}{\mathcal{D}}}}}(\mathcal{S}(\mathcal{B}_0)_w)$ is nonzero (see Definition 3.16 and Lemma 9.2 ((i))). By Lemma 9.5, we have

$$\begin{aligned} 0 \neq \text{Hom}_{\widetilde{K}}(\widetilde{\eta_{\overset{\circ}{\mathcal{D}}}}, \mathcal{S}(\mathcal{B}_0)_{\Omega_K}) &= \text{Hom}_K(\eta_{\overset{\circ}{\mathcal{D}}}, \mathcal{S}(\mathcal{B}_0)_{\Omega_K}) \\ &= \text{Hom}_K(\eta_{\overset{\circ}{\mathcal{D}}}, \text{Ind}_{S_K}^K \bar{\omega}_{{}^{\square}S_K}) = \text{Hom}_{{}^{\square}S_K}(\eta_{\overset{\circ}{\mathcal{D}}}, \bar{\omega}_{{}^{\square}S_K}) \\ &= \text{Hom}_{{}^{\square}S_K \ltimes {}^{\square}J}({}^{\square}\eta_{\overset{\circ}{\mathcal{D}}} \otimes {}^{\square}K, \bar{\omega}_{{}^{\square}S_K} \otimes {}^{\square}\bar{\omega}) \\ &= (\text{Hom}_{{}^{\square}S_K}({}^{\square}\eta, \bar{\omega}_{{}^{\square}S_K}))^{\oplus c} \\ &= \left(\text{Hom}_{{}^bS_K}(\eta_{{}^b\overset{\circ}{\mathcal{D}}}, \bar{\omega}_{{}^bS_K}) \boxtimes \text{Hom}_{{}^aS_K}(\eta_{{}^a\overset{\circ}{\mathcal{D}}}, \bar{\omega}_{{}^aS_K}) \right)^{\oplus c}. \end{aligned}$$

In particular, $0 \neq \text{Hom}_{{}^aS_K}(\eta_{{}^a\overset{\circ}{\mathcal{D}}}, \bar{\omega}_{{}^aS_K}) \subseteq \text{Hom}_{{}^aK}(\eta_{{}^a\overset{\circ}{\mathcal{D}}}, \mathcal{S}({}^a\mathcal{B}_0))$.

Let ${}^a\overset{\circ}{\mathcal{D}} := ({}^ax, {}^a\Gamma, {}^a\phi, {}^a\overset{\circ}{\rho}, \xi_{{}^ax, {}^ax''})$. Then ${}^a\overset{\circ}{\mathcal{D}}$ has $b - 1$ blocks. Applying the induction hypothesis to $({}^a\overset{\circ}{\mathcal{D}}, {}^aV, {}^aV')$, we get ${}^a\Sigma$ and ${}^a\Sigma' = \vartheta_{{}^aV, {}^aV'}({}^a\Sigma)$.

Now we define

- (a) $x' := ({}^bx'', {}^ax') \in \mathcal{B}({}^bG', F) \times \mathcal{B}({}^aG', F) \subseteq \mathcal{B}(G', F)$;
- (b) ${}^b\Sigma := ({}^bx, {}^b\Gamma, {}^b\phi, {}^b\overset{\circ}{\rho})$ where ${}^b\overset{\circ}{\rho} := {}^b\rho \otimes (\mu_{\xi, \xi_{x, x'}}|_{{}^bK}) = {}^b\overset{\circ}{\rho} \otimes (\mu_{\xi_{x, x''}, \xi_{x, x'}}|_{{}^bK})$;
- (c) $\Sigma := {}^b\Sigma \oplus {}^a\Sigma$, ${}^b\Sigma' := \vartheta^+({}^b\Sigma)$ with respect to bw and $\Sigma' := {}^b\Sigma' \oplus {}^a\Sigma'$.

Obviously, Σ' is a lift of Σ and x' occur as a part of the datum Σ' . By the functoriality of the construction of lattice model, one can see that

$$\mu_{\xi_{{}^ax, {}^ax''}, \xi_{{}^ax, {}^ax'}} = \mu_{\xi_{({}^bx, {}^ax), ({}^bx'', {}^ax'')}, \xi_{({}^bx, {}^ax), ({}^bx'', {}^ax')}}|_{{}^aG_{{}^ax}} = \mu_{\xi_{x, x''}, \xi_{x, x'}}|_{{}^aG_{{}^ax}}.$$

Hence, we conclude that $(\Sigma, \xi_{x, x'}) \sim \widetilde{\mathcal{D}}$. Applying the argument in Section 8, we conclude that

$$0 \neq \text{Hom}_{K \times K'}(\eta_{\Sigma} \boxtimes \eta'_{\Sigma'}, \mathcal{S}(\mathcal{B}_{x, x'; 0})_{KK \cdot w}).$$

This finishes the proof of Proposition 9.1 and hence also completes the proof of 1.3 (ii) of the Main Theorem. □

Appendix A

Heisenberg-Weil representations

In this appendix, we collect some facts about Heisenberg-Weil representations.

A.1. Heisenberg-Weil representation after Gérardin

Let \mathfrak{f} be a finite field with q elements and let $\bar{\psi}$ be a nontrivial character of \mathfrak{f} . Let \mathbf{W} be a non-degenerate symplectic space over \mathfrak{f} . Let $\mathbf{H}(\mathbf{W}) = \mathbf{W} \times \mathfrak{f}$ and $\mathbf{Sp}(\mathbf{W})$ denote the corresponding Heisenberg group and symplectic group as usual. We let $(\bar{\omega}_{\mathbf{W}}, \mathbb{S}(\mathbf{W}))$ denote the space of the Heisenberg-Weil representation of $\mathbf{SH}(\mathbf{W}) := \mathbf{Sp}(\mathbf{W}) \ltimes \mathbf{H}(\mathbf{W})$ with central character $\bar{\psi}$ realizing on $\mathbb{S}(\mathbf{W})$. In [10], Gérardin carefully studied the isomorphism class of $\bar{\omega}_{\mathbf{W}}$. We now recall the mixed model of this representation. See [10, § 2] for details.

For any subspace $\mathbf{V} \subseteq \mathbf{W}$, let $\mathbf{H}(\mathbf{V})$ be its inverse image in $\mathbf{H}(\mathbf{W})$ under the projection $\mathbf{H}(\mathbf{W}) \rightarrow \mathbf{H}(\mathbf{W})/\mathfrak{f} = \mathbf{W}$. Let \mathbf{W}_+ be a non-trivial totally isotropic subspace of \mathbf{W} . Then $\mathbf{W}_0 := \mathbf{W}_+^\perp/\mathbf{W}_+$ is naturally a non-degenerate symplectic space. Let $\mathbf{P}(\mathbf{W}_+)$ be the parabolic subgroup stabilizing \mathbf{W}_+ . By an abuse of notation, we let $\bar{\omega}_{\mathbf{W}_0}$ denote the pull back of $\bar{\omega}_{\mathbf{W}_0}$ to $\mathbf{P}(\mathbf{W}_+) \ltimes \mathbf{H}(\mathbf{W}_+^\perp)$ via the natural quotient

$$\mathbf{P}(\mathbf{W}_+) \ltimes \mathbf{H}(\mathbf{W}_+^\perp) \longrightarrow \mathbf{Sp}(\mathbf{W}_0) \ltimes \mathbf{H}(\mathbf{W}_0).$$

Let $\chi^{\mathbf{W}_+}$ be the (unique real) character of $\mathbf{P}(\mathbf{W}_+)$ given by $g \mapsto (\det g|_{\mathbf{W}_+})^{(q-1)/2} \in \{\pm 1\}$ for all $g \in \mathbf{P}(\mathbf{W}_+)$. Then

- (i) $\bar{\omega}_{\mathbf{W}}$ is the unique $\mathbf{SH}(\mathbf{W})$ -module extending $\text{Ind}_{\mathbf{P}(\mathbf{W}_+) \ltimes \mathbf{H}(\mathbf{W}_+^\perp)}^{\mathbf{P}(\mathbf{W}_+) \ltimes \mathbf{H}(\mathbf{W})} (\chi^{\mathbf{W}_+} \otimes \bar{\omega}_{\mathbf{W}_0})$.
- (ii) Fix a totally isotropic subspace \mathbf{W}_- such that $\mathbf{W} = \mathbf{W}_- \oplus \mathbf{W}_+^\perp$, then the induced module in ((i)) could be identified with the set of functions on \mathbf{W}_- with values in $\mathbb{S}(\mathbf{W}_0)$. The group actions could be easily work out.
- (iii) The space $(\bar{\omega}_{\mathbf{W}})^{\mathbf{W}_+}$ of \mathbf{W}_+ -invariants in $\bar{\omega}_{\mathbf{W}}$ is isomorphic to $\bar{\omega}_{\mathbf{W}_0}$ as an $\mathbf{H}(\mathbf{W}_+^\perp)$ -module. Moreover $\mathbf{P}(\mathbf{W}_+)$ acts by $\chi^{\mathbf{W}_+} \otimes \bar{\omega}_{\mathbf{W}_0}$.
- (iv) The module $\bar{\omega}_{\mathbf{W}}$ has dimension $\sqrt{\#\mathbf{W}} = q^{\frac{1}{2} \dim_{\mathfrak{f}} \mathbf{W}}$.

Note that when \mathbf{W}_+ is a maximal isotropic subspace in \mathbf{W} , we have $\mathbf{H}(\mathbf{W}_0) = \mathfrak{f}$ and $\bar{\omega}_{\mathbf{W}_0} = \bar{\psi}$ so that we get the Schrödinger model of $\bar{\omega}_{\mathbf{W}}$.

A.2. Construction of κ .

Following [37], we discuss the construction of the K^i -module κ^i which extends $\psi_\Gamma|_{\mathcal{K}_+^i}$.

A.2.1. *Special isomorphism.* – As Yu [37] has pointed out, the extension of a Heisenberg representation to a “Weil representation” of K is subtle. The problem is that, $\mathbf{H}(\mathbf{W})$ has a large subgroup of the automorphism group (isomorphic to \mathbf{W}) whose action on the center \mathfrak{f} and on $\mathbf{H}(\mathbf{W})/\mathfrak{f}$ are identity. Therefore, $J \rightarrow \mathbf{H}(\mathbf{W})$ in [37, § 11] is far from unique and, Yu gives a canonical construction from root datum.

We retain the notation and situation in [37, § 11] and [17]:

- (i) $\Gamma \in \mathfrak{g}$ is a good element of depth $-r$;
- (ii) (\check{G}, G) is a tamely ramified twisted Levi sequence with $\check{G} = Z_G(\Gamma)$;
- (iii) $\mathcal{B}(\check{G}) \hookrightarrow \mathcal{B}(G)$ is a fixed embedding of buildings;
- (iv) $x \in \mathcal{B}(\check{G})$;
- (v) $\mathfrak{g} = \check{\mathfrak{g}} \oplus \check{\mathfrak{g}}^\perp$ is an orthogonal decomposition with respect to the form \mathbb{B} in Section 2.1.1;
- (vi) $J = (\check{G}, G)(F)_{x,(r,s)}$ and $J_+ = (\check{G}, G)(F)_{x,(r,s^+)}$. See [37, p. 586]⁽¹⁹⁾.

⁽¹⁹⁾ In our cases, $(\check{G}, G)(F)_{x,(r_1,r_2)} = \exp(\check{\mathfrak{g}}_{x,r_1} \oplus \check{\mathfrak{g}}_{x,r_2}^\perp)$ for $0 < \frac{1}{2}r_1 \leq r_2$.

Taking a clue from Lemma 2.4, we could define a “canonical” morphism for J below. The symplectic space $\mathbf{W} = J/J_+$ is identified with $\check{\mathfrak{g}}_{x,s;s+}^\perp$ via the exponential map. Suppose $\bar{X}_1, \bar{X}_2 \in \check{\mathfrak{g}}_{x,s;s+}^\perp$ with lifts $X_1, X_2 \in \check{\mathfrak{g}}_{x,s}^\perp$ respectively. Then we have a non-degenerate symplectic form on \mathbf{W} given by $\langle \bar{X}_1, \bar{X}_2 \rangle = \overline{\mathbb{B}([X_1, X_2], \Gamma)} \in \mathfrak{f}$ (cf. [37, Lemma 11.1]). By the Baker-Campbell-Hausdorff formula, we have a group homomorphism ⁽²⁰⁾

$$(A.1) \quad \zeta: J \longrightarrow \mathbf{H}(\mathbf{W}) = \mathbf{W} \times \mathfrak{f} \quad \text{given by} \quad \exp(X) \mapsto (\bar{X}, \overline{\mathbb{B}(X, \Gamma)}).$$

Note that ζ agrees with the special morphism defined in [37, Section 11] since they agree on root subgroups. By an abuse of notation, we also let ζ denote its natural extension

$$\zeta: \check{G}_x \ltimes J \longrightarrow \mathbf{Sp}(\mathbf{W}) \ltimes \mathbf{H}(\mathbf{W}) = \mathbf{SH}(\mathbf{W}).$$

A.2.2. – We fix a good factorization $\Gamma = \sum_{i=0}^d \Gamma_i$ and therefore get a sequence of subgroups G^i as in Definition 3.3. We follow the notation in [37, p.591]: $K^i := K \cap G^i = G_x^0 G_{x,s_0}^1 \cdots G_{x,s_{i-1}}^i$, $J^i := (G^{i-1}, G^i)_{r_{i-1}, s_{i-1}}$, $J_+^i := (G^{i-1}, G^i)_{r_{i-1}, s_{i-1}^+}$ and $\mathbf{W}^i := J^i/J_+^i$.

Now we define a sequence of representations κ^i of K^i inductively such that K_+^i acts by the character ψ_Γ . This is essentially Yu’s construction in [37, § 4].

0. First we set $\kappa^0 = \phi$ (cf. Definition 3.5 ((f))).

Suppose we have constructed κ^{i-1} . We now construct the K^i -module κ^i :

1. Note that $K^i = K^{i-1} J^i$. Let $\zeta^i: K^{i-1} \ltimes J^i \longrightarrow \mathbf{SH}(\mathbf{W}^i)$ denote the special homomorphism with respect to the Levi sequence (G^{i-1}, G^i) and the good element Γ_{i-1} (cf. Appendix A.2.1). Let $\bar{\omega}_{\Gamma_{i-1}}^i$ denote the $K^{i-1} \ltimes J^i$ -module obtained by pulling back of $\bar{\omega}_{\mathbf{W}^i}$ (cf. Appendix A.1) via ζ^i .
2. We set $\Gamma^i := \sum_{l=i}^d \Gamma_l$ which is in the center of \mathfrak{g}^i . We see that ψ_{Γ^i} is a character of $G_{x,0+}^i \supseteq J^i$. Let $\mathbf{1} \times \psi_{\Gamma^i}$ be its extension to $K^{i-1} \ltimes J^i$ such that K^{i-1} acts trivially. As a subgroup of J^i , J_+^{i-1} acts by the character ψ_Γ on $\bar{\omega}_{\Gamma_{i-1}}^i \otimes (\mathbf{1} \times \psi_{\Gamma^i})$.
3. We inflate κ^{i-1} to a $K^{i-1} \ltimes J^i$ -module. Since $K^{i-1} \cap J^i = G_{x,r_{i-1}}^{i-1} \subseteq J_+^i \cap K_+^{i-1}$, the $K^{i-1} \ltimes J^i$ -module $\kappa^{i-1} \otimes \bar{\omega}_{\Gamma_{i-1}}^i \otimes (\mathbf{1} \times \psi_{\Gamma^i})$ factors through $K^{i-1} \ltimes J^i \longrightarrow K^{i-1} J^i = K^i$. Let κ^i be the corresponding K^i -module. It is clear that $K_+^i = K_+^{i-1} J_+^i$ acts by ψ_Γ .

Appendix B

A quick proof of a result of Pan

As the reader may notice, (9.1) is a generalization of [27, Proposition 6.3]. Our proof follows Pan’s idea. Although we use the exponential map to identify $\mathfrak{g}_{x,r}$ with $G_{x,r}$, the statements and proofs in this appendix also hold if we replace the exponential map by the “Cayley transform” and in which case we only assume the residual characteristic $p \neq 2$. To ease notation, we normalize the valuation map such that $\text{val}(\varpi_D) = 1$.

⁽²⁰⁾ We check that $\zeta|_J$ is a group homomorphism. Indeed by the Baker-Campbell-Hausdorff formula $\log(\exp(X)\exp(Y)) \equiv X + Y + \frac{1}{2}[X, Y] \pmod{\mathfrak{g}_{l_x, r+}^i}$ and

$$\zeta(e^X)\zeta(e^Y) = (\bar{X}, \overline{\mathbb{B}(X, \Gamma)}) \cdot (\bar{Y}, \overline{\mathbb{B}(Y, \Gamma)}) = (\bar{X} + \bar{Y}, \overline{\mathbb{B}(X + Y + \frac{1}{2}[X, Y], \Gamma)}) = \zeta(e^X e^Y).$$

B.1. Invariant vectors under the action of lattices

Let \mathcal{S} be any realization of the oscillator representation of $\widetilde{\mathrm{Sp}}(W) \times \mathrm{H}(W)$. Here $\mathrm{H}(W) := W \times F$ denotes the Heisenberg group of W and we identify W as a subset of $\mathrm{H}(W)$.

Suppose L is a lattice in W such that $L \supseteq A^*$ for a certain good lattice A . Let \mathcal{S}^{L^*} denote the space of L^* -fixed vectors in \mathcal{S} under the Heisenberg group $\mathrm{H}(W)$ action. Let $\mathcal{S}(A^*)$ be the generalized lattice model of the oscillator representation with respect to A^* and let $\mathcal{S}(A^*)_L$ be the subspace of functions in $\mathcal{S}(A^*)$ supported on L . We identify \mathcal{S} with $\mathcal{S}(A^*)$ via a fixed intertwining map. It is easy to see that \mathcal{S}^{L^*} is exactly the image of $\mathcal{S}(A^*)_L$ (cf. [27, Lemma 8.2]). Since \mathcal{S}^{L^*} neither depends on the choice of A nor the choice of intertwining map, it makes sense to let \mathcal{S}_L denote \mathcal{S}^{L^*} to emphasis that it is the space of functions with support on L under the generalized lattice module with respect to any $A^* \subseteq L$. In particular, for a self-dual lattice function \mathcal{B} in W , we identify $\mathcal{S}^{\mathcal{B},+}$ with $\mathcal{S}(\mathcal{B}_0)_{\mathcal{B}-s}$ and denote it by $\mathcal{S}_{\mathcal{B}-s}$.

B.2. Proof of (9.1) and depth preservation

We only need to consider rational points in the building for our study of minimal K -types. These points correspond to lattice functions with rational jumps.

In the rest of this section, we will prove the following theorem which is a slightly stronger version of (9.1).

THEOREM B.1. – *Suppose $\mathrm{Jump}(\mathcal{L}_x) \subseteq \frac{1}{m}\mathbb{Z}$ so that $\mathrm{Jump}(\mathfrak{g}_x) \subseteq \frac{1}{m}\mathbb{Z}$ for certain positive integer m . Let*

$$\mathcal{B}(G')_{2m} := \{y \in \mathcal{B}(G') \mid \mathrm{Jump}(\mathcal{L}'_y) \subseteq \frac{1}{2m}\mathbb{Z}\}.$$

Then for all $0 \leq r \in \frac{1}{m}\mathbb{Z}$,

$$\mathcal{S}^{G_{x,r+}} = \sum_{y \in \mathcal{B}(G')_{2m}} \mathcal{S}_{\mathcal{B}_{x,y,-r/2}}.$$

B.2.1. – We will call \mathcal{L} an \mathfrak{o}_D -module function in V if \mathcal{L}_s is only an \mathfrak{o}_D -submodule in V in Definition 2.3. In this case, $\mathcal{L}_s \otimes_{\mathfrak{o}_D} D$ may not equal to V and

$$\mathcal{L}_s^* := \{v \in V \mid \langle v, \mathcal{L}_s \rangle_V \subseteq \mathfrak{p}_D\}$$

may not be a lattice.

The following are the key lemmas:

LEMMA B.2 (cf. [27, Lemma 10.1]). – *Suppose \mathcal{N} is an \mathfrak{o}_D -module function in V such that $\mathrm{Jump}(\mathcal{N}) \subseteq \frac{1}{2m}\mathbb{Z}$ and $\langle \mathcal{N}_{t_1}, \mathcal{N}_{t_2} \rangle_V \subseteq \mathfrak{p}_D^{\lceil t_1+t_2+\frac{1}{m} \rceil}$. Then there is a self-dual lattice function \mathcal{L} such that $\mathcal{N}_t \subseteq \mathcal{L}_{t+\frac{1}{2m}}$ and $\mathrm{Jump}(\mathcal{L}) \subseteq \frac{1}{2m}\mathbb{Z}$.*

Proof. – Since $\left\langle \mathcal{N}_{\frac{i}{2m}}, \mathcal{N}_{-\frac{i-1}{2m}} \right\rangle_V \subseteq \mathfrak{p}_D^{\lceil \frac{i-i-1}{2m} + \frac{1}{m} \rceil} = \mathfrak{p}_D$, we have $\mathcal{N}_{\frac{i}{2m}} \subseteq \mathcal{N}_{-\frac{i-1}{2m}}^*$. In particular, we have $\mathcal{N}_0 \subseteq \mathcal{N}_{-\frac{1}{2m}} \subseteq \mathcal{N}_0^*$. Fix a good lattice R which contains \mathcal{N}_0 . Define

$$(B.1) \quad \mathcal{L}_{\frac{i}{2m}} := \begin{cases} \mathcal{N}_{\frac{i-1}{2m}} + R^* & \text{when } -\frac{1}{2} < \frac{i}{2m} \leq 0; \\ \mathcal{L}_{-\frac{i+1}{2m}}^* = \mathcal{N}_{\frac{i}{2m}}^* \cap R & \text{when } 0 < \frac{i}{2m} \leq \frac{1}{2}. \end{cases}$$

Observe that $\langle \mathcal{N}_{-\frac{1}{2}}, R^* \rangle_V = \varpi_D^{-1} \langle \mathcal{N}_{\frac{1}{2}}, R^* \rangle_V \subseteq \varpi_D^{-1} \langle \mathcal{N}_0, R^* \rangle_V \subseteq \varpi_D^{-1} \langle R, R^* \rangle_V = \mathfrak{o}_D$.
 Therefore we have $\langle \mathcal{L}_{\frac{-m+1}{2m}}, \mathcal{L}_{\frac{-m+1}{2m}} \rangle_V = \langle \mathcal{N}_{-\frac{1}{2}} + R^*, \mathcal{N}_{-\frac{1}{2}} + R^* \rangle_V \subseteq \mathfrak{o}_D$ which is equivalent to

$$\mathcal{L}_{\frac{-m+1}{2m}} \subseteq (\mathcal{L}_{\frac{-m+1}{2m}})^* \varpi_D^{-1} = \mathcal{L}_{\frac{m}{2m}} \varpi_D^{-1}.$$

Hence (B.1) determines a lattice function \mathcal{L} such that $\text{Jump}(\mathcal{L}) \subseteq \frac{1}{2m}\mathbb{Z}$. Moreover, \mathcal{L} is self-dual since $\mathcal{L}_{\frac{i}{2m}}^* = \mathcal{L}_{\frac{-i+1}{2m}}$ by definition.

Note that

$$\begin{aligned} \mathcal{L}_{\frac{i+1}{2m}} &= \mathcal{N}_{\frac{i}{2m}} + R^* \supseteq \mathcal{N}_{\frac{i}{2m}} && \text{when } -\frac{1}{2} \leq \frac{i}{2m} < 0; \\ \mathcal{L}_{\frac{i+1}{2m}} &= \mathcal{L}_{\frac{i}{2m}}^* = \mathcal{N}_{\frac{-i-1}{2m}} \cap R \supseteq \mathcal{N}_{\frac{i}{2m}} \cap \mathcal{N}_0 = \mathcal{N}_{\frac{i}{2m}} && \text{when } 0 \leq \frac{i}{2m} < \frac{1}{2}. \end{aligned}$$

Therefore $\mathcal{L}_{t+\frac{1}{2m}} \supseteq \mathcal{N}_t$ for all $t \in \mathbb{R}$ by the definition of \mathcal{L} . □

LEMMA B.3 (cf. [27, Proposition 10.5]). – *Suppose x' is a point in $\mathcal{B}(G')$ and j is a positive integer. Then*

$$\mathcal{S}_{\mathcal{B}_{x,x'}, -j/2m}^{G_{x,j/m}} \subseteq \sum_{y \in \mathcal{B}(G')_{2m}} \mathcal{S}_{x,y, \frac{-j+1}{2m}}.$$

Proof. – Let $r := j/m$ and $s := r/2 = j/2m$. Note that $G_{x,r} \subseteq G_{x,s+}$ and

$$\mathcal{S}_{\mathcal{B}_{x,x'}, -j/2m} = \bigoplus_w \mathcal{S}(\mathcal{B}_{x,x'}, 0)_w$$

as $G_{x,r}$ -module where w is running over representatives of $\mathcal{B}_{x,x',-s;0}$. By Remark 2. of Lemma 2.4, the summand $\mathcal{S}(\mathcal{B}_{x,x'}, 0)_w$ is $G_{x,r+}$ -isotypic and $\exp(X) \in G_{x,r}$ acts by the scalar $\psi(\mathbb{B}(X, w^\star w))$. Now fix a w such that $\mathcal{S}(\mathcal{B}_{x,x'}, 0)_w^{G_{x,r}} \neq 0$. Since $\psi|_{\mathfrak{o}}$ is non-degenerate, $\mathbb{B}(\mathfrak{g}_{x,r}, w^\star w) \in \mathfrak{p}$, i.e., $M(w) = w^\star w \in \mathfrak{g}_{x,-r+}$. Clearly, $\mathcal{S}(\mathcal{B}_{x,x'}, 0)_w \subseteq \mathcal{S}^{G_{x,r}}$.

Define $\mathcal{N}_t := (w + \mathcal{B}_{x,x'}, 0) \mathcal{L}_{x,t+s}$. It is clear that \mathcal{N}_t is an \mathfrak{o}_D -module function in V' and $\text{Jump}(\mathcal{N}_t) \subseteq \frac{1}{2m}\mathbb{Z}$. On the other hand, $w_1^\star w_2 \equiv M(w) \equiv 0 \pmod{\mathfrak{gl}(V)_{x,-r+}}$ for any $w_1, w_2 \in w + \mathcal{B}_{x,x'}, 0$ and $\mathfrak{gl}(V)_{x,-r+} = \mathfrak{gl}(V)_{x,-r+\frac{1}{m}}$. Therefore,

$$\begin{aligned} \langle \mathcal{N}_{t_1}, \mathcal{N}_{t_2} \rangle_{V'} &\subseteq \langle \mathcal{L}_{x,t_1+s}, \mathfrak{gl}(V)_{x,-r+\frac{1}{m}} \cdot \mathcal{L}_{x,t_2+s} \rangle_V \\ &\subseteq \langle \mathcal{L}_{x,t_1+s}, \mathcal{L}_{x,t_2-s+\frac{1}{m}} \rangle_V \subseteq \mathfrak{p}_D^{[t_1+t_2+\frac{1}{m}]}. \end{aligned}$$

By Lemma B.2, there is a self-dual lattice function \mathcal{L}'_y such that $\mathcal{N}_t \subseteq \mathcal{L}'_{y, t+\frac{1}{2m}}$. Hence we have

$$w + \mathcal{B}_{x,x'}, 0 \subseteq \bigcap_{t \in \frac{1}{2m}\mathbb{Z}} \text{Hom}_{\mathfrak{o}_D}(\mathcal{L}_{x,t+s}, \mathcal{L}'_{y, t+\frac{1}{2m}}) = \mathcal{B}_{x,y, -s+\frac{1}{2m}}$$

and $\mathcal{B}_{x,x'}, 0 = \{w_1 - w_2 \mid w_1, w_2 \in w + \mathcal{B}_{x,x'}, 0\} \subseteq \mathcal{B}_{x,y, -s+\frac{1}{2m}}$.

This means $\mathcal{S}(\mathcal{B}_{x,x'}, 0)_w \subseteq \mathcal{S}_{x,y, \frac{-j+1}{2m}}$ and proves the lemma. □

B.2.2. *Proof of Theorem B.1.* – The “ \supseteq ” direction is obvious. We now prove the “ \subseteq ” direction. Let $r = k/m$ and fix any $x' \in \mathcal{B}(G')_{2m}$. For each integer $j > k$, we have

$$\begin{aligned} \mathcal{S}_{\mathcal{B}_{x,x'}, \frac{-j}{2m}}^{G_{x,r^+}} &= (\mathcal{S}_{\mathcal{B}_{x,x'}, \frac{-j}{2m}})^{G_{x, \frac{k+1}{m}}} = ((\mathcal{S}_{\mathcal{B}_{x,x'}, \frac{-j}{2m}})^{G_{x, \frac{j}{m}}})^{G_{x, \frac{k+1}{m}}} \\ &\subseteq \sum_{y \in \mathcal{B}(G')_{2m}} (\mathcal{S}_{\mathcal{B}_{x,y}, \frac{-(j-1)}{2m}})^{G_{x, \frac{k+1}{m}}} \subseteq \dots \subseteq \sum_{y \in \mathcal{B}(G')_{2m}} \mathcal{S}_{\mathcal{B}_{x,y}, \frac{-k}{2m}} \end{aligned}$$

by Lemma B.3. Now the theorem follows from the fact that $\mathcal{S} = \bigcup_{j \geq k} \mathcal{S}_{\mathcal{B}_{x,x'}, \frac{-j}{2m}}$. \square

B.2.3. – By the argument in the proof of [27, Theorem 6.6], “depth preservation” of theta correspondence is an immediate consequence of Theorem B.1. Indeed let $\Phi: \mathcal{S} \rightarrow \pi \boxtimes \theta(\pi)$ be the $\tilde{G} \times \tilde{G}'$ -intertwining map. Suppose π has depth r . Then there is a certain $x \in \mathcal{B}(G)$ such that $\text{Jump}(\mathcal{L}_x) \in \mathbb{Q}$ and $\pi^{G_{x,r^+}} \neq 0$. By Theorem B.1, $\Phi(\mathcal{S}_{\mathcal{B}_{x,y}, -r/2}) \neq 0$ for some $y \in \mathcal{B}(G')$ so $(\theta(\pi))^{G'_{y,r^+}} \neq 0$. Hence $\text{depth}(\theta(\pi)) \leq \text{depth}(\pi)$. Since the roles of π and $\theta(\pi)$ are symmetric, we have $\text{depth}(\pi) \leq \text{depth}(\theta(\pi))$ as well which proves the “depth preservation”.

We remark that in proving his result [29, Theorem 5.5], Pan uses the fact that an irreducible representation of a classical group of positive depth has an unrefined minimal K -types of the form $(G_{\mathcal{L},r}, \zeta)$ where ζ is a character of $G_{\mathcal{L},r,r^+}$ and \mathcal{L} is some regular small admissible lattice chain. See [29, Proposition 3.4]. By the result in this appendix, this could be circumvented and we could replace “a regular small admissible lattice chain \mathcal{L} in V ” by “a rational point in the building of $U(V)$ ” or simply “a point in the building of $U(V)$ ” (since unrefined minimal K -type always could be achieved at a rational point) in the statement of [29, Theorem 5.5].

BIBLIOGRAPHY

- [1] J. D. ADLER, S. DEBACKER, Some applications of Bruhat-Tits theory to harmonic analysis on the lie algebra of a reductive p -adic group, *Michigan Math. J.* **50** (2002), 263–286.
- [2] A.-M. AUBERT, Conservation de la ramification modérée par la correspondance de howe, *Bull. Soc. Math. France* **117** (1989), 297–303.
- [3] P. BROUSSOUS, S. STEVENS, Buildings of classical groups and centralizers of Lie algebra elements, *Journal of Lie Theory* **19** (2009), 55–78.
- [4] F. BRUHAT, J. TITS, Schémas en groupes et immeubles des groupes classiques sur un corps local, II. Groupes unitaires, *Bull. Math. Soc. France* **115** (1987), 141–195.
- [5] C. J. BUSHNELL, P. C. KUTZKO, *The admissible dual of $GL(N)$ via compact open subgroups*, **129**, Princeton Univ. Press, 1993.
- [6] S. DEBACKER, Parametrizing nilpotent orbits via Bruhat-Tits theory, *Ann. of math.* **156** (2002), 295–332.
- [7] S. DEBACKER, M. REEDER, Depth-zero supercuspidal L -packets and their stability, *Ann. of math.* **169** (2009), 795–901.
- [8] J. DIEUDONNÉ, *La géométrie des groupes classiques*, Springer, 1963.
- [9] W. T. GAN, J.-L. KIM, Tame types of nonlinear covering groups, in preparation.

- [10] P. GÉRARDIN, Weil representations associated to finite fields, *J. of Algebra* **46** (1977), 54–101.
- [11] J. HAKIM, F. MURNAGHAN, Distinguished tame supercuspidal representations, *Int. Math. Res. Pap.* **2008** (2008).
- [12] T. K. HOWARD, M. H. WEISSMAN, Depth-zero representations of nonlinear covers of p -adic groups, *Int. Math. Res. Not.* **21** (2009), 3979–3995.
- [13] R. HOWE, θ -series and invariant theory, in *Automorphic Forms, Representations and L-functions*, Proc. Symp. Pure Math **33**, 1979, 275–285.
- [14] R. HOWE, Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond, in *The Schur lectures* (I. Piatetski-Shapiro et al., eds.), Isr. Math. Conf. Proc. **8**, Ramat-Gan: Bar-Ilan University, 1995, 1–182.
- [15] R. HOWE, Tame ramified supercuspidal representations of \mathfrak{gl}_n , *Pacific J. Math.* **73** (73), 437–460.
- [16] R. HOWE, A. MOY, *Harish-Chandra homomorphisms for p -adic groups*, CBMS Regional Conference Series in Mathematics **59**, Amer. Math. Soc., 1985.
- [17] J.-L. KIM, Supercuspidal representations: An exhaustion theorem, *J. Amer. Math. Soc.* **20** (2007), pp. 273–320.
- [18] J.-L. KIM, F. MURNAGHAN, Character expansions and unrefined minimal k -types, *Amer. J. Math.* **125** (2003), pp. 1199–1234.
- [19] J.-L. KIM, F. MURNAGHAN, k -types and Γ -asymptotic expansions, *J. reine angew. Math.* **592** (2006), 189–236.
- [20] J.-S. LI, Singular unitary representations of classical groups, *Invent. math.* **97** (1989), 237–255.
- [21] J.-S. LI, Minimal representations and reductive dual pairs, in *Proceedings of Representation theory of Lie groups, Park City, UT, 1998*, IAS/Park City Math. Ser. **8**, Amer. Math. Soc., 2000, 293–340.
- [22] H. Y. LOKE, J.-J. MA, G. SAVIN, Local theta correspondences between epipelagic supercuspidal representations, *Math. Z.* **283**, 169–196.
- [23] C. MÆGLIN, M.-F. VIGNÉRAS, J.-L. WALDSPURGER, *Correspondances de howe sur un corps p -adique*, Lecture Notes in Math. **1291**, Springer.
- [24] A. MOY, G. PRASAD, Unrefined minimal k -types for p -adic groups, *Invent. math.* **116** (1994), 393–408.
- [25] A. MOY, G. PRASAD, Jacquet functors and unrefined minimal k -types, *Comment. Math. Helvetici* **71** (1996), 98–121.
- [26] S.-Y. PAN, Splittings of the metaplectic covers of some reductive dual pairs, *Pacific J. Math.* **199** (2001), 163–226.
- [27] S.-Y. PAN, Depth preservation in local theta correspondence, *Duke Math. J.* **113** (2002), 531–592.
- [28] S.-Y. PAN, Local theta correspondence of depth zero representations and theta dichotomy, *J. Math. Soc. Japan* **54** (2002), 793–845.
- [29] S.-Y. PAN, Local theta correspondence and minimal k -types of positive depth, *Israel J. Math.* **138** (2003), 317–352.

- [30] S.-Y. PAN, Supercuspidal representations and preservation principle of theta correspondence, *J. reine angew. Math.* (2016).
- [31] S.-Y. PAN, Supercuspidal representations and theta correspondence, preprint <http://ir.lib.nthu.edu.tw/handle/987654321/52348>.
- [32] S. STEVENS, Intertwining and supercuspidal types for p -adic classical groups, *Proc. London Math. Soc.* **83** (2001), 120–140.
- [33] S. STEVENS, Semisimple characters for p -adic classical groups, *Duke Math. J.* **127** (2005), 123–173.
- [34] S. STEVENS, The supercuspidal representations of p -adic classical groups, *Invent. math.* **172** (2008), 289–352.
- [35] B. SUN, C. ZHU, Conservation relations for local theta correspondence, *J. Amer. Math. Soc.* **28** (2015), 939–983.
- [36] J.-L. WALDSPURGER, Démonstration d’une conjecture de dualité de Howe dans le cas p -adique, $p \neq 2$, *Israel Math. Conf. Proc.* **2** (1990), 267–324.
- [37] J.-K. YU, Construction of tame supercuspidal representations, *J. Amer. Math. Soc.* **14** (2001), 579–622.
- [38] J.-K. YU, Bruhat-Tits theory and buildings, in *Ottawa lectures on admissible representations of reductive p -adic groups*, **26**, Amer. Math. Soc., 2009.

(Manuscrit reçu le 28 décembre 2015 ;
 accepté, après révision, le 17 mars 2017.)

Hung Yean LOKE
 Department of Mathematics
 National University of Singapore
 2 Science Drive 2, Singapore 117543
 E-mail: mat1hy@nus.edu.sg

Jia-Jun MA
 School of Mathematical Sciences
 Shanghai Jiao Tong University
 800 Dongchuan RD
 Shanghai, China 200240
 E-mail: hoxide@sjtu.edu.cn

