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SPLITTING PROPERTIES OF HIGH SUBGROUPS;

BY

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In this paper we continue the investigation of high subgroups of Abelian groups. (*See* [3], [4] and [5].) All groups considered here will be Abelian. If G is a group, G^1 denotes the subgroup of elements of infinite height in $G\left(G^1=\bigcap_{n=1}^{\infty}(nG)\right)$, and G_t denotes the torsion subgroup of G. A *high* subgroup of G is any subgroup of G maximal disjoint from G^1. A group G *splits* if G_t is a summand of G. In general, we adopt the notation used in [1].

One of the fundamental problems in Abelian group theory is to find decent necessary and sufficient conditions for a group to split. We investigate here the relation between the splitting of a group G and the splitting of high subgroups of G.

Our main result states (theorem 2) that a reduced group G splits if and only if G/G_t is reduced and some high subgroup of G splits. In a sense, this reduces the splitting problem for arbitrary groups to groups with no elements of infinite height.

LEMMA. — *If H is a high subgroup of G, then H/H_t is a summand of G/H_t.*

PROOF. — By [5], H_t is high in G_t and G_t/H_t is divisible. Since a divisible subgroup is an absolute direct summand,

$$G/H_t = G_t/H_t \oplus R/H_t$$

with $H \subseteq R$. Let

$$H/H_t = D/H_t \oplus F/H_t,$$

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where D/H_t is divisible and F/H_t is reduced. Now we may write

$$R/H_t = E/H_t \oplus D/H_t \oplus S/H_t,$$

where $E/H_t \oplus D/H_t$ is divisible, S/H_t is reduced, and $F \subseteq S$. Now

$$D + F = H \subseteq D + S.$$

Assume $(d + s) \in G^1$, $d \in D$, $s \in S$. Then

$$(d + s) + H_t \in ((D + S)/H_t)^1 = (D/H_t)^1 \oplus (S/H_t)^1 = D/H_t,$$

since S/H_t is a reduced subgroup of the torsion free group R/H_t. Hence $(d + s) \in D \cap G^1 \subseteq H \cap G^1 = 0$. Since H is high in G, we get $H = D + S$, and it follows that $S = F$. Therefore

$$H/H_t = D/H_t \oplus S/H_t,$$

and so

$$G/H_t = G_t/H_t \oplus E/H_t \oplus H/H_t.$$

Theorem 1. — *Let H be a high subgroup of G, and suppose $H = H_t \oplus L$. Then $G = M \oplus L$, where M/G_t is the divisible part of G/G_t.*

Proof. — From the lemma, we have that

$$G/H_t = G_t/H_t \oplus E/H_t \oplus H/H_t,$$

with $G_t/H_t \oplus E/H_t$ divisible. Let $M = G_t + E$. Then

$$G/H_t = M/H_t \oplus (H_t \oplus L)/H_t,$$

and hence $G = M \oplus L$ by [6], lemma 6. Now

$$G/G_t = M/G_t \oplus (L \oplus G_t)/G_t.$$

Since $M/H_t = G_t/H_t \oplus E/H_t$ is divisible, M/G_t is divisible.

But $(L \oplus G_t)/G_t \cong L$ is reduced, so that M/G_t is the divisible part of G/G_t.

It is interesting to note that M is the only summand of G complementary to L. In fact, if

$$G = N \oplus L = M \oplus L,$$

then $G_t \subseteq N$, and

$$G/G_t = N/G_t \oplus (L \oplus G_t)/G_t = M/G_t \oplus (L \oplus G_t)/G_t.$$

Since M/G_t is the divisible part of G/G_t, N/G_t must also be the divisible part of G/G_t, and hence $N = M$.

In trying to determine necessary and sufficient conditions for a group G to split, one may as well assume G is reduced. If G is reduced, then a necessary condition for G to split is that G/G_t be reduced. Examples are easy to find which show that this condition is not sufficient. Howewer, from the provious theorem we obtain the following interesting necessary and sufficient condition that a reduced group split.

THEOREM 2. — *Let G be reduced. Then G splits if and only if G/G_t is reduced and some high subgroup of G splits.*

PROOF. — Suppose G splits. Let $G = G_t \oplus L$, and let H_t be a high subgroup of G_t. Now L is torsion free reduced, so that $H_t \oplus L$ has no elements of infinite height in G. It follows readily that $H_t \oplus L$ is high is G, and hence that G has high subgroup that splits. Since $G/G_t \cong L$, G/G_t is reduced.

Suppose G/G_t is reduced and that G has a high subgroup H that splits. Let $H = H_t \oplus L$. From the previous theorem we have $G = M \oplus L$, where M/G_t is the divisible part of G/G_t. But G/G_t is reduced, and hence $M = G_t$. Therefore $G = G_t \oplus L$ and G splits.

In theorem 2 the hypothesis that G/G_t be reduced is required, and we now give an example to show this. Suppose G is a reduced group with the properties :

(1) $G/G_t \cong Q$, the group of rational numbers;

(2) G^1 is torsion free and not zero.

Let H be a high subgroup of G. Then since H_t is high in G_t and $(G_t)^1 = 0$, $H_t = G_t$. Thus, from the lemma, H/G_t is a summand of G/G_t. This means $H/G_t = 0$ since $H \neq G$ and G/G_t is indecomposable. Thus $H = G_t$ is the only high subgroup of G. G does not split since G is reduced and G/G_t is divisible, but G_t, the only high subgroup of G, does split trivially.

To demonstrate the existence of such a group we employ homological methods and results. Let $\overline{B}$ be an unbounded closed p-group. (*See* [1], p. 114.) Let B be a basic subgroup of $\overline{B}$. We may write $\overline{B}/B = \sum (C_\alpha/B)$, where $C_\alpha/B \cong Z(p^\infty)$ for all α. Let $C/B = \sum_{\alpha \neq \alpha_0} (C_\alpha/B)$ Then

$$\overline{B}/B = C/B \oplus C_{\alpha_0}/B,$$

so that C is pure in $\overline{B}$ and $\overline{B}/C \cong Z(p^\infty)$. Let Q denote the group of rational numbers and Z the group of integers. $G = \operatorname{Ext}(Q/Z, C)$ is a reduced group such that $G_t \cong C$ and G/G_t is divisible and not 0. (*See* [2], p. 370-376.) Since C is a p-group, $\operatorname{Ext}(Z(q^\infty), C) = 0$ for all primes $q \neq p$. Thus if P = set of all primes,

$$G = \operatorname{Ext}(Q/Z, C) = \operatorname{Ext}\left(\sum_{q \in P} Z(q^\infty),\ C\right)$$

$$\cong \prod_{q \in P} \operatorname{Ext}(Z(q^\infty), C) \cong \operatorname{Ext}(Z(p^\infty), C).$$

The pure exact sequence $0 \to C \to \overline{B} \to Z(p^\infty) \to 0$ yields the exact sequence

$$\mathrm{Hom}\big(Z(p^\infty), \overline{B}\big) \to \mathrm{Hom}\,(Z(p^\infty), Z(p^\infty)) \\ \to \mathrm{Pext}\,(Z(p^\infty), C) \to \mathrm{Pext}\big(Z(p^\infty), \overline{B}\big).$$

But $\mathrm{Hom}\big(Z(p^\infty), \overline{B}\big) = 0$ since $\overline{B}$ is reduced and $Z(p^\infty)$ is divisible, and $\mathrm{Pext}\big(Z(p^\infty), \overline{B}\big) = 0$ since $\overline{B}$ is a closed group and $Z(p^\infty)$ is torsion. (*See* [1], p. 117.) Thus from the exactness of the above sequence,

$$\mathrm{Pext}\,(Z(p^\infty), C) \cong \mathrm{Hom}\,(Z(p^\infty), Z(p^\infty)).$$

This is the group of p-adic integers, (*see* [1], p. 211), so is torsion free and not zero. Moreover Pext $(Z(p^\infty), C) = G^1$. (*See* [1], p. 246.) Thus we have a group G such that G/G_t is divisible and not zero and G^1 is torsion free and not zero.

Now let g be a non-zero element of infinite height in G. Since G/G_t is torsion free divisible we may write

$$G/G_t = A/G_t \oplus B/G_t$$

where $g \in A$ and $A/G_t \cong Q$. A is pure in G so $A^1 = A \cap G^1 \neq 0$. The group A has the desired properties.

It is not known whether or not high subgroups of torsion groups T are endomorphic images of T. The group $G = \mathrm{Ext}(Q/Z, C)$ constructed above is an example of a mixed group such that none of its high subgroups are endomorphic images. This group is cotorsion, i. e., $\mathrm{Ext}(X, G) = 0$ for all torsion free groups X and G is reduced. (*See* [2].) A homomorphic image of a cotorsion group is the direct sum of a cotorsion group and a divisible group. (*See* [7].) Thus if a high subgroup of G is an endomorphic image, it must be cotorsion. Any high subgroup H of G contains G_t and is pure so that G/H is torsion free divisible, and not zero. Let Q denote the group of rational numbers. Then the exact sequence

$$0 \to H \to G \to G/H \to 0$$

yields the exact sequence

$$0 \to \mathrm{Hom}\,(Q, G/H) \to \mathrm{Ext}\,(Q, H) \to 0,$$

G being cotorsion. But Hom $(Q, G/H) \neq 0$, and hence H is not cotorsion. Thus H is not an endomorphic image of G.

It would be interesting to know the class of groups whose high subgroups are endomorphic images. Torsion groups whose high subgroups have this property include those torsion groups whose high subgroups are direct sums of cyclic groups. (*See* [5].) If every high subgroup of a group G is a direct sum of cyclic groups, is every high subgroup of G an endomorphic image of G?

We proceed now to discuss high subgroups in reduced groups G that split. First we give a characterization of the high subgroups of such groups.

THEOREM 3. — *Let $G = G_t \oplus S$, with G reduced. Then there is a one-to-one correspondence between the set of all high subgroups of G and the set $\bigcup_{\beta \in I} \mathrm{Hom}(S, G_t/K_\beta)$, where $\{K_\beta\}_{\beta \in I}$ is the set of all high subgroups of G_t.*

PROOF. — Let K_β be high in G_t, and let $\alpha \in \mathrm{Hom}(S, G_t/K_\beta)$. Then α induces an isomorphism δ from $S/\mathrm{Ker}(\alpha)$ onto T/K_β, where $K_\beta \subseteq T \subseteq G_t$. Let

$$K = \{ t + s \mid (s + \mathrm{Ker}(\alpha))\, \delta = t + K_\beta \}.$$

We show that K is high in G. Since G is reduced, L is torsion free reduced, and hence $G^1 = (G_t)^1$. Since $K \cap G_t = K_\beta$, we have that $K \cap G^1 = 0$. Suppose $g \notin K$. Then $g = g_t + s$, $g_t \in G_t$, $s \in S$, and $g_t \notin K$. There exists $t \in T$ such that $(t + s) \in \mathrm{K}$. Hence $(g_t + s) - (t + s) = (g_t - t) \notin K$. Since K_β is high in G_t,

$$0 \neq \langle g_t - t, K_\beta \rangle \cap G^1 \subseteq \langle g, K \rangle \cap G^1,$$

whence K is high in G. Clearly distinct $\alpha' s$ in $\bigcup_{\beta \in I} \mathrm{Hom}(S, G_t/K_\beta)$ give rise to distinct high subgroups.

Now let K be high in $G = G_t \oplus S$. Then K_t is high in G_t. (*See* [5].) Suppose $s \in S$, $s \notin K$. Then

$$0 \neq ns + k = g_1 \in G^1 = (G_t)^1$$

for some integer n and $k \in K$. Since $g_1 \in G^1$, $g_1 = ng$, $g \in G_t$, and from the purity of K (*see* [5]), we have $k = nk_1$ with $k_1 \in K$. Thus

$$s + k_1 - g = g_t \in G_t.$$

So

$$s + (g - g_t) = -k_1 \in K.$$

Hence the group of S components of the elements of K is S. Let T be the group of G_t components of the elements of K. Then $K_t \subseteq T \subseteq G_t$, K is a subdirect sum of T and S, and

$$S/(K \cap S) \cong T/(K \cap T) = T/K_t.$$

The theorem follows.

A natural question to ask is the following. If one high subgroup of a group G splits, do all high subgroups of G split? The answer is negative, even if G itself splits, as the following example shows.

EXAMPLE. — Let G_t be any reduced p-group with non-zero elements of infinite height. (e. g., let G_t be the Prüfer group. *See* [1], p. 105.) Let S be the group of rational numbers with denominators powers of p, and let $G = G_t \oplus S$. Let H_t be high in G_t. Since G_t/H_t is divisible there exists a subgroup T of G_t such that $T/H_t \cong Z(p^\infty)$. Let R be any subgroup of S such that $S/R \cong Z(p^\infty)$. (Let R be the integers for example.) Let δ be an isomorphism from T/H_t onto S/R. From the proof of theorem 3, we have that $H = \{ t + s \mid (t + H_t)\,\delta = s + R \}$ is a high subgroup of G, and H_t is the torsion subgroup of H. Suppose H splits and $H = H_t \oplus V$. Now V is a subdirect sum of T_1 and S (*see* proof of theorem 3), where $H_t \subseteq T_1 \subseteq T$. Furthermore $T_1/(T_1 \cap V) \cong S/(S \cap V)$. But V is torsion free so that $T_1 \cap V = 0$. Hence T_1 is a homomorphic image of S. But T_1 is reduced and the only reduced p-group that is a homomorphic image of S is 0. Therefore $T_1 = 0$, and so $0 = H_t \subseteq T_1$. But no high subgroup of T is 0. This contradiction establishes that H does not split.

However, G does have a high subgroup that splits, namely $H_t \oplus S$. (*See* theorem 3.) In particular, we have that *two high subgroups of a group are not necessarily isomorphic.* It is still not known whether or not two high subgroups of a torsion group are isomorphic.

Although two high subgroups H and K of a group G are not necessarily isomorphic, and in fact H may split and K not split, it is true that high subgroups do provide several invariants, in the following sense.

THEOREM 4. — *Let H and K be high subgroups of G. Then*

(a) $$G/H \cong G/K;$$

(b) $$H/H_t \cong K/K_t;$$

(c) $$G/H_t \cong G/K_t.$$

PROOF. — The proof of (a) may be found in [5].

If A is a subgroup of G, let $\tilde{A} = (A + G_t)/G_t$. Then $\tilde{H}$ is maximal disjoint from $\tilde{G}^1$ in $\tilde{G}$. To see that $\tilde{H} \cap \tilde{G}^1 = 0$, suppose that $h + G_t = g_1 + G_t$ with $h \in H$, $g_1 \in G^1$. Then from $h - g_1 \in G_t$, it follows that for some integer m, $mh = mg_1 = 0$, whence $h + G_t = 0$. Next suppose there exists $g + G_t \notin \tilde{H}$, and that $\langle g + G_t, \tilde{H} \cap \tilde{G}^1 \rangle = 0$. Since H is high in G, there exists $h \in H$ and an integer m such that $0 \neq h + mg = g_1 \in G^1$. There exists $h_1 \in H$, $g_2 \in G$ such that $mh_1 + mg = mg_2$. Now $g_1 \in G_t$, and hence $g_2 \in G_t$. Thus

$$h_1 + g - g_2 = g_t \in G_t,$$

and so

$$g + G_t = -h_1 + (g_2 + g_t) + G_t = -h_1 + G_t \in \tilde{H}.$$

But $g + G_t \notin \tilde{H}$. We conclude that $\tilde{H}$ is maximal disjoint from $\tilde{G}^1$. Since $\tilde{G}^1 \subseteq (\tilde{G})^1$ which is divisible, $\tilde{G}$ contains a (unique) minimal divisible subgroup $\tilde{D}$ which contains $\tilde{G}^1$ and $\tilde{H} \cap \tilde{D} = 0$. (*See* [3].) But $\tilde{D}$ is an absolute summand so that $\tilde{G} = \tilde{H} \oplus \tilde{D}$, for any high subgroup H of G. Thus $\tilde{H} \cong \tilde{K}$. Finally,

$$H/H_t = H/(H \cap G_t) \cong (H + G_t)/G_t = \tilde{H} \cong \tilde{K} \cong K/K_t,$$

and (b) is proved.

To prove (c), first notice that

$$G/H_t = G_t/H_t \oplus R/H_t, \qquad \text{and} \qquad G/K_t \cong G_t \oplus S/H_t.$$

Since H_t and K_t are high in G_t (*see* [5]), by (a), $G_t/H_t \cong G_t/K_t$. Also $R/H_t \cong G/G_t \cong S/K_t$. Hence $G/H_t \cong G/K_t$ as stated.

We remark that in the case where H/H_t is reduced for some high subgroup H of G, and in particular when some high subgroup of G splits, that all high subgroups of G may be obtained as follows. Let K_t be high in G_t and K/K_t high in G/K_t. Then K is high in G and every high subgroup of G with torsion subgroup K_t is such a K.

In [5] the notion of Σ-group was introduced. A group G is a *Σ-group* if and only if each high subgroup of G is a direct sum of cyclic groups. As a corollary to the preceding theorem we obtain the following result, proved in the torsion case in [5] and in the general case by Paul Hill (Abstract 582-47, *Notices of the American Mathematical Society*, August 1961).

Corollary. — *If one high subgroup of a group G is a direct sum of cyclic groups, then G is a Σ-group, and any two high subgroups of G are isomorphic.*

Proof. — Let H and K be high subgroups of G with H a direct sum of cyclic groups. Since H_t and K_t are high in G_t, we have by theorem 7 in [5], that K_t is a direct sum of cyclic groups and $H_t \cong K_t$. But $H/H_t \cong K/K_t$ is free so that K is a direct sum of cyclic groups aud $H \cong K$.

Clearly Σ-groups form a class of groups in which all high subgroups split and are isomorphic. However the class of such groups properly contains the class of all Σ-groups. In fact, let P be the Prüfer group for the prime p. (*See* [1], p. 105.) Let S be the group of rational numbers with denominators a power of the prime q, $q \neq p$. Then every high subgroup of $G = P \oplus S$ splits and any two high subgroups of G are isomorphic. We merely outline the proof of this fact. Let H be high in G, and let P_H be the group of P components of the elements of H. Then H is a subdirect sum of P_H and S, and

$$P_H/(H \cap P) = P_H/H_t \cong S/(H \cap S).$$

Thus $S/(H \cap S)$ is a p-group, and if n is the smallest positive integer such that $1/q^n \notin H \cap S$, then

$$S/(H \cap S) = \langle 1/q^n + H \cap S \rangle \cong P_H/H_t$$

is finite cyclic. Since H_t is pure in P_H, we have that

$$P_H = H_t \oplus R.$$

Thus

$$H \subseteq H_t \oplus R \oplus S, \qquad H = H_t \oplus (H \cap (R \oplus S)),$$

and so H splits. Since $H/H_t \cong S$, we have $H \cong H_t \oplus S$. Any subgroup high in P is a basic subgroup of P (*see* [1], p. 98), whence any two high subgroups of P are isomorphic. It follows that every high subgroup of G is isomorphic to $H_t \oplus S$.

It would be interesting to know the class of groups in which all high subgroups split, the class of groups in which all high subgroups are isomorphic, and the intersection of these two classes.

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