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## FORMAL MODULI FOR ONE-PARAMETER FORMAL LIE GROUPS

BY

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In this paper we study formal Lie groups using methods introduced by LAZARD [2]. This material was exposed in a preliminary form in a seminar at the Woods Hole Institute on Algebraic Geometry in July 1964. All formal groups discussed here are commutative formal Lie groups on *one* parameter, which we will frequently refer to as "group laws". The reader is referred to [2] and [3] for all basic definitions.

Suppose that  $\mathfrak{o}$  is a complete noetherian local ring with maximal ideal  $\mathfrak{m}$  and residue field  $k = \mathfrak{o}/\mathfrak{m}$  of characteristic  $p > 0$ . If  $f$  is a power series with coefficients in  $\mathfrak{o}$ , let us call  $f^*$  the power series over  $k$  whose coefficients are those of  $f$ , reduced modulo  $\mathfrak{m}$ . Let us say that two group laws, i. e. one-parameter formal Lie groups,  $F$  and  $G$ , over  $\mathfrak{o}$ , are  $\star$ -isomorphic if  $F^* = G^*$  and there is an  $\mathfrak{o}$ -isomorphism  $\varphi$  between  $F$  and  $G$  such that  $\varphi^*(x) = x$ . We shall show that if  $\Phi$  is a group law of height  $h < \infty$  over  $k$ , the set  $\mathcal{G}_{\mathfrak{o}}(\Phi)$  of  $\star$ -isomorphism classes of group laws  $F$  over  $\mathfrak{o}$  such that  $F^* = \Phi$  can be put into one-to-one correspondence with the (set-theoretic) product of  $\mathfrak{m}$  with itself  $(h-1)$  times, in a way that is compatible with extension of the ring  $\mathfrak{o}$ .

### 1. Generic group laws of height $h$ .

We give here a construction of a group law  $\Gamma$  which will turn out to be (theorem 3.1) a generic lifting of a given group law  $\Phi$  of height  $h$ . We recall that if  $F(x, y)$  is an abelian  $(r-1)$ -bud over a ring  $R$ , i. e. a

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polynomial that behaves modulo degree  $r$  like a group law over  $R$  (see [2], p. 255) then there is an abelian  $r$ -bud  $F'$  defined over  $R$  such that  $F \equiv F' \pmod{\deg r}$ ; and if  $F''$  is another such  $r$ -bud, then  $F' \equiv F'' + aC_r \pmod{\deg(r+1)}$  for some  $a \in R$ , where  $C_r$  is the modified binomial form, see [2], definition 2.5 or [3], definition 3.2.1. We point out that if  $\Phi$  is a group law defined over a field  $k$  of characteristic  $p \neq 0$  and if  $\Phi$  is of height  $h < \infty$ , then there is  $\Phi'$  isomorphic to  $\Phi$  over  $k$  such that

$$\Phi'(x, y) \equiv x + y + aC_q(x, y) \pmod{\deg(q+1)}$$

where  $q = p^h$  and  $a$  is a non-zero element of  $k$ . This can be proved directly from [2], lemma 6 or by applying [3], lemma 3.2.2 to any group law  $F$  defined over an appropriate discrete valuation ring  $\mathfrak{o}$  with residue field  $k$ , such that  $F^* = \Phi$ .

PROPOSITION 1.1. — *Let  $k$  be a field of characteristic  $p \neq 0$ , and let  $\Phi(x, y) \in k[[x, y]]$  be a group law of height  $h < \infty$ , with  $\Phi(x, y) \equiv x + y \pmod{\deg p^h}$ . Let  $R$  be a ring with maximal ideal  $I$ , such that  $R/I \cong k$ , and let  $R[[t]] = R[[t_1, \dots, t_{h-1}]]$  be the ring of formal power series in  $h-1$  letters  $t_1, \dots, t_{h-1}$  over  $R$ . Then there is a group law  $\Gamma(t_1, \dots, t_{h-1})(x, y)$  defined over  $R[[t_1, \dots, t_{h-1}]]$  such that :*

1.  $\Gamma(\mathfrak{o}, \dots, \mathfrak{o})^*(x, y) = \Phi(x, y)$ ,
2. For each  $i (1 \leq i \leq h-1)$ ,

$$\Gamma(\mathfrak{o}, \dots, \mathfrak{o}, t_i, \dots, t_{h-1})(x, y) \equiv x + y + t_i C_{p^i}(x, y) \pmod{\deg(p^i+1)}.$$

*Proof.* — We start with the abelian 1-bud  $x + y$  defined over  $R[[t]]$  and complete it to a group law with the desired properties. Suppose for  $r > 1$  that we have an abelian  $(r-1)$ -bud  $\Gamma_{r-1}(t_1, \dots, t_{h-1})$  such that :

1.  $\Gamma_{r-1}(\mathfrak{o}, \dots, \mathfrak{o})^*(x, y) \equiv \Phi(x, y) \pmod{\deg r}$ ,
2. For each  $i$ ,

$$\begin{aligned} &\Gamma_{r-1}(\mathfrak{o}, \dots, \mathfrak{o}, t_i, \dots, t_{h-1})(x, y) \\ &\equiv x + y + t_i C_{p^i}(x, y) \pmod{\deg(\min(r, p^i+1))}. \end{aligned}$$

Now let  $\Gamma'_r$  be any abelian  $r$ -bud defined over  $R[[t]]$  such that  $\Gamma'_r \equiv \Gamma_{r-1} \pmod{\deg r}$ .

CASE 1 :  $r > p^{h-1}$ . — Then

$$\Gamma'_r(\mathfrak{o}, \dots, \mathfrak{o})^*(x, y) \equiv \Phi(x, y) + a^*C_r(x, y) \pmod{\deg(r+1)}$$

for some  $a \in R$ , by [2], proposition 2, and so we set  $\Gamma_r = \Gamma'_r - aC_r$ .

CASE 2 :  $p^{j-1} < r \leq p^j$  for some  $j \leq h-1$ . — Then our hypotheses on  $\Gamma_{r-1}$  imply that

$$\Gamma'_r(o, \dots, o, t_j, \dots, t_{h-1})(x, y) \equiv x + y + bC_r(x, y) \pmod{\deg(r+1)} \quad \text{for } b \in R[[t_j, \dots, t_{h-1}]]$$

and in this case we let  $\Gamma_r = \Gamma'_r - bC_r$  if  $r \neq p^j$  and  $\Gamma_r = \Gamma'_r + (t_j - b)C_r$  if  $r = p^j$ .

In either case,  $\Gamma_r$  is an abelian  $r$ -bud congruent to  $\Gamma_{r-1} \pmod{\deg r}$  such that :

1.  $\Gamma_r(o, \dots, o)^*(x, y) \equiv \Phi(x, y) \pmod{\deg(r+1)}$ ,
2. For each  $i$ ,

$$\Gamma_r(o, \dots, o, t_i, \dots, t_{h-1})(x, y) \equiv x + y + t_i C_{p^i}(x, y) \pmod{\deg(\min(r+1, p^i+1))}.$$

Then if we let  $\Gamma = \lim \Gamma_r$ , we see that  $\Gamma$  has the desired properties.

## 2. The 2-cohomology group of a formal group.

DEFINITION 2.1. — Let  $R$  be a ring and  $M$  an  $R$ -module. We denote by  $M[[x_1, \dots, x_n]]$  the module  $M \hat{\otimes}_R R[[x_1, \dots, x_n]]$ .

By this we mean the completion of  $M \otimes_R R[[x_1, \dots, x_n]]$  with respect to the family of submodules  $M \otimes_R J^r$ , where  $J$  is the ideal  $(x_1, \dots, x_n)$  of  $R[[x_1, \dots, x_n]]$ . An element of  $M[[x_1, \dots, x_n]]$  can be represented as  $\sum \alpha_\mu \mu$ , where  $\mu$  runs through all the monomials in the  $x$ 's, and each  $\alpha_\mu$  belongs to  $M$ .

It should be observed that  $M[[x_1, \dots, x_n]]$  is not only an  $R[[x_1, \dots, x_n]]$ -module, but also has a substitution operation : if  $f(x_1, \dots, x_n) \in M[[x_1, \dots, x_n]]$  and if  $g_1, \dots, g_n \in R[[y_1, \dots, y_m]]$  are such that  $g_i(o, o, \dots, o) = o$  for each  $i$ , then  $f(g_1, \dots, g_n) \in M[[y_1, \dots, y_m]]$ .

DEFINITION 2.2. — Let  $F(x, y) \in R[[x, y]]$  be a group law and  $M$  be an  $R$ -module. If  $f \in M[[x]]$ , then  $\partial_F f \in M[[x, y]]$  is defined by

$$(\partial_F f)(x, y) = f(y) - f(F(x, y)) + f(x).$$

If  $f \in M[[x, y]]$ , then  $\partial_F f \in M[[x, y, z]]$  is defined by

$$(\partial_F f)(x, y, z) = f(y, z) - f(F(x, y), z) + f(x, F(y, z)) - f(x, y).$$

Also,  $B_M^2(F)$  is the set of all  $f \in M[[x, y]]$  such that  $f = \partial g$  for some  $g \in M[[x]]$  and  $Z_M^2(F)$  is the set of all  $f \in M[[x, y]]$  such that  $f(x, y) = f(y, x)$  and such that  $\partial f = o$ . Since  $B_M^2(F) \subset Z_M^2(F)$ , we can define  $H_M^2(F)$  as  $Z_M^2(F)/B_M^2(F)$ . Elements of  $B^2$  and  $Z^2$  are called coboundaries and cocycles respectively.

2.3. — In case  $F$  is defined over a field  $k$  and  $M$  is a finite-dimensional  $k$ -vector space,  $M[[x_1, \dots, x_n]]$  is canonically isomorphic to  $M \otimes_k k[[x_1, \dots, x_n]]$ . Also,  $Z_M^2(F) \cong M \otimes_k Z_k^2(F)$ , and similarly for  $B_M^2(F)$  and  $H_M^2(F)$ .

Suppose  $f(x, y) \in Z_M^2(F)$  and  $f(x, y) \equiv 0 \pmod{\deg r}$ . Then

$$\begin{aligned} 0 = (\partial f)(x, y, z) &\equiv f(y, z) - f(x + y, z) \\ &+ f(x, y + z) - f(x, y) \pmod{\deg(r + 1)} \end{aligned}$$

so that by [2], lemma 3,  $f(x, y) \equiv a C_r(x, y) \pmod{\deg(r + 1)}$  for some  $a \in R$ . Similarly, if  $M$  is a finite-dimensional vector space over a field  $k$  over which  $F$  is defined, for each nonzero  $f(x, y) \in Z_M^2(F)$ , there is an integer  $r$  and a nonzero element  $a$  of  $M$  such that

$$f(x, y) \equiv a C_r(x, y) \pmod{\deg(r + 1)}.$$

In the next proposition, we show how the second cohomology group  $H^2$  measures the “infinitesimal deformations” of a formal group. If  $\mathfrak{o}$  is a local ring with maximal ideal  $\mathfrak{m}$  and residue field  $k = \mathfrak{o}/\mathfrak{m}$ , let us call  $\nu_r$  the canonical homomorphism of  $\mathfrak{m}^r$  onto the  $k$ -vector space  $M_r = \mathfrak{m}^r/\mathfrak{m}^{r+1}$ , and we will use the same symbol,  $\nu_r$ , for the corresponding homomorphism between the power-series modules in  $n$  variables, over  $\mathfrak{m}^r$  and  $M_r$ , respectively. We will be dealing with a group law  $\Phi(x, y) \in k[[x, y]]$ , and we will denote by  $\Phi_1$  and  $\Phi_2$  the first partial derivatives of  $\Phi$  with respect to the left- and the right-hand arguments, respectively. Observe that  $\Phi_1$  has constant term 1, so that  $\Phi_1(\mathfrak{o}, x)$  has a reciprocal in  $k[[x]]$ .

PROPOSITION 2.4. — *Let  $\mathfrak{o}$ ,  $\mathfrak{m}$ ,  $M_r$ , and  $\Phi$  be as above. Let  $F$  and  $G$  be group laws over  $\mathfrak{o}$  such that  $F^* = G^* = \Phi$ . Suppose  $\varphi(x) \in \mathfrak{o}[[x]]$  is a power series such that :*

1.  $\varphi^*(x) = x$ ,
2.  $\varphi(F(x, y)) \equiv G(\varphi x, \varphi y) \pmod{\mathfrak{m}^r}$ .

Let  $\Delta(x, y) \in M_r[[x, y]]$  be defined by

$$\Delta(x, y) = [\Phi_1(\mathfrak{o}, \Phi(x, y))]^{-1} \cdot \nu_r[\varphi(F(x, y)) - G(\varphi x, \varphi y)].$$

Then  $\Delta(x, y) \in Z_{M_r}^2(\Phi)$ . Furthermore,  $\Delta(x, y) \in B_{M_r}^2(\Phi)$  if and only if there is  $\varphi'(x) \in \mathfrak{o}[[x]]$  such that :

1.  $\varphi'(x) \equiv \varphi(x) \pmod{\mathfrak{m}^r}$ ,
2.  $\varphi'(F(x, y)) \equiv G(\varphi' x, \varphi' y) \pmod{\mathfrak{m}^{r+1}}$ .

Finally, such a  $\varphi'$  is unique modulo  $\mathfrak{m}^{r+1}$ , if  $\Phi$  is of finite height.

*Proof.* — We will use the simplifying notation  $x \star y$  for  $\Phi(x, y)$  and make use of the facts that  $\Phi_1(\mathfrak{o}, x) = \Phi_2(x, \mathfrak{o})$  and  $\Phi_1(x, y) \cdot \Phi_1(\mathfrak{o}, x) = \Phi_1(\mathfrak{o}, x \star y)$ ,

which are proved by differentiating the identities expressing the commutativity and associativity of  $\Phi$ , and then setting one of the variables equal to zero.

By abuse of notation, we can say, modulo  $\mathfrak{m}^{r+1}$ ,

$$\varphi(F(x, y)) \equiv G(\varphi x, \varphi y) + \Delta(x, y) \Phi_1(o, x \star y) \pmod{\mathfrak{m}^{r+1}}.$$

Hence, computing modulo  $\mathfrak{m}^{r+1}$  we have :

$$\begin{aligned} \varphi(F(F(x, y), z)) &\equiv G(G(\varphi x, \varphi y) + \Delta(x, y) \cdot \Phi_1(o, x \star y), \varphi z) \\ &\quad + \Delta(x \star y, z) \cdot \Phi_1(o, x \star y \star z) \\ &\equiv G(G(\varphi x, \varphi y), \varphi z) + \Phi_1(x \star y, z) \\ &\quad \times \Delta(x, y) \cdot \Phi_1(o, x \star y) + \Delta(x \star y, z) \cdot \Phi_1(o, x \star y \star z) \\ &\equiv G(G(\varphi x, \varphi y), \varphi z) + \Phi_1(o, x \star y \star z) \\ &\quad \times [\Delta(x, y) + \Delta(x \star y, z)]. \end{aligned}$$

Symmetrically,

$$\varphi(F(x, F(y, z))) \equiv G(\varphi x, G(\varphi y, \varphi z)) + \Phi_1(o, x \star y \star z) \cdot [\Delta(y, z) + \Delta(x, y \star z)].$$

Then, since both  $F$  and  $G$  are associative, we see immediately that  $\Delta \in Z_{\mathfrak{M}_r}^2(\Phi)$ .

If we have  $\varphi'(x) \in \mathfrak{o}[[x]]$  such that  $\varphi'(x) \equiv \varphi(x) \pmod{\mathfrak{m}^r}$ , let us set  $\psi(x) = \Phi_1(o, x)^{-1} \cdot \nu_r(\varphi x - \varphi' x)$ . Then, again by abuse of notation, we have, modulo  $\mathfrak{m}^{r+1}$ ,

$$\varphi(x) \equiv \varphi'(x) - \Phi_1(o, x) \psi(x),$$

and

$$\begin{aligned} \Phi_1(o, x \star y) \cdot \Delta(x, y) &\equiv \varphi'(F(x, y)) - \Phi_1(o, x \star y) \cdot \psi(x \star y) \\ &\quad - G(\varphi' x - \Phi_1(o, x) \cdot \psi(x), \varphi' y - \Phi_1(o, y) \cdot \psi(y)) \\ &\equiv \varphi'(F(x, y)) - G(\varphi' x, \varphi' y) - \Phi_1(o, x \star y) \psi(x \star y) \\ &\quad + \Phi_1(o, x) \cdot \psi(x) \cdot \Phi_1(x, y) \\ &\quad + \Phi_1(y, o) \cdot \psi(y) \cdot \Phi_1(x, y) \pmod{\mathfrak{m}^{r+1}}. \end{aligned}$$

Thus  $\Delta(x, y) = \Phi_1(o, x \star y)^{-1} \cdot \nu_r[\varphi'(F(x, y)) - G(\varphi' x, \varphi' y)] + (\delta\psi)(x, y)$ .

This shows that  $\Delta \in B_{\mathfrak{M}_r}^2(\Phi)$  is a necessary and sufficient condition for the existence of a series  $\varphi'(x)$  satisfying conditions 1 and 2 of the proposition. It remains only to prove the unicity of such a  $\varphi'$  in case  $\Phi$  is of finite height. If  $\varphi''$  is another such series, then the difference of  $\varphi'$  and  $\varphi''$  in  $\text{Hom}_{\mathfrak{o}/\mathfrak{m}^{r+1}}(F, G)$  is a homomorphism  $\rho \equiv \mathfrak{o} \pmod{\mathfrak{m}^r}$ . Such a  $\rho$  satisfies

$$\rho(F(x, y)) \equiv G(\rho x, \rho y) \equiv \rho x + \rho y \pmod{\mathfrak{m}^{r+1}}.$$

Hence the series  $h(x) = \nu_r(\rho(x))$  satisfies

$$h(\Phi(x, y)) = h(x) + h(y).$$

By iteration, this implies  $h([p](x)) = ph(x) = 0$ , where

$$[p](x) = x \star x \dots \star x$$

is the  $p$ -fold endomorphism for the group  $\Phi$ . Since  $[p](x) \neq 0$  for  $\Phi$  of finite height, we can conclude  $h = 0$ , and consequently  $\varphi' \equiv \varphi'' \pmod{\mathfrak{m}^{r+1}}$  in that case.

2.5 REMARK. — It should be noted that under the hypotheses of the preceding proposition,  $\Delta$  is congruent modulo degree  $n$  to a coboundary if and only if there is  $\varphi(x) \in \mathfrak{o}[[x]]$  such that :

1.  $\varphi'(x) \equiv \varphi(x) \pmod{\mathfrak{m}^r}$ , and
2.  $\varphi'(F(x, y)) \equiv G(\varphi'x, \varphi'y) \pmod{\mathfrak{m}^{r+1}, \text{ mod deg } n}$ .

We are now in a position to compute  $H_k^2(\Phi)$  for  $\Phi$  a group law of finite height over a field  $k$  of characteristic  $p \neq 0$  :

PROPOSITION 2.6. — *If  $\Phi$  is a group law of height  $h < \infty$ , defined over a field  $k$  of characteristic  $p \neq 0$ , then  $H_k^2(\Phi)$  is a  $k$ -vector space of dimension  $h - 1$ . If  $\Phi(x, y) \equiv x + y \pmod{\text{deg } p^h}$ , and  $\Gamma(t)(x, y)$  is any group law over  $k[[t_1, \dots, t_{h-1}]]$  satisfying the conditions of proposition 1.1 with  $R = k$ , then the functions*

$$f_i(x, y) = (\Phi_1(0, x \star y))^{-1} \frac{\partial \Gamma}{\partial t_i}(0, \dots, 0)(x, y) \quad (1 \leq i \leq h - 1),$$

are cocycles satisfying

$$f_i(x, y) \equiv C_{p^i}(x, y) \pmod{\text{deg } p^i + 1},$$

whose classes form a base for  $H_k^2(\Phi)$ .

Let  $\Phi(x, y)$  and  $\Gamma(t)(x, y)$  be as in proposition 1.1, with  $R = k$ . Apply proposition 2.4 with  $\mathfrak{o} = k[\tau]/(\tau^2)$ , with  $r = 1$ , with  $\varphi(x) = x$ , with  $G(x, y) = \Phi(x, y) = \Gamma(0, \dots, 0)(x, y)$  and with  $F(x, y) = \Gamma(0, \dots, 0, \tau, 0, \dots, 0)(x, y)$ , where the  $\tau$  is in the  $i$ -th place. Since then

$$F(x, y) = G(x, y) + \tau \frac{\partial \Gamma}{\partial t_i}(0, \dots, 0)(x, y),$$

we conclude that  $f_i(x, y)$  is a cocycle. The fact that

$$f_i(x, y) \equiv C_{p^i}(x, y) \pmod{\text{deg } p^i + 1}$$

is obvious from the definition of  $f_i$ , and using this we will now show that the classes of the  $f_i$  form a base for  $H_k^2(\Phi)$ .

For each  $j$ , let  $g_j(x) = x^j$ . Then if  $j$  is not a power of  $p$ ,

$$(\delta g_j)(x, y) \equiv B_j(x, y) \pmod{\deg(j+1)}$$

where  $B_j = \lambda C_j$  for  $\lambda$  some nonzero element of  $k$ . And if  $j = p^s$  for  $s \geq 0$ , then

$$(\delta g_j)(x, y) \equiv y^j - (\Phi(x, y))^j + x^j \equiv -\alpha^j (C_{p^h}(x, y))^j \pmod{\deg(jp^h+1)},$$

since  $\Phi(x, y) \equiv x + y + \alpha C_{p^h}(x, y) \pmod{\deg(p^h+1)}$  for some  $\alpha \neq 0$ . But  $(C_{p^j}(x, y))^{p^s} = C_{p^{js}}(x, y)$  in characteristic  $p$ , so that  $(\delta g_j)(x, y) \equiv \lambda C_{jp^h}(x, y) \pmod{\deg(jp^h+1)}$ , for  $\lambda \neq 0$ , if  $j$  is a power of  $p$ . With these facts, we can now show that if  $\psi \in Z_k^2(\Phi)$ ,  $\psi$  is equal to a linear combination of the  $f_i$ , ( $1 \leq i < h$ ), plus a coboundary.

Indeed, suppose

$$\psi \equiv \sum \lambda_i f_i + \delta \gamma_{n-1} \pmod{\deg n},$$

for  $\lambda_i \in k$  and  $\gamma_{n-1} \in k[[x]]$ . It then follows that

$$\psi \equiv \sum \lambda_i f_i + \delta \gamma_{n-1} + a C_n \pmod{\deg(n+1)},$$

for  $a \in k$ , by 2.3.

CASE 1 :  $n = p^j$  for  $j < h$ . — Then since

$$a C_n \equiv a f_j \pmod{\deg(n+1)},$$

$\psi \equiv a f_j + \sum \lambda_i f_i + \delta \gamma_{n-1}$  so that we can let  $\gamma_n = \gamma_{n-1}$ .

CASE 2 :  $n = p^j$  for  $j \geq h$ . — Let  $m = n/p^h = p^{j-h}$ . Then

$$a C_n \equiv b \delta g_m \pmod{\deg(n+1)} \text{ for some } b \in k,$$

and so we let  $\gamma_n = \gamma_{n-1} + b g_m$ .

CASE 3 :  $n$  is not a power of  $p$ . — Then

$$a C_n \equiv b \delta g_n \pmod{\deg(n+1)} \text{ for some } b \in k$$

and so we let  $\gamma_n = \gamma_{n-1} + b g_n$ .

Since  $\gamma = \lim \gamma_n$  exists in  $k[[x]]$ , we see that  $\psi$  is equal to  $\delta \gamma$  plus a linear combination of the  $f_i$ , which shows that  $H_k^2(\Phi)$  is spanned by the classes  $\xi_1, \dots, \xi_{h-1}$  of  $f_1, \dots, f_{h-1}$ . But since  $\sum \lambda_i f_i(x, y) = (\delta g)(x, y)$  is impossible unless each  $\lambda_i$  is zero, as one sees by considering the equation  $\pmod{\deg(p^i+1)}$  successively for  $i = 1, 2, \dots, h-1$ , the  $\xi_i$  are linearly independent and so form a basis for  $H_k^2(\Phi)$ .



2.7. — In the above proposition, we showed that  $\dim(H_k^2(\Phi)) \cong h - 1$  by using  $\Gamma(t)$  to find for each  $i < h$  a cocycle

$$f_i(x, y) \equiv C_{p^i}(x, y) \pmod{\deg(p^i + 1)}.$$

Such cocycles can be constructed by another method, which we outline here :

If  $f$  is a cocycle modulo degree  $r$ , then the  $r$ -degree form  $\varphi$  of  $\delta f$  is a polynomial 3-cocycle in the sense of [1], i. e.

$$\begin{aligned} \varphi(y, z, w) - \varphi(x + y, z, w) + \varphi(x, y + z, w) \\ - \varphi(x, y, z + w) + \varphi(x, y, z) = 0, \end{aligned}$$

and furthermore,  $\varphi$  is “ symmetric ” in the sense that

$$\varphi(x, y, z) - \varphi(x, z, y) + \varphi(z, x, y) = 0.$$

By [1], page 272, any such 3-cocycle is the coboundary of a symmetric form  $\psi(x, y)$  :

$$\varphi(x, y, z) = (\delta\psi)(x, y, z) = \psi(y, z) - \psi(x + y, z) + \psi(x, y + z) - \psi(x, y),$$

so that  $\delta(f - \psi) \equiv 0 \pmod{\deg(r + 1)}$ . Thus  $f$  can be completed to a cocycle in  $Z_k^2(\Phi)$ , and to construct our  $f_i$ , we start off with  $C_{p^i}(x, y)$  which is a cocycle modulo degree  $(p^i + 1)$ .

### 3. The formal moduli.

**THEOREM 3.1.** — *Let  $R, I, k, \Phi$ , and  $\Gamma$  be as in proposition 1.1. Let  $\mathfrak{o}$  be a complete noetherian local  $R$ -algebra, with maximal ideal  $\mathfrak{m}$  containing  $I\mathfrak{o}$  and residue field  $K \supset k$ . Let  $F(x, y) \in \mathfrak{o}[[x, y]]$  be a group law such that  $F^* = \Phi$ . Then there is a unique  $(h - 1)$ -tuple  $(\alpha_1, \dots, \alpha_{h-1})$  of elements of  $\mathfrak{m}$ , such that  $F$  is  $\star$ -isomorphic to  $\Gamma(\alpha)$ . Furthermore, there is only one  $\star$ -isomorphism  $\varphi : F \rightarrow \Gamma(\alpha)$ .*

*Proof.* — By induction on  $r$  we will show that the conclusion is true for the ring  $\mathfrak{o}/\mathfrak{m}^r$  : there is a unique vector  $(\alpha^{(r)})$  of elements of  $\mathfrak{m}/\mathfrak{m}^r$  such that  $F$  is  $\star$ -isomorphic modulo  $\mathfrak{m}^r$  to  $\Gamma(\alpha^{(r)})$ , and there is only one  $\star$ -isomorphism  $\varphi^{(r)} : F \rightarrow \Gamma(\alpha^{(r)})$ ,  $\varphi^{(r)} \in (\mathfrak{o}/\mathfrak{m}^r)[[x]]$ . Uniqueness then implies immediately that  $(\alpha) = \lim (\alpha^{(r)})$  and  $\varphi = \lim \varphi^{(r)}$  exist and are unique, so that the conclusion is true for the ring  $\mathfrak{o}$ .

For  $r = 1$  there is nothing to be proved. Suppose now that we have  $(\alpha) \in (\mathfrak{m})^{h-1}$  and  $\varphi \in \mathfrak{o}[[x]]$  such that

$$\varphi^*(x) = x \quad \text{and} \quad \varphi(F(x, y)) \equiv \Gamma(\alpha)(\varphi x, \varphi y) \pmod{\mathfrak{m}^r},$$

and that such  $(\alpha)$  and  $\varphi$  are unique modulo  $\mathfrak{m}'$ . We will now construct  $\varphi'$  and  $(\alpha')$  such that  $\varphi'(x) \equiv \varphi(x) \pmod{\mathfrak{m}'}$ , for each  $i$ ,  $\alpha'_i \equiv \alpha_i \pmod{\mathfrak{m}'}$ , and

$$\varphi'(F(x, y)) \equiv \Gamma(\alpha')(\varphi'x, \varphi'y) \pmod{\mathfrak{m}'^{r+1}}.$$

For each  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{h-1}) \in (\mathfrak{m}^r)^{h-1}$ , let  $\Delta_\varepsilon$  be the cocycle

$$\Delta_\varepsilon(x, y) = (\Phi_1(\mathfrak{o}, x \star y))^{-1} \nu_r[\varphi(F(x, y)) - \Gamma(\alpha + \varepsilon)(\varphi x, \varphi y)],$$

as in proposition 2.4, where  $\nu_r$  is the canonical projection of  $\mathfrak{m}^r$  onto  $M_r = \mathfrak{m}^r/\mathfrak{m}^{r+1}$ . Since

$$\Gamma(\alpha + \varepsilon)(\varphi x, \varphi y) - \Gamma(\alpha)(\varphi x, \varphi y) \equiv \sum_{i=1}^{h-1} \frac{\partial \Gamma}{\partial t_i}(\alpha)(\varphi x, \varphi y) \varepsilon_i \pmod{\mathfrak{m}'^{r+1}},$$

we have, on subtracting, and noting  $\alpha^* = \mathfrak{o}$ , and  $\varphi^*x = x$ ,

$$\begin{aligned} \Delta_0(x, y) - \Delta_\varepsilon(x, y) &= (\Phi_1(\mathfrak{o}, x \star y))^{-1} \sum_{i=1}^{h-1} \frac{\partial \Gamma^*}{\partial t_i}(\alpha^*)(\varphi^*x, \varphi^*y) \nu_r(\varepsilon_i) \\ &= \sum_{i=1}^{h-1} f_i(x, y) \nu_r(\varepsilon_i), \end{aligned}$$

where the  $f_i(x, y)$  are cocycles by proposition 2.6 applied to  $\Gamma^*$ . The same proposition shows that there is a family  $\varepsilon = (\varepsilon_i)$  such that  $\Delta_\varepsilon = \mathfrak{o}$ , and that such an  $\varepsilon$  is unique modulo  $\mathfrak{m}'^{r+1} = \text{Ker } \nu_r$ . Putting  $\alpha' = \alpha + \varepsilon$  and applying proposition 2.4 we see then that there is a  $\varphi'$  such that  $\varphi' \equiv \varphi \pmod{\mathfrak{m}'^{r+1}}$  and

$$\varphi'(F(x, y)) \equiv \Gamma(\alpha')(\varphi'x, \varphi'y) \pmod{\mathfrak{m}'^{r+1}}$$

and that such a  $\varphi'$  is unique mod  $\mathfrak{m}'^{r+1}$ .

3.2. — Thus we see that if  $\Phi$  is a one-parameter formal group over  $k$ , of height  $h < \infty$ , the set  $\mathfrak{G}_\mathfrak{o}(\Phi)$  of all  $\star$ -isomorphism classes of group laws  $F$  over  $\mathfrak{o}$  such that  $F^* = \Phi$  is in one-to-one correspondence with the set-theoretic product of  $\mathfrak{m}$  with itself  $(h - 1)$  times.

This correspondence is obviously functorial; the functor  $\mathfrak{o} \mapsto \mathfrak{G}_\mathfrak{o}(\Phi)$  is isomorphic to the functor  $\mathfrak{o} \mapsto (\mathfrak{m})^{h-1}$ , for  $\mathfrak{o}$  running through the category of complete local noetherian  $R$ -algebras,  $R$  being a fixed local ring with residue field  $k = R/I$ .

PROPOSITION 3.3. — *Under the hypotheses of theorem 3.1, if  $u \in \text{Aut}_k(\Phi)$ , there is a unique  $(h - 1)$ -tuple  $(\alpha)$  of elements of  $\mathfrak{m}$  and a unique isomorphism  $\varphi \in \text{Hom}_\mathfrak{o}(F, \Gamma(\alpha))$  such that  $\varphi^*(x) = u(x)$ .*

*Proof.* — Let  $g(x) \in \mathfrak{o}[[x]]$  be any power series such that  $g^*(x) = u^{-1}(x)$ . Let  $G(x, y) = g^{-1}(F(gx, gy))$ . Then since  $G^* = \Phi$ , we can use theorem 3.1 to get an  $(h-1)$ -vector  $(\alpha)$  of elements of  $\mathfrak{m}$  and a  $\star$ -isomorphism  $\psi$  from  $G$  to  $\Gamma(\alpha)$ . Then  $\psi \circ g^{-1} = \varphi$  is the isomorphism we want. Uniqueness is clear.

3.4. — If in particular  $R$  is a complete noetherian local ring and  $\mathfrak{o}$  is  $R[[t_1, \dots, t_{n-1}]]$ , then for each  $u \in \text{Aut}_k(\Phi)$  there is a unique substitution

$$u' : t_i \mapsto u'_i(t_1, \dots, t_{n-1})$$

where each  $u'_i(t)$  is in the maximal ideal of  $R[[t]]$ , and a unique isomorphism  $\varphi_u \in \text{Hom}_{\mathfrak{o}}(\Gamma(t), \Gamma(u'(t)))$  such that  $\varphi_u^* = u$ . One sees readily, using uniqueness, that if  $u$  and  $v$  are  $k$ -automorphisms of  $\Phi$ , then  $u'(v'(t)) = (u \circ v)'(t)$  so that  $\text{Aut}_k(\Phi)$  has a representation by analytic transformations of the “analytic variety”  $\mathfrak{G}_R(\Phi)$ . By our construction,  $\Gamma(\alpha)$  has an automorphism reducing to  $u$  modulo the maximal ideal if and only if for each  $i$ , we have  $u'_i(\alpha) = \alpha_i$ . Thus  $u'$  is the identity substitution if and only if  $u \in \mathbf{Z}_p$ , since by [3], 5.2.1 there are group laws of all heights with endomorphism ring  $\mathbf{Z}_p$ .

3.5. — We can use this operation of  $\text{Aut}_k(\Phi)$  on  $\mathfrak{G}_R(\Phi)$  to find an elliptic curve  $E$  without complex multiplications but whose associated formal group does have complex multiplications, i. e. endomorphisms not in  $\mathbf{Z}_p$ .

Take the case  $p = 2$ ,  $R =$  the ring of integers of the quadratic unramified extension of  $\mathbf{Q}_2$ ,  $k =$  the field with four elements. Consider the elliptic curve  $E_t$  defined over  $R[[t]]$  which is given by  $Y^2 + tXY + Y = X^3$ , which has  $j$ -invariant equal to  $t^3(t^3 - 24)^3/(t^3 - 27)$ . The point  $(0, 0)$  is an inflection point of  $E_t$ , and we can take this as zero-point to make  $E_t$  an Abelian variety. If the function  $X$  is used as local uniformizing parameter at  $(0, 0)$ , the group law associated with  $E_t$  turns out to be congruent modulo degree 5 to  $x + y + txy + 2x^3y + 3x^2y^2 + 2xy^3$  and is therefore a  $\Gamma(t)(x, y)$  as in paragraph 1, if we call  $\Phi$  the height-two group law  $\Gamma(0)^*(x, y) \in k[[x, y]]$ .

Now consider  $E_0$  which is an Abelian variety with endomorphism ring isomorphic to  $\mathbf{Z}[\omega]$  where  $\omega$  is a primitive cube root of 1. The endomorphism ring of the group law  $\Gamma(0)$  contains a subring isomorphic to  $\mathbf{Z}[\omega]$  and thus  $\text{End}(\Gamma(0)) \cong R$ ; in other words  $\Gamma(0)$  is full in the sense of [3].

Now for  $u \in \text{Aut}_k(\Phi)$ , we have  $u'(0) = 0$  if and only if there is  $\varphi \in \text{Aut}_R(\Gamma(0))$  such that  $\varphi^* = u$ . Thus under the action of  $\text{Aut}_k(\Phi)$  on the set  $pR \cong \mathfrak{G}_R(\Phi)$ , the orbit of 0 is in one-to-one correspondence with the set of left cosets of  $(\text{Aut}_R(\Gamma(0)))^*$  in  $\text{Aut}_k(\Phi)$ . But  $\text{Aut}_k(\Phi)$  is isomorphic to the group  $U$  of invertible elements in the maximal order

of a central division algebra  $D$  of rank four over  $\mathbf{Q}_2$ , and  $(\text{Aut}_\bullet(\Gamma(\circ)))^*$  corresponds to the intersection of  $U$  with a commutative subfield of  $D$ , so that the index is uncountable. Therefore, there are uncountably many distinct values of  $u'(\circ)$ , and so (in virtue of the  $j$ -invariant) uncountably many non-isomorphic elliptic curves  $E_{u'(\circ)}$  whose formal groups  $\Gamma(u'(\circ))$  are full. But of course only countably many of these elliptic curves can have complex multiplications.

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