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DIAGONALIZABLY LINEARIZED COHERENT SHEAVES

BY

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SUMMARY. — Let X denote a smooth projective scheme with an action of the smooth diagonalizable group D. The Grothendieck group $K_D(X)$ on the category of D-linearized coherent sheaves on X is studied.

The main result is a localization theorem for K_D, an algebraic analogue of the Atiyah-Segal theorem.

Applications are given to Lefschetz formulas of various types.

RÉSUMÉ. — Soient X un schéma projectif lisse, muni d'une action du groupe diagonalisable, lisse, D. On fait une étude du groupe de Grothendieck, $K_D(X)$ sur la catégorie des faisceaux cohérent, D-linéarisé sur X.

Notre résultat principal est un théorème de localisation pour le foncteur K_D, variante algébrique de celui de Atiyah-Segal.

Comme application des formules de Lefschetz de types variés sont données.

The paper is concerned with equivariant K-theory of a smooth projective scheme X, equipped with an action of a smooth diagonalizable group D.

Our main result is a localization theorem for the equivariant K-functor, K_D. Namely, the inclusion $i : X^D \to X$ induces a map

$$i^! : \quad K_D(X) \to K_D(X^D)$$

which, considered as a linear map over the representation ring of D, becomes an isomorphism after a suitable localization.

The localization theorem combined with the Riemann-Roch formula yields a Lefschetz fixed point formula of the type,

$$\sum_i (-1)^i \operatorname{Tr} H^i(X, \mathscr{F}) = \int_{X^D} \frac{\operatorname{ct}(i^* \mathscr{F}) \operatorname{Todd}(X^D)}{\operatorname{ct}(\lambda_{-1} N)}$$

valid in a localization of the representation ring of D, *see* (4.10). By various specializations of the coefficients, we obtain results of more classical type, among others the Woods Hole fixed point formula and those of [2], [4], [5], [9].

It should be mentioned that the localization theorem is inspired by a similar topological theorem, *see* [1] for reference.

In case of a torus action, the above form of the Lefschetz fixed point formula was conjectured by Birger IVERSEN, whom I thank for indispensable guidance not only in this subject.

CONTENTS :

NOTATION. — Throughout we fix an algebraically closed field k and a smooth diagonalizable k-group scheme D. Δ denotes the character group of D, and we put $R(D) = \mathbf{Z}[\Delta]$. For $\varkappa \in \Delta$, we let $e^{\varkappa}$ denote the corresponding element in $R(D)$.

For a k-linear representation E of D, we put

$$\operatorname{tr}(E) = \textstyle\sum_{\varkappa \in \Delta} (\operatorname{rank}_k E_{\varkappa}) e^{\varkappa},$$

where $E_{\varkappa}$ is the space of semi-invariants of D of weight $\varkappa$ in E.

As is well known tr induces an isomorphism from the representation ring of D to $R(D)$.

Let $S \subseteq R(D)$ be the multiplicative subset generated by elements of the form $1-e^{\varkappa}$, $\varkappa$ a non-trivial character of D. An easy consideration shows $0 \notin S$.

1. Equivariant K-theory

DEFINITION 1.1. — Let X be a scheme [1] with a D-action. Then we let $K_D(X)$ denote the Grothendieck ring of the category of D-linearized [12] locally free sheaves on X, the multiplication being induced by $\otimes$. The image of a D-linearized locally free sheaf $\mathscr{F}$ in $K_D(X)$ is denoted $\operatorname{cl}\mathscr{F}$. Let **D-Sch** denote the category of k-schemes with D-actions, the morphisms

(1) Scheme = k-scheme throughout the paper.

being D-equivariant morphisms of k-schemes. The pullback functor makes K_D into a functor

$$K_D : \mathbf{D\text{-}Sch}^{\mathrm{op}} \to \mathbf{Rings}.$$

As is well known the trace tr gives an isomorphism $K_D(\mathrm{Spec}(k)) \xrightarrow{\sim} R(D)$. In the following, we shall always view

$$K_D : \mathbf{D\text{-}Sch}^{\mathrm{op}} \to \mathbf{R(D)\text{-}Rings}.$$

Put $f^! = K_D(f)$, f a D-equivariant scheme morphism.

1.2. *λ-operations.* – We have naturel equivariant operations

$$\lambda^i : K_D(X) \to K_D(X), \qquad i \geqq 0,$$

satisfying

(★) For $\mathscr{F}$ a D-linearized locally free sheaf on X

$$\lambda^i(\mathrm{cl}\,\mathscr{F}) = \mathrm{cl}\,\Lambda^i\mathscr{F}.$$

(★★)
$$\lambda_t : K_D(X) \to 1 + tK_D(X)[[t]],$$
$$x \mapsto 1 + \textstyle\sum_{i=0}^{\infty} \lambda^i(x)\,t^i,$$

is a group homomorphism.

$K_D(X)$ is actually a λ-ring in the sense of SGA 6 ([14], V, 2.4).

1.3. *The trivial action.* – Suppose D acts trivially on X. A *D-linearized* locally free sheaf $\mathscr{F}$ decomposes $\mathscr{F} = \oplus_{\varkappa \in \Delta} \mathscr{F}_{\varkappa}$, where D acts on $\mathscr{F}_{\varkappa}$ through $\varkappa$, *see* [3]. $\mathscr{F} \mapsto \sum_{\varkappa \in \Delta} \mathrm{cl}(\mathscr{F}_{\varkappa}) \otimes e^{\varkappa}$ induces a natural map

$$\mathrm{tr}_X : K_D(X) \to K(X) \otimes_{\mathbf{Z}} R(D)$$

which is an $R(D)$-linear isomorphism (*loc. cit.*).

1.4. *Linear action on projective space.* – Let E be a rank $r+1$ k-linear representation of D. Put

$$E = \oplus_{\varkappa \in \Delta} E_{\varkappa}, \qquad \mathrm{rank}_k E_{\varkappa} = n_{\varkappa}, \qquad \textstyle\sum_{\varkappa \in \Delta} n_{\varkappa} = r+1.$$

The action of D on E induces an action of D on $\mathbf{P}(E) \xrightarrow{\pi} \mathrm{Spec}\, k$ together with a linearization of $\mathscr{O}_{\mathbf{P}}(1)$.

THEOREM 1.5 ([2]). — *We have an $R(D)$-linear isomorphism*

$$K_D(\mathbf{P}(E)) \xleftarrow{\sim} R(D)[T]/\textstyle\prod_{\varkappa \in \Delta}(T-e^{\varkappa})^{n_\varkappa},$$
$$\mathrm{cl}(\mathcal{O}_{\mathbf{P}}(1)) \leftarrow\!\dashv T.$$

Proof. – Fix notation for the proof

$$l = \mathrm{cl}(\mathcal{O}_{\mathbf{P}}(1)),$$
$$w = \mathrm{cl}(\mathrm{Ker}\,\Pi^*(E) \to \mathcal{O}_{\mathbf{P}}(1)),$$
$$v = \mathrm{cl}(\Pi^*(E)).$$

The proof consists in three steps (1.6), (1.7), (1.8).

(1.6) $K_D(\mathbf{P}(E))$ is generated over $R(D)$ by $\{l^n;\ n \in \mathbf{Z}\}$.

Let us first make some considerations over graded modules.

D acts on $A = \mathrm{Sym}_k E$ through E. By a graded D-A-module we understand a graded A-module M together with a k-linear action of D on each graded piece of M subjected to

$$\sigma(am) = (\sigma a)(\sigma m); \qquad \sigma \in D, \quad a \in A, \quad m \in M.$$

The morphisms in the category of graded D-A-modules are graded of degree 0 and as well A- as D-linear.

If $\varkappa \in \Delta$ and M is a graded D-A-module, then $M_\varkappa$ denotes the graded D-A-module obtained from M by twisting the D-action as follows :

$$\sigma m : \ = \varkappa(\sigma)\sigma m; \qquad \sigma \in D, \quad m \in M.$$

Note that if N is a graded D-A-module $\varkappa \in \Delta$ and $n \in \mathbf{Z}$, then

$$\mathrm{Hom}_{\mathrm{gr}-D-A}(A_\varkappa(-n), N) \simeq (N_n)_\varkappa$$

where $(N_n)_\varkappa$ denotes the semi-invariants of D of weight $\varkappa$ in N_n.

Let us call a graded D-A-module free if it is a finite direct sum of graded D-A-modules of the form $A_\varkappa(-n)$, $\varkappa \in \Delta$, $n \in \mathbf{Z}$.

In virtue of the above remark, it is clear that if M is a finitely generated D-A-module then there exists a surjective morphism of graded D-A-modules $L \to M$ with L a free graded D-A-module.

([2]) This is a particular case of the theorem giving the structure of $K(P(E))$ for E a locally free sheaf on a ringed topos ([14], VII.1.4).

We are going to prove :

A finitely generated graded D-A-module M has a resolution

$$(\bigstar) \qquad 0 \to L_{r+1} \to L_r \to \ldots \to L_0 \to M \to 0,$$

where the L_i's are free graded D-A-modules.

First it is clear from the preceding remarks that we can find a resolution as above where L_0, L_1, ..., L_r are free graded D-A-modules. By Hilbert's syzygy theorem, L_{r+1} is a free gradded A-Module. Thus it suffices to prove.

($\bigstar\bigstar$) A finitely generated graded D-A-module M which is free as a graded A-module is free as a graded D-A-module.

Proof. — Pick a family (m_i) of semi-invariant homogeneous elements of M such that $(m_i \otimes 1_k)$ form a basis for $M \otimes_A k$. Let m_i have degree d_i and weight $\varkappa_i$, and put $L = \otimes A_{\varkappa_i}(-d_i)$. We have a morphism $f : L \to M$ whose reduction mod A_+ is an isomorphism. From this and the assumption that M is a free graded A-module follows that f is an isomorphism (*see* [8], Lemma 2.2).

Returning to the proof of (1.6). The sheafification functor lifts to a functor $(\tilde{\ })$ from the category of finitely generated D-A-modules to D-linearized coherent sheaves on $\mathbf{P}(E)$. $(\tilde{\ })$ is exact and onto objects. If L is a free graded D-A-module then $\operatorname{cl}\tilde{L}$ is an $R(D)$-linear combination of $\{l^n; n \in \mathbf{Z}\}$. Now (1.6) follows from ($\bigstar$),

$$(1.7) \qquad \textstyle\prod_{\varkappa \in \Delta}(l-e^{\varkappa})^{n_\varkappa} = 0.$$

The sequence $0 \to \operatorname{Ker} \to \Pi^*(E) \to \mathcal{O}_{\mathbf{P}}(1) \to 0$ gives, with the introduced notation, $v.l^{-1} = w.l^{-1}+1$. Applying λ_t gives

$$\lambda_t(vl^{-1}) = (1+t)\lambda_t(wl^{-1}).$$

Now substitute $v = \sum_{\varkappa \in \Delta} n_\varkappa e^\varkappa$, and use (1.2) ($\bigstar\bigstar$), then

$$\textstyle\prod_{\varkappa \in \Delta}\lambda_t(e^\varkappa l^{-1})^{n_\varkappa} = (1+t)\lambda_t(wl^{-1}).$$

For $t = -1$, we get the relation

$$\textstyle\prod_{\varkappa \in \Delta}(1-e^\varkappa l^{-1})^{n_\varkappa} = 0 \ .$$

(1.7) follows after multiplication with l^{r+1} :

(1.8) $\qquad$ 1, l, ..., l^r are linearly independent over $R(D)$.

Let $\chi_D(\mathbf{P}, x) : K_D(\mathbf{P}(E)) \to R(D)$ denote the Lefschetz trace, *see* (4.2) for details.

Suppose $\sum_{i=0}^{r} a_i l^i = 0$. Let a_s be the biggest non-trivial coefficient. $a_s = \sum_{i=0}^{s-1} -a_i l^{i-s}$. By Serre's calculations [13], $\chi_D(\mathbf{P}, l^{i-s}) = 0, i = 0, \ldots, s-1$. Now apply $\chi_D(\mathbf{P}, x)$ to the above relation, use that $\chi_D(\mathbf{P}, l)$ is $R(D)$-linear and conclude $a_s = 0$.

2. The Gysin morphism

In this paragraph, we introduce a Gysin morphism ($i_!$) for equivariant K-theory, and give three formulas interrelating $i_!$ and $i^!$.

PROPOSITION 2.1. — *Let D act on the smooth projective scheme X. Then the natural map of $K_D(X)$ into the Grothendieck group of the category of D-linearized coherent sheaves on X is an isomorphism.*

Proof. — The category of D-linearized locally free sheaves on X is a full subcategory of the abelian category of D-linearized coherent sheaves on X. So by standard theory, e. g. [2] or [6], we are easily reduced to prove the following lemma.

LEMMA 2.2. — *Let* $X \xrightarrow{\Pi} \operatorname{Spec} k$ *be a smooth projective scheme on which D acts.*

(2.3) *There exists a D-linearized ample sheaf $\mathscr{L}$ on X.*

(2.4) *Every D-linearized coherent sheaf $\mathscr{F}$ on X is an equivariant quotient of a D-linearized locally free sheaf on X.*

Proof. — (2.3) is contained in the results of Kambayashi (*see* [10]). For (2.4) choose m so large that $\mathscr{F} \otimes \mathscr{L}^m$ is generated by its global sections. $V = H^0(X, \mathscr{F} \otimes \mathscr{L}^m)$ is a k-linear representation of D, (4.1), hence we have a D-equivariant surjection $\Pi^* V \to \mathscr{F} \otimes \mathscr{L}^m$ and therefore $\mathscr{F}$ is a quotient of $\Pi^* V \otimes \mathscr{L}^{-m}$.

DEFINITION 2.5. — *Let $i : Y \to X$ be a D-equivariant closed immersion of smooth projective schemes with D-action. By* (2.1), *the direct image functor i_* induces an Abelian group homomorphism*

$$i_! : \; K_D(Y) \to K_D(X)$$

such that for $Z \xrightarrow{i} Y \xrightarrow{j} X$, *we have*

$$(i \circ j)_! = i_! \circ j_!,$$

Three formulas. — Notation as in (2.5). $i : Y \to X$.

(2.6) *The projection formula :* For every $x \in K_D(X)$, $y \in K_D(Y)$:

$$i_!(y.i^!(x)) = i_!(y).x.$$

(2.7) *The self-intersection formula :* Put $N = \mathrm{cl}(\mathcal{N}_{Y/X})$, $\mathcal{N}_{Y/X}$ being the conormal bundle on Y with its canonical linearization. For every $y \in K_D(Y)$:

$$i^!(i_!(y)) = y.\lambda_{-1}(N).$$

(2.8) *The cartesian formula :* Let

$$\begin{array}{ccc} T & \xrightarrow{j'} & Y \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ Z & \xrightarrow[j]{} & X \end{array}$$

be a cartesian square of D equivariant closed immersions between smooth projective schemes with D-action. Then there exists $\gamma_T \in K_D(T)$ such that for every $y \in K_D(Y)$:

$$j^!(i_!(y)) = i'_!(\gamma_T . j'^!(y)), \qquad y \in K_D(Y).$$

Remark on proof. — (2.6) follows from a natural isomorphism. (2.7) follows from a closer look at the " unlinearized " proof (*see* MANIN [11] or SGA 6 ([14], VII, 2.7)). (2.8) is proved as follows :

Let t denote the inclusion $T \to X$. Put (**Tor** is short for $\mathbf{Tor}^{\mathcal{O}_X}$) :

$$\gamma_T = \sum (-1)^i \mathrm{cl}\, t^* \mathbf{Tor}_i(\mathcal{O}_Y, \mathcal{O}_Z).$$

Let now $y = \mathrm{cl}(\mathcal{F})$, where $\mathcal{F}$ is a locally free sheaf on Y :

$$j^! i_! y = \sum (-1)^i \mathrm{cl}\, j^* \mathbf{Tor}_i(\mathcal{O}_Z, i_* \mathcal{F}).$$

Now

$$\mathbf{Tor}_i(\mathcal{O}_Z, i_* \mathcal{F}) = \mathbf{Tor}_i(\mathcal{O}_Z, \mathcal{O}_Y) \otimes i_* \mathcal{F},$$

gives

$$\begin{aligned} j^! i_! y &= \sum (-1)^i \mathrm{cl}\, j^* \mathbf{Tor}_i(\mathcal{O}_Z, \mathcal{O}_Y) \otimes j^* i_* \mathcal{F} \\ &= \sum (-1)^i \mathrm{cl}\, i'_*(t^* \mathbf{Tor}_i(\mathcal{O}_Z, \mathcal{O}_Y) \otimes j'^* \mathcal{F}) \\ &= i'_!(\sum (-1)^i \mathrm{cl}\, t^* \mathbf{Tor}_i(\mathcal{O}_Z, \mathcal{O}_Y) . j'^! y) \\ &= i'_!(\gamma_T . j'^! y). \end{aligned}$$

3. The localization theorem

Let D act on the smooth projective scheme X. The fixed point scheme X^D is smooth [9]. The inclusion $i : X^D \to X$ induces an $R(D)$-linear map

$$i^! : \quad K_D(X) \to K_D(X^D).$$

We show that this map becomes an isomorphism after localization with respect to the multiplicative subset $S \subseteq R(D)$ generated by elements of the form $1-e^{\varkappa}$, $\varkappa$ a nontrivial character of D. Note $0 \notin S$.

LEMMA 3.1. — *Let $N = \mathrm{cl}(\mathcal{N}_{X^D/X})$ the class in $K_D(X^D)$ of the conormal bundle of X^D in X. Then $\lambda_{-1} N$ becomes a unit in $S^{-1} K_D(X^D)$.*

Proof. – It is enough to prove that $\lambda_{-1} N$ is a unit when we restrict to every connected component Z of X^D. Now choose a closed point $z \in Z$ and let $j_z : \{ z \} \to Z$ be the inclusion. By MANIN ([11], § 8 and 9) :

$$K(Z) = \mathbf{Z} \oplus \mathrm{Ker}(j_z^*)$$

and $\mathrm{Ker}(j_z^*)$ is nilpotent. Tensoring this with $R(D)$ and using (1.3), we obtain a decomposition

$$K_D(Z) \xrightarrow{\sim} R(D) \oplus \mathrm{Ker}(j_z^!),$$

with $\mathrm{Ker}(j_z^!)$ nilpotent. Clearly it suffices to prove that the component of $\lambda_{-1} N$ after $R(D)$ belongs to S. Now the component of $\lambda_{-1} N$ after $R(D)$ equals

$$\mathrm{tr}_{\{z\}} j_z^! (\lambda_{-1} N) = \mathrm{tr}_{\{z\}} \lambda_{-1} (j_z^! N).$$

All weights of D in the fibre of $\mathcal{N}_{X^D/X}$ at z are nontrivial as it follows from the fact that the fixed point scheme is smooth [9], hence we can write $\mathrm{tr}_{\{z\}} j_z^! N = \sum_{\varkappa \neq 0} m_{\varkappa} e^{\varkappa}$. By (1.2) (★★),

$$\mathrm{tr}_{\{z\}} j_z^! \lambda_{-1} N = \prod_{\varkappa \neq 0} (1-e^{\varkappa})^{m_{\varkappa}}$$

which belongs to S.

THEOREM 3.2. — *The inclusion $i : X^D \to X$ induces an $R(D)$-linear map*

$$i^! : \quad K_D(X) \to K_D(X^D)$$

which is an isomorphism after localization with respect to S. The inverse map is given by

$$y \mapsto S^{-1} i_!(y.(\lambda_{-1} N)^{-1}), \qquad y \in S^{-1} K_D(X^D).$$

Proof. – Localizing the formulas (2.6) and (2.7), we get

$$S^{-1} i_!(y.S^{-1} i^!(x)) = S^{-1} i_!(y).x,$$
$$S^{-1} i^!(S^{-1} i_!(y)) = y.\lambda_{-1}(N)$$

for $x \in S^{-1} K_D(X)$, $y \in S^{-1} K_D(X^D)$.

Using (3.1) and these formulas, it remains to prove the following two equivalent statements :

(3.3) $\quad S^{-1} i^! : \; S^{-1} K_D(X) \to S^{-1} K_D(X^D)$ is injective,

(3.4) $\quad S^{-1} i_! : \; S^{-1} K_D(X^D) \to S^{-1} K_D(X)$ takes the value 1.

We proceed by two lemmas.

LEMMA 3.5. — (3.3) *is true for a linear action on a projective space* **P** (E) [*cf.* (1.4) for notation].

Proof. – By the calculations in (1.4), we get $\mathbf{P}(E)^D = \amalg_{\varkappa \in \Delta} \mathbf{P}(E_\varkappa)$ and

$$S^{-1} i^! : \; (S^{-1} R(D)[T]/\textstyle\prod_{\varkappa \in \Delta}(T - e^\varkappa)^{n_\varkappa}) \to (\textstyle\prod_{\varkappa \in \Delta} S^{-1} R(D)[T]/(T - e^\varkappa)^{n_\varkappa})$$
$$(T \mapsto \textstyle\prod T).$$

Using that

$$(T - e^\varkappa) = e^{\varkappa'}(1 - e^{\varkappa - \varkappa'}) + (T - e^{\varkappa'})$$

is a unit in $S^{-1} R(D)[T]/(T - e^{\varkappa'})^{n_{\varkappa'}}$ for $\varkappa' \neq \varkappa$ this map is easily seen to be injective.

LEMMA 3.6. — (3.4) *is true for any X.*

Proof. – By (2.3), we can find a k-linear representation E of D and a D-equivariant closed immersion $j : X \to \mathbf{P}(E)$. The following diagram

$$\begin{array}{ccc} X^D & \xrightarrow{\;j^D\;} & \mathbf{P}(E)^D \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_P} \\ X & \xrightarrow[\;j\;]{} & \mathbf{P}(E) \end{array}$$

is cartesian as it follows from the definition of the fixed point scheme.

According to (3.5), we can find $z_P \in S^{-1} K_D(\mathbf{P}(E)^D)$ such that $S^{-1} i_{P!} z_P) = 1$. Put

$$z_X = S^{-1} j^{D!}(z_P) . \gamma_{X^D}, \qquad S^{-1} i_! (z_X) = 1$$

as it follows from the localized version of (2.8).

4. Applications

The applications of the localization theorem we are going to discuss are based on the Lefschetz trace :

$$\chi_D(X, x) : \quad K_D(X) \to R(D),$$

X a projective scheme with D-action.

4.1. *Construction of k-linear representations.* — Let us recall that if $\mathscr{F}$ is a D-linearized sheaf on X, then we have a canonical action of D on the cohomology groups $H^i(X, \mathscr{F})$. Namely for $\sigma \in D(k)$ the D-linearization of $\mathscr{F}$ provides a morphism $\sigma^* \mathscr{F} \to \mathscr{F}$ which induces a linear map $H^i(X \sigma^* \mathscr{F}) \to H^i(X, \mathscr{F})$. Composing this and the canonical map $H^i(X, \mathscr{F}) \to H^i(X, \sigma^* \mathscr{F})$ gives the action of σ on $H^i(X, \mathscr{F})$.

4.2. *The Lefschetz trace.* — Let X be a smooth projective scheme with a D-action. The functor

$$\mathscr{F} \mapsto \textstyle\sum_i (-1)^i \operatorname{tr} H^i(X, \mathscr{F})$$

from the category of D-linearized locally free sheaves to $R(D)$ is additive. This functor induces the Lefschetz trace :

$$\chi_D(X, x) : \quad K_D(X) \to R(D),$$

$\chi_D(X, x)$ is an $R(D)$-linear map satisfying :

(4.3) For a D-equivariant closed immersion $j : Y \to X$:

$$\chi_D(Y, y) = \chi_D(X, x) \circ j_!(y).$$

(4.4) If D acts trivially on X then he following diagram commutes :

$$\begin{array}{ccc} K_D(X) & \xrightarrow{\ \mathrm{tr}_X\ } & K(X) \otimes_{\mathbf{Z}} R(D) \\ & {\scriptstyle \chi_D(X,\, x)} \searrow \quad \swarrow {\scriptstyle \chi(X,\, x) \otimes \mathrm{id}_{R(D)}} & \\ & R(D) & \end{array}$$

The next proposition shows how to compute the total χ_D by means of χ_D on the fixed point scheme. Moreover (4.4) shows that χ_D on the fixed point scheme may be computed from the unlinearized χ.

PROPOSITION 4.5. — *Let* $i : X^D \to X$, $x \in K_D(X)$, *then*

$$\chi_D(X, x) = S^{-1}\chi_D(X^D, i^!(x).(\lambda_{-1} N)^{-1}) \quad \textit{in } S^{-1}R(D).$$

Proof. — (3.2), (4.3).

Example 4.6. — From the exact sequence

$$0 \to \mathcal{N}_{X^D/X} \to i^*\Omega_X \to \Omega_{X^D} \to 0,$$

we get $\lambda_{-1}\,\mathrm{cl}\,(i^*\Omega_X) = \lambda_{-1} N.\lambda_{-1}(\mathrm{cl}\,\Omega_{X^D})$ in $K_D(X^D)$. By (4.5),

$$\chi_D(X, \lambda_{-1}\mathrm{cl}\,\Omega_X) = \chi_D(X^D, \lambda_{-1}\mathrm{cl}\,\Omega_{X^D}) \quad \text{in } S^{-1}R(D).$$

For $D = T$ an algebraic torus, the equality above holds in $R(D)$. So we may specialize the characters to 1, and get

$$\chi(X, \lambda_{-1}\Omega_X) = \chi(X^T, \lambda_{-1}\Omega_{X^T})$$

in **Z** proved by Birger IVERSEN [9].

4.7. *Isolated fixed points.* — Assume X^D finite. Then (4.5) gives for a D-linearized sheaf $\mathcal{F}$ on X :

$$\sum_i (-1)^i \,\mathrm{tr}\, H^i(X, \mathcal{F}) = \sum_{z \in X^D} \frac{\mathrm{tr}\,\mathcal{F}_z}{\sum(-1)^i \mathrm{tr}(\Lambda^i\, T_z(X)^\vee)}.$$

4.8. *H. Weyl's character formula.* — An interesting application of (4.7) is the case where $X = G/B$, B is a Borel subgroup of the reductive linear algebraic group G and $D = T$ a maximal torus contained in B (*see* [2] and [4]). In characteristic zero, this leads to a proof of Weyl's character formula (*loc. cit.*).

4.9. *The Woods Hole formula.* — Let $\sigma \in D(k)$. The evaluation map $\Delta \to k^*$, $\varkappa \mapsto \varkappa(\sigma)$ gives rise to a ring homomorphism $\mathrm{ev}_\sigma : R(D) \to k$ such that for a k-linear representation E we have $\mathrm{ev}_\sigma(\mathrm{tr}\, E) = \mathrm{Tr}(\sigma, E) \in k$ the usual trace for the operation of σ on E.

Let $\sigma \in D(k)$ be a dense element, i. e. $\varkappa(\sigma) \neq 1$ for all nontrivial characters $\varkappa \in \Delta$, then ev_σ factors through $R(D) \to S^{-1}R(D)$.

Applying ev_σ to the formula (4.7) gives the following formula in k :

$$\textstyle\sum_i (-1)^i \operatorname{Tr}(\sigma, H^i(X, \mathscr{F})) = \sum_{z \in X^D} \frac{\operatorname{Tr}(\sigma, \mathscr{F}_z)}{\operatorname{Det}(1 - d_z \sigma)}.$$

4.10. *The cohomological formula.* — Assume we have a cohomology theory in the sense of Grothendieck [7] such that its Chern-character satisfies the Riemann-Roch theorem

$$\chi(X, \mathscr{F}) = \int_X \operatorname{ch} \mathscr{F} . \operatorname{Todd}(X).$$

For the trivial action of D on X put

$$\mathrm{ct}_D : \quad K_D(X) \xrightarrow{\mathrm{tr}_X} K(X) \otimes R(D) \xrightarrow{\mathrm{ch} \otimes \mathrm{id}_{R(D)}} A(X) \otimes \mathbf{Q} \otimes R(D),$$

$$\mathrm{Todd}_D = \mathrm{Todd} \otimes 1_{R(D)},$$

$$\int_X \text{“}=\text{”} \int_X \otimes \mathrm{id}_{R(D)}.$$

Now (4.4), (4.5) gives the formula

$$\chi_D(X, \mathscr{F}) = \int_{X^D} \frac{\mathrm{ct}_D(i^* \mathscr{F}) . \mathrm{Todd}_D(X^D)}{\mathrm{ct}_D(\lambda_{-1} N)}$$

in $\mathbf{Q} \otimes S^{-1} R(D)$ for $\mathscr{F}$ a D-linearized coherent sheaf on X.

4.11. *Specialization to the Witt ring.* — Assume $\mathrm{char}(k) = p \neq 0$. For an element $\sigma \in D(k)$ the composite of the evaluation map (4.9) $\mathrm{ev}_\sigma : \Delta \longrightarrow k^*$ and the Teichmüller lifting $w : k^* \longrightarrow W(k)$ gives the map $b_\sigma : R(D) \longrightarrow W(k)$ such that for a k-linear representation E, we have $b_\sigma(\mathrm{tr}\, E) = B \operatorname{Tr}(\sigma, E)$, the Brauer trace for the operation of σ on E.

Now assume D to be finite cyclic with generator $d \in D(k)$. d is “ dense ” (4.9), so b_d factors

$$R(D) \rightarrow S^{-1} R(D) \rightarrow W(k).$$

The cohomological formula (4.10) specializes through this map to the formula of Donovan ([2] and [5]).

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