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THE LAGRANGE COMPLEX

BY

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RÉSUMÉ. — Nous définissons le complexe de co-chaînes (Λ, δ), et nous prouvons le lemme de Poincaré pour l'opérateur δ. L'opérateur δ est utilisé dans le calcul des variations en vue de déduire les équations d'Euler-Lagrange. Le lemme de Poincaré fournit alors le critère suivant lequel un système d'équations est un système d'Euler-Lagrange.

ABSTRACT. — A cochain complex (Λ, δ) is defined, and the δ-Poincaré lemma is proved. The work is motivated by applications to the calculus of variations. The operator δ is used in the calculus of variations to construct the Euler-Lagrange equations, and the δ-Poincaré lemma provides criteria for partial differential equations to be Euler-Lagrange equations.

The present paper generalizes results contained in earlier publications ([6], [8]) which were applicable to ordinary differential equations of the Euler-Poisson type.

1. Jets and tangent vectors

Let M be a C^∞-manifold. We denote by $T^{(k)} M$ the manifold $J_0^k(\mathbf{R}^p, M)$ of jets of order k from $\mathbf{R}^p$ to M with source 0 called by EHRESMANN [1] *p^k-vitesses* in M. Elements of $T^{(k)} M$ are equivalence classes of smooth mappings of $\mathbf{R}^p$ into M. Two mappings γ and γ' are equivalent if $D^n(f \circ \gamma)(0) = D^n(f \circ \gamma')(0)$ for each C^∞-function f on M and each $n = (n_1, \ldots, n_p) \in \mathbf{N}^p$ such that $|n| = n_1 + \ldots + n_p \leqslant k$. The symbol $D^n g(0)$ is used to denote the partial derivative of a function g :

$$\mathbf{R}^p \to \mathbf{R} : \quad (t_1, \ldots, t_p) \mapsto g(t_1, \ldots, t_p)$$

of orders $n_1, \ldots, n_p$ with respect to the arguments $t_1, \ldots, t_p$ respectively at $(t_1, \ldots, t_p) = (0, \ldots, 0)$. We denote by $j_0^k(\gamma)$ the jet of the mapping γ. For each $k \in N$, there is the projection

$$\tau_{(k)} : \quad T^{(k)} M \to M : \quad j_0^k(\gamma) \mapsto \gamma(0)$$

and, if $k' \leqslant k$, then there is the projection

$$\rho_{(k')(k)} : \quad T^{(k)} M \to T^{(k')} M : j_0^k(\gamma) \mapsto j_0^{k'}(\gamma).$$

The manifold $T^{(0)} M$ is identified with M, and $T^{(1)} M$ is the tangent bundle TM of M. For each $n \in \mathbf{N}^p$ such that $|n| \leqslant k$ and each C^∞-function f on M there is a C^∞-function f_n defined on $T^{(k)} M$ by $f_n(j_0^k(\gamma)) = D^n(f \circ \gamma)(0)$.

For each $k \in \mathbf{N}$, we introduce an equivalence relation in the set of smooth mappings of $\mathbf{R}^{p+1}$ into M. Two mappings χ and χ' will be considered equivalent if $D^{(r,n)}(f \circ \chi)(0) = D^{(r,n)}(f \circ \chi')(0)$ for each C^∞-function f on M, each $n \in \mathbf{N}^p$ such that $|n| \leqslant k$ and $r = 0,1$. The symbol $D^{(r,n)} g(0)$ denotes the partial derivative of a function g :

$$\mathbf{R}^{p+1} \to \mathbf{R} : \quad (s, t_1, \ldots, t_p) \mapsto g(s, t_1, \ldots, t_p)$$

of orders $r, n_1, \ldots, n_p$ with respect to the arguments $s, t_1, \ldots, t_p$ respectively at $(s, t_1, \ldots, t_p) = (0, 0, \ldots 0)$. We denote the equivalence class of the mapping χ by $j_0^{(1,k)}(\chi)$. The set of equivalence classes can be canonically identified with the tangent bundle $TT^{(k)} M$ in such a way that

$$\langle j_0^{(1,k)}(\chi), df_n \rangle = D^{(1,n)}(f \circ \chi)(0)$$

for each function f on M and each $n \in \mathbf{N}^p$ such that $|n| \leqslant k$ and also

$$\tau_{T^{(k)}M}(j_0^{(1,k)}(\chi)) = j_0^k(\chi_0),$$

where $\tau_{T^{(k)}M} : TT^{(k)} M \to T^{(k)} M$ is the tangent bundle projection, and χ_0 is the mapping

$$\chi_0 : \quad \mathbf{R}^p \to M : \quad (t_1, \ldots, t_p) \mapsto \chi(0, t_1, \ldots, t_p) \quad [7].$$

The tangent mapping $T\rho_{(k')(k)} : TT^{(k)} M \to TT^{(k')} M$ is given by

$$T\rho_{(k')(k)}(j_0^{(1,k)}(\chi)) = j_0^{(1,k')}(\chi).$$

For each $k \in \mathbf{N}$ and each $m \in \mathbf{N}^p$ there is the mapping

$$\mathbf{F}_m : \quad TT^{(k)} M \to TT^{(k)} M : \quad j_0^{(1,k)}(\chi) \mapsto j_0^{(1,k)}(\chi_m),$$

where χ_m is the mapping

$$\chi_m : \quad \mathbf{R}^{p+1} \to M : \quad (s, t_1, \ldots, t_p) \mapsto \chi(st^m, t_1, \ldots, t_p),$$

and $t^m = t_1^{m_1} \ldots t_p^{m_p}$. Diagrams

$$\begin{array}{ccc} TT^{(k)} M & \xrightarrow{\mathbf{F}_m} & TT^{(k)} M \\ \downarrow{\scriptstyle \tau_{T^{(k)}M}} & & \downarrow{\scriptstyle \tau_{T^{(k)}M}} \\ T^{(k)} M & = & T^{(k)} M \end{array}$$

and

$$\begin{array}{ccc} TT^{(k)}M & \xrightarrow{\mathbf{F}_m} & TT^{(k)}M \\ {\scriptstyle T\rho_{(k')(k)}}\downarrow & & \downarrow{\scriptstyle T\rho_{(k')(k)}} \\ TT^{(k')}M & \xrightarrow{\mathbf{F}_m} & TT^{(k')}M \end{array}$$

are commutative.

For each $\alpha = 1, \ldots, p$ and each $k \in \mathbf{N}$, there is the mapping

$$\mathbf{T}^\alpha : \quad T^{(k+1)}M \to TT^{(k)}M : \quad j_0^{k+1}(\gamma) \mapsto j_0^{(1,k)}(\gamma^\alpha),$$

where γ^α is the mapping

$$\gamma^\alpha : \quad \mathbf{R}^{p+1} \to M : \quad (s, t_1, \ldots, t_p) \mapsto \gamma(t_1, \ldots, t_\alpha + s, \ldots, t_p) \quad (^1).$$

Diagrams

$$\begin{array}{ccc} T^{(k+1)}M & \xrightarrow{\mathbf{T}^\alpha} & TT^{(k)}M \\ {\scriptstyle \rho_{(k)(k+1)}}\downarrow & & \downarrow{\scriptstyle \tau_{T^{(k)}M}} \\ T^{(k)}M & = & T^{(k)}M \end{array}$$

and

$$\begin{array}{ccc} T^{(k+1)}M & \xrightarrow{\mathbf{T}^\alpha} & TT^{(k)}M \\ {\scriptstyle \rho_{(k'+1)(k+1)}}\downarrow & & \downarrow{\scriptstyle T\rho_{(k')(k)}} \\ T^{(k'+1)}M & \xrightarrow{\mathbf{T}^\alpha} & TT^{(k')}M \end{array}$$

are commutative.

2. Forms and derivations

Let $\Omega_k^{(q)}$ denote the $\mathbf{R}$-linear space of q-forms on $T^{(k)}M$, and let $\Omega_{(k)}$ be the nonnegative graded linear space $\{ \Omega_{(k)}^q \}$. The exterior differential d is a collection $\{ d^q \}$ of linear mappings

$$d^q : \quad \Omega_{(k)}^q \to \Omega_{(k)}^{q+1}$$

and the exterior product $\wedge$ is a collection $\{ \wedge^{(q,q')} \}$ of operations $\wedge^{(q,q')} : \Omega_{(k)}^q \times \Omega_{(k)}^{q'} \to \Omega_{(k)}^{q+q'}$. For each $k' \leqslant k$ and each q, there is the cotangent mapping $\rho_{(k')(k)}^* : \Omega_{(k')}^q \to \Omega_{(k)}^q$ corresponding to the mapping $\rho_{(k')(k)} : T^{(k)}M \to T^{(k')}M$, and, if $k'' \leqslant k' \leqslant k$, then

$$\rho_{(k')(k)}^* \circ \rho_{(k'')(k')}^* = \rho_{(k'')(k)}^*.$$

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(1) The mappings $\mathbf{T}^\alpha$ are related to the *holonomic lift* λ defined by KUMPERA [3].

Hence $(\Omega^q_{(k)}, \rho^*_{(k')\,(k)})$ is a directed system. Let Ω^q denote the direct limit of this system, and let Ω be the graded linear space $\{ \Omega^q \}$. The underlying set of Ω^q is the quotient set of $\bigcup_k \Omega^q_{(k)}$ by the equivalence relation according to which two forms $\mu \in \Omega^q_{(k)}$ and $\nu \in \Omega^q_{(k')}$ are equivalent if $k' \leqslant k$ and $\mu = \rho^*_{(k')\,(k)}\,\nu$, or $k' \geqslant k$ and $\nu = \rho^*_{(k)\,(k')}\,\mu$. The exterior differential d and the exterior product $\wedge$ extend in a natural way to the direct limits giving the graded linear space Ω the structure of both a cochain complex and a commutative graded algebra. We write $\mu \in \Omega^q_{(k)}$ for an element μ of Ω^q if μ has a representative in $\Omega^q_{(k)}$. This notation could be justified by identifying $\Omega^q_{(k)}$ with the image of the canonical injection $\Omega^q_{(k)} \to \Omega^q$. A collection $a = \{ a^q \}$ of linear mappings $a^q : \Omega^q \to \Omega^{q+r} : \mu \to a^q\,\mu$ is called a graded linear mapping of degree r. We write a instead of a^q if this can be done without causing any confusion. The exterior differential d is a graded linear mapping of degree 1.

DEFINITION 2.1. — A graded linear mapping $a = \{ a^q \}$ of degree r is called a *derivation* of Ω of degree r if

$$a(\mu \wedge \nu) = a\,\mu \wedge \nu + (-1)^{qr}\,\mu \wedge a\,\nu, \qquad \text{where} \quad q = \text{degree}\,\mu.$$

The exterior differential d is a derivation of Ω of degree 1. If a and b are derivations of Ω of degrees r and s respectively, then

$$[a, b] = \{ a^{q+s}\,b^q - (-1)^{rs}\,b^{q+r}\,a^q \}$$

is a derivation of Ω of degree $r+s$ called the commutator of a and b.

It follows from the general theory of derivations [2] that derivations of Ω are completely characterized by their action on Ω^0 and Ω^1. In fact, a derivation is completely determined by its action on equivalence classes of f_n and df_n for each function f on M and each $n \in \mathbf{N}^p$. Following FRÖLICHER and NIJENHUIS [2], we call a derivation a a derivation of type i_* if it acts trivially on Ω^0. We call a a derivation of type d_* if $[a, d] = 0$.

For each $m \in \mathbf{N}^p$, each $k \in \mathbf{N}$ and each $q > 0$ there is a linear mapping

$$i_{\mathbf{F}_m} : \quad \Omega^q_{(k)} \to \Omega^q_{(k)} : \quad \mu \mapsto i_{\mathbf{F}_m}\,\mu,$$

defined by

$$\begin{aligned} &\langle w_1 \wedge \ldots \wedge w_q,\ i_{\mathbf{F}_m}\,\mu \rangle \\ &\quad = \langle \mathbf{F}_m(w_1) \wedge w_2 \wedge \ldots \wedge w_q,\ \mu \rangle \\ &\qquad + \langle w_1 \wedge \mathbf{F}_m(w_2) \wedge \ldots \wedge w_q,\ \mu \rangle + \ldots + \langle w_1 \wedge w_2 \wedge \ldots \wedge \mathbf{F}_m(w_q),\ \mu \rangle, \end{aligned}$$

where $w_1, \ldots, w_q$ are vectors in $TT^{(k)}M$ such that $\tau_{T^{(k)}M}(w_1) = \ldots = \tau_{T^{(k)}M}(w_q)$ and $\mathbf{F}_m : TT^{(k)}M \to TT^{(k)}M$ is the mapping defined in Section 1. Due to commutativity of diagrams

$$\begin{array}{ccc} \Omega^q_{(k')} & \xrightarrow{i_{\mathbf{F}_m}} & \Omega^q_{(k')} \\ {\scriptstyle \rho^*_{(k')(k)}} \downarrow & & \downarrow {\scriptstyle \rho^*_{(k')(k)}} \\ \Omega^q_{(k)} & \xrightarrow{i_{\mathbf{F}_m}} & \Omega^q_{(k)} \end{array}$$

the mappings $i_{\mathbf{F}_m}$ extend to a derivation $i_{\mathbf{F}_m}$ of Ω of type i_* and degree 0. If $\mu \in \Omega^q_{(k)}$, then $i_{\mathbf{F}_m}\mu \in \Omega^q_{(k)}$ and $i_{\mathbf{F}_m}\mu = 0$ if $\mu \in \Omega^q_{(k)}$ and $|m| > k$.

For each $\alpha = 1, \ldots, p$, each $k \in \mathbf{N}$, and each $q \in \mathbf{N}$, there is a linear mapping

$$i_{\mathbf{T}^\alpha} : \ \Omega^{q+1}_{(k)} \to \Omega^q_{(k+1)} : \ \mu \mapsto i_{\mathbf{T}^\alpha}\mu,$$

defined by

$$\langle w_1 \wedge \ldots \wedge w_q, i_{\mathbf{T}^\alpha}\mu \rangle = \langle x \wedge u_1 \wedge \ldots \wedge u_q, \mu \rangle,$$

where

$$x = \mathbf{T}^\alpha(v), \qquad v = \tau_{T^{(k+1)}M}(w_1) = \ldots = \tau_{T^{(k+1)}M}(w_q),$$

$$u_1 = T\rho_{(k+1),(h)}(w_1), \qquad \ldots, \qquad u_q = T\rho_{(k+1),(h)}(w_q),$$

and $\mathbf{T}^\alpha : T^{(k+1)}M \to TT^{(k)}M$ is the mapping defined in Section 1. Due to commutativity of diagrams

$$\begin{array}{ccc} \Omega^{q+1}_{(k')} & \xrightarrow{i_{\mathbf{T}^\alpha}} & \Omega^q_{(k'+1)} \\ {\scriptstyle \rho^*_{(k')(k)}} \downarrow & & \downarrow {\scriptstyle \rho^*_{(k'+1)(k+1)}} \\ \Omega^{q+1}_{(k)} & \xrightarrow{i_{\mathbf{T}^\alpha}} & \Omega^q_{(k+1)} \end{array}$$

the mappings $i_{\mathbf{T}^\alpha}$ extend to a derivation $i_{\mathbf{T}^\alpha}$ of Ω of type i_* and degree -1. A derivation $d_{\mathbf{T}^\alpha}$ of Ω of type d_* and degree 0 is defined by $d_{\mathbf{T}^\alpha} = [i_{\mathbf{T}^\alpha}, d]$. If $\mu \in \Omega^{q+1}_{(k)}$, then $i_{\mathbf{T}^\alpha}\mu \in \Omega^q_{(k+1)}$, and $d_{\mathbf{T}^\alpha}\mu \in \Omega^{q+1}_{(k+1)}$.

For each $\alpha = 1, \ldots, p$ let e^α denote the element $(e^\alpha_1, \ldots, e^\alpha_p)$ of $\mathbf{N}^p$ defined by $e^\alpha_\beta = 1$ if $\alpha = \beta$, and $e^\alpha_\beta = 0$ if $\alpha \neq \beta$. Let $\geqslant$ denote the partial ordering relation in $\mathbf{N}^p$ defined by $(n_1, \ldots, n_p) \geqq (n'_1, \ldots, n'_p)$ if

$$n_1 \geqslant n'_1, \ \ldots, \ n_{p-1} \geqslant n'_{p-1} \qquad \text{and} \qquad n_p \geqslant n'_p.$$

For each $m \in \mathbf{N}^p$, let $m!$ denote $m_1! \ldots m_p!$.

PROPOSITION 2.1. — *If $m \geqslant e^\alpha$ then*

$$[i_{F_m}, d_{T^\alpha}] = \frac{m!}{(m-e^\alpha)!} i_{F_{m-e^\alpha}}, \qquad \textit{and} \qquad [i_{F_m}, d_{T\alpha}] = 0$$

in all cases other than $m \geqslant e^\alpha$.

Proof. — The commutator $[i_{F_m}, d_{T^\alpha}]$ is a derivation and it is of type i_* since it acts trivially on Ω. It can be easily shown for each $n \in \mathbf{N}^p$ and each function f on M that $i_{F_m} df_n = (n!/(n-m)!)\, df_{n-m}$ if $n \geqslant m$, and $i_{F_m} df_n = 0$ in all other cases. Also $d_{T^\alpha} f_n = f_{n+e^\alpha}$. It follows that

$$[i_{F_m}, d_{T^\alpha}]\, df_n = \frac{m!}{(m-e^\alpha)!} i_{F_{m-e^\alpha}} df_n \qquad \text{if} \quad m \geqslant e^\alpha,$$

and $[i_{F_m}, d_{T^\alpha}]\, df_n = 0$ in all cases other than $m \geqslant e^\alpha$. This completes the proof since a derivation of type i_* is completely determined by its action on equivalence classes of df_n for each f and each $n \in \mathbf{N}^p$.

PROPOSITION 2.2. — *For each $\alpha, \beta = 1, \ldots, p$, $[d_{T^\alpha}, d_{T^\beta}] = 0$.*

Proof. — Obvious.

3 The Lagrange complex (Λ, δ) (2)

Let $\tau = \{ \tau^q \}$ be the graded linear mapping of Ω into Ω of degree 0 defined by $\tau^0 = 1$ and

$$\tau^q \mu = \frac{1}{q} \sum_{|m| \leqslant k} (-1)^{|m|} (m!)^{-1} d_T^m i_{F_m} \mu,$$

where $q > 0$, $\mu \in \Omega^p_{(k)}$ and $d_T^m = (d_{T^1})^{m_1} \ldots (d_{T_p})^{m_p}$. The sum in the above definition contains all nonzero terms $(-1)^{|m|} (m!)^{-1} d_T^m i_{F_m} \mu$ since $i_{F_m} \mu = 0$ unless $|m| \leqslant k$. We write

$$\tau^q = \frac{1}{q} \sum_m (-1)^{|m|} (m!)^{-1} d_T^m i_{F_m}$$

without explicitely restricting the summation range which is understood to be wide enough to include in the sum all nonzero terms when τ^q is applied to an element of Ω^q.

PROPOSITION 3.1. — *If $q > 0$, then $\tau^q d_{T^\alpha} = 0$ for each $\alpha = 1, \ldots, p$.*

(2) For definitions of algebraic topology terms used in this and the following sections, *see* reference [5].

Proof :

$$\tau^q d_{T^\alpha} = \frac{1}{q}\sum_m (-1)^{|m|}(m!)^{-1} d_T^m i_{F_m} d_{T^\alpha}$$

$$= \frac{1}{q}\sum_m (-1)^{|m|}(m!)^{-1} (d_T^{m+e^\alpha} i_{F_m} + d_T^m [i_{F_m}, d_{T^\alpha}])$$

$$= \frac{1}{q}\sum_m (-1)^{|m|}(m!)^{-1} d_T^{m+e^\alpha} i_{F_m}$$

$$+ \frac{1}{q}\sum_{m \geqslant e^\alpha} (-1)^{|m|}((m-e^\alpha)!)^{-1} d_T^m i_{F_{m-e^\alpha}} = 0.$$

It follows from proposition 3.1, that $\tau\tau = \tau$ and $\tau\, d\tau = \tau\, d$.

PROPOSITION 3.2. – *The graded linear mapping* $\tau\, d = \{ \tau^{q+1} d^q \}$ *is a differential of degree* 1.

Proof. – $\tau\, d\tau\, d = \tau\, dd = 0$ and degree $(\tau\, d)$ = degree τ + degree $d = 1$.

We introduce the graded linear space $\Lambda = \{ \Lambda^q \}$, where $\Lambda^q = \operatorname{im} \tau^q$. The differential $\tau\, d$ can be restricted to Λ due to $\tau\, d\tau = \tau\, d$.

The restriction of $\tau\, d$ to Λ is a differential of degree 1 denoted by δ.

DEFINITION 3.1. – The differential $\delta = \{ \delta^q \}$ is called the *Lagrange differential,* and the cochain complex $\{ \Lambda^q, \delta^q \}$ is called the *Lagrange complex.*

THEOREM 3.1 (δ-*Poincaré lemma*). – *If the manifold M is contractible then the Lagrange complex* $\{ \Lambda^q, \delta^q \}$ *is acyclic for* $q > 0$.

Let $\mathbf{R}$ denote the subspace of $\Lambda^0 = \Omega^0$ consisting of equivalence classes of constant functions and let $\gamma : G \to \Lambda^0$ be the canonical injection of the subspace $G = \mathbf{R} \oplus (d_{T^1}(\Omega^0) + \ldots + d_{T^p}(\Omega^0))$.

THEOREM 3.2. – *The mapping* $\gamma : G \to \Lambda^0$ *is an augmentation of the Lagrange complex and the sequence*

$$0 \to G \xrightarrow{\gamma} \Lambda^0 \xrightarrow{\delta^0} \Lambda^1 \xrightarrow{\delta^1} \ldots \xrightarrow{\delta^{q-1}} \Lambda^q \xrightarrow{\delta^q} \ldots$$

is a resolution of G.

We give proofs of the two theorems in the following section after having constructed a resolution of the graded linear space $\Lambda' = \{ \Lambda^q \}_{q>0}$.

4. A resolution of Λ'

Let K be the simplicial complex with vertices $1, \ldots, p$, and let $\Delta_r(K)$ denote the free abelian group generated by the ordered r-simplexes of K [5].

We introduce a bigraded linear space $\Phi = \{ \Phi^q_r \}$, where $\Phi^q_r = \Delta_{r-1}(K) \otimes \Omega^q$ for $r > 0$, $\Phi^p_0 = \Omega^q$, and $\Phi^p_r = 0$ for $r < 0$. Elements of Φ^p_r are said to be of bidegree (q, r). The exterior differential in Ω is extended to a bigraded linear mapping $d = \{ d^q_r \}$ of bidegree $(1, 0)$ by the formula

$$d^q_r((\alpha_1, \ldots, \alpha_r) \otimes \mu) = (\alpha_1, \ldots, \alpha_r) \otimes d\mu,$$

where $(\alpha_1, \ldots, \alpha_r)$ is an ordered $r+1$-simplex and $\mu \in \Omega^q$. A bigraded linear mapping $\partial = \{ \partial^p_r \}$ of bidegree $(0, -1)$ is defined by

$$\partial^q_r((\alpha_1, \ldots, \alpha_r) \otimes \mu) = \sum_{1 \leqslant i \leqslant r} (-1)^{i-1} (\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_r) \otimes d_{\mathbf{T}^{\alpha_i}} \mu.$$

For each fixed r, $\{ \Phi^q_r, d^q_r \}$ is a cochain complex, and for each fixed q, $\{ \Phi^q_r, \partial^q_r \}$ is a chain complex. Since $\partial^{q+1}_r d^q_r = d^q_{r-1} \partial^q_r$, for each fixed r the collection $\{ \partial^q_r : \Phi^q_r \to \Phi^q_{r-1} \}$ is a cochain mapping, and for each fixed q the collection $\{ d^q_r : \Phi^q_r \to \Phi^{q+1}_r \}$ is a chain mapping.

PROPOSITION 4.1. — *For each fixed $q > 0$ the chain complex $\{ \Phi^q_r, \partial^q_r \}$ is acyclic for $r > 0$.*

Proof. — For each $\alpha = 1, \ldots, p$, let a graded linear mapping

$$\sigma_\alpha = \{ \sigma^q_a : \Omega^q \to \Omega^q \}$$

be defined by $\sigma^0_\alpha = 0$ and

$$\sigma^q_\alpha = -\frac{1}{q} \sum_{m \in I_\alpha} (-1)^{|m|} (m!)^{-1} d^m_{\mathbf{T}} i_{\mathbf{F}_m}, \qquad \text{where} \quad q > 0,$$

$I_\alpha = \{ m \in \mathbf{N}^p ; m_\alpha > 0, m_\beta = 0 \text{ for } \beta > \alpha \}$ and the summation range is governed by a convention similar to the one used in the definition of τ in Section 3. From Proposition 2.1, it follows easily for $q > 0$ that $\sigma^q_\alpha d_{\mathbf{T}^\beta} = 0$ if $\beta < \alpha$, $\sigma^q_\alpha d_{\mathbf{T}^\alpha} = 1 - \sum_{\gamma < \alpha} d_{\mathbf{T}^\gamma} \sigma^q_\gamma$, and $\sigma^q_\alpha d_{\mathbf{T}^\beta} = d_{\mathbf{T}^\beta} \sigma^q_\alpha$ if $\beta > \alpha$. A bigraded linear mapping $D = \{ D^q_r \}$ is defined by $D^q_0 \mu = \sum_\beta (\beta) \otimes \sigma^q_\beta \mu$ and

$$D^q_r((\alpha_1, \ldots, \alpha_r) \otimes \mu) = \sum_{\beta < \alpha_1} (\beta, \alpha_1, \ldots, \alpha_r) \otimes \sigma^q_\beta \mu,$$

where $\mu \in \Omega^q$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_r$. Relations $\partial^q_{r+1} D^q_r + D^q_{r-1} \partial^q_r = 1$ for $r > 0$, $q > 0$ are readily verified using the above stated properties of σ_α. It follows that for each fixed $q > 0$ the graded mapping $D^q = \{ D^q_r \}$ defines a chain contraction of $\{ \Phi^q_r, \partial^q_r \}$ for $r > 0$. Hence $\{ \Phi^q_r, \partial^q_r \}$ is acyclic for $r > 0$.

PROPOSITION 4.2. — *For each $q > 0$, the mapping $\tau^q : \Phi^q_0 \to \Lambda^q$ is an augmentation of the chain complex $\{ \Phi^q_r, \partial^q_r \}$ and the sequence*

$$\ldots \to \Phi^q_r \xrightarrow{\delta^q_r} \Phi^q_{r-1} \xrightarrow{\delta^q_{r-1}} \ldots \xrightarrow{\delta^q_1} \Phi^q_0 \xrightarrow{\tau^q} \Lambda^q \to 0$$

is a resolution of Λ^q.

Proof. – The mapping $\tau^q : \Omega^q \to \Lambda^q$ is an epimorphism, and $\tau^q \partial_1^q = 0$ follows from Proposition 3.1. Further $\tau^q + \partial_1^q D_0^q = 1$, where D_0^q is the mapping defined in the proof of Proposition 4.1. Hence $\tau^q \mu = 0$ implies $\mu = \partial_1^q D_0^q \mu$ for each $\mu \in \Omega^q$. It follows that $\ker \tau^q = \operatorname{im} \partial_1^q$.

Proof of Theorems 3.1 *and* 3.2. – We define a nonnegative graded linear space $C = \{ C_r \}$ by $C_0 = \mathbf{R}$ and $C_r = \Delta_{r-1}(K) \otimes \mathbf{R}$ for $r > 0$, and a collection $\eta = \{ \eta_r : C_r \to \Phi_r^0 \}$ by $\eta_r = 1 \otimes \eta_0$, where $\eta_0 : \mathbf{R} \to \Omega^0$ is the canonical injection of the space $\mathbf{R} \subset \Omega^0$ of equivalence classes of constant functions identified with the field $\mathbf{R}$ of constants. If the manifold M is contractible, then all rows except the bottom row of the commutative diagram

$$
\begin{array}{ccccccccccccc}
 & & & & 0 & & 0 & & & & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \to & C_p & \xrightarrow{\eta_p} & \Phi_p^0 & \xrightarrow{a_p} & \Phi_p^1 & \xrightarrow{d_p^1} & \dots & \to & \Phi_p^q & \xrightarrow{d_p^q} & \dots \\
 & & & & \downarrow{\scriptstyle \partial_p^0} & & \downarrow{\scriptstyle \partial_p^1} & & & & \downarrow{\scriptstyle \partial_p^q} & & \\
 & & & & \vdots & & \vdots & & & & \vdots & & \\
 & & & & \downarrow{\scriptstyle \partial_1^0} & & \downarrow{\scriptstyle \partial_1^1} & & & & \downarrow{\scriptstyle \partial_q^1} & & \\
0 & \to & C_0 & \xrightarrow{\eta_0} & \Phi_0^0 & \xrightarrow{d_0^0} & \Phi_0^1 & \xrightarrow{d_0^1} & \dots & \to & \Phi_0^q & \xrightarrow{d_0^p} & \dots \\
 & & & & \downarrow{\scriptstyle \tau^0} & & \downarrow{\scriptstyle \tau^1} & & & & \downarrow{\scriptstyle \tau^q} & & \\
0 & \to & G & \xrightarrow{\gamma} & \Lambda^0 & \xrightarrow{\delta^0} & \Lambda^1 & \xrightarrow{\delta^1} & \dots & \to & \Lambda^q & \xrightarrow{\delta^q} & \dots \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & & & 0 & & 0 & & & & 0 & &
\end{array}
$$

are known to be exact and all columns for $q > 0$ are exact. For each $q > 0$, the top tatement in the sequence

$$\ker(\partial_p^{q+p+1} d_p^{q+p}) = \operatorname{im} d_p^{q+p-1},$$

$$\ker(\partial_{p-1}^{q+p} d_{p-1}^{q+p-1}) = \operatorname{im} d_{p-1}^{q+p-2} + \operatorname{im} \partial_p^{q+p-1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\ker(\partial_1^{q+2} d_1^{q+1}) = \operatorname{im} d_1^q + \operatorname{im} \partial_2^{q+1},$$

$$\ker(\tau^{q+1} d_0^q) = \operatorname{im} d_0^{q-1} + \operatorname{im} \partial_1^q,$$

is true, and each of the remaining statements follows from the one immediately above. Hence the bottom statement is true. The same holds for $q = 0$ if the bottom statement is replaced by

$$\ker(\tau^1 d_0^0) = \operatorname{im} \eta_0 \otimes \operatorname{im} \partial_1^0.$$

If $q > 0$ and μ is an element of $\Lambda^q \subset \Omega^q$, then $\tau^q \mu = \mu$, and $\delta^q \mu = \tau^{q+1} d_0^q \mu$. If $\delta^q \mu = 0$, then there are elements $\varkappa \in \Phi_0^{q-1}$ and $\lambda \in \Phi_1^q$ such that $\mu = d_0^{q-1} \varkappa + \partial_1^q \lambda$. It follows that

$$\mu = \tau^q \mu = \tau^q d_0^{q-1} \varkappa = \tau^q d_0^{q-1} \tau^{q-1} \mathrm{q} = \delta^{q-1} \tau^{q-1} \mathrm{q}.$$

Hence $\ker \delta^q = \operatorname{im} \delta^{q-1}$ and the Lagrange complex is acyclic for $q > 0$. We note that $\delta^0 = \tau^1 d_0^0$ and

$$G = \mathbf{R} \otimes (d_{\mathbf{T}^1}(\Omega^0) + \ldots + d_{\mathbf{T}^p}(\Omega^0)) = \operatorname{im} \chi_0 \otimes \operatorname{im} \partial_1^0.$$

Hence $\ker \delta^0 = G$. It follows that the sequence

$$0 \to G \xrightarrow{\gamma} \Lambda^0 \xrightarrow{\delta^0} \Lambda^1 \xrightarrow{\delta^1} \ldots \to \Lambda^q \xrightarrow{\delta^q} \ldots$$

is exact.

5 . Applications of the δ-Poincaré lemma in the calculus of variations

A smooth mapping $\chi : \mathbf{R}^{p+1} \to M : (s, t_1, \ldots, t_p) \mapsto \chi(s, t_1, \ldots, t_p)$ will be called a *homotopy*. For each $s \in \mathbf{R}$, we denote by χ_s the mapping

$$\chi_s : \quad \mathbf{R}^p \to M : (t_1, \ldots, t_p) \mapsto \chi(s, t_1, \ldots, t_p).$$

The mapping $\gamma = \chi_0$ will be called the *base* of the homotopy χ. We say that the homotopy χ is *constant* on $A \subset \mathbf{R}^p$ if $\chi(s, t_1, \ldots, t_p) = \chi(0, t_1, \ldots, t_p)$ for each $s \in \mathbf{R}$ and each $(t_1, \ldots, t_p) \in A$. For each mapping

$$\varphi : \quad \mathbf{R}^p \to M : (t_1, \ldots, t_p) \mapsto \varphi(t_1, \ldots, t_p),$$

we denote by $\varphi^{(k)}$ the mapping

$$\varphi^{(k)} : \quad \mathbf{R}^p \to T^{(k)} M : (t_1, \ldots, t_p) \mapsto j^{(k)}_{(t_1, \ldots, t_p)}(\varphi).$$

For each homotopy χ, we denote by $\chi'^{(k)}$ the mapping

$$\chi'^{(k)} : \quad \mathbf{R}^p \to TT^{(k)} M : (t_1, \ldots, t_p) \mapsto j^{(1,k)}_{(0, t_1, \ldots, t_p)}(\chi),$$

where $j^{(1,k)}_{(0, t_1, \ldots, t_p)}(\chi)$ is a jet-like object similar to $j_0^{(1,k)}(\chi)$ defined in terms of partial derivatives at $(0, t_1, \ldots, t_p)$ instead of $(0, 0, \ldots, 0)$ and identified with an element of $TT^{(k)} M$.

Each element $L \in \Omega^0_{(k)}$ gives rise to a family of functions

$$\gamma \mapsto \int_V L \circ \gamma^{(k)},$$

defined on the set of smooth mappings of $\mathbf{R}^p$ into M for each domain $V \subset \mathbf{R}^p$.

DEFINITION 5.1. — A mapping $\gamma : \mathbf{R}^p \to M$ is called an *extremal* of the family of functions

$$\gamma \mapsto \int_V L \circ \gamma^{(k)} \qquad \text{if} \quad \frac{d}{ds} \int_V L \circ \chi_s^{(k)} \bigg|_{s=0} = 0,$$

for each domain $V \subset \mathbf{R}^p$ and each homotopy χ with base γ constant on the boundary ∂V of V.

DEFINITION 5.2. — A form $\lambda \in \Omega^1_{(k')}$ is called an *Euler-Lagrange* form associated with $L \in \Omega^0_{(k)}$ if $i_{\mathbf{F}_m} \lambda = 0$ for each $m > 0$ and if

$$\int_V \langle \chi'^{(k)}, dL \rangle = \int_V \langle \chi'^{(k')}, \lambda \rangle$$

for each domain $V \subset \mathbf{R}^p$ and each homotopy χ constant on ∂V.

It is clear from the definition of $\mathbf{F}_m$ that if $\lambda \in \Omega^1_{(k')}$ satisfies $i_{\mathbf{F}_m} \lambda = 0$ for each $m > 0$, then λ can be interpreted as a mapping $\lambda : T^{(k')} M \to T^* M$. If λ is an Euler-Lagrange form associated with L then

$$\begin{aligned} \frac{d}{ds} \int_V L \circ \chi_s^{(k)} \bigg|_{s=0} &= \int_V \langle \chi'^{(k)}, dL \rangle \\ &= \int_V \langle \chi'^{(k')}, \lambda \rangle \\ &= \int_V \langle \chi'^{(0)}, \lambda \circ \gamma^{(k')} \rangle, \end{aligned}$$

for each homotopy χ with base γ constant on ∂V. It follows that $\gamma : \mathbf{R}^p \to M$ is an extremal of the family

$$\gamma \to \int_V L \circ \gamma^{(k)},$$

if, and only if, γ satisfies the equation $\lambda \circ \gamma^{(k')} = 0$ called the *Euler-Lagrange equation.*

We show that $\lambda = \delta^0 L$ is the unique Euler-Lagrange form associated with $L \in \Omega^0$. We also show that $i_{\mathbf{F}_m} \lambda = 0$ for each $m > 0$ means that $\lambda \in \Omega^1$ is in Λ^1. These statements imply applications of the δ-Poincaré lemma. A form $\lambda \in \Omega^1$ is an Euler-Lagrange form if, and only if, $\lambda \in \Lambda^1$ and $\delta^1 \lambda = 0$. Euler-Lagrange forms associated with two elements L and L' of Ω^0 are the same if, and only if, $L' - L \in \mathbf{R} \oplus (d_{\mathbf{T}^1}(\Omega^0) + \ldots + d_{\mathbf{T}^p}(\Omega^0))$.

PROPOSITION 5.1. — *A form* $\lambda \in \Omega^1$ *belongs to* Λ^1 *if, and only if,* $i_{\mathbf{F}_m} \lambda = 0$ *for each* $m > 0$.

Proof. — If $i_{\mathbf{F}_m} \lambda = 0$ for each $m > 0$, then

$$\tau^1 \lambda = \sum_m (-1)^{|m|} (m!)^{-1} d_{\mathbf{T}}^m i_{\mathbf{F}_m} \lambda = i_{\mathbf{F}_0} \lambda = \lambda.$$

Hence $\lambda \in \operatorname{im} \tau^1 = \Lambda^1$. From Proposition 2.1, it follows that

$$i_{\mathbf{F}_{e^\alpha}} d_{\mathbf{T}}^m = d_{\mathbf{T}}^m i_{\mathbf{F}_{e^\alpha}} + (m!/(m-e^\alpha)!)\, d_{\mathbf{T}}^{m-e^\alpha} i_{\mathbf{F}_0}$$

if $m \geqslant e^\alpha$ and $i_{\mathbf{F}_{e^\alpha}} d_{\mathbf{T}}^m = d_{\mathbf{T}}^m i_{\mathbf{F}_{e^\alpha}}$ in all other cases. Since $i_{\mathbf{F}_m} i_{\mathbf{F}_n} \mu = i_{\mathbf{F}_{m+n}} \mu$ for each $\mu \in \Omega^1$, it follows that

$$\begin{aligned} i_{\mathbf{F}_{e^\alpha}} \tau^1 &= \sum_m (-1)^{|m|} (m!)^{-1} i_{\mathbf{F}_{e^\alpha}} d_{\mathbf{T}}^m i_{\mathbf{F}_m} \\ &= \sum_m (-1)^{|m|} (m!)^{-1} d_{\mathbf{T}}^m i_{\mathbf{F}_{m+e^\alpha}} \\ &\quad + \sum_{m \geqslant e^\alpha} (-1)^{|m|} ((m-e^\alpha)!)^{-1} d_{\mathbf{T}}^{m-e^\alpha} i_{\mathbf{F}_m} = 0. \end{aligned}$$

Consequently, $i_{\mathbf{F}_m} \tau^1 = 0$ for each $m > 0$, and if $\lambda \in \Lambda^1$ then $i_{\mathbf{F}_m} \lambda = 0$ for each $m > 0$.

PROPOSITION 5.2. — *The space* Ω^1 *is the direct sum of* Λ^1 *and*

$$d_{\mathbf{T}^1}(\Omega^1) + \ldots + d_{\mathbf{T}^p}(\Omega^1).$$

Proof. — Let μ be an element of Ω^1. Then $\mu = \lambda + \nu$, where $\lambda = \tau^1 \mu \in \Lambda^1$, and

$$\nu = -\sum_{m>0} (-1)^{|m|} (m!)^{-1} d_{\mathbf{T}}^m i_{\mathbf{F}_m} \mu \in d_{\mathbf{T}^1}(\Omega^1) + \ldots + d_{\mathbf{T}^p}(\Omega^1).$$

It follows from $\tau^1 \tau^1 = \tau^1$ and $\tau^1 d_{\mathbf{T}^\alpha} = 0$ that this decomposition of μ into elements of Λ^1 and $d_{\mathbf{T}^1}(\Omega^1) + \ldots + d_{\mathbf{T}^p}(\Omega^1)$ is unique.

PROPOSITION 5.3. — *Let* μ *be an element of* $\Omega^1_{(k)}$. *Then*

$$\int_V \langle \chi'^{(k)}, \mu \rangle = 0,$$

for each domain $V \subset \mathbf{R}^p$ *and each homotopy* $\chi : \mathbf{R}^{p+1} \to M$ *constant on* ∂V *if, and only if,* $\mu \in d_{\mathbf{T}^1}(\Omega^1) + \ldots + d_{\mathbf{T}^p}(\Omega^1)$.

Proof. — If $\mu = \sum_\alpha d_{\mathbf{T}^\alpha} \omega^\alpha$ then

$$\int_V \langle \chi'^{(k)}, \mu \rangle = \sum_\alpha \int_V \frac{\partial}{\partial t^\alpha} \langle \chi'^{(k)}, \omega^\alpha \rangle = \sum_\alpha \int_{\partial V} n_\alpha \langle \chi'^{(k)}, \omega^\alpha \rangle,$$

where n_α are the components of the normal vector. If χ is constant on ∂V, then

$$\int_V \langle \chi'^{(k)}, \mu \rangle = 0.$$

Let $\mu = \lambda + \nu$ be the unique decomposition of $\mu \in \Omega^1$ used in the proof of proposition 5.2. If $\int_V \langle \chi'^{(k)}, \mu \rangle = 0$, then

$$\int_V \langle \chi'^{(k)}, \lambda \rangle = \int_V \langle \chi'^{(0)}, \lambda \circ \gamma^{(k')} \rangle = 0,$$

where γ is the base of χ, and λ is interpreted as a mapping $\lambda : T^{(k)} M \longrightarrow T^* M$. It follows that $\lambda = 0$ and $\mu = \nu$. Hence $\mu \in d_{\mathbf{T}^1}(\Omega^1) + \ldots + d_{\mathbf{T}^p}(\Omega^1)$.

COROLLARY. — *If L is an element of Ω^0, then $\lambda = \delta^0 L$ is the unique element of Λ^1 such that $dL - \lambda \in d_{\mathbf{T}^1}(\Omega^1) + \ldots + d_{\mathbf{T}^p}(\Omega^1)$. It follows that λ is the unique Euler-Lagrange form associated with L.*

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