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**ELEMENTARY PROOF OF A THEOREM
OF BRUHAT-TITS-ROUSSEAU AND OF A THEOREM OF TITS**

BY

GOPAL PRASAD (*)

RÉSUMÉ. — Nous donnerons une démonstration élémentaire d'un théorème de Bruhat, Tits et Rousseau, et aussi d'un théorème de Tits.

ABSTRACT. — We give an elementary proof of a theorem of Bruhat, Tits and Rousseau, and also of a theorem of Tits.

Let k be a field with a non-trivial non-archimedean valuation v . We shall assume that the valuation v has a (up to equivalence) unique extension to any finite field extension of k , or, equivalently, k is *henselian* for v (i. e. the Hensel's lemma holds in k with respect to v). We fix an algebraic closure \mathcal{K} of k and shall denote the unique valuation on it, which extends the given valuation on k , again by v . Let K be the separable closure of k in \mathcal{K} ; the extended valuation on K is obviously invariant under the Galois group $\text{Gal}(K/k)$.

Let V be a finite dimensional k -vector space. Let G be a connected reductive k -subgroup of $\text{SL}(V)$. For any extension L of k contained in \mathcal{K} , let $G(L)$ be the group of L -rational points of G endowed with the Hausdorff topology and the bornology induced by the valuation on L . Let $G(k)^+$ be the normal subgroup of $G(k)$ generated by the k -rational points of the unipotent radicals of parabolic k -subgroups of G .

G is said to be *isotropic* over k if G contains a non-trivial k -split torus, and *k -anisotropic* (or *anisotropic over k*) otherwise.

The object of this note is to give a simple proof of the following theorem proved first by F. Bruhat and J. Tits in case k is a discretely valued complete field with perfect residue field and then in general by G. Rousseau in his thesis (Orsay, 1977).

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THEOREM (BTR). — $G(k)$ is bounded if and only if G is anisotropic over k .

Remark. — Thus in case k is a non-discrete locally compact field, $G(k)$ is compact if and only if G is anisotropic over k .

We shall also give a simple proof of the following (unpublished) theorem of J. Tits:

THEOREM (T). — Let G be semi-simple and almost k -simple. Then any proper open subgroup of $G(k)^+$ is bounded.

Acknowledgment. — The proof, given below, of Theorem (BTR) is based on a suggestion of G. A. Margulis that it should be possible to use the following lemma (Lemma 1) to prove Theorem (BTR). I had originally used the lemma to give a simple proof of Theorem (T). The comments of J. Tits on an earlier version have led to further simplifications in the proofs of both the theorems. I thank Margulis and Tits heartily.

LEMMA 1. — Let H be a subgroup of $G(k)$ which is dense in G in the Zariski topology. Assume that H is unbounded. Then there is an element h of H which has an eigenvalue α with $v(\alpha) < 0$.

Proof. — Let :

$$V = V_0 \supset V_1 \supset \dots \supset V_r \supset V_{r+1} = \{0\},$$

be a flag of G -invariant vector subspaces (not necessarily defined over k) such that for $0 \leq i \leq r$, the natural representation ρ_i of G on $W_i = V_i/V_{i+1}$ is absolutely irreducible. Let $\rho (= \bigoplus_i \rho_i)$ be the natural representation of G on $\bigoplus W_i$; ρ is defined over a finite Galois extension of k . The kernel of ρ is obviously a unipotent normal subgroup of G , and as G is reductive, we conclude that ρ is faithful. Now, as H is an unbounded subgroup of $G(k)$, $\rho(H(k))$ is unbounded, and hence there is a non-negative integer a , $a \leq r$, such that $\rho_a(H(k))$ is unbounded.

Now assume, if possible, that the eigenvalues of all the elements of H lie in the local ring of the valuation on \mathcal{K} . Then, the trace form of ρ_a , restricted to H , also takes values in the local ring of the valuation (this ring is bounded!). But since W_a is an absolutely irreducible G -module, and since H is dense in G in the Zariski topology, $\rho_a(H)$ spans $\text{End}(W_a)$. So, in view of the non-degeneracy of the trace form, we conclude that $\rho_a(H)$ is bounded (see TITS [5], Lemma 2.2). This is a contradiction, which proves the lemma.

Proof of Theorem (BTR). — If T is a one-dimensional k -split torus, then $T(k)$ is isomorphic to k^* and hence it is unbounded. This implies that if G is isotropic over k , then $G(k)$ is unbounded. We shall now assume that $G(k)$ is unbounded and prove the converse.

It is well known that $G(k)$ is dense in G in the Zariski topology ([1], 18.3), hence, according to the preceding lemma, there is an element $g \in G(k)$ which has an eigenvalue α with $v(\alpha) \neq 0$. Now, in case k is of positive characteristic, after replacing g by a suitable power, we shall assume that g is semi-simple. In case k is of characteristic zero, let $g = u \cdot s = s \cdot u$ be the Jordan decomposition of g , with u (resp. s) unipotent (resp. semi-simple). Then $u, s \in G(k)$, and the eigenvalues of g are the same as that of s . Thus we may (and we shall), after replacing g by s , again assume that g is semi-simple.

Now there is a maximal torus S in G defined over k , such that $g \in S(k)$. (See BOREL-TITS [2], Proposition 10.3 and Theorem 2.14a; note that according to Theorem 11.10 of [1], g is contained in a maximal torus of G .) Since any absolutely irreducible representation of a torus is 1-dimensional, there is a character χ of S , χ defined over a finite galois extension \mathfrak{R} of k , such that $\chi(g) = \alpha$. Let $m = [\mathfrak{R}:k]$. Then:

$$v\left(\sum_{\gamma \in \text{Gal}(\mathfrak{R}/k)} \gamma \chi\right)(g) = mv(\chi(g)) = mv(\alpha) \neq 0.$$

Thus the character $\sum_{\gamma \in \text{Gal}(\mathfrak{R}/k)} \gamma \chi$ is non-trivial. On the other hand, it is obviously defined over k . Thus S admits a non-trivial character defined over k , and hence it contains a non-trivial k -split torus. This proves that in case $G(k)$ is unbounded, G is isotropic over k .

We shall now assume that G is semi-simple and almost k -simple.

NOTATION. — For $g \in G(k)$, let \mathcal{P}_g be the subset of $G(K)$ consisting of those x in $G(K)$ for which the sequence $\{g^i x g^{-i}\}_{i>0}$ is contained in a bounded subset of $G(K)$, and let \mathcal{U}_g be the subset consisting of those x in $G(K)$ for which the sequence $\{g^i x g^{-i}\}_{i>0}$ converges to the identity. It is obvious that \mathcal{P}_g is a subgroup of $G(K)$, and \mathcal{U}_g is a normal subgroup of \mathcal{P}_g .

We let \mathcal{P}_g^- denote $\mathcal{P}_{g^{-1}}$ and \mathcal{U}_g^- denote $\mathcal{U}_{g^{-1}}$.

In the sequel we shall denote the adjoint representation of an algebraic group on its Lie algebra by Ad .

LEMMA 2. — Let t be an element of $G(k)$ such that $\text{Ad } t$ has an eigenvalue α with $v(\alpha) \neq 0$. Then:

(i) \mathcal{P}_t is the group of K -rational points of a proper parabolic k -subgroup P_t of G and \mathcal{U}_t is the group of K -rational points of the unipotent radical U_t of P_t ;

(ii) P_t^- ($:= P_{t^{-1}}$) is opposed to P_t .

Proof. — Since for any integer $n > 0$, $\mathcal{P}_{t^n} = \mathcal{P}_t$ and $\mathcal{U}_{t^n} = \mathcal{U}_t$, in case k is of positive characteristic, after replacing t by a suitable (positive) power of t , we shall assume that t is semi-simple. In case k is of characteristic zero, let $t = u \cdot s = s \cdot u$ be the Jordan decomposition of t with u (resp. s) unipotent (resp. semi-simple). Then $s, u \in G(k)$ and the eigenvalues of $\text{Ad } t$ are the same as that of $\text{Ad } s$. Since the cyclic group generated by a unipotent element is bounded, we see easily that $\mathcal{P}_t = \mathcal{P}_s$ and $\mathcal{U}_t = \mathcal{U}_s$. Thus we may (and we shall) assume, after replacing t by s , that t is semi-simple.

Now there is a maximal torus T of G , defined over K , such that $t \in T(K)$. Since K is separably closed, any K -torus splits over K . Let $\mathfrak{t} + \sum_{\varphi \in \Phi} \mathfrak{g}^\varphi$ be the root space decomposition of the Lie algebra \mathfrak{g} of G with respect to T ; where \mathfrak{t} is the Lie algebra of T and Φ is the set of roots. According to BOURBAKI [4], Chapitre VI, paragraphe 1, Proposition 22, there is an ordering on Φ such that the subset $\{\varphi \mid \varphi \in \Phi, v(\varphi(t)) > 0\}$ is contained in the set Φ^+ of roots positive with respect to this ordering; let $\Delta \subset \Phi$ be the set of simple roots.

For a subset Θ of Δ , let T_Θ be the identity component of $\bigcap_{\theta \in \Theta} \text{Ker } \theta$ and let M_Θ be the centralizer of T_Θ in G . Let $\mathfrak{u}_\Theta = \sum_{\varphi \in \Phi^+ - \langle \Theta \rangle} \mathfrak{g}^\varphi$ (resp. $\mathfrak{u}_\Theta^- = \sum_{\varphi \in \Phi^+ - \langle \Theta \rangle} \mathfrak{g}^{-\varphi}$), and U_Θ (resp. U_Θ^-) be the connected unipotent K -subgroup of G , normalized by T , and with Lie algebra \mathfrak{u}_Θ (resp. \mathfrak{u}_Θ^-). Let $P_\Theta = M_\Theta \cdot U_\Theta$ and $P_\Theta^- = M_\Theta \cdot U_\Theta^-$. Then P_Θ and P_Θ^- are opposed parabolic K -subgroups of G , and if $\Theta \neq \Delta$, these subgroups are proper. Moreover, U_Θ (resp. U_Θ^-) is the unipotent radical of P_Θ (resp. P_Θ^-).

Now let $\Pi = \{\delta \in \Delta \mid v(\delta(t)) = 0\}$. Then since $\text{Ad } t$ has an eigenvalue α with $v(\alpha) \neq 0$, Π is a proper subset of Δ . It is obvious that $P_\Pi(K) \subset \mathcal{P}_t$, $P_\Pi^-(K) \subset \mathcal{P}_t^-$ and $U_\Pi(K) \subset \mathcal{U}_t$. Since \mathcal{P}_t contains $P_\Pi(K)$, it equals $P_\Theta(K)$ for a subset Θ of Δ , containing Π . But since the action of $\text{Ad } t$ on $\mathfrak{u}_\Pi^-(K)$ is "expanding", we conclude at once that $\Theta = \Pi$ and hence, $P_\Pi(K) = \mathcal{P}_t$. A similar argument shows that $P_\Pi^-(K) = \mathcal{P}_t^-$. We set $P_t = P_\Pi$ and $P_t^- = P_\Pi^-$.

To prove the second assertion of (i) we need to show that $U_\Pi(K) = \mathcal{U}_t$. For this purpose we observe that $U_\Pi(K) \subset \mathcal{U}_t$ and since

$$\mathcal{P}_t = P_\Pi(K) = M_\Pi(K) \cdot U_\Pi(K):$$

$$\mathcal{U}_t = (M_\Pi(K) \cap \mathcal{U}_t) \cdot U_\Pi(K);$$

also \mathcal{U}_t and hence $M_\Pi(K) \cap \mathcal{U}_t$ are normalized by $T(K)$. We now note that the Lie algebra of M_Π is $t + \sum_{\varphi \in \pm \langle \Pi \rangle} \mathfrak{g}^\varphi$, and $v(\varphi(t)) = 0$ for $\varphi \in \langle \Pi \rangle$. From these observations it is evident that $M_\Pi(K) \cap \mathcal{U}_t$ is trivial, and hence, $\mathcal{U}_t = U_\Pi(K)$.

Now to complete the proof of the lemma it only remains to show that both P_t and P_t^- are defined over k . But this is obvious, from the Galois criteria, in view of the fact that \mathcal{P}_t and \mathcal{P}_t^- are stable under $\Gamma = \text{Gal}(K/k)$ since t is a k -rational element, and $\mathcal{P}_t (= P_t(K))$ is dense in P_t , whereas $\mathcal{P}_t^- (= P_t^-(K))$ is dense in P_t^- in the Zariski topology.

LEMMA 3. — Let $t \in G(k)$ be such that $\text{Ad } t$ has an eigenvalue α with $v(\alpha) \neq 0$. Then $\mathcal{U}_t(k) (= \mathcal{U}_t \cap G(k))$ and $\mathcal{U}_t^-(k) (= \mathcal{U}_t^- \cap G(k))$ together generate $G(k)^+$.

Proof. — According to the preceding lemma, $\mathcal{U}_t(k)$ and $\mathcal{U}_t^-(k)$ are the groups of k -rational points of the unipotent radicals of two opposed proper parabolic k -subgroups of G . Hence, according to BOREL-TITS [3], Proposition 6.2v, $\mathcal{U}_t(k)$ and $\mathcal{U}_t^-(k)$ together generate $G(k)^+$.

Proof of Theorem (T). — Let \mathcal{G} be the adjoint group of G and $\pi: G \rightarrow \mathcal{G}$ be the natural (central) isogeny. Then since π is a finite morphism, the induced map $G(k) \rightarrow \mathcal{G}(k)$ is a proper map, i. e., the inverse image of a bounded subset of $\mathcal{G}(k)$ is bounded.

Now let H be an unbounded open subgroup of $G(k)^+$. Then, clearly, H is dense in G in the Zariski topology and hence, $\pi(H)$ is an unbounded Zariski-dense subgroup of $\mathcal{G}(k)$. Now Lemma 1 (applied to $\pi(H) (= \mathcal{G}(k))$) implies that there is an element h of H such that $\text{Ad } h$ has an eigenvalue α with $v(\alpha) \neq 0$.

Let $\mathcal{U}_h(k) = \mathcal{U}_h \cap G(k)$ and $\mathcal{U}_h^-(k) = \mathcal{U}_h^- \cap G(k)$. Then as H is open in $G(k)^+$, $H \cap \mathcal{U}_h(k)$ is an open subgroup of $\mathcal{U}_h(k)$, and obviously

$$\bigcup_{n>0} h^n (H \cap \mathcal{U}_h^-(k)) h^{-n} = \mathcal{U}_h^-(k).$$

Thus H contains both $\mathcal{U}_h(k)$ and $\mathcal{U}_h^-(k)$. But according to Lemma 3, $\mathcal{U}_h(k)$ and $\mathcal{U}_h^-(k)$ together generate $G(k)^+$. Therefore, $H = G(k)^+$. This proves the theorem.

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