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**ON THE BOREL CLASS
OF THE DERIVED SET OPERATOR, II**

BY

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RÉSUMÉ. — Soit X un espace non-énumérable topologique métrisable compact, 2^X l'espace topologique des compacts de X avec la topologie de Hausdorff et soit D la dérivation de Cantor. KURATOWSKI a démontré que D est borélienne et précisément de la deuxième classe, et a posé le problème de trouver la classe précise des dérivés successifs D^n . Nous démontrons que si n est fini, alors D^n est précisément de la classe $2n$ et si λ est un ordinal de seconde espèce et n fini, alors $D^{\lambda+n}$ est précisément de la classe $\lambda+2n+1$.

ABSTRACT. — KURATOWSKI showed that the derived set operator D , acting on the space of closed subsets of the Cantor space 2^N , is a Borel map of class exactly two and posed the problem of determining the precise classes of the higher order derivatives D^n . In part I of our work [*Bull. Soc. Math. France*, 110, 4, 1982, p. 357-380], we obtained upper and lower bounds for the Borel class of D^n and in particular showed that for limit ordinals λ , D^λ is exactly of class $\lambda+1$. The first author recently showed, using different methods (*cf.* [1]) that for finite n , D^n is exactly of Borel class $2n$. We now complete the solution of KURATOWSKI'S problem by showing that for any limit ordinal λ and any finite n , the operator $D^{\lambda+n}$ is of Borel class exactly $\lambda+2n+1$.

In this paper, we determine the exact Borel classes of the iterated derived set operators D^n , acting on the space \mathcal{H} of closed subsets of the Cantor space 2^N with the usual Vietoris topology. This completes the solution of the problem of KURATOWSKI [3] which was begun in part I of our work [2].

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The results in the present paper depend strongly on those of its predecessor. We begin with some basic definitions and results from [2].

The derived set operator D maps \mathcal{K} into \mathcal{K} and is defined by :

$$D(F) = F' = \{x : x \in \text{Cl}(F - \{x\})\}.$$

The α 'th iterate D^α of the derived set operator map be defined for all ordinals α by letting $D^0(F) = F$, $D^{\alpha+1}(F) = D(D^\alpha(F))$ for all α and $D^\lambda(F) = \bigcap \{D^\alpha(F) : \alpha < \lambda\}$ for limit ordinals λ . The set F is said to be scattered if $D^{\alpha+1}(F) = \emptyset$ for some α ; the derived set order $o(F)$ of F is the least such ordinal α .

The countable subset S of 2^N is defined to be $\{x : (\exists m) (\forall n > m), x(n) = 0\}$. If 2^N is identified with the family $\mathcal{P}(N)$ of subsets of N , then S corresponds to the family of finite sets. Let $\bar{0} = (0, 0, 0, \dots)$. The stitching operator Φ mapping \mathcal{K}^N into \mathcal{K} is defined as follows:

$$\Phi(F_0, F_1, F_2, \dots)$$

$$= \{\bar{0}\} \cup \overbrace{\{(0, 0, \dots, 0, 1, x(0), x(1), \dots) : x \in F_n\}}^n.$$

Note that Φ preserves both finite intersections and unions, that is:

$$\Phi(F_0 \cup G_0, F_1 \cup G_1, \dots) = \Phi(F_0, F_1, \dots) \cup \Phi(G_0, G_1, \dots)$$

and similarly for intersections. This also implies that Φ is monotone, that is, whenever $F_i \subset H_i$ for all i , then $\Phi(F_1, F_1, \dots) \subset \Phi(H_0, H_1, \dots)$. The two fundamental results on the stitching operator, Lemmas 3.7 and 3.8 of [2] concern the derived set order of the stitched set and the continuity of the stitching map. We actually need an extension of the former lemma to infinite ordinals; the proof goes through without difficulty.

LEMMA 1. — For any sequence (F_0, F_1, \dots) of sets from $\mathcal{K} \cap \mathcal{P}(S)$ and any ordinal α :

$$D^\alpha(\Phi(F_0, F_1, \dots)) = \begin{cases} \Phi(D^\alpha(F_0), D^\alpha(F_1), \dots), \\ \text{if } (\forall \beta < \alpha) \{n : D^\beta(F_n) \neq \emptyset\} \text{ is infinite,} \\ \Phi(D^\alpha(F_0), D^\alpha(F_1), \dots) - \{\bar{0}\}, \text{ otherwise. } \quad \square \end{cases}$$

LEMMA 2. — Let (H_0, H_1, \dots) be a sequence of continuous functions mapping a topological space X into the space \mathcal{K} of closed subsets of 2^N

such that each $H_n(x) \subset S$. Then the function H , defined by $H(x) = \Phi(H_0(x), H_1(x), \dots)$ is also continuous. \square

Calculation of the exact Borel classes of the iterated derived set operators begins with Theorem 1.3 of [2].

THEOREM 3. — For any finite n and any limit ordinal λ :

- (a) D^n is of Borel class $2n$;
- (b) $D^{\lambda+n}$ is of Borel class $\lambda + 2n + 1$. \square

Proofs that the Borel classes cited in Theorem 3 are exact proceed as follows. First we note that $\{\emptyset\}$ is both a closed and an open subset of \mathcal{H} . Thus if D^n were of class $2n - 1$, then $T_n = (D^n)^{-1}(\{\emptyset\})$ would have to be a Borel subset of \mathcal{H} of both additive and multiplicative class $2n - 1$; similarly, if $D^{\lambda+n}$ were of class $\lambda + 2n$, then $T_{\lambda+n} = (D^{\lambda+n})^{-1}(\{\emptyset\})$ would be of both additive and multiplicative class $\lambda + 2n$. To show that T_n is not of multiplicative class $2n - 1$, we prove that T_n is actually universal for Borel sets of additive class $2n - 1$; a similar result is given for $T_{\lambda+n}$. Both results will be proved by induction on n . We need two more propositions from [2]; the first is Proposition 4.1:

THEOREM 4. — For any F_σ subset B of N^N , there is a continuous function H mapping N^N into $\mathcal{H} \cap \mathcal{P}(S) \cap T_2$ such that, for all $x, x \in B$ if and only if $H(x) \in T_1$. \square

We actually need the following improvement of Theorem 6.2 of [2].

THEOREM 5. — For any countable limit ordinal λ and any Borel subset B of N^N of additive class λ , there is a continuous function H mapping N^N into $\mathcal{H} \cap \mathcal{P}(S) \cap T_{\lambda+1}$ such that, for all $x, x \in B$ if and only if $H(x) \in T_\lambda$.

Proof. — Let B be a Borel subset of N^N of additive class λ . By Theorem 6.2 of [2], there is a continuous function G from N^N into $\mathcal{H} \cap \mathcal{P}(S) \cap T_{\lambda+2}$ such that, for all $x, x \in B$ if and only if $G(x) \in T_\lambda$; furthermore, $G(x)$ is also normal, as defined in §. 1 of [2]. Now let $C = C_\lambda$ be some canonical normal set with $o(C) = \lambda$ (see 5.10 of [2]). Define the function H by:

$$H(x) = G(x) \cap C_\lambda.$$

Recall from Lemma 5.2 of [2] that, for two normal sets F and G :

$$o(F \cap G) = \min(o(F), o(G)).$$

It follows that:

$$o(H(x)) = \min(o(G(x)), \lambda).$$

This implies that H maps into $T_{\lambda+1}$ and that, for any x , $x \in B$ if and only if $H(x) \in T_\lambda$. Recall from Lemma 5.12 of [2] that the intersection map is continuous for normal sets. Of course the constant map $F(x) = C_\lambda$ is continuous. It follows that H is also continuous. \square

It should be pointed out that the proof of Theorem 6.2 in [2] required the introduction of a more complex stitching operator acting on the family of normal sets.

L. Pigtkiewicz has pointed out that in Proposition 5.8 of [2] $\theta(\hat{F})$ is actually normal if and only if $\gamma = \lim_{n \rightarrow \infty} (o(F_n) + 1)$; this does not affect the proof of Theorem 6.2.

The induction step in the proofs that T_n and $T_{\lambda+n}$ are universal depends on Lemmas 1 and 2 and the following well-known result (a version of which can be found in LUSIN's classic book [5]).

LEMMA 6. — *Let X be a topological space with a countable basis of clopen sets (such as 2^N and N^N). Then for any countable ordinal α and any Borel subset B of X of additive class α , B can be written as the disjoint countable union of Borel sets B_m , each of multiplicative class $< \alpha$.*

THEOREM 7. — (a) *For any natural number k and any Borel subset B of N^N of additive class $2k-1$, there is a continuous function H mapping N^N into $\mathcal{H} \cap \mathcal{P}(S) \cap T_{k+1}$ such that, for all x , $x \in B$ if and only if $H(x) \in T_k$.* (b) *For any countable limit ordinal λ , any natural number k and any Borel subset B of N^N of additive class $\lambda+2k$, there is a continuous function H mapping N^N into:*

$$\mathcal{H} \cap \mathcal{P}(S) \cap T_{\lambda+k+1}$$

such that, for all x , $x \in B$ if and only if $H(x) \in T_{\lambda+k}$.

Proof. — The proofs of parts (a) and (b) proceed from, respectively, Theorems 4 and 5 in a similar manner. We will give the proof of (b), which is of course by induction on k . Theorem 5 covers the case $k=0$. Suppose therefore that the result is true for k and let B be a Borel subset of N^N of additive class $\lambda+2k+2$. Since $N^N \setminus B$ is of multiplicative class $\lambda+2k+2$, there is a decreasing sequence $\{C_n : n \in N\}$ of sets of additive class $\lambda+2k+1$ such that $N^N \setminus B = \bigcap_n C_n$. Now by Lemma 6, there exists for each n a disjoint sequence $\{C_{n,m} : m \in N\}$ of sets of

multiplicative class $\lambda + 2k$ such that $C_n = \bigcup_m C_{n,m}$. It is now easy to see that, for all x :

(i) $x \in B \leftrightarrow \{(n, m) : x \in C_{n,m}\}$ is finite.

Let $(n_0, m_1), (n_1, m_1), \dots$ be some one-to-one enumeration of $N \times N$ and let $A_i = N^N \setminus C_{n_i, m_i}$. By the induction hypothesis, there exists a sequence $\{H_i : i \in N\}$ of continuous functions from N^N into:

$$\mathcal{H} \cap \mathcal{P}(S) \cap T_{\lambda+k+1}$$

such that, for all x :

(ii) $x \in A_i \leftrightarrow H_i(x) \in T_{\lambda+k}$.

The desired reduction H of B to $T_{\lambda+k}$ is now defined by:

(iii) $H(x) = \Phi(H_0(x), H_1(x), \dots)$.

H is continuous by Lemma 2. We must now calculate the possible derived set order of $H(x)$. First of all, from the induction hypothesis $D^{\lambda+k+1}(H_i(x)) = \emptyset$; it follows from Lemma 1 that $D^{\lambda+k+2}(H(x)) = \Phi(\emptyset, \emptyset, \dots) \setminus \{\bar{0}\} = \emptyset$. Thus $H(x) \in T_{\lambda+k+2}$ for any x . Next suppose that $x \in B$. Then by (i) and the definition of the A_i , $\{i : x \notin A_i\}$ is finite. It follows from (ii) that:

$$\{i : D^{\lambda+k}(H_i(x)) \neq \emptyset\}$$

is finite. Then by Lemma 1, $D^{\lambda+k+1}(H(x)) = \emptyset$ as desired. Finally, suppose that $x \notin B$. Then again using (i) and (ii), it follows that:

$$\{i : D^{\lambda+k}(H_i(x)) \neq \emptyset\}$$

is infinite. Applying Lemma 1 and the fact that each $D^{\lambda+k+1}(H_i(x)) = \emptyset$, we obtain:

$$D^{\lambda+k+1}(H(x)) = \Phi(\emptyset, \emptyset, \dots) = \{\bar{0}\},$$

so that $H(x) \notin T_{\lambda+k+1}$. \square

THEOREM 8. — (a) For any natural number k , T_k is a Borel subset of \mathcal{H} of additive class $2k - 1$ but not of multiplicative class $2k - 1$. (b) For any countable limit ordinal λ and any finite k , $T_{\lambda+k}$ is a Borel subset of \mathcal{H} of additive class $\lambda + 2k$ but not of multiplicative class $\lambda + 2k$.

Proof. — The positive direction is proved by induction, as follows. $T_1 = \{F : F \text{ is finite}\}$ is an F_σ set by Lemma 1.1 of [2]. For any limit ordinal λ , $T_\lambda = \bigcup_{\alpha < \lambda} T_\alpha$ and will therefore be of additive class λ

if the result is assumed for $\alpha < \lambda$. Finally, $T_{\alpha+1} = D^{-1}(T_{\alpha})$; since D is a mapping of Borel class 2, the result can always be extended from α to $\alpha+1$. The other direction has similar proofs for parts (a) and (b); we give the proof of (b). Let B be an arbitrary subset of N^N which is of additive class $\lambda+2k$ but not of multiplicative class $\lambda+2k$ (see [4], p. 425). By Theorem 7, there is a continuous function H such that $B = H^{-1}(T_{\lambda+k})$. Now if $T_{\lambda+k}$ were of multiplicative class $\lambda+2k$, it would follow that B must also be of multiplicative class $\lambda+2k$, contradicting our choice of B .

We can now give the complete solution of the problem of Kuratowski.

THEOREM 9. — (a) For any natural number k , the iterated derived set operator D^k is of Borel class exactly $2k$. (b) For any countable limit ordinal λ and any natural number k , $D^{\lambda+k}$ is of Borel class exactly $\lambda+2k+1$.

Proof. — One direction is given by Theorem 3. The other direction has similar proofs for parts (a) and (b); we give the proof of (b). Recall that $\{\emptyset\}$ is a closed subset of \mathcal{H} . Thus if $D^{\lambda+k}$ were of Borel class $\lambda+2k$, then:

$$T_{\lambda+k} = (D^{\lambda+k})^{-1}(\{\emptyset\})$$

would have to be a Borel set of multiplicative class $\lambda+2k$, which would contradict Theorem 8.

The finite cases of Theorems 7, 8 and 9 were previously obtained by the first author in [1] using different methods.

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