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ON THE COHOMOLOGY OF THE CLASSIFYING SPACE OF THE GAUGE GROUP OVER SOME 4-COMPLEXES

BY

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RÉSUMÉ. — Nous étudions l'algèbre de cohomologie de l'espace classifiant du groupe de jauge d'un SU(2)-fibré sur certains espaces de dimension 4. En particulier, nous obtenons des renseignements sur les propriétés de divisibilité, et de non-divisibilité, des classes obtenues par l'application μ introduite par S. Donaldson. Ces résultats ont été annoncés dans [M3].

ABSTRACT. — We study the cohomology algebra of the classifying space of the gauge group of a SU(2)-bundle over some 4-dimensional spaces. In particular, we obtain information on divisibility and indivisibility properties of classes obtained via the map μ introduced by S. Donaldson. These results were announced in [M3].

1. Introduction

We consider pairs $(X, [X])$, where X is a space having the homotopy type of a bouquet of a finite number of 2-spheres with one 4-cell attached, and $[X]$ is a generator of $H_4(X; \mathbb{Z}) \approx \mathbb{Z}$. For example, it is well known (see for instance [MH]) that any oriented closed simply-connected 4-manifold X, with fundamental class $[X]$, is of this type. The algebraic invariants of the pair $(X, [X])$ are (L, φ), where $L = H_2(X; \mathbb{Z})$ is a free $\mathbb{Z}$-module of finite rank, and $\varphi \in \mathrm{BS}(L^*)$ is the symmetric bilinear form on $L^* = H^2(X; \mathbb{Z})$ given by the cup product and evaluation on $[X]$. We call φ the "intersection form" of X, even though X in general cannot be realized as a manifold.

Consider a principal SU(2)-bundle $P \to X$, with second Chern number k. Let $\mathcal{G}_k(X)$ be the *gauge group* of P, that is the group of automorphisms of the bundle inducing the identity on X. It is well known [D2]

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that the classifying space $\mathrm{B}\mathcal{G}_k(X)$ has the (weak) homotopy type of the function space $\mathcal{C}(X, \mathrm{B}S^3)_k$ of continuous maps $f : X \to \mathrm{B}S^3 = \mathrm{BSU}(2)$ of degree k, *i.e.* such that $\langle f^*(c_2), [X]\rangle = k$. We are interested in the cohomology of this space.

As in [D2], consider the linear map

$$\mu : H_i(X; \mathbb{Z}) \to H^{4-i}\big(\mathcal{C}(X, \mathrm{B}S^3)_k; \mathbb{Z}\big)$$

defined by the slant product $\mu(\alpha) = \mathrm{ev}^*(c_2)/\alpha$, where :

$$\mathrm{ev} : X \times \mathcal{C}(X, \mathrm{B}S^3)_k \to \mathrm{B}S^3$$

is the evaluation map. As observed by DONALDSON, the map μ generates all of the *rational* cohomology of $\mathcal{C}(X, \mathrm{B}S^3)_k$. More precisely, the rational cohomology of $\mathcal{C}(X, \mathrm{B}S^3)_k$ is isomorphic to the polynomial algebra

$$\mathbb{Q}\big[\mu([\text{base point}]),\ \mu(\alpha_1), \dots, \mu(\alpha_s)\big],$$

where $\alpha_1, \dots, \alpha_s$ is a basis of L.

To analyze the situation, and study integral cohomology, we can proceed as follows. There is a natural isomorphism $\mathrm{BS}(L^*) \approx \pi_3(M(L,2))$, where $M(L,2)$ denotes the 2-dimensional Moore space over L. Viewing φ as an element of $\pi_3(M(L,2))$ via this isomorphism, we can replace X, up to (oriented) homotopy, by the cofibre of $\varphi : X \sim M(L,2) \cup_\varphi D^4$. This induces a fibration :

$$\Omega^4 \widehat{B} \to \mathcal{C}(X, \mathrm{B}S^3)_k \xrightarrow{r} \mathcal{C}\big(M(L,2), \mathrm{B}S^3\big). \tag{1}$$

Here r denotes restriction of maps, $\widehat{B}$ is the 4-connective covering of $\mathrm{B}S^3$, and Ω is the loop space functor.

Set $A(L) = H^*(\mathcal{C}(M(L,2), \mathrm{B}S^3); \mathbb{Z})$. This algebra is a covariant functor of L, and was determined in [M1].

THEOREM 1.1.

$$\begin{aligned} A(L) &= \bigoplus_{i \geq 0} A_i(L) \\ &= \mathbb{Z}[p]\big[\{\mu_i(\alpha) \mid i \geq 0,\ \alpha \in L\}\big]/I. \end{aligned}$$

Here p has degree 4, $\mu_i(\alpha)$ has degree $2i$, and the ideal I is given by the following relations :

$$\text{(i)} \qquad \mu_0(\alpha) = 1\,;$$

$$\text{(ii)} \quad \mu_n(\alpha+\alpha') = \sum_{i+j=n} \mu_i(\alpha)\mu_j(\alpha')\,;$$

$$\text{(iii)} \quad \mu_i(\alpha)\mu_j(\alpha) = \sum_k \binom{i+j-2k}{i-k}\binom{i+j-k-1}{k}\mu_{i+j-2k}(\alpha)p^k.$$

Moreover, we have μ ([base point])$=r^*(p)$, and $\mu(\alpha) = r^*(\mu_1(\alpha))$, $\alpha \in L = H_2(X;\mathbb{Z})$. Consider then Serre's spectral sequence of fibration (1) :

$$E_2^{**} = A(L)\otimes H^*(\Omega^4\widehat{B};\mathbb{Z}) \Longrightarrow H^*\big(\mathcal{C}(X,\mathrm{B}S^3)_k;\mathbb{Z}\big).$$

Note that the E_2-terms is independent of φ and k. Moreover, $A(L)$ has no torsion, whereas $\widetilde{H}^*(\Omega^4\widehat{B};\mathbb{Z})$ is torsion since $\pi_i(\Omega^4\widehat{B}) = \pi_{i+3}(S^3)$ is finite for $i \geq 1$. Thus the restriction map r induces an inclusion

$$r^* : A(L) \hookrightarrow H^*\big(\mathcal{C}(X,\mathrm{B}S^3)_k;\mathbb{Z}\big)$$

whose cokernel is torsion. From now on, we will identify $A(L)$ with its image under r^*.

Here is a brief outline of this paper.

In paragraph 2, we define and study some "natural" cohomology classes on the space $\mathcal{C}(X,\mathrm{B}S^3)_k$. In particular, the intersection form φ defines an integral class Ω of degree 4, and as a corollary we show that the class $(kp+n\Omega)p^{n-1} \in H^{4n}(\mathcal{C}(X,\mathrm{B}S^3)_k;\mathbb{Z})$ is divisible by $2n+1$. This also shows that in general $A(L)$ is not a direct summand in the integral cohomology of the space $\mathcal{C}(X,\mathrm{B}S^3)_k$.

In paragraph 3, we use some results on Dyer-Lashof-operations to describe explicitely the homology of $\Omega^4\widehat{B}$, the fiber of fibration (1).

Paragraph 4 is devoted to studying a certain map $j : \Omega^4\widehat{B} \to \mathrm{BO}$ in homology, which will be used later. We also describe the mod 2 cohomology algebra of $\Omega^4\widehat{B}$ as a quotient of $H^*(\mathrm{BO};\mathbb{F}_2)$.

In paragraph 5, we put together the results of the previous sections to obtain some divisibility properties in the cohomology of $\mathcal{C}(X,\mathrm{B}S^3)_k$ that depend heavily on the second Chern number k. For example, in

PROPOSITION 5.4 we show that in the integral cohomology of the space $\mathcal{C}(S^4, \mathrm{B}S^3)_k$, for any odd prime ℓ, the element $p^{(\ell-1)/2}$ is divisible by ℓ if and only if $k \not\equiv 0$ (ℓ). The results of this section allow to distinguish some of the topological group extensions :

$$1 \to \mathcal{G}_\bullet \approx \mathcal{C}_\bullet(X, S^3) \to \mathcal{G}_k(X) \to S^3 \to 1,$$

where $\mathcal{G}_\bullet$ is the subgroup of gauge transformations that act as the identity on one fibre (see REMARK 5.6).

In paragraph 6, we study integral cohomology modulo torsion in the special case $X = S^4$, $k = 1$. The main result of this section is stated in PROPOSITION 6.1, where we completely determine the subring of $H^*(\mathcal{C}(S^4, \mathrm{B}S^3)_1; \mathbb{Z})/\text{torsion}$ generated by p and the natural classes of paragraph 2. It is possible that this subring is actually equal to $H^*(\mathcal{C}(S^4, \mathrm{B}S^3)_1; \mathbb{Z})/\text{torsion}$. We show this to be the case at least in low degrees, and after inverting 2 (see COROLLARY 6.3).

Finally, the main result of paragraph 7 is THEOREM 7.1 where we show that in the case of base-point-preserving maps, the analogue of fibration (1) is a product when localised at a prime ≥ 5. This gives an upper bound on divisibility of classes of the form $\mu(\alpha)^n$ (see COROLLARY 7.2).

REMARK. — Gauge Theory has been used by DONALDSON to prove striking results on smooth 4-manifolds (see [D1] for an overview). These results are obtained by studying moduli spaces of anti-self-dual connections, using non-linear analysis and algebraic geometry. The definition of Donaldson's "polynomial invariants" [D3] makes use, at least formally, of the cohomology of the moduli space of *all* (irreducible) connections on a SU(2)-bundle over a compact smooth 4-manifold X. This space has the (weak) homotopy type of the classifying space of the group $\mathcal{G}'_k(X)$, the quotient of the gauge group $\mathcal{G}_k(X)$ of the bundle by its center $\{\pm 1\}$ (*cf.* [D2]). Hence this space is at odd primes the same as the space $\mathrm{B}\mathcal{G}_k(X) \approx \mathcal{C}(X, \mathrm{B}S^3)_k$ studied in this paper. This relationship originally motivated our interest in divisibility properties in the cohomology ring of $\mathrm{B}\mathcal{G}_k(X)$.

2. Natural cohomology classes on $\mathcal{C}(\mathbf{X}, \mathbf{B}S^3)_k$

Suppose we can associate to each $(X, [X])$ a cohomology class $\omega(X)$ on $\mathcal{C}(X, \mathrm{B}S^3)_k$ such that for any degree one map $f : X \to X'$ (*i.e.* such that $f_*[X] = [X']$) we have $F^*(\omega(X)) = \omega(X')$, where

$$F : \mathcal{C}(X', \mathrm{B}S^3)_k \to \mathcal{C}(X, \mathrm{B}S^3)_k$$

is composition with f. Then we will call $\omega(X)$ a *natural* cohomology class. For example, $p = \mu$ ([base point]) is natural. The intersection form φ of X defines another natural class Ω as follows.

Recall that the universal quadratic module $\Gamma_2(L)$ is defined as F/R, where F is the free $\mathbb{Z}$-module generated by L, and R is the smallest submodule such that the map $\gamma_2 : L \to \Gamma_2(L)$ defined in the obvious way satisfies :

1) $\gamma_2(n\alpha) = n^2\gamma_2(\alpha)$ for $n \in \mathbb{Z}$;
2) the map $(\alpha,\beta) \mapsto \gamma_2(\alpha+\beta) - \gamma_2(\alpha) - \gamma_2(\beta)$ is bilinear.

There is a well known natural isomorphism $\Gamma_2(L) \approx BS(L^*)$, given by sending $\gamma_2(\alpha)$ to the bilinear form $(\ell_1,\ell_2) \mapsto \ell_1(\alpha)\ell_2(\alpha)$. Next observe that $\Gamma_2(L)$ is also the degree 4 part of the classical divided power algebra

$$\Gamma(L) = \bigoplus_{i\geq 0}\Gamma_i(L) = \mathbb{Z}\big[\{\gamma_i(\alpha) \mid i \geq 0,\ \alpha \in L\}\big]/J,$$

where $\gamma_i(\alpha)$ has degree $2i$, and the ideal J is given by relations (i), (ii) and (iii) of THEOREM 1.1 with μ_i replaced by γ_i, and $p = 0$. (Note that (iii) becomes simply $\gamma_i(\alpha)\gamma_j(\alpha) = \binom{i+j}{i}\gamma_{i+j}(\alpha)$.) The correspondence $\mu_n(\alpha) \mapsto \gamma_n(\alpha)$ defines a ring homomorphism $A(L) \to \Gamma(L)$, whose kernel is the ideal generated by p (*cf.* [M1]). Moreover, the exact sequence

$$0 \to \mathbb{Z}\cdot p \to A_2(L) \to \Gamma_2(L) \to 0$$

is canonically split, upon lifting $\gamma_2(\alpha)$ to $\mu_2(\alpha)$. Here is then the promised definition : the class $\Omega \in A_2(L) \subset H^4(\mathcal{C}(X, \mathrm{B}S^3)_k; \mathbb{Z})$ is the canonical lift of the intersection form $\varphi \in BS(L^*)$, where the latter group is identified with $\Gamma_2(L)$ as explained above.

Here is the main result of this section :

THEOREM 2.1

(i) *There are natural classes* $\tilde{p}_n(X) \in H^{4n}\big(\mathcal{C}(X, \mathrm{B}S^3)_k; \mathbb{Z}\big[\frac{1}{2}\big]\big)$, *verifying :*

$$2(2n+1)s_n\big(\tilde{p}_1(X), \tilde{p}_2(X), \ldots\big) = (-1)^{n+1}(kp + n\Omega)p^{n-1}.$$

(ii) *If the intersection form of X is even, there are natural classes* $\tilde{w}_i(X) \in H^i(\mathcal{C}(X, \mathrm{B}S^3)_k; \mathbb{F}_2)$, *verifying :*

$$s_n\big(\tilde{w}_1(X), \tilde{w}_2(X), \ldots\big)^4 = (k\bar{p} + n\bar{\Omega})\bar{p}^{n-1}.$$

Moreover in this case the $\tilde{p}_n(X)$ are integral classes, and they verify the relations given above in integral cohomology modulo an element of order 2.

Here s_n is the n-th Newton polynomial, and "$\bar{\ }$" means reduction mod 2.

Before defining these classes and proving their properties, let us point out the following corollary :

COROLLARY 2.2. — *The class* $(kp+n\Omega)p^{n-1} \in H^{4n}(\mathcal{C}(X,\mathrm{B}S^3)_k;\mathbb{Z})$ *is divisible by* $2n+1$.

Note that if $\varphi \in \mathrm{BS}(L^*)$ is indivisible (*e.g.* if φ is non-degenerate), and if $(k,n)=1$, then $(kp+n\Omega)p^{n-1}$ is indivisible in $A_{2n}(L)$. (Indeed, it is obvious from the definition of the class Ω that $kp+n\Omega$ is indivisible in $A_2(L)$ if $(k,n)=1$. Moreover, it is not hard to see that $A(L)$ is isomorphic as a $\mathbb{Z}[p]$-*module* (but not as a ring, *cf.* [M1]), to $\mathbb{Z}[p]\otimes\Gamma(L)$. Hence multiplication by p preserves indivisible elements, and the statement follows.)

Thus the corollary implies that the subalgebra

$$A(L) \subset H^*(\mathcal{C}(X,\mathrm{B}S^3)_k;\mathbb{Z})$$

is not a direct summand in this case.

REMARK 2.3. — Note that $H^*(\mathcal{C}(X;\mathrm{B}S^3)_k;\mathbb{Z})/\mathrm{torsion}$ injects into $A(L)\otimes\mathbb{Q}$. A calculation shows that modulo torsion, we have :

$$\begin{aligned} 1-\tilde p_1+\tilde p_2-\cdots & \\ &= (1+p)^{-k/2}\exp\Big[\Big(k-\frac{\Omega}{2p}\Big)\Big(1-\frac{\arctan\sqrt{p}}{\sqrt{p}}\Big)\Big] \\ &= 1-\frac{1}{6}(kp+\Omega)+\frac{1}{360}\big[(18k+5k^2)p^2+(10k+36)p\Omega+5\Omega^2\big]+\cdots \end{aligned}$$

To define the classes appearing in THEOREM 2.1, we need the following lemma, whose proof is left to the reader.

LEMMA 2.4. — *The homology Chern character of* X *is injective. Moreover, for all* X *of the considered type, we have*

$$\mathrm{ch}_*\big(K_0(X)\big)\otimes\mathbb{Z}\big[\tfrac12\big] \approx H_*\big(X;\mathbb{Z}\big[\tfrac12\big]\big) \subset H_*(X;\mathbb{Q}),$$

and if X *has even intersection form, then*

$$\mathrm{ch}_*\big(K_0(X)\big) = H_*(X;\mathbb{Z}) \subset H_*(X;\mathbb{Q}).$$

We introduce the following notation. Let :

$$[X]_K = (\mathrm{ch}_*)^{-1}[X] \in K_0\big(X;\mathbb{Z}\big[\tfrac12\big]\big).$$

Define $\eta_X \in \widetilde{K}^0\big(X\times\mathcal{C}(X,\mathrm{B}S^3)_k;\mathbb{Z}\big[\tfrac12\big]\big)$ by the evaluation map

$$X\times\mathcal{C}(X,\mathrm{B}S^3)_k \to \mathrm{B}S^3 = \mathrm{BSU}(2) \to \mathrm{BSU}$$

and put $\xi_X = \eta_X/[X]_K \in K^0\big(\mathcal{C}(X,\mathrm{B}S^3)_k;\mathbb{Z}[\frac{1}{2}]\big)$. We now define :

$$\tilde{p}_n(X) = (-1)^n c_{2n}(\xi_X) \in H^{4n}\big(\mathcal{C}(X,\mathrm{B}S^3)_k;\mathbb{Z}[\tfrac{1}{2}]\big).$$

Note that, by LEMMA 2.4, we have $[X]_K \in K_0(X) \subset K_0\big(X;\mathbb{Z}[\frac{1}{2}]\big)$ if X has even intersection form. Hence $\xi_X \in K^0(\mathcal{C}(X,\mathrm{B}S^3)_k)$ in this case, and $\tilde{p}_n(X) \in H^{4n}(\mathcal{C}(X,\mathrm{B}S^3)_k;\mathbb{Z})$. Moreover, we can then define $\tilde{w}_i(X) = w_i(\xi_X) \in H^i(\mathcal{C}(X,\mathrm{B}S^3)_k;\mathbb{F}_2)$.

It is not hard to see that ξ_X qualifies as natural in our sense, hence the classes $\tilde{p}_n(X)$ and $\tilde{w}_i(X)$ are natural. Moreover, after inverting 2, a space X which is the cofiber of $\varphi \in \pi_3(M(L,2))$ has the same homotopy type as a space X' which is the cofiber of 4φ because there is an obvious degree one map $X \to X'$ induced by multiplication by 2 on L. Hence, to prove THEOREM 2.1 we may suppose that X has even intersection form.

Consider $S_g = M(\mathbb{Z}^{2g},2) \cup_{\varphi_g} D^4$, where $\varphi_g = \sum[e_i,e_i']$, the standard basis of $\mathbb{Z}^{2g}$ being $(e_1,e_1',\dots,e_g,e_g')$. (Here, $[\alpha,\beta] = \gamma_2(\alpha+\beta) - \gamma_2(\alpha) - \gamma_2(\beta)$ is the Whitehead product.) Note that S_g has the homotopy type of a connected sum of g copies of $S^2 \times S^2$. If X has even intersection form φ, then we can write $\varphi = \sum[\alpha_i,\alpha_i']$ where $\alpha_i,\alpha_i' \in L$. Clearly the map $f : \mathbb{Z}^{2g} \to L$, defined by $f(e_i) = \alpha_i$, $f(e_i') = \alpha_i'$, extends to a degree one map $f : S_g \to X$. Since the classes p, Ω, $\tilde{p}_n$, $\tilde{w}_i$ are all natural, this shows that it suffices to prove THEOREM 2.1 in the case $X = S_g$.

From now one, we consider $X = S_g$. The idea of proof is as follows. The stabilisation map $j : S^3 = \mathrm{SU}(2) \to \mathrm{SU}$ induces a commutative diagram :

$$\begin{array}{ccc} S_g \times \mathcal{C}(S_g,\mathrm{B}S^3)_k & \xrightarrow{\ \eta\ } & \mathrm{B}S^3 \\ \Big\downarrow{\scriptstyle 1\times j} & & \Big\downarrow{\scriptstyle j} \\ S_g \times \mathcal{C}(S_g,\mathrm{BSU})_k & \xrightarrow{\ \tilde{\eta}\ } & \mathrm{BSU}. \end{array} \tag{2}$$

Here η and $\tilde{\eta}$ are the evaluation maps. Let $c_n \in H^{2n}(\mathrm{BSU};\mathbb{Z})$ be the n-th Chern class. For $n \geq 3$ we have $j^*(c_n) = 0$, hence $(1 \times j)^*(c_n(\tilde{\eta})) = 0$. Writing this equation explicitely will prove the theorem.

In order to calculate the total Chern class of $\tilde{\eta}$, we will first decompose the space $\mathcal{C}(S_g,\mathrm{BSU})_k$ as a product. Let $\mathcal{C}_\bullet(S_g,\mathrm{BSU})_k$ be the subspace formed by the base-point preserving maps. The restriction map

$$r : \mathcal{C}_\bullet\big(S_g,\mathrm{BSU}\big)_0 \to \mathcal{C}_\bullet\big(M(\mathbb{Z}^{2g},2),\mathrm{BSU}\big)$$

admits a canonical section s defined as follows : thinking of $M(\mathbb{Z}^{2g},2)$ as a bouquet of $2g$ copies of the 2-sphere, we have :

$$\mathcal{C}_\bullet(M(\mathbb{Z}^{2g},2),\mathrm{BSU}) = (\Omega^2\,\mathrm{BSU})^{2g}.$$

Let ε_i, $\varepsilon_i' : S^2 \to M(\mathbb{Z}^{2g},2) \hookrightarrow S_g$ correspond to e_i, $e_i' \in \mathbb{Z}^{2g}$, and define retractions r_i, $r_i' : S_g \to S^2 \times S^2 \to S^2$ by first contracting to the base point those parts of the 2-skeleton corresponding to an index different from i, identifying the result in a standard way with $S^2 \times S^2$, and then projecting onto one of the two factors. Then the section s is defined by the formula

$$s(f_1,f_1',\dots,f_g,f_g')(x) = f_1\big(r_1(x)\big)\cdot f_1'\big(r_1'(x)\big)\cdots f_g\big(r_g(x)\big)\cdot f_g'\big(r_g'(x)\big).$$

(Here we use the multiplication on BSU induced by Whitney sum of bundles.) Next, define a map $\widetilde{Q} : \mathcal{C}(S_g,\mathrm{BSU})_k \to \mathcal{C}(S_g,\mathrm{BSU})_k$ by the formula

$$\widetilde{Q}(f) = \big(s\big(r\big(f(pt)^{-1}\cdot f\big)\big)\big)^{-1}\cdot f(pt)^{-1}\cdot f.$$

We may suppose that the multiplication on BSU has a strict identity. Then the restriction of $\widetilde{Q}(f)$ to $M(\mathbb{Z}^{2g},2)$ is the trivial map, hence $\widetilde{Q}$ factors in the obvious way over a map $Q : \mathcal{C}(S_g,\mathrm{BSU})_k \to \Omega^4_k\,\mathrm{BSU}$. Moreover, the following is a homotopy equivalence :

$$\begin{aligned}\mathcal{C}(S_g,\mathrm{BSU})_k &\xrightarrow{\approx} \mathrm{BSU}\times\mathcal{C}_\bullet\big(M(\mathbb{Z}^{2g},2),\mathrm{BSU}\big)\times\Omega^4_k\,\mathrm{BSU}\\ f &\longmapsto \big(f(pt), r\big(f(pt)^{-1}\cdot f\big), Q(f)\big).\end{aligned}$$

Let $F : S^2\times\mathrm{BU}\to\mathrm{BSU}$, $\widetilde{F} : S^4\times\mathrm{BU}\times k\to\mathrm{BSU}$ be adjoint to the Bott equivalences $\mathrm{BU}\approx\Omega^2\,\mathrm{BSU}$, $\mathrm{BU}\times k\approx\Omega^4_k\,\mathrm{BSU}$. Using the inverse of the above homotopy equivalence, the evaluation map $\tilde{\eta}$ becomes :

$$\begin{aligned}S_g\times\mathrm{BSU}\times(\mathrm{BU})^{2g}\times\mathrm{BU}\times k \approx S_g\times\mathcal{C}(S_g,\mathrm{BSU})_k &\longrightarrow \mathrm{BSU}\\ \big(x,z,(y_1,y_1',\dots,y_g,y_g'),y\big) &\longmapsto z\cdot F\big(r_1(x),y_1\big)\cdot F\big(r_1'(x),y_1'\big)\cdots\\ &\qquad\cdots F\big(r_g(x),y_g\big)\cdot F\big(r_g'(x),y_g'\big)\cdot\widetilde{F}([x],y).\end{aligned}$$

(Here, $[x]$ means the image of $x\in S_g$ in $S_g/M(L,2)\approx S^4$.) Let c be the total Chern class. A standard calculation using the splitting principle shows :

$F^*(c) = 1+\sigma_2\otimes A$, where $A = \sum_{n\geq 1}(-1)^{n+1}s_n(c_1,c_2,\dots)$;

$\widetilde{F}^*(c) = 1+\sigma_4\otimes B$, where $B = k+\sum_{n\geq 1}(-1)^{n+1}(n+1)s_n(c_1,c_2,\dots)$.

(Here, σ_i is the standard generator of $H^i(S^i;\mathbb{Z})$.) Let $(a_1,a_1',\dots,a_g,a_g')$ be the basis of $H^2(S_g;\mathbb{Z})=(\mathbb{Z}^{2g})^*$ dual to $(e_1,e_1',\dots,e_g,e_g')$, and let $\sigma=[S_g]^*\in H^4(S_g;\mathbb{Z})$ be the standard generator of $H^4(S_g;\mathbb{Z})$. Since our multiplication on BSU is induced by Whitney sum of bundles, the total Chern class of $\tilde\eta$ is given by :

$$\begin{aligned} c(\tilde\eta) &= (1\otimes c)\big(1+a_1\otimes A_1\big)\big(1+a_1'\otimes A_1'\big)\cdots \\ &\qquad\cdots\big(1+a_g\otimes A_g\big)\big(1+a_g'\otimes A_g\big)(1+\sigma\otimes B) \\ &= 1\otimes c+\sum a_i\otimes cA_i+\sum a_i'\otimes cA_i'+\sigma\otimes c\big(B+\textstyle\sum A_iA_i'\big). \end{aligned}$$

(Here, the classes c, A_i, A_i' and $B\in H^*(\mathcal{C}(S_g,\mathrm{BSU})_k;\mathbb{Z})$ are meant to correspond in the obvious way to the different components of $\mathcal{C}(S_g,\mathrm{BSU})_k\approx\mathrm{BSU}\times(\mathrm{BU})^{2g}\times\mathrm{BU}\times k$. We also used $a_ia_j'=\delta_{ij}\sigma$ and $a_ia_j=0=a_i'a_j'$.)

Now consider diagram (2). Clearly the total Chern class of η is of the form :

$$c(\eta)=1\otimes(1+p)+\sum a_i\otimes b_i+\sum a_i'\otimes b_i'+\sigma\otimes k.$$

Since $H^*(S_g;\mathbb{Z})$ has no torsion, we deduce :

$$j^*(c)=1+p,\quad j^*(cA_i)=b_i,\quad j^*(cA_i')=b_i',\quad j^*\big(c(B+\textstyle\sum A_iA_i')\big)=k.$$

Multiplying by $\sum_{n\geq 0}(-p)^n=1/(1+p)$, we deduce $j^*(A_i)=b_i/(1+p)$, $j^*(A_i')=b_i'/(1+p)$. Hence

$$j^*(B)=\frac{k}{1+p}-\frac{\Omega}{(1+p)^2}=k+\sum_{n\geq 1}(-1)^n(kp+n\Omega)p^{n-1},$$

where we used $\sum b_ib_i'=\Omega$. Thus, the following lemma immediately implies THEOREM 2.1.

LEMMA 2.5. — *We have :*

(i) $2j^*(B)=2\Big(k-2\sum_{n\geq 1}(2n+1)s_n(\tilde p_1,\tilde p_2,\dots)\Big)\in H^*\big(\mathcal{C}(S_g,\mathrm{B}S^3)_k;\mathbb{Z}\big)$;

(ii) $\overline{j^*(B)}=\sum_{n\geq 1}s_n\big(\tilde w_1(X),\tilde w_2(X),\dots\big)^4\in H^*\big(\mathcal{C}(S_g,\mathrm{B}S^3)_k;\mathbb{F}_2\big)$.

Proof. — The main point here is that $\tilde{\eta}/[S_g] \in \widetilde{K}^0(\mathcal{C}(S_g, \mathrm{BSU})_k)$ is represented by the map $Q : \mathcal{C}(S_g, \mathrm{BSU})_k \to \Omega^4_k \mathrm{BSU} \approx \mathrm{BU}$. This can be seen as follows. Put $\pi_i = \varepsilon_i \circ r_i$, $\pi'_i = \varepsilon'_i \circ r'_i$, and let $\pi : S_g \to S_g$ be the constant map to the base point. Then for $f \in \mathcal{C}(S_g, \mathrm{BSU})_k$, $\widetilde{Q}(f)$ can be written :

$$\Big((f\circ\pi)^{-1}\cdot(f\circ\pi_1)\cdot(f\circ\pi)^{-1}\cdot(f\circ\pi'_1)\cdots \\ (f\circ\pi)^{-1}\cdot(f\circ\pi_g)\cdot(f\circ\pi)^{-1}\cdot(f\circ\pi')_g)\Big)^{-1}\cdot(f\circ\pi)^{-1}\cdot f.$$

Define $\Phi : S_g \times \mathcal{C}(S_g, \mathrm{BSU})_k \to \mathrm{BSU}$ by the formula

$$\Phi(x,f) = \widetilde{Q}(f)(x) = \tilde{\eta}(x, \widetilde{Q}(f)).$$

Since $\tilde{\eta}(x, f\circ\pi_i) = f(\pi_i(x)) = \tilde{\eta}(\pi_i(x), f)$, we see that in K-theory we can write :

$$\Phi = (q\times 1)(\tilde{\eta}) \in \widetilde{K}^0\big(S_g \times \mathcal{C}(S_g, \mathrm{BSU})_k\big),$$

where $q = K^0(S_g) \to K^0(S_g)$ is given by $q = 1 - \sum \pi_i^* - \sum \pi_i'^* + (2g-1)\pi^*$.

Clearly, q is a projector onto $\widetilde{K}^0(S^4) \subset K^0(S_g)$. Applying the Chern character, it is not hard to see that q corresponds to $[S_g]_K = \mathrm{ch}_*^{-1}([S_g])$ under the canonical isomorphism :

$$\mathrm{Hom}\big(K^0(S_g), \widetilde{K}^0(S^4)\big) \approx \mathrm{Hom}\big(K^0(S_g), \mathbb{Z}\big) \approx K_0(S_g).$$

It follows

$$\Phi = \theta \otimes \big(\tilde{\eta}/[S_g]_K\big),$$

where $\theta \in \widetilde{K}^0(S^4) \subset K^0(S_g)$ denotes the canonical generator. Since Φ is essentially the adjoint of Q, this shows $Q = \tilde{\eta}/[S_g]_K$ as required.

Thus, we have from the very definition of B :

$$B = k + \sum_{n\geq 1}(-1)^{n+1}(n+1)s_n\big(c_1(\tilde{\eta}/[S_g]_K), c_2(\tilde{\eta}/[S_g]_K), \ldots\big).$$

Since $\xi_{S_g} = \eta_{S_g}/[S_g]_K = j^*(\tilde{\eta}/[S_g]_K) \in K^0(\mathcal{C}(X, \mathrm{B}S^3)_k)$, it follows :

$$j^*(B) = k + \sum_{n\geq 1}(-1)^{n+1}(n+1)s_n\big(c_1(\xi_{S_g}), c_2(\xi_{S_g}), \ldots\big).$$

Now recall that we have defined $\tilde{p}_n = (-1)^n c_{2n}(\xi_{S_g})$, $\tilde{w}_i = w_i(\xi_{S_g})$. Of course, the reason for this definition is that ξ_{S_g} is in the image of the complexification $KO^0 \to K^0$, since the stabilisation map $S^3 \to \mathrm{SU}$ factors over Sp. Thus, it follows from the well known description of the complexification map BO $\to$ BU in integral cohomology that the odd Chern classes of ξ_{S_g} are torsion of order 2. This implies :

$$s_{2n}\big(c_1(\xi_{S_g}), c_2(\xi_{S_g}), \ldots, \big) = 2s_n\big(\tilde{p}_1, \tilde{p}_2, \ldots\big) + \text{an element of order 2},$$

whence part (i) of the lemma. Part (ii) is proved similarly.

This completes the proof of THEOREM 2.1.

REMARK 2.6. — Let M_g a closed orientable (real) surface of genus g. Note that M_g has the homotopy type of a bouquet of circles with one 2-cell attached. The analogy of this with the homotopy type of S_g may be used to apply the above method to study the cohomology algebra of $\mathcal{C}(M_g, \mathrm{B}S^3) \approx B\mathcal{G}(M_g)$, the classifying space of the gauge group of a (necessarily trivial) SU(2)-bundle over M_g. This generalizes [M1]. Here we only state the result; details may be found in [M2].

Let $\alpha_1, \ldots, \alpha_g, \alpha'_1, \ldots, \alpha'_g$ be a symplectic basis of $H_1(M_g; \mathbb{Z})$. Define

$$p = \mu([\text{base point}]), \quad \beta_i = \mu(\alpha_i), \quad \beta'_i = \mu(\alpha'_i), \quad t = \mu([M_g]),$$

where $\mu : H_i(M_g; \mathbb{Z}) \to H^{4-i}(\mathcal{C}(M_g, \mathrm{B}S^3); \mathbb{Z})$ is defined as in paragraph 1. Set $\Phi = \sum \beta_i \beta'_i \in H^6(\mathcal{C}(M_g, \mathrm{B}S^3); \mathbb{Z})$. Let $\eta \in K^0(M_g \times \mathcal{C}(M_g, \mathrm{B}S^3))$ correspond to the evaluation map, set $[M_g]_K = \mathrm{ch}_*^{-1}[M_g]$, and define $x_i = c_i(\eta/[M_g]_K) \in H^{2i}(\mathcal{C}(M_g, \mathrm{B}S^3); \mathbb{Z})$. Note $x_1 = t$. Then

$$\begin{aligned} H^*\big(\mathcal{C}(M_g, \mathrm{B}S^3); \mathbb{Z}\big) &\subset H^*(\mathcal{C}_g; \mathbb{Q}) \\ &\approx \mathbb{Q}[p] \otimes \Lambda_{\mathbb{Q}}\big(\beta_1, \ldots, \beta_g, \beta'_1, \ldots, \beta'_g\big) \otimes \mathbb{Q}[t] \end{aligned}$$

is the subalgebra generated p, $\beta_1, \ldots, \beta_g$, $\beta'_1, \ldots, \beta'_g$, and the x_i. (This fact was already shown in [AB].) Calculating as in [M1], we find :

$$\sum_{n=0}^{\infty} x_n = \exp\Big[\Big(t - \frac{\Phi}{2p}\Big)\frac{\arctan\sqrt{p}}{\sqrt{p}} + \frac{\Phi}{2p(1+p)}\Big].$$

(This power series can be written $\exp(tf(p) + \Phi f'(p))$, where $f(p) = \arctan(\sqrt{p})/(\sqrt{p})$.)

Here is a description of this algebra analoguous to THEOREM 1.1. As an algebra over $\mathbb{Z}[p] \otimes \Lambda_{\mathbb{Z}}(\beta_1, \ldots, \beta'_g)$ (which is the cohomology algebra corresponding to the 1-skeleton of M_g), $H^*(\mathcal{C}(M_g, \mathrm{B}S^3); \mathbb{Z})$ is isomorphic to the algebra generated by the x_i, divided by an ideal of relations of the form :

$$x_i x_j = \sum_{k,\ell=0}^{\infty} A_{ijk\ell}\, x_{i+j-2k-3\ell}\, p^k \frac{\Phi^\ell}{\ell!}.$$

(Note that Φ^ℓ is divisible by $\ell\,!$ in $\Lambda_{\mathbf{Z}}(\beta_1, \ldots, \beta_g')$.) Here is a formula for the numbers $A_{ijk\ell}$:

$$A_{ijk\ell} = \sum_{s=0}^{k} (-1)^s \binom{i+j-k-s-3\ell-1}{k-s} \times$$
$$\sum_{\substack{-s \le h \le s \\ h \equiv s \bmod 2}} \binom{i+j-2k-2\ell}{i-k-\ell+h} \binom{\ell+\frac12(s-h)-1}{\frac12(s-h)} \binom{\ell+\frac12(s+h)-1}{\frac12(s+h)}.$$

Note that, as they must, the numbers A_{ijk0} coincide with the A_{ijk} given in THEOREM 1.1. It also follows from this description that $x_1^n \in H^{2n}(\mathcal{C}(M_g, \mathrm{B}S^3); \mathbb{Z})$ is divisible precisely by the power of 2 contained in $n\,!$. This generalizes Corollary 1 of [M1].

3. The classifying space of the based gauge group on $\mathbf{S^4}$

The subgroup of the gauge group formed by those gauge transformations whose restriction to the fiber over the base point is the identity, is called the *based* gauge group, and denoted by $\mathcal{G}_\bullet(X)$. It is well known that for any S^3-bundle, it is isomorphic to the group $\mathcal{C}_\bullet(X, S^3)$ of base-point preserving maps $X \to S^3$. Hence the classifying space of the based gauge group on S^4 has the homotopy type of the space $\Omega^4 \widehat{B}$, the fiber of fibration (1).

The space $\Omega^4 \widehat{B}$ is the zero component of $\Omega^4\,\mathrm{B}S^3 \approx \Omega^3 S^3 \approx \Omega^3 \Sigma^3 S^0$, and it is well known how to describe the homology of the latter in terms of Dyer-Lashof-operations acting on $[1] \in H_0(\Omega_1^3 S^3)$ (see for example [CLM]). However, since we are ultimately interested in cohomology, it is more convenient to restrict attention to the zero component. We proceed as follows. From the definition of $\widehat{B}$, we deduce a fibration :

$$S^1 \approx K(\mathbb{Z}, 1) \to \Omega^2 \widehat{B} \to \Omega^2\,\mathrm{B}S^3 \approx \Omega S^3.$$

An easy calculation with the Serre spectral sequence then shows :

$$H_*(\Omega^2 \widehat{B}; \mathbb{F}_\ell) \approx P(z_{2\ell}) \otimes E(\beta z_{2\ell}).$$

(Here, ℓ is a prime, P means polynomial algebra, E means exterior algebra, z_n is an element of degreee n, and β is the Bockstein operator in (mod ℓ) homology.) Proceeding as in [CLM, p. 229], we see that $H_*(\Omega^4\widehat{B};\mathbb{F}_\ell)$ is the free graded commutative algebra on generators obtained by certain Dyer-Lashof-operations acting on an element $y_{2\ell-2} \in H_*(\Omega^4\widehat{B};\mathbb{F}_\ell)$ obtained from $z_{2\ell}$ by transgression. (Note however that if $\ell = 2$, y_2 is well defined only modulo $(\beta y_2)^2$.) Here is the result :

PROPOSITION 3.1.

a) $H_*(\Omega^4\widehat{B};\mathbb{F}_2) \approx P\big[(Q_1)^i\beta y_2, (Q_1)^i(Q_2)^j y_2\,;\, i,j \geq 0\big]$;

b) *for* $\ell \geq 3$, $H_*(\Omega^4\widehat{B};\mathbb{F}_\ell)$ *is the free graded commutative algebra on generators* $\beta^\varepsilon(Q_{\ell-1})^j\beta^{\bar\varepsilon}(Q_{2(\ell-1)})^i y_{2\ell-2}$, *where* i, $j \geq 0$, ε, $\bar\varepsilon \in \{0,1\}$, $\varepsilon \leq j$ *and* $(j \geq 1 \Rightarrow \bar\varepsilon = 1)$.

(See [CLM, p. 7] for a definition of the operations Q_n. Compare also [Mi].)

Note that $|(Q_1)^i\beta y_2| = 2^{i+1}-1$, $|(Q_1)^i(Q_2)^j y_2| = 2^{i+j+2}-2^i-1$, and that $|\beta^\varepsilon(Q_{\ell-1})^j\beta^{\bar\varepsilon}(Q_{2(\ell-1)})^i y_{2\ell-2}| = 2\ell^j(\ell^{i+1}-1)-\varepsilon-\bar\varepsilon$.

For $\ell \geq 3$, $y_{2\ell-2}$ is clearly primitive, hence it follows from the Cartan formula that $H_*(\Omega^4\widehat{B};\mathbb{F}_\ell)$ is primitively generated. This implies that the mod ℓ *cohomology* algebra $H^*(\Omega^4\widehat{B};\mathbb{F}_\ell)$ is simply a tensor product of an exterior algebra (on odd-dimensional generators) with a divided power algebra (on even-dimensional generators), the generators being the duals of the homology generators given above. The analoguous statement is not true for mod 2 cohomology. In the next section, we will obtain a presentation of $H^*(\Omega^4\widehat{B};\mathbb{F}_2)$.

The relations between Dyer-Lashof-operations and the higher Bockstein operators can also be found in [CLM]. This allows to determine the additive structure of $H_*(\Omega^4\widehat{B};\mathbb{Z})$ as follows. Set $n(i,j;\ell) = 2\ell^j(\ell^i-1)$ and $\varphi_n(t) = (1+t^{n-1})/(1-t^n)$.

PROPOSITION 3.2. — *For any prime* ℓ, *the Poincaré series of*

$$E^r H_*(\Omega^4\widehat{B};\mathbb{F}_\ell), \qquad r \geq 2,$$

is given by

$$f_r(t) = \prod_{i\geq 1} \varphi_{n(i,r-1\,;\,\ell)}(t).$$

We leave it to the reader to write down $f_1(t)$, *i.e.* the Poincaré series of $E^1H_*(\Omega^4\widehat{B};\mathbb{F}_\ell) = H_*(\Omega^4\widehat{B};\mathbb{F}_\ell)$, using PROPOSITION 3.1.

Now recall that $\widetilde{H}_*(\Omega^4\widehat{B};\mathbb{Z}) = \bigoplus_\ell \widetilde{H}_*(\Omega^4\widehat{B};\mathbb{Z}_{(\ell)})$, since the space $\Omega^4\widehat{B}$ is rationally contractible. Moreover, if we write

$$H_n(\Omega^4\widehat{B};\mathbb{Z}_{(\ell)}) \approx \bigoplus_{r\geq 1}(\mathbb{Z}/\ell^r)^{a_{nr}},$$

then the a_{nr} are given by

$$\sum_{n\geq 1} a_{nr}t^n = \frac{f_r(t) - f_{r+1}(t)}{1+t}.$$

This determines the additive structure of $H_*(\Omega^4\widehat{B};\mathbb{Z})$. For later use, we record the following

COROLLARY 3.3. — *Let* $\ell = 2m+1$ *be an odd prime, and set* $N(\ell) = \ell^2 - \frac{1}{2}(\ell+3)$ *if* $\ell \geq 5$, *and* $N(3) = 536$. *Suppose* $1 \leq n < N(\ell)$. *If* $n \equiv 0 \ (m)$, *then* $H^{4n}(\Omega^4\widehat{B};\mathbb{Z}_{(\ell)})$ *has exponent* $\ell^{1+\nu_\ell(n/m)}$. *If* $n \not\equiv 0 \ (m)$, *then* $H^{4n}(\Omega^4\widehat{B};\mathbb{Z}_{(\ell)}) = 0$.

Here $\nu_\ell : \mathbb{Q}^* \to \mathbb{Z}$ is ℓ-adic valuation.

Sketch of proof. — The Bockstein spectral sequence of $H_*(\Omega^4\widehat{B};\mathbb{F}_\ell)$ has a direct summand of the form $P(y_{2\ell-2}) \otimes E(\beta y_{2\ell-2})$, with $\beta_{r+1}y_{2\ell-2}^{\ell^r} = y_{2\ell-2}^{\ell^r-1}\beta y_{2\ell-2}$. The $\mathbb{Z}_{(\ell)}$-cohomology corresponding to this direct summand verifies the statement of the corollary for all n. Moreover, it turns out that for $n < N(\ell)$, the exponent of $H^{4n}(\Omega^4\widehat{B};\mathbb{Z}_{(\ell)})$ stems from this direct summand. Details are left to the reader.

4. The map $j : \Omega^4\widehat{B} \to BO$

The stabilisation map $S^3 = \mathrm{SU}(2) \to \mathrm{SU}$ factors over the inclusion $\mathrm{Sp} \subset \mathrm{SU}$. Thus, the induced map $\mathcal{C}(X, \mathrm{B}S^3) \to \mathcal{C}(X, \mathrm{BSU})$ factors over $\mathcal{C}(X, \mathrm{BSp})$. Restricting to base-point preserving maps, and using real Bott periodicity, we have a map $\Omega^4\,\mathrm{B}S^3 \to \Omega^4\mathrm{BSp} \approx \mathrm{BO}\times\mathbb{Z}$. In this section, let us denote by $j : \Omega^4\widehat{B} \to \mathrm{BO}$ the map obtained by restricting to the zero degree component. Clearly, this is a morphism of 4-fold loop spaces.

PROPOSITION 4.1. — $j_* : H_*(\Omega^4\widehat{B};\mathbb{F}_2) \to H_*(\mathrm{BO};\mathbb{F}_2)$ *is injective.*

Proof. — Recall $H_*(\mathrm{BO};\mathbb{F}_2) = P(a_1, a_2, \ldots)$, where $|a_i| = i$. Since the inclusion $S^3 \to \mathrm{Sp}$ is 6-connected, j_* is an isomorphism in degrees ≤ 2. Replacing, if necessary, y_2 by $y_2 + (\beta y_2)^2$, it follows $j_*(y_2) = a_2$, $j_*(\beta y_2) = a_1$. From [K], THEOREM 36, we know that in $H_*(\mathrm{BO};\mathbb{F}_2)$, we have $Q_n(a_k) =$

$\binom{n+k-1}{k} a_{n+2k}$ modulo decomposable elements. Since j_* commutes with Q_1 and Q_2, it follows that j_* sends the generators of $H_*(\Omega^4 \widehat{B}; \mathbb{F}_2)$ given in PROPOSITION 3.1 to indecomposable elements. This implies the proposition.

COROLLARY 4.2. — $H^*(\Omega^4 \widehat{B}; \mathbb{F}_2) \approx H^*(\mathrm{BO}; \mathbb{F}_2)/(\ker j^*)$.

The Hopf algebra structure of $H^*(\mathrm{BO}; \mathbb{F}_2)$ is given by

$$\Delta a_n = \sum a_i \otimes a_{n-i}.$$

Since j_* is injective, it follows $\Delta y_2 = 1 \otimes y_2 + \beta y_2 \otimes \beta y_2 + y_2 \otimes 1$. This and the Cartan formula for Dyer-Lashof-operations completely determine the Hopf algebra structure of $H_*(\Omega^4 \widehat{B}; \mathbb{F}_2)$. Note that generators of the form $(Q_1)^i \beta y_2$ are primitive, whereas those of the form $(Q_1)^i (Q_2)^j y_2$ are not.

For n a positive integer, let $\varepsilon_0(n)$ be the number of zeros of n when written in binary form. Note that $H_*(\Omega^4 \widehat{B}; \mathbb{F}_2)$ has a generator precisely in those degrees n such that $\varepsilon_0(n) \leq 1$. Recall that $H^*(\mathrm{BO}; \mathbb{F}_2)$ is a polynomial algebra on the Stiefel-Whitney-classes w_i. The following proposition will be proved in the appendix :

PROPOSITION 4.3

(i) *For each n such that $\varepsilon_0(n) \geq 2$, the ideal* $\ker(j^*) \subset H^*(\mathrm{BO}; \mathbb{F}_2)$ *contains an element r_n of degree n, such that if $n = 2^\ell m$ where m is odd, then r_n is indecomposable if $\varepsilon_0(m) \geq 2$, and r_n is the square (the fourth power) of an indecomposable element if $\varepsilon_0(m) = 1$ ($\varepsilon_0(m) = 0$).*

(ii) *The ideal* $\ker(j^*) \subset H^*(\mathrm{BO}; \mathbb{F}_2)$ *is freely generated by any system of elements r_n verifying the indecomposability properties of part* (i).

Note that the proposition implies $w_n^4 \in \ker(j^*)$ for all n. Here are generators for $\ker(j^*)$ in degrees ≤ 16 : w_1^4, w_2^4, s_9, s_5^2, w_3^4, w_4^4. (s_n means the n-th Newton polynomial of the w_i.) In the appendix, we will give an algorithm to construct generators r_n in terms of Stiefel-Whitney-classes.

We now study the map j at an odd prime ℓ. Recall that $H_*(\Omega^4 \widehat{B}; \mathbb{F}_\ell)$ is the free graded commutative algebra on certain elements of the form $\beta^\varepsilon (Q_{\ell-1})^j \beta^{\bar{\varepsilon}} (Q_{2(\ell-1)})^i y_{2\ell-2}$.

PROPOSITION 4.4. — *The kernel of $j_* : H_*(\Omega^4 \widehat{B}; \mathbb{F}_\ell) \to H_*(\mathrm{BO}; \mathbb{F}_\ell)$ is the ideal generated by those of the above elements whose degree is not divisible by* 4.

Note that these are precisely the generators *not* of the form $(Q_{2(\ell-1)})^i y_{2\ell-2}$, $i \geq 0$.

Proof. — Clearly these elements are in the kernel of j_*, since $H_n(\mathrm{BO};\mathbb{F}_\ell)$ is zero unless n is divisible by 4. To complete the proof, it suffices to show that the subalgebra of $H_*(\Omega^4\widehat{B};\mathbb{F}_\ell)$ generated by the classes $(Q_{2(\ell-1)})^i y_{2\ell-2}$ injects into $H_*(\mathrm{BO};\mathbb{F}_\ell)$. To see this, we proceed as follows. Write

$$H_*(\Omega^3S^3;\mathbb{F}_\ell) = H_*(\Omega^4\widehat{B};\mathbb{F}_\ell)\otimes\mathbb{F}_\ell[\mathbb{Z}],$$

$$H_*(\mathrm{BO}\times\mathbb{Z};\mathbb{F}_\ell) = H_*(\mathrm{BO};\mathbb{F}_\ell)\otimes\mathbb{F}_\ell[\mathbb{Z}].$$

From [CLM] we know that in $H_*(\Omega^3S^3;\mathbb{F}_\ell)$, one has $Q_1(1\otimes[1])\neq 0$. Hence $y_{2\ell-2}$ may be chosen such that $Q_1(1\otimes[1]) = y_{2\ell-2}\otimes[\ell]$. From [K], THEOREM 33, we know that in $H_*(\mathrm{BO}\times\mathbb{Z};\mathbb{F}_\ell)$, we have $Q_1(1\otimes[1]) = \mathfrak{p}_{(\ell-1)/2}\otimes[\ell]$. Here $\mathfrak{p}_n\in H_{4n}(\mathrm{BO};\mathbb{F}_\ell)$ is the dual of $\bar p_n$, the mod ℓ reduction of the n-th Pontryagin class. (The dual is taken with respect to the obvious basis of $H^{4n}(\mathrm{BO};\mathbb{F}_\ell)$ given by monomials in the $\bar p_j$, $j\le n$.) Since the map $\Omega^3S^3\to\mathrm{BO}\times\mathbb{Z}$ is a morphism of 3-fold loop spaces, and respects components, it follows $j_*(y_{2\ell-2}) = \mathfrak{p}_{(\ell-1)/2}$. From [K], THEOREM 25, it follows :

$$j_*\big((Q_{2(\ell-1)})^i y_{2\ell-2}\big) = (Q_{2(\ell-1)})^i\mathfrak{p}_{(\ell-1)/2} = \pm\mathfrak{p}_{(\ell^{i+1}-1)/2}.$$

It is well known that $\mathfrak{p}_n$ is, up to scalar multiples, the unique primitive element in $H_{4n}(\mathrm{BO};\mathbb{F}_\ell)$. (Recall that $H_*(\mathrm{BO};\mathbb{F}_\ell)\approx P(a_n\ ;\ n\ge 1)$, with $|a_n| = 4n$, and $\Delta a_n = \sum a_i\otimes a_{n-i}$.) From the Newton formula, we see that $\mathfrak{p}_{(\ell^i-1)/2}$ is indecomposable, since $\frac{1}{2}(\ell^i-1)$ is not divisible by ℓ. Thus, $\mathrm{Im}(j_*)$ is freely generated by $\{\mathfrak{p}_n \mid n=\frac{1}{2}(\ell^i-1),\ i\ge 1\}$. This implies the proposition.

5. Divisibility properties depending on k

In this section, we study the fibration (1) in cohomology. First, we study the situation at the prime 2.

PROPOSITION 5.1. — *If X has even intersection form, then the* mod 2 *cohomology spectral sequence of fibration* (1) *degenerates at the E_2-level*

Proof. — It suffices to prove this in the case $X = S_g$, since there is a degree one map $S_g\to X$ (*cf.* the proof of THEOREM 2.1). The stabilisation map $S^3\to\mathrm{Sp}$ induces a morphism of fibrations $\mathcal{C}(S_g,\mathrm{B}S^3)_k\to\mathcal{C}(S_g,\mathrm{BSp})_k$ whose restriction to the fiber is the map $j:\Omega^4\widehat{B}\to\mathrm{BO}$ studied in paragraph 4. Proceeding as in the proof of THEOREM 2.1, we can decompose $\mathcal{C}(S_g,\mathrm{BSp})_k$ as a product $\mathrm{BSp}\times(\Omega\mathrm{Sp})^{2g}\times\mathrm{BO}$. Hence the spectral sequence of this fibration degenerates at the E_2-level. Since $j^*: H^*(\mathrm{BO};\mathbb{F}_2)\to H^*(\Omega^4\widehat{B};\mathbb{F}_2)$ is surjective by PROPOSITION 4.1, the result follows.

COROLLARY 5.2. — *If X has even intersection form, then*

$$H^*(\mathcal{C}(X,\mathrm{B}S^3)_k;\mathbb{F}_2)$$

is an extension of the algebra $H^*(\mathrm{BO};\mathbb{F}_2)/\ker(j^*)$ determined in Proposition 4.3 by $A(L)\otimes\mathbb{F}_2$.

Note that $\tilde{w}_1^4 = k\bar{p}+\overline{\Omega}$ by THEOREM 2.1, hence the above extension of algebras is non-trivial if k is odd.

COROLLARY 5.3. — *If X has even intersection form, then*

$$H^*\big(\mathcal{C}(X,\mathrm{B}S^3)_k;\mathbb{Z}_{(2)}\big) \approx A(L)\otimes\mathbb{Z}_{(2)} \oplus \text{torsion}.$$

Here, $\mathbb{Z}_{(2)}$ is $\mathbb{Z}$ localized at 2. Note that this is not true at odd primes, *cf.* COROLLARY 2.2.

Now let ℓ be an odd prime. Consider first the case $X = S^4$.

PROPOSITION 5.4. — *In $H^*(\mathcal{C}(S^4,\mathrm{B}S^3)_k;\mathbb{Z})$, the element $p^{(\ell-1)/2}$ is divisible by ℓ if and only if $k \not\equiv 0$ (ℓ).*

Proof. — To simplify notation, set $\mathcal{C}_k = \mathcal{C}(S^4,\mathrm{B}S^3)_k$ and $m=\frac{1}{2}(\ell-1)$. From PROPOSITION 3.1, it follows $H^i(\Omega^4\widehat{B};\mathbb{F}_\ell) = 0$ for $1\le i\le 4m-2$, $H^{4m-1}(\Omega^4\widehat{B};\mathbb{F}_\ell) \approx \mathbb{F}_\ell$, $H^{4m}(\Omega^4\infty\widehat{B};\mathbb{F}_\ell)\approx\mathbb{F}_\ell$. Moreover, the latter is generated by $i^*(\tilde{p}_m)$ where $i:\Omega^4\widehat{B}\to\mathcal{C}_k$ is the inclusion of the fiber. This follows from PROPOSITION 4.4 since $i^*(\tilde{p}_m) = j^*(p_m)$ where $j:\Omega^4\widehat{B}\to\mathrm{BO}$ is the map studied in paragraph 4. Also, in the mod ℓ cohomology spectral sequence of the fibration $\Omega^4\widehat{B}\to\mathcal{C}_k\to\mathrm{B}S^3$, the first non-trivial differential is :

$$\mathrm{d}_{4m}: H^{4m-1}\big(\Omega^4\widehat{B};\mathbb{F}_\ell\big)\to H^{4m}(\mathrm{B}S^3;\mathbb{F}_\ell).$$

Clearly, p^m is divisible by ℓ if and only if $\mathrm{d}_{4m}\neq 0$.

If $k\not\equiv 0$ (ℓ), then it follows immediately from COROLLARY 2.2 that p^m is divisible by ℓ. Now suppose $k=\ell k'$. Consider :

$$z = 4(-1)^{m+1}s_m\big(\tilde{p}_1,\tilde{p}_2,\dots\big) - 2k'p^m \in H^{4m}(\mathcal{C}_k;\mathbb{Z}).$$

Since $i^*(z) = \pm 4s_m(i^*(\tilde{p}_1), i^*(\tilde{p}_2),\dots) = \pm 4mi^*(\tilde{p}_m)$, we have $\bar{z}\neq 0\in H^{4m}(\mathcal{C}_k;\mathbb{F}_\ell)$. On the other hand, THEOREM 2.1 implies $\ell z = 0$. It follows that $\bar{z}$ is in the image of the mod ℓ cohomology Bockstein operator. In particular, we have $H^{4m-1}(\mathcal{C}_k;\mathbb{F}_\ell)\neq 0$. This implies $d_{4m}=0$ in the spectral sequence, hence p^m is not divisible by ℓ.

This completes the proof of PROPOSITION 5.4.

For general X, we have :

PROPOSITION 5.5. — *Suppose $\mathcal{C}(X, \mathrm{B}S^3)_k$ and $\mathcal{C}(X, \mathrm{B}S^3)_{k'}$ have isomorphic cohomology algebras. Then for each prime $\ell \geq 5$, one has :*

$$k \equiv 0 \ (\ell) \Longleftrightarrow k' \equiv 0 \ (\ell).$$

Moreover, if the intersection form of X is even, or divisible by 3, then this is also true for $\ell = 2$, or $\ell = 3$, respectively.

As an example where the last condition is satisfied, one may take $X = S^4$.

Proof. — Let $\varphi \in \mathrm{BS}(L^*)$ be the intersection form of X, and set $\mathcal{C}_k = \mathcal{C}(X, \mathrm{B}S^3)_k$. As before, set $m = \frac{1}{2}(\ell - 1)$. We distinguish three cases.

Case 1 : $\ell = 2$. Then φ is even by hypothesis, hence by THEOREM 2.1, we have :

$$\tilde{w}_1^8 = s_2\big(\tilde{w}_1, \tilde{w}_2\big)^4 = k\bar{p}^2.$$

PROPOSITION 5.1 implies $\bar{p}^2 \neq 0$. Hence we have $\tilde{w}_1^8 = 0$ if and only if $k \equiv 0 \ (2)$. Since $\tilde{w}_1$ generates $H^1(\mathcal{C}_k; \mathbb{F}_2) \approx \mathbb{F}_2$, the result follows.

Case 2 : ℓ an odd prime, and $\varphi \equiv 0 \ (\ell)$. Then $\overline{\Omega}$, the mod ℓ reduction of Ω, is zero. In this case, we proceed as in the case $X = S^4$ to see that p^m is divisible by ℓ if and only if $k \not\equiv 0 \ (\ell)$. Actually the proof shows that $H^{4m-1}(\mathcal{C}_k; \mathbb{F}_\ell) = 0$ if and only if $k \not\equiv 0 \ (\ell)$. The result follows.

Case 3 : ℓ a prime ≥ 5, and $\varphi \not\equiv 0 \ (\ell)$. We consider again the fibration $\Omega^4 \widehat{B} \to \mathcal{C}_k \xrightarrow{r} \mathcal{C}(M(L,2), \mathrm{B}S^3)$. In this proof, all cohomology classes will be reduced modulo ℓ. But here we will distinguish between $\bar{p}$, $\overline{\Omega}$ as cohomology classes on $\mathcal{C}(M(L,2), \mathrm{B}S^3)$, and their images $r^*(\bar{p})$, $r^*(\overline{\Omega})$ on $\mathcal{C}_k$. THEOREM 2.1 implies $r^*((k\bar{p} + m\overline{\Omega})\bar{p}^{m-1}) = 0$. Since $\varphi \not\equiv 0 \ (\ell)$, it follows easily from the description of $A(L)$ that for all k, the element $(k\bar{p} + m\overline{\Omega})\bar{p}^{m-1}$ is non-zero (compare the reasoning following COROLLARY 2.2). Arguing as in the proof of PROPOSITION 5.4, we see from the spectral sequence that in degree $4m$, $\ker(r^*) = \mathrm{Im}(\mathrm{d}_{4m})$ is one-dimensional. Hence $\ker(r^*)$ is generated by $(k\bar{p} + m\overline{\Omega})\bar{p}^{m-1}$.

Now let $\alpha^* : H^*(\mathcal{C}_{k'}; \mathbb{F}_\ell) \approx H^*(\mathcal{C}_k; \mathbb{F}_\ell)$ be a (graded) algebra isomorphism. Affect all objects concerning $\mathcal{C}_{k'}$ with a. Since $\ell \geq 5$, r^* and r'^* are isomorphisms in degree 4. Hence, there are elements $\bar{q}$, $\overline{\Lambda} \in A(L)$ of degree 4 such that :

$$\alpha^*\big(r'^*(\bar{p})\big) = r^*(\bar{q}), \quad \alpha^*\big(r'^*(\overline{\Omega})\big) = r^*(\overline{\Lambda}).$$

Again, THEOREM 2.1 implies $r'^*((k'\bar{p} + m\overline{\Omega})\bar{p}^{m-1}) = 0$. Applying α^*, it follows $(k'\bar{q} + m\overline{\Lambda})\bar{q}^{m-1} \in \ker r^*$. Since $\ker r^*$ is one-dimensional, there is $\lambda \neq 0$ such that $(k'\bar{q} + m\overline{\Lambda})\bar{q}^{m-1} = \lambda \ (k\bar{p} + m\overline{\Omega})\bar{p}^{m-1}$. It then follows easily from the description of $A(L)$ that $k \equiv 0 \ (\ell)$ if and only if $k' \equiv 0 \ (\ell)$.

This completes the proof.

REMARK 5.6. — PROPOSITION 5.5 was motivated by the following amusing application. Consider the family of topological group extensions

$$1 \to \mathcal{G}_\bullet \approx \mathcal{C}_\bullet(X, S^3) \to \mathcal{G}_k(X) \to S^3 \to 1$$

depending on the second Chern number k. Here the map $\mathcal{G}_k(X) \to S^3$ is given by restriction to the fiber over the base point. One wants to conjecture that these extensions are distinguished by k. Since $B\mathcal{G}_k(X) \approx \mathcal{C}(X, \mathrm{B}S^3)_k$, PROPOSITION 5.5 gives a partial answer. In the literature, there seems to be only the following invariant : if X has even intersection form, then the central element $-1 \in \mathcal{G}_k(X)$ is homotopic to 1 if and only if k is even [FU].

6. The classifying space $\mathrm{B}\mathcal{G}_1(S^4)$

We now consider the special case

$$X = S^4, \quad k = 1.$$

Set $\mathcal{C}_1 = \mathcal{C}(S^4, \mathrm{B}S^3)_1 \approx \mathrm{B}\mathcal{G}_1(S^4)$. THEOREM 2.1 implies that

$$H^*(\mathcal{C}_1; \mathbb{Z})/\,\mathrm{torsion} \ \subset H^*(\mathcal{C}_1; \mathbb{Q}) = \mathbb{Q}[p]$$

contains classes $\tilde{p}_i$ such that :

$$1 + \tilde{p}_1 + \tilde{p}_2 + \cdots = \exp\Big(\sum_{i=1}^{\infty} \frac{p^i}{2i(2i+1)}\Big) = 1 + \frac{1}{6}p + \frac{23}{360}p^2 + \frac{1493}{45360}p^3 + \cdots$$

We introduce the following notation. If ℓ is a prime, set $m = \frac{1}{2}(\ell - 1)$ if ℓ is odd, and $m = 1$ if $\ell = 2$. For $n \in \mathbb{N}$, set $\mu_\ell(n) = \nu_\ell([(\ell/m) \cdot n]\,!)$, where $\nu_\ell : \mathbb{Q}^* \to \mathbb{Z}$ is ℓ-adic valuation, and $[x]$ means the greatest integer $\leq x$. The main result of this section is :

PROPOSITION 6.1. — *The subring of* $H^*(\mathcal{C}_1; \mathbb{Z})/\,\mathrm{torsion}$ *generated by* p *and the* $\tilde{p}_i$ *is generated in degree* $4n$ *by* p^n/α_n, *where*

$$\alpha_n = \prod_\ell \ell^{\mu_\ell(n)}.$$

Before giving the proof, we point out that it is tempting to conjecture that $H^*(\mathcal{C}_1; \mathbb{Z})/\,\mathrm{torsion}$ is actually equal to this subring. Here is a proof for this conjecture in low degrees, and after inverting 2. From fibration (1), we have an exact sequence :

$$0 \to \mathbb{Z} \cdot p^n \hookrightarrow H^{4n}(\mathcal{C}_1; \mathbb{Z}) \to Q^{4n} \to 0,$$

where Q^{4n} is torsion. Moreover, it follows easily from the spectral sequence that the exponent of Q^{4n} is less or equal than the product of the exponents of $H^{4i}(\Omega^4\widehat{B};\mathbb{Z})$ for $1 \leq i \leq n$. Now let ℓ be an odd prime. An easy calculation using COROLLARY 3.3 shows that for $n < N(\ell)$, the exponent of the ℓ-primary part of Q^{4n} is less or equal than $\ell^{\mu_\ell(n)}$. (Recall $N(3) = 536$, and $N(\ell) = \ell^2 - \frac{1}{2}(\ell+3)$ if $\ell \geq 5$.) On the other hand, PROPOSITION 6.1 implies that p^n is divisible by $\ell^{\mu_\ell(n)}$ in $H^{4n}(\mathcal{C}_1;\mathbb{Z})/$ torsion. Putting things together, one easily deduces the following corollary.

COROLLARY 6.2. — *Let ℓ be an odd prime, and $n < N(\ell)$. Then $p^n \in H^{4n}(\mathcal{C}_1;\mathbb{Z})$ is divisible by $\ell^{\mu_\ell(n)}$, and $H^{4n}(\mathcal{C}_1;\mathbb{Z}_{(\ell)})/$* torsion *is generated by $p^n/\ell^{\mu_\ell(n)}$.*

Note that the smallest $N(\ell)$ is $N(5) = 21$. Since by PROPOSITION 5.1 $p^n \in H^{4n}(\mathcal{C}_1;\mathbb{Z})$ is not divisible by 2, it follows :

COROLLARY 6.3. — *For $n < 21$, $p^n \in H^{4n}(\mathcal{C}_1;\mathbb{Z})$ is divisible precisely by $\prod_{\ell\geq 3}\ell^{\mu_\ell(n)}$. Moreover, in degrees less than $4\times 21 = 84$, $H^*\big(\mathcal{C}_1;\mathbb{Z}\big[\frac{1}{2}\big]\big)/$* torsion *coincides with the subring generated by p and the $\tilde{p}_i$.*

We now prove PROPOSITION 6.1. Write $\tilde{p}_n = b_n p^n \in H^{4n}(\mathcal{C}_1;\mathbb{Q})$. We leave it to the reader to deduce PROPOSITION 6.1 from the following lemma, using the easily verified inequality $\mu_\ell(n_1)+\mu_\ell(n_2) \leq \mu_\ell(n_1+n_2)$.

LEMMA 6.4. — *For $n \geq 1$, one has $\nu_\ell(b_n) \geq -\mu_\ell(n)$. Moreover, equality holds if $n \equiv 0\ (m)$.*

To prove LEMMA 6.4, recall that by definition :

$$\exp\Big(\sum_{i=1}^{\infty}\frac{p^i}{2i(2i+1)}\Big) = \sum_{n=0}^{\infty} b_n p^n.$$

Differentiating this expression, we obtain :

$$b_{n+1} = \frac{1}{2(n+1)}\sum_{i=0}^{\infty}\frac{b_i}{2n-2i+3}.$$

Using the well known fact $\nu_\ell(x+y) = \min(\nu_\ell(x),\nu_\ell(y))$ whenever $\nu_\ell(x) \neq \nu_\ell(y)$, it is not hard to deduce $\nu_2(b_n) = -\nu_2((2n)!)$ and $\nu_3(b_n) = -\nu_3((3n)!)$ by induction on n. This proves LEMMA 6.4 for $\ell \in \{2,3\}$.

In the general case, we proceed as follows. We have the following expression :

$$b_n = \sum_k \sum_{n_1+2n_2+\cdots+kn_k=n}\frac{1}{\prod_{i=1}^{k} n_i!\big(2i\,(2i+1)\big)^{n_i}}.$$

Let E_n denote the set of sequences $(n_1, n_2, \ldots)$ such that $n_1+2n_2+\cdots \le n$. Define $f : E_n \to \mathbb{Z}$ by the formula :

$$f(n_1, n_2, \ldots) = \nu_\ell\Big(\prod_i n_i\,!\big(2i\,(2i+1)\big)^{n_i}\Big).$$

Note that E_n contains the sequence $(n_1^{(0)}, n_2^{(0)}, \ldots)$ defined by $n_m^{(0)} = [n/m]$, $n_i^{(0)} = 0$ for $i \neq m$. Moreover,

$$f(n_1^{(0)}, n_2^{(0)}, \ldots) = \nu_\ell\big([n/m]\,!\big) + [n/m] = \mu_\ell(n).$$

Clearly, it follows from the expression for the b_n given above that the following LEMMA 6.5 implies LEMMA 6.4.

LEMMA 6.5. — *For all sequences $(n_1, n_2, \ldots) \in E_n$, one has the inequality $f(n_1, n_2, \ldots) \le \mu_\ell(n)$. Moreover, if $n \equiv 0\ (m)$, then equality holds if and only if :*

$$(n_1, n_2, \ldots) = \big(n_1^{(0)}, n_2^{(0)}, \ldots\big).$$

We now prove LEMMA 6.5. Consider $(n_1, n_2, \ldots) \in E_n$. Set $h_i = [in_i/m]$.

SUBLEMMA 1. — *If $n_i \ge 0$, then $\nu_\ell(n_i\,!) < h_i$ unless $h_i = 0$.*

Indeed, $\nu_\ell(n_i\,!) \le \dfrac{n_i - 1}{2m} < \dfrac{n_i}{2m} \le \dfrac{mh_i + m - 1}{2mi} < \dfrac{h_i + 1}{2} \le h_i$.

SUBLEMMA 2. — *If $n_i > 0$ and $i > m$, then $n_i\nu_\ell(2i(2i+1)) < h_i$.*

Since $h_i = [in_i/m] \ge n_i > 0$, this is obvious unless $i \equiv 0\ (\ell)$ or $2i+1 \equiv 0\ (\ell)$. First, suppose $i \equiv 0\ (\ell)$. Then we have

$$n_i\nu_\ell\big(2i(2i+1)\big) = n_i\nu_\ell(i) \le n_i \log_\ell(i) \le \frac{mh_i + m - 1}{i}\log_\ell(i),$$

hence it suffices to show $((mh + m - 1)/i)\log_\ell(i) < h_i$, which is equivalent to

$$(*) \qquad i^{mh_i+m-1} < \ell^{ih_i}.$$

We will show this inequality by induction on h_i, keeping i fixed. Observe that we may suppose $h_i \ge 2$. Indeed, since $i \ge \ell$, we have

$$1 \le n_i \le \frac{mh_i + m - 1}{i} \le \frac{mh_i + m - 1}{\ell},$$

which is impossible if $h_i \leq 1$.

Letting $h_i = 2$ in (*), we obtain :

$$(**) \qquad i^{3m-1} < \ell^{2i}.$$

Observe that once we know (**), it follows $i^m \leq i^{(3m-1)/2} < \ell^i$, which implies the induction. Thus, it only remains to show (**), which is equivalent to :

$$i < \ell^{2i/(3m-1)} = \ell^{4i/(3\ell-5)}.$$

Now this is obvious if $i = \ell$, moreover, differentiating with respect to i yields :

$$1 < \frac{4}{3\ell-5}\log(\ell)\,\ell^{4i/(3\ell-5)} = \frac{4\ell}{3\ell-5}\log(\ell)\,\ell^{(4i-3\ell+5)/(3\ell-5)}$$

which is true for $i \geq \ell$. This implies (**), hence SUBLEMMA 2 in the case $i \equiv 0 \ (\ell)$.

The case $2i+1 \equiv 0 \ (\ell)$ is similar and left to the reader.

SUBLEMMA 3. — $\sum_{i\geq 1} h_i \leq [n/m]$, $\sum_{i\geq m} n_i \leq [n/m]$.

This is obvious since $n = \sum_{i\geq 1} i n_i$.

Applying these sublemmas, we have :

$$\begin{aligned} f(n_1,n_2,\ldots) &= \sum_{i\geq 1}\big(\nu_\ell(n_i\,!) + n_i\,\nu_\ell(2i(2i+1))\big) \leq \sum_{i\geq m}\nu_\ell(n_i!) + \sum_i h_i \\ &\leq \nu_\ell\big([n/m]\,!\big) + [n/m] = \mu_\ell(n) = f\big(n_1^{(0)}, n_2^{(0)},\ldots\big). \end{aligned}$$

This implies the first part of LEMMA 6.5. Now suppose we have equality here. Then it follows from sublemmas 1 and 2 that $n_i = 0$ for all $i > m$, and $h_i = 0$ for all $i < m$. But this implies :

$$f\big(n_1,n_2,\ldots\big) = n_m + \nu_\ell(n_m!),$$

hence $n_m = [n/m]$. If $n \equiv 0 \ (m)$, then this is impossible unless :

$$\big(n_1,n_2,\ldots\big) = \big(n_1^{(0)}, n_2^{(0)},\ldots\big).$$

This completes the proof of LEMMA 6.5.

7. The classifying space of the based gauge group

For $\varphi \in \Gamma_2(L) = \pi_3(M(L,2))$, define $F_\varphi : \mathcal{C}_\bullet(M(L,2), \mathrm{B}S^3) \to \Omega^3 \mathrm{B}S^3$ by $F_\varphi(f) = f \circ \varphi$. Clearly, the map :

$$F : \Gamma_2(L) \to \big[\mathcal{C}_\bullet(M(L,2), \mathrm{B}S^3), \Omega^3 \mathrm{B}S^3\big], \quad \varphi \mapsto F_\varphi$$

is a homomorphism of abelian groups. (Here, the notation [A,B] means based homotopy classes of based maps $A \to B$.) The main result of this section is the following theorem.

THEOREM 7.1. — $\ker F = 12\,\Gamma_2(L)$.

We apply this as follows. It is not hard to see that, up to homotopy, $\mathcal{C}_\bullet(X, \mathrm{B}S^3)$ is the total space of the fibration induced by F_φ from the path fibration over $\Omega^3 \mathrm{B}S^3$. Thus THEOREM 7.1 implies that for any prime $\ell \geq 5$, we have an ℓ-equivalence :

$$\mathcal{C}_\bullet(X, \mathrm{B}S^3) \sim_{(\ell)} \mathcal{C}_\bullet\big(M(L,2), \mathrm{B}S^3\big) \times \Omega^4 \mathrm{B}S^3.$$

Moreover, this is still true for $\ell = 3$, or $\ell = 2$, if we suppose $\varphi \equiv 0 \ (3)$, or $\varphi \equiv 0 \ (4)$, respectively. On the other hand, if $\varphi \not\equiv 0 \ (3)$, then $\mathcal{C}_\bullet(X, \mathrm{B}S^3)_{(3)}$ is not a product, as follows from THEOREM 2.1. Similarly, if φ is odd, then $\mathcal{C}_\bullet(X, \mathrm{B}S^3)_{(2)}$ is not a product (see also REMARK 7.8).

Since $H*(\mathcal{C}_\bullet(M(L,2), \mathrm{B}S^3); \mathbb{Z})$ is the divided power algebra $\Gamma(L)$, we deduce :

COROLLARY 7.2. — *Let $\alpha \in L$ be indivisible. If*

$$\mu(\alpha)^n \in H^{2n}\big(\mathcal{C}(X, \mathrm{B}S^3)_k; \mathbb{Z}\big[\tfrac{1}{6}\big]\big)$$

is divisible by N, then N divides $n\,!$. Moreover, if $\varphi \equiv 0 \ (3)$, then this is true with coefficients in $\mathbb{Z}\big[\frac{1}{2}\big]$.

Note that if φ is even (as a bilinear form), then COROLLARY 5.3 together with COROLLARY 1 of [M1] imply that $\mu(\alpha)^n \in H^{2n}(\mathcal{C}(X, \mathrm{B}S^3)_k; \mathbb{Z}_{(2)})$ is divisible exactly by $n\,!$.

We now prove THEOREM 7.1. We start with two lemmas whose proof is left to the reader.

LEMMA 7.3. — *The suspension $\Sigma\mathcal{C}_\bullet(M(L,2), \mathrm{B}S^3)$ has the homotopy type of a bouquet of spheres.*

LEMMA 7.4. — *There is a natural filtration (induced by a Postnikov decomposition of $\mathrm{B}S^3$) :*

$$\big[\mathcal{C}_\bullet\big(M(L,2), \mathrm{B}S^3\big), \Omega^3 \mathrm{B}S^3\big] = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots,$$

where $\mathcal{F}_{n-1}/\mathcal{F}_n \approx \Gamma_n(L) \otimes \pi_{2n+2}(S^3)$.

Using this filtration, the map F defines natural linear maps :

$$\theta_1 : \Gamma_2(L) \longrightarrow \mathcal{F}_0/\mathcal{F}_1 \approx \Gamma_1(L) \otimes \pi_4(S^3) = L \otimes \mathbb{Z}/2,$$

$$\theta_2 : \ker(\theta_1) \to \mathcal{F}_1/\mathcal{F}_2 \approx \Gamma_2(L) \otimes \pi_6(S^3),$$

where $\theta_i(\varphi) = F_\varphi \bmod \mathcal{F}_i$. It is not hard to see that θ_1 corresponds to the suspension $\Gamma_2(L) = \pi_3(M(L,2)) \xrightarrow{\Sigma} \pi_4(\Sigma M(L,2)) = L \otimes \mathbb{Z}/2$. Alternatively, θ_1 is given by the formula $\theta_1(\gamma_2(x)) = \bar{x}$, $(x \in L)$. Thus, $\ker(\theta_1)$ consists exactly of the even forms.

The following two lemmas will imply THEOREM 7.1.

LEMMA 7.5. — *Let* $w = [i_1, i_2] \in \pi_3(S^2 \vee S^2) = \Gamma_2(\mathbb{Z} \oplus \mathbb{Z})$ *be the Whitehead product of the obvious inclusions* i_1, i_2. *If we localize at a prime* $\ell \geq 5$, *then* F_w *becomes null homotopic.*

LEMMA 7.6. — *Let* $h \in \pi_3(S^2) = \Gamma_2(\mathbb{Z})$ *be a generator. Then* $\theta_2(2h)$ *is the double of a generator of* $\Gamma_2(\mathbb{Z}) \otimes \pi_6(S^3) \approx \mathbb{Z}/12$.

Granting these lemmas, here is a proof of the theorem.

First, we show $12\,\Gamma_2(L) \subset \ker F$. Suppose $\varphi \in 12\,\Gamma_2(L)$. It follows from LEMMA 7.3 that the abelian group $[\mathcal{C}_\bullet(M(L,2), \mathrm{B}S^3), \Omega^3\,\mathrm{B}S^3]$ is (non-naturally) isomorphic to $\prod_{j \in J} \pi_{n_j}(S^3)$ for some integers n_j. It clearly suffices to show that the image of F_φ in each of the $\pi_{n_j}(S^3)$ is zero. Now it is well known [S] that the ℓ-primary part of $\pi_i(S^3)$ $(i \geq 4)$ has exponent ℓ, for ℓ an odd prime, and exponent 4, for $\ell = 2$. Thus, the 2- and 3-primary parts of F_φ are zero. To study the ℓ-primary part for $\ell \geq 5$, we may as well localise at ℓ. The image of $w = [i_1, i_2]$ in $\pi_3(S^2)$ under the obvious sum map $S^2 \vee S^2 \to S^2$ is $2h$, where h is a generator of $\pi_3(S^2)$. Thus, LEMMA 7.5 implies that F_{2h} is null-homotopic (after localization at ℓ). But F_{2h} is homotopic to $2 \circ F_h$, where 2 means the self-map of $\Omega^3\,\mathrm{B}S^3_{(2)}$ induced by multiplication by 2 on S^3. Since this map is a homotopy equivalence, it follows that F_h is null-homotopic. By naturality, this implies that (the ℓ-primary part of) F_φ is null-homotopic for any $\varphi \in \Gamma_2(L)$. This shows $12\,\Gamma_2(L) \subset \ker F$.

Next, we show $\ker F \subset 12\,\Gamma_2(L)$. By naturality, LEMMA 7.6 implies that there is a generator $\varepsilon \in \pi_6(S^3)$ such that $\theta_2(2\varphi) = 2\varphi \otimes \varepsilon \in \Gamma_2(L) \otimes \pi_6(S^3)$ for any $\varphi \in \Gamma_2(L)$. Now, suppose we have $\varphi \in \ker F$. Then $2\varphi \in \ker F$, whence $2\varphi \otimes \varepsilon = \theta_2(2\varphi) = 0$. Thus φ must be divisible by 6. In particular, we have $\varphi = 2\varphi'$, thus we can repeat the argument to find $\varphi \otimes \varepsilon = \theta_2(\varphi) = 0$. Thus φ must be divisible by 12.

It remains to prove LEMMAS 7.5 and 7.6.

Proof of Lemma 7.5. — Set $G = S^3_{(\ell)}$. We must show that

$$F_w : \mathcal{C}_\bullet(S^2 \vee S^2, \mathrm{B}G) = \Omega^2 BG \times \Omega^2 \mathrm{B}G \to \Omega^3 \mathrm{B}G$$

is null-homotopic.

Recall that the join $X * Y$ of two spaces X, Y is defined as the quotient of the product $X \times I \times Y$ by the identifications $(x,0,y) = (x',0,y)$, $(x,1,y) = (x,1,y')$. Think of S^3 as $S^1 * S^1$. Think of S^2 as $S^1 \wedge S^1$. For $t \in I = [0,1]$, let $[t]$ be its image in $S^1 = I/(0=1)$. Then the map $w : S^3 = S^1 * S^1 \to S^2 \vee S^2$, defined by :

$$w(x,t,y) = \begin{cases} i_1\big([2t] \wedge x\big) & \text{if } t \le \frac{1}{2}, \\ i_2\big([2-2t] \wedge y\big) & \text{if } t \ge \frac{1}{2}, \end{cases}$$

represents the Whithehead product $[i_1, i_2]$.

Similarly, define $\tilde{w} : G * G \to \Sigma G = S^1 \wedge G$ by the formula :

$$\tilde{w}(a,t,b) = \begin{cases} [2t] \wedge a & \text{if } t \le \frac{1}{2}, \\ [2-2t] \wedge b & \text{if } t \ge \frac{1}{2}. \end{cases}$$

Let $s : G \to \Omega\Sigma G$ be the canonical map, sending $x \in G$ to the loop $t \mapsto t \wedge x$. Let $i : \Sigma G \to \mathrm{B}G$ be the map classifying the principal G-bundle whose clutching function is the identity $G \to G$. Then it is well known that the composition

$$G \xrightarrow{s} \Omega\Sigma G \xrightarrow{\Omega i} \Omega\,\mathrm{B}G$$

is a homotopy equivalence.

The key observation is that the following diagram is homotopy commutative :

$$(3) \qquad \begin{array}{ccccc} \Omega G \times \Omega G & \xrightarrow{\Omega s \times \Omega s} & \Omega^2\Sigma G \times \Omega^2 \Sigma G & \xrightarrow{\Omega^2 i \times \Omega^2 i} & \Omega^2 \mathrm{B}G \times \Omega^2 \mathrm{B}G \\ \Big\downarrow{*} & & \Big\downarrow & & \Big\downarrow{F_w} \\ \Omega^3(G * G) & \xrightarrow{\Omega^3 \tilde{w}} & \Omega^3 \Sigma G & \xrightarrow{\Omega^3 i} & \Omega^3 \mathrm{B}G \end{array}$$

Here, the map $*$ is the join, that is $*(f,g) = f * g$, where $f * g(x,t,y) = (f(x),t,g(y))$. Now $i \circ \tilde{w} \in [G * G, \mathrm{B}G] \approx \pi_7(\mathrm{B}S^3)_{(\ell)}$. But this group is zero, since $\pi_7(\mathrm{B}S^3) = \pi_6(S^3) = \mathbb{Z}/12$, and $\ell \geq 5$. This implies that F_w is null-homotopic, since $(\Omega i) \circ s$ is a homotopy equivalence. This proves LEMMA 7.5.

Proof of Lemma 7.6 : recall that the image of $w = [i_1, i_2]$ in $\pi_3(S^2)$ under the obvious sum map $S^2 \vee S^2 \to S^2$ is $2h$. Thus, diagram (3) gives a homotopy commutative diagram :

$$\begin{tikzcd}
\Omega S^3 \arrow[r, "\Delta"] \arrow[d, "\sim"] & \Omega S^3 \times \Omega S^3 \arrow[r, "*"'] & \Omega^3(S^3 * S^3) = \Omega^3 S^7 \arrow[d, "\Omega^3(i\circ\tilde{w})"] \\
\Omega^2 \mathrm{B}S^3 \arrow[rr, "F_{2h}"] & & \Omega^3 \mathrm{B}S^3.
\end{tikzcd}$$

Here, Δ is the diagonal map, and $*$ is the join. As in LEMMA 7.4, we have a filtration :

$$[\Omega S^3, \Omega^3 S^7] = \mathcal{F}'_0 \supset \mathcal{F}'_1 \supset \mathcal{F}'_2 \supset \cdots,$$

where $\mathcal{F}'_{n-1}/\mathcal{F}'_n \approx \Gamma_n(\mathbb{Z}) \otimes \pi_{2n+3}(S^7)$. Since $\mathcal{F}'_0/\mathcal{F}'_1 = 0$, the map $* \circ \Delta$ defines an element

$$\eta \in \mathcal{F}'_1/\mathcal{F}'_2 \approx \pi_7(S^7).$$

Moreover, identifying $\pi_6(S^3) = \pi_7(\mathrm{B}S^3)$, we have by naturality :

$$\theta_2(2h) = (i \circ \tilde{w})_*(\eta).$$

As is well known [T], $i \circ \tilde{w}$ is a generator of $\pi_7(\mathrm{B}S^3)$. Thus, identifying $\pi_7(S^7) = \mathbb{Z}$, we are reduced to prove the following :

Claim : $\eta = \pm 2$.

To prove the claim, let $A : \Sigma^3 \Omega S^3 \to S^7$ be the the map adjoint to $* \circ \Delta$. Note that the induced map $H_7(A;\mathbb{Z})$ is of the form $\mathbb{Z} \to \mathbb{Z}$, and it is not hard to see that this is actually multiplication by η.

I owe P. VOGEL the following argument. Represent a generator of $H_4(\Omega S^3;\mathbb{Z}) \approx \mathbb{Z}$ by a map $g : M^4 \to \Omega S^3$, where M^4 is a closed oriented 4-manifold. Call F the composition

$$F : S^3 \times M \to \Sigma^3 M \xrightarrow{\Sigma^3 g} \Sigma^3 \Omega S^3 \xrightarrow{A} S^7.$$

Clearly η is equal to $\mathrm{d}^\circ(F)$, where $\mathrm{d}^\circ(F)$ means the degree of F as a map between smooth compact oriented manifolds. Now let f be the map $S^1 \times M \to \Sigma M \to S^3$ adjoint to g. Identifying $S^3 = S^1 * S^1$, $S^7 = S^3 * S^3$, we see that F is given by the formula

$$F\big((a,t,b,),x\big) = \big(f(a,x),t,f(b,x)\big).$$

We may suppose f is smooth. Then F is also smooth, and has a regular value of the form $(z,t_0,z') \in S^3 * S^3$, where $0 < t_0 < 1$. Thus

$$\begin{aligned}\mathrm{d}^\circ(F) &= \#\Big\{\big((a,t_0,b),x\big) \mid f(a,x) = z,\ f(b,x) = z'\Big\}\\ &= \pm \mathrm{d}^\circ(\widetilde{F}),\end{aligned}$$

where $\widetilde{F} : S^1 \times S^1 \times M \to S^3 \times S^3$ is given by $\widetilde{F}(a,b,x) = (f(a,x), f(b,x))$.

Finally, we can calculate $\mathrm{d}^\circ(\widetilde{F})$ as follows. Let $\sigma \in H^3(S^3;\mathbb{Z})$ and $\theta \in H^1(S^1;\mathbb{Z})$ be the standard generators. Then $f^*(\sigma) = \theta \otimes g^*(\alpha)$, with α a generator of $H^2(\Omega S^3;\mathbb{Z})$. Hence $\widetilde{F}^*(\sigma\otimes\sigma) = \pm\theta\otimes\theta\otimes g^*(\alpha)^2$, and since $\frac{1}{2}\alpha^2$ generates $H^4(\Omega S^3;\mathbb{Z})$, we see $\mathrm{d}^\circ(\widetilde{F}) = \pm 2$.

This proves LEMMA 7.6, and completes the proof of THEOREM 7.1.

COROLLARY 7.7. — *Let $\varphi \in \Gamma_2(L)$. Then $F_\varphi : \mathcal{C}_\bullet(M(L,2),\mathrm{B}S^3) \to \Omega^3\,\mathrm{B}S^3$ is homotopy linear if and only if F_φ is null-homotopic.*

Proof. — Let $i_1, i_2 : M(L,2) \to M(L\oplus L,2)$ be induced by the obvious inclusions $L \to L\oplus L$. For $\varphi \in \Gamma_2(L)$, define

$$\mathrm{d}(\varphi) = (i_1+i_2)\circ\varphi - i_1\circ\varphi - i_2\circ\varphi \in \pi_3\big(M(L\oplus L,2)\big) = \Gamma_2(L\oplus L).$$

Then F_φ is homotopy linear if and only if

$$F_{\mathrm{d}(\varphi)} \in [\mathcal{C}_\bullet(M(L\oplus L,2),\mathrm{B}S^3),\Omega^3\,\mathrm{B}S^3]$$

is zero. But the linear map $\Gamma_2(L) \to \Gamma_2(L\oplus L)$, $\varphi \mapsto \mathrm{d}(\varphi)$ is injective. This implies the corollary.

REMARK 7.8. — Consider the fibration

$$\Omega^4\widehat{B} \to \mathcal{C}_\bullet(X,\mathrm{B}S^3)_k \to \mathcal{C}_\bullet(M(L,2),\mathrm{B}S^3)$$

obtained from fibration (1) by restricting to base point preserving maps. It is not hard to see that in the homology spectral sequence, the differential :

$$\begin{aligned}\mathrm{d}^2_{2,0} : E^2_{2,0} \approx H_2\big(\mathcal{C}_\bullet(M(L,2),\mathrm{B}S^3);\mathbb{Z}\big) \approx L^*\\ \longrightarrow E^2_{0,1} \approx H_1\big(\Omega^4_k\,\mathrm{B}S^3;\mathbb{Z}\big) \approx \pi_4(S^3) \approx \mathbb{Z}/2\end{aligned}$$

corresponds to $\theta_1(\varphi)$ via the natural isomorphism

$$\mathrm{Hom}(L^*, \mathbb{Z}/2) \approx L \otimes \mathbb{Z}/2.$$

Recall that $H_*(\mathcal{C}_\bullet(M(L,2), \mathrm{B}S^3); \mathbb{Z})$ is a polynomial algebra on 2-dimensional generators. Thus, if the homology spectral sequence were *multiplicative*, then the condition $\theta_1(\varphi) = 0$ would imply that the whole spectral sequence degenerates at the E^2-level. However, the only geometric condition to ensure multiplicativity of the spectral sequence we can think of is that F_φ be homotopy linear. Curiously enough, if $\theta_1(\varphi) = 0$, then the mod 2 spectral sequence *does* degenerate by PROPOSITION 5.1, although F_φ, even localised at 2, need not be homotopy linear as follows from COROLLARY 7.7.

Appendix : proof of Proposition 4.3. — Write :

$$A_* = H_*(\mathrm{BO}; \mathbb{F}_2) = P(a_i; i \geq 1), \quad B_* = \mathrm{Im}(j_*) = P(b_n; \varepsilon_0(n) \leq 1).$$

Here b_n is the image of the generator of degree n appearing in PROPOSITION 3.1. Recall that the b_n are indecomposable, and their expression in terms of the a_i can be found in [K]. We will use the following notation. When $I = (i_1, i_2, \ldots, i_s)$ is a partition of n, then $a(I) = a_{i_1} a_{i_2} \ldots a_{i_s}$, $b(I) = b_{i_1} b_{i_2} \ldots b_{i_s}$, and $a(I)^*$ is the dual of $a(I)$ with respect to the basis of A_* given by the monomials in the a_i. We need the following lemma :

LEMMA. — *Let $I = (i_1, i_2, \ldots, i_s)$ be a partition of $2^\lambda m$, where $\lambda \geq 1$ and m is an odd integer. Suppose all $i_\nu \equiv 0(m)$. Then $a(I)^*$ is indecomposable if and only if $I = (m, m, \ldots, m)$.*

Proof. — Recall $H^*(\mathrm{BO}; \mathbb{F}_2) = P(w_i\,;\ i \geq 1)$, where w_i is the mod 2 reduction of the i^{th} symmetric polynomial σ_i in formal indeterminates $t_1, t_2, \ldots$ In terms of symmetric polynomials, the element $a(I)^*$ can be written

$$a(I)^* = s_{i_1, \ldots, i_s} = \sum t_1^{i_1} \ldots t_s^{i_s}$$

(*cf.* [MS] for this notation). We will also use the notation

$$S_m^{(i)} = s_{m, \ldots, m} = (a_m^i)^*.$$

Observe that $s_m^{(i)} = \sigma_i(t_1^m, t_2^m, \ldots)$. Finally, recall the Newton formula :

$$s_n - \sigma_1 s_{n-1} + \cdots + (-1)^{n-1} \sigma_{n-1} s_1 + (-1)^n n \sigma_n = 0.$$

Consider a partition of $2^\lambda m$ of the form $I = (j_1 m, \dots, j_r m)$. First, suppose $r < 2^\lambda$. Define an algebra homomorphism $\Phi : H^*(\mathrm{BO}) \to \mathbb{F}_2$ by setting $\Phi(t_\nu) = 1$ for $1 \le \nu \le 2^\lambda$, $\Phi(t_\nu) = 0$ for $\nu > 2^\lambda$. Then $\Phi(w_j) = \binom{2^\lambda}{j}$, hence $\Phi(w_j) = 0$ for $j < 2^\lambda$, and $\Phi(w_{2^\lambda}) = 1$. Similarly, $\Phi(s_{j_1,\dots,j_r}) = \binom{2^\lambda}{r} = 0$, since $r < 2^\lambda$. Hence $s_{j_1,\dots,j_r}$ is a polynomial in $w_1, \dots, w_{2^\lambda - 1}$. This implies that $s_{j_1 m,\dots,j_r m}$ is a polynomial in $s_m, s_m^{(2)}, \dots, s_m^{(2^\lambda - 1)}$. Thus $a(I)^* = s_{j_1 m,\dots,j_r m}$ is decomposable.

Now suppose $r = 2^\lambda$, that is $a(I)^* = (a_m^{2^\lambda})^* = s_m^{(2^\lambda)}$. We must show that this is indecomposable. To see this, we work in the ring of symmetric polynomials with integral coefficients. By the Newton formula, $s_{2^\lambda m} + 2^\lambda m\, \sigma_{2^\lambda m}$ is decomposable. Applying the Newton formula with the formal variables t_i replaced by t_i^m shows that $s_{2^\lambda m} + 2^\lambda s_m^{(2^\lambda)}$ is also decomposable. Hence $s_m^{(2^\lambda)} \equiv m\, \sigma_{2^\lambda m}$ modulo decomposable elements. Since m is odd, the result follows.

This completes the proof of our lemma.

We now prove PROPOSITION 4.3. For each n such that $\varepsilon_0(n) \ge 2$, we define $r_n \in \ker(j^*)$ as follows. Write $n = 2^\ell m$ where m is odd. Also, write $n = 2^\lambda \mu$ where $\mu = 4m$ if $\varepsilon_0(m) = 0$, $\mu = 2m$ if $\varepsilon_0(m) = 1$, and $\mu = m$ if $\varepsilon_0(m) \ge 2$. Set $r_n^{(0)} = (a_\mu^{2^\lambda})^*$. Define inductively

$$r_n^{(i)} = r_n^{(i-1)} + \sum \langle r_n^{(i-1)}, b(I) \rangle\, a(I)^*$$

where the sum is over all partitions $I = (i_1, i_2, \dots, i_s)$ of n such that $s \ge 2^\lambda - i$ and all $i_\nu \equiv 0 \ (\mu)$. Then set $r_n = r_n^{(2^\lambda)}$.

We now show $r_n \in \ker(j^*)$. It suffices to show that $\langle r_n, b(I) \rangle = 0$ for all possible monomials $b(I)$ of degree n. By the very definition of r_n, it is clear that we only have to consider those monomials $b(I)$ where the partition $I = (i_1, i_2, \dots, i_s)$ is such that all $i_\nu \equiv 0 \ (\mu)$. Call these partitions *admissible*, and call s the *length* of such a partition. Observe that since $\varepsilon_0(\mu) \ge 2$, there is no generator b_μ. Hence there is no admissible partition of length 2^λ. It then follows from the definition of $r_n^{(1)}$ that $\langle r_n^{(1)}, b(I) \rangle = 0$ for all admissible partitions of length $\ge 2^\lambda - 1$. Similarly, since $\langle a(I)^*, b(I') \rangle = 0$ whenever the length of I' is greater than the length of I, we see by induction on i that $\langle r_n^{(i)}, b(I) \rangle = 0$ for all admissible partitions of length $\ge 2^\lambda - i$. This shows $r_n \in \ker(j^*)$.

Next we show that the r_n verify the indecomposability properties claimed in PROPOSITION 4.3. First suppose $\varepsilon_0(m) \ge 2$. Then $\mu = m$ is odd, and the above lemma implies that r_n is indecomposable. Second, suppose $\varepsilon_0(m) = 1$. Then $\mu = 2m$, hence r_n admits a unique square root $x_{n/2}$. (Indeed, r_n is a sum of terms of the form $s_{j_1\mu,\dots,j_r\mu}$, and we have $s_{j_1\mu,\dots,j_r\mu} = (s_{j_1 m,\dots,j_r m})^2$). Moreover, the lemma implies that $x_{n/2}$ is indecomposable. Similarly, if $\varepsilon_0(m) = 0$, then $\mu = 4m$, and r_n is the fourth power of an indecomposable element $x_{n/4}$.

This completes the proof of part (i) of PROPOSITION 4.3. For part (ii), suppose given a system of elements $r_n \in \ker(j^*)$ with the above indecomposability properties. For $n = 2^\ell m$ where m is odd, define $x_n = r_n$ if $\varepsilon_0(m) \geq 2$, $x_n = (r_{2n})^{1/2}$ if $\varepsilon_0(m) = 1$, and $x_n = (r_{4n})^{1/4}$ if $\varepsilon_0(m) = 0$. Since all x_n are indecomposable, we have $H^*(\mathrm{BO};\mathbb{F}_2) = P(x_n; n \geq 1)$. This shows that no r_n is in the ideal generated by the r_i with $i < n$. Using this, an easy calculation shows that that the Poincaré series of $\ker(j^*)$ coincides with the Poincaré series of the ideal freely generated by the r_n. This proves part (ii) of PROPOSITION 4.3.

Remarks :

1) If $\varepsilon_0(m) = 0$, then r_n is a fourth power, and we may replace r_n by $w_{n/4}^4$.

2) If $\varepsilon_0(m) \leq 1$, then $r_n = r_n^{(0)} = (a_\mu^{2^\lambda})^*$. This is obvious if $\varepsilon_0(m) = 0$, since in this case $\mu \equiv 0\ (4)$, and there are no generators b_n in degrees divisible by 4. If $\varepsilon_0(m) = 1$, the argument is as follows. We must show $\langle (a_\mu^{2^\lambda})^*, b(I)\rangle = 0$ for all possible monomials $b(I)$ of degree $n = 2^\ell m = 2^{\ell-1}\mu$. Suppose $I = (j_1, \dots, j_r)$ is a partition of n such that there is a monomial $b(I)$. Then $\sum j_\nu = n$, and all $\varepsilon_0(j_\nu) \leq 1$. If $\langle (a_\mu^{2^\lambda})^*, b(I)\rangle = 1$, then all j_ν must be divisible by $\mu = 2m$. But we will show that this is impossible. Indeed, suppose that all j_ν are divisible by $\mu = 2m$. Set $k_\nu = \frac{1}{2} j_\nu$. Then $\sum k_\nu = \frac{1}{2} n = 2^\lambda m$. Moreover, we have $\varepsilon_0(k_\nu) = 0$, hence we can write $k_\nu = 2^{\ell_\nu} - 1$. Let ℓ_0 denote the order of 2 in $(\mathbb{Z}/m)^*$. Since each k_ν is divisible by m, each ℓ_ν is divisible by ℓ_0. This implies that each k_ν is divisible by $2^{\ell_0} - 1$, hence so is $\sum k_\nu = \frac{1}{2} n = 2^\lambda m$. It follows that m is divisible by $2^{\ell_0} - 1$. On the other hand, m divides $2^{\ell_0} - 1$ by definition. Thus $m = 2^{\ell_0} - 1$. But this implies $\varepsilon_0(m) = 0$, thus contradicting our hypothesis.

3) It turns out that the smallest n such that $r_n \neq (a_\mu^{2^\lambda})^*$, is $n = 144$. In this case, the algorithm yields $r_{144} = (a_9^{16})^* + (a_{27}^3 a_{63})^*$.

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