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The explicit reciprocity law and the cohomology of Fontaine-Messing

THE EXPLICIT RECIPROCITY LAW AND THE

COHOMOLOGY OF FONTAINE–MESSING

BY

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Résumé. — En utilisant la cohomologie de Fontaine-Messing, on prouve une loi de réciprocité explicite pour des anneaux locaux réguliers de certain type. Ceci étend la loi de réciprocité explicite classique pour le symbole de Hilbert (pour la caractéristique résiduelle $p \neq 2$) et aussi les lois de réciprocité explicites de Brückner, Vostokov et Kirillov sous une condition additionnelle faible sur p.

Abstract. — By using the cohomology theory of Fontaine-Messing, we prove an explicit reciprocity law for regular local rings of a certain type. This extends the classical explicit reciprocity law for Hilbert symbols (for the residue characteristic $p \neq 2$), and also the explicit reciprocity laws of Brückner, Vostokov and Kirillov under an additional weak condition on p.

0. Introduction

0.1. — Recently J.-M. FONTAINE and W. MESSING defined a new cohomology theory for schemes X of mixed characteristic $(0, p)$, which is closely related to the crystalline cohomology theory of $X \otimes \mathbb{Z}/p\mathbb{Z}$ and also to the p-adic etale cohomology theory of $X \otimes \mathbb{Z}\left[\frac{1}{p}\right]$. In this paper, as an application of their cohomology theory, we obtain "explicit reciprocity laws" for various regular local rings.

The explicit reciprocity law is classically a mysterious relation between Hilbert symbols and differential forms. In this paper, we regard it as the mysterious relation between p-adic etale cohomology of $X \otimes \mathbb{Z}\left[\frac{1}{p}\right]$ and the Fontaine-Messing cohomology of X, in which the latter is expressed "explicitly" in terms of differential forms.

Under a certain weak assumption on p (the assumption $p > r + s + 1$ in (0.3)), our result extends the generalized explicit reciprocity laws of

(*) Texte reçu le 25 novembre 1986. (Cf. the note at the end of this paper.)
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Vostokov in [V1], [V2], [V3], and also of Brückner [B1], [B2], Vostokov-Kirillov [VK], and our method gives new proofs and new interpretations to them.

0.2. — In this paper, our objective is a regular local ring A satisfying the following conditions (0.2.1) and (0.2.2).

0.2.1. — The field of fractions of A is of characteristic zero, and the residue field of A is of characteristic $p > 0$.

0.2.2. — Let $p = u\pi_1^{e_1} \cdots \pi_s^{e_s}$ ($u \in A^x, e_1, \ldots, e_s \geq 1$) be a factorization of p in (non-equivalent) irreducible elements of A, and let $\mathfrak{a}$ be the ideal $(\pi_1, \ldots, \pi_s)$. Then, $(\pi_i)_{1 \leq i \leq s}$ forms a part of a regular system of parameters of A, $(A, \mathfrak{a})$ is a henselian couple ([R], chap. XI), and $A/\mathfrak{a}$ has a finite p-basis over $\mathbb{F}_p$ ([Gr], chap. 0, § 21.1).

Note that the last condition on p-basis is not so restrictive. Indeed, most regular local rings of characteristic p which appear in arithmetic or in algebraic geometry have finite p-bases over $\mathbb{F}_p$. Note also that a henselian discrete valuation ring of mixed characteristic $(0, p)$ has properties (0.2.1), (0.2.2) if the residue field k satisfies $[k : k^p] < \infty$.

Let r be the cardinal number of a p-basis of R over $\mathbb{F}_p$. Fix $n \geq 1$ and assume that :

0.2.3. — A contains a primitive p^n-th root ζ of 1.

Under these conditions, our explicit reciprocity law gives an explicit description of the cohomological symbol map

$$K_{r+s+1}^m\left(A\left[\tfrac{1}{p}\right]\right) \longrightarrow H^{r+s+1}\left(\mathrm{Spec}\left(A\left[\tfrac{1}{p}\right]\right)_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z}(r + s + 1)\right)$$

on the assumption $p > r + s + 1$. Here for a ring S and for $q \geq 0$, we define the q-th Milnor K-group of S by

$$K_q^M(S) = \big(\underbrace{S^x \otimes \cdots \otimes S^x}_{r \text{ times}} \big)/N$$

where N is the subgroup of the tensor product generated by elements of the form $x_1 \otimes \cdots \otimes x_q$ such that $x_i + x_j$ is 0 or 1 for some $i \neq j$. (An element $a_1 \otimes \cdots \otimes a_q \mod N$ of $K_q^M(S)$ is denoted by $\{a_1, \ldots, a_q\}$).

THEOREM 0.3. — *Let A, r and s be as in (0.2.1)–(0.2.3) and let*

$$H = \mathrm{Coker}\big(F - 1 : W_n\Omega_R^r \longrightarrow W_n\Omega_R^r / \mathrm{d}W_n\Omega_R^{r-1}\big)$$

where $W_n\Omega_R^\bullet$ is the de Rham-Witt complex [I1]. *Assume $p > r + s + 1$. Then :*

(1) *There is a canonical homomorphism*

$$\tau_A : K_{r+s+1}^M\left(A\left[\tfrac{1}{p}\right]\right) \longrightarrow H$$

having the "explicit" characterization (0.7.3) *stated below.*

(2) *We have a commutative diagram*

$$K_{r+s+1}^M\left(A\left[\tfrac{1}{p}\right]\right) \xrightarrow{\ \ h\ \ } H^{r+s+1}\left(\mathrm{Spec}(A\left[\tfrac{1}{p}\right])_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z}(r+s+1)\right)$$

$$\searrow{\scriptstyle -\tau_A} \qquad \nearrow{\scriptstyle i_A}$$

$$H$$

where h is the cohomological symbol map and i_A is the canonical homomorphism given in (5.3).

Here, the definition of i_A is rather simple. The definition of τ_A by using the cohomology of Fontaine-Messing, and by using differentials, logarithms and residues, is the central point of this paper.

By this theorem and [BK], § 5, we have :

COROLLARY 0.4. — *Assume further that A is a discrete valuation ring (so $s = 1$) and let $K = A\left[\tfrac{1}{p}\right]$ be the field of fractions of A. Then we have a commutative diagram of isomorphisms*

$$K_{r+2}^M(K)/p^n K_{r+2}^M \xrightarrow[\cong]{\ \ h\ \ } H^{r+2}\left(\mathrm{Gal}(\overline{K}/K), \mathbb{Z}/p^n\mathbb{Z}(r+2)\right)$$

$$\searrow{\scriptstyle -\tau_A} \qquad \nearrow{\scriptstyle \cong}$$

$$H$$

where the right sloping arrow is the isomorphism of [K3], *th.* 3.

In (0.4), if the residue field k of A is perfect, the map

$$\tau_A : K_2(K) \longrightarrow H = W_n(k)/(F-1)W_n(k)$$

(F is the frobenius) is already given in [B1], [V2] by using differentials, logarithms and residues. In this case, the above diagram becomes :

$$
\begin{array}{ccc}
K_2(K)/p^n K_2(K) & \xrightarrow[\cong]{\ h\ } & {}_{p^n}\mathrm{Br}(K) \\[2em]
{\scriptstyle -\tau_A}\searrow & & \nearrow{\scriptstyle \cong} \\[1em]
& W_n(k)/(F-1)W_n(k) &
\end{array}
$$

($\mathrm{Br}(K)$ is the Brauer group of K, ${}_{p^n}(\ )$ means the kernel of p^n, the horizontal arrow is the norm residue symbol, and the right sloping arrow is the isomorphism of Witt), whose commutativity is proved in [B1], [V2]. If the residue field k of A is a "$(d-1)$-dimensional local field" so that K is a "d-dimensional local field" in the sense of [V3], then $r = d-1$, $H = \mathbb{Z}/p^n\mathbb{Z}$ and the map τ_A are given in [V3]. (In fact, since we assume $p > r + s + 1$, we can not cover the case $p = 2$ of [B1] and the case $2 < p \le d+1$ of [V3]. But cf. (5.12).)

0.5. — We explain how our results come from the theory of Fontaine-Messing. The cohomology of Fontaine-Messing, which we denote by $H^\bullet(X, \mathcal{S}_n(r))$, is defined for a scheme X satisfying a certain condition and has the following properties if $r < p$. It fits into the long exact sequence

$$(0.5.1) \quad \cdots \longrightarrow H^q\big(X, \mathcal{S}_n(r)\big) \longrightarrow M^q \xrightarrow{\ f_r - 1\ } N^q$$

$$\longrightarrow H^{q+1}\big(X, \mathcal{S}_n(r)\big) \longrightarrow M^{q+1} \xrightarrow{\ f_r - 1\ } N^{q+1} \longrightarrow \cdots$$

where

$$M^\bullet = H^\bullet\big(X_n/(\mathbb{Z}/p^n\mathbb{Z})_{\mathrm{crys}}, J^{[r]}_{X_n/(\mathbb{Z}/p^n\mathbb{Z})}\big),$$

$$N^\bullet = H^\bullet\big(X_n/(\mathbb{Z}/p^n\mathbb{Z})_{\mathrm{crys}}, \mathcal{O}_{X_n/(\mathbb{Z}/p^n\mathbb{Z})}\big).$$

Here $X_n = X \otimes \mathbb{Z}/p^n\mathbb{Z}$, $\mathcal{O}_{X_n/(\mathbb{Z}/p^n\mathbb{Z})}$ is the structural sheaf on the crystalline site, $J^{[r]}_{X_n/(\mathbb{Z}/p^n\mathbb{Z})}$ is the r-th divided power of the ideal

$$J_{X_n/(\mathbb{Z}/p^n\mathbb{Z})} = \mathrm{Ker}\big(\mathcal{O}_{X/(\mathbb{Z}/p^n\mathbb{Z})} \longrightarrow \mathcal{O}_{X_n,\mathrm{zar}}\big),$$

and f_r is a certain homomorphism, which is, roughly speaking, p^{-r} times the Frobenius. Furthermore, FONTAINE and MESSING related their cohomology to the p-adic etale cohomology of $X \otimes \mathbb{Z}[\frac{1}{p}]$.

For $X = \text{Spec}(A)$ with A satisfying (0.2.1), (0.2.2), they can define a canonical homomorphism for $r < p - 1$

$$(0.5.2) \qquad c : H^{\bullet}(X, \mathcal{S}_n(r)) \longrightarrow H^{\bullet}((X \otimes \mathbb{Z}[\tfrac{1}{p}])_{\text{et}}, \mathbb{Z}/p^n\mathbb{Z}(r))$$

which makes the following diagram commutative.

$$(0.5.3)$$

$$
\begin{array}{ccc}
K_q^M(A) & \xrightarrow{\quad\quad h \quad\quad} & H^q\big(\text{Spec}(A[\tfrac{1}{p}])_{\text{et}}, \mathbb{Z}/p^n\mathbb{Z}(q)\big) \\
 & & \\
\ {}^{h'}\searrow & \nearrow{}_{c} & \quad (q < p - 1) \\
 & H^q(X, \mathcal{S}_n(q)) &
\end{array}
$$

Here h is the cohomological symbol map and h' is the symbol map for the cohomology of Fontaine-Messing (cf. § 2). We prove in § 4 :

THEOREM 0.6. — *Let A be as in Theorem (0.3) and assume $p > r + s + 1$ and that A is complete with respect to the $\mathfrak{a}$-adic topology. Let $X = \text{Spec}(A)$. Then there exists a canonical isomorphism*

$$H^{r+s+1}(X, \mathcal{S}_n(r + s + 1)) \cong H.$$

This result with the following comments explains THEOREM (0.3).

1) The symbol map h' for the cohomology of Fontaine-Messing is described by using differentials and logarithms, and such map is already constructed in special cases in the above mentioned works on explicit reciprocity laws of course without the view point of a new cohomology theory. In these works, residue maps play roles. In this paper, a residue map appears in the definitions of the isomorphism (0.6) and of τ_A.

2) This symbol map h' on $K_q^M(A)$ is extended to $K_q^M(A[\tfrac{1}{p}])$ if we slightly modify the target group $H^q(X, \mathcal{S}_n(q))$ (cf. § 3).

3) The definition of the map c (0.5.2) seems to be an extremely difficult one. In fact we do not use c in this paper, though we introduced it in (0.5) to make our philosophy clear. In THEOREM (0.3) (2), its role is played by a map i_A which has a simple "explicit" definition and is defined also when $q = p - 1$.

0.7. — We give further explanation of the above point 1) together with the characterization of the map τ_A in (0.3).

A fundamental property of the crystalline cohomology is that it is related to differential forms. For example, let C be a ring such that p is a non-zero-divisor in C, and such that C/pC is reduced and has a p-basis over $\mathbf{F}_p$ ([Gr], chap. 0, § 21.1). Then the crystalline cohomology of $C_n = C \otimes \mathbf{Z}/p^n\mathbf{Z}$ coincides with the de Rham cohomology (cf. § 1). That is, $N^q \cong H^q(\Omega^\bullet_{C_n/\mathbf{Z}})$ in (0.5.1) for $X = \operatorname{Spec}(C)$. If the cardinal number of a p-basis of C/pC is $q < \infty$, (0.5.1) induces

$$H^{q+1}\big(X, \mathcal{S}_n(q+1)\big) \cong H^q(\Omega^\bullet_{C_n/\mathbf{Z}}) = \Omega^q_{C_n/\mathbf{Z}}/\,\mathrm{d}\Omega^{q-1}_{C_n/\mathbf{Z}}.$$

If $q + 1 < p$, the composite map

$$\theta_C : K^M_{q+1}(C) \xrightarrow{\quad h' \quad} H^{q+1}\big(H, \mathcal{S}_n(q+1)\big) \cong \Omega^q_{C_n/\mathbf{Z}}/\,\mathrm{d}\Omega^{q-1}_{C_n/\mathbf{Z}}$$

is the homomorphism having the following characterization (cf. § 2).

0.7.1. — For any ring homomorphism $f : C_{n+1} \to C_{n+1}$ which lifts $C_1 \to C_1$; $x \mapsto x^p$, $\theta_C(\{a_1, \ldots, a_{q+1}\})$ is equal to

$$\sum_{i=1}^{q+1} (-1)^{i-1} \Big(\frac{1}{p}\log\Big(\frac{f(a_i)}{a_i^p}\Big)\Big) \frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_{i-1}}{a_{i-1}}$$

$$\wedge \Big(\frac{1}{p}\frac{\mathrm{d}f(a_{i+1})}{f(a_{i+1})}\Big) \wedge \cdots \wedge \Big(\frac{1}{p}\frac{\mathrm{d}f(a_{q+1})}{f(a_{q+1})}\Big)$$

for any $a_1, \ldots, a_{q+1} \in C^x$. (The homomorphism f as above always exists).

Here log is the map $1 + pC_{n+1} \to pC_{n+1}$,

$$\log(1 + px) = \sum_{i \leq 1} \frac{(-1)^{i-1}p^i}{i}\, x^i.$$

Since $\Omega^1_{C_{n+1}/\mathbf{Z}}$ is a free C_{n+1}-module (see (1.8), (1)), and $p^{-1}\,\mathrm{d}f(a_i)/f(a_i)$ makes sense in $\Omega^1_{C_n/\mathbf{Z}}$ as $\mathrm{d}f(a_i) \in p\Omega^1_{C_{n+1}/\mathbf{Z}}$. This map θ_C is defined in special cases in the above mentioned works on explicit reciprocity laws (cf. especially [V3], (6), (7)).

Now the characterization of the map τ_A in (0.3) is stated as follows. The map τ_A in the general case is induced from the case A is $\mathfrak{a}$-adically complete, via the map $K^M_q(A[\frac{1}{p}]) \to K^M_q(A_\mathfrak{a}[\frac{1}{p}])$ where $A_\mathfrak{a}$ is the $\mathfrak{a}$-adic completion of A. Assume now that A is $\mathfrak{a}$-adically complete. Take a ring homomorphism $P \to A$ such that P is a ring in which p is a non-zero-divisor satisfying $P \xrightarrow{\cong} \varprojlim P/p^nP$, and such that the induced map

$P/pP \rightarrow A/\mathfrak{a}$ is an isomorphism. (The existence of P and $P \rightarrow A$ will be shown in §4 (see (4.13))). Let $P[[T_1, \ldots, T_s]] \rightarrow A$ be the surjective homomorphism $T_i \mapsto \pi_i$, take a lifting $\xi \in P[[T_1, \ldots, T_s]]$ of ζ such that $\xi - 1$ is contained in the ideal $(T_1, \ldots, T_s)$ of $P[[T_1, \ldots, T_s]]$, and let $\beta = (\xi^{p^n} - 1) - \frac{1}{2}(\xi^{p^n} - 1)^2$. Let

$$C = P[[T_1, \ldots, T_s]][\frac{1}{T_1 \cdots T_s}],$$

and let $\mathrm{Res} : \Omega^{r+s}_{C_n/\mathbf{Z}} \longrightarrow \Omega^r_{P_n/\mathbf{Z}}$ be the residue homomorphism

$$\sum_{(i_1, \ldots, i_s)} w_{i_1, \ldots, i_s} \wedge T_1^{i_1} \cdots T_s^{i_s} \frac{\mathrm{d}T_1}{T_1} \wedge \cdots \wedge \frac{\mathrm{d}T_s}{T_s} \longmapsto w_{0, \ldots, 0}$$

($w_{i_1, \ldots, i_s} \in \Omega^r_{P_n/\mathbf{Z}}$ and P_n denotes $P/p^n P$). We regard H as a quotient of $\Omega^r_{P_n/\mathbf{Z}}$ via the canonical isomorphism [IR], chap. III, (1.5)

(0.7.2) $$W_n \Omega^r_R \xrightarrow{\cong} \Omega^r_{P_n/\mathbf{Z}}/\Omega^{r-1}_{P_n/\mathbf{Z}},$$

$$w\, \mathrm{d}\log(y_1) \cdots \mathrm{d}\log(y_r) \longmapsto \sum_{i=0}^{n-1} p^i\, \tilde{a}_i^{p^{n-i}} \frac{\mathrm{d}\tilde{y}_1}{\tilde{y}_1} \wedge \cdots \wedge \frac{\mathrm{d}\tilde{y}_r}{\tilde{y}_r},$$

$$\mathrm{d}\big(w\, \mathrm{d}\log(y_1) \cdots \mathrm{d}\log(y_{r-1})\big) \longmapsto \sum_{i=0}^{n-1} \tilde{a}_i^{p^{n-i}-1}\, \mathrm{d}\tilde{a}_i \frac{\mathrm{d}\tilde{y}_1}{\tilde{y}_1} \wedge \cdots \wedge \frac{\mathrm{d}\tilde{y}_{r-1}}{\tilde{y}_{r-1}},$$

($w = (a_0, \ldots, a_{n-1}) \in W_n(R)$ with $a_i \in R$, $y_i \in R^x$, and $\tilde{\ }$ means a lifting to P_n). Then, τ_A is characterized by the following (0.7.3).

0.7.3. — For any choices of P, $P \rightarrow A\left[\frac{1}{p}\right]$, and ξ as above, the following diagram is commutative

$$
\begin{array}{ccccc}
K_{r+s+1}^M(C) & \xrightarrow{\theta_C} & \Omega^{r+s}_{C_n/\mathbf{Z}}/\mathrm{d}\Omega^{r+s-1}_{C_n/\mathbf{Z}} & & w \\
\downarrow & & \downarrow & & \downarrow \\
K_{r+s+1}^M(A[\frac{1}{p}]) & \xrightarrow{\tau_A} & H & & \mathrm{class}(\mathrm{Res}(\beta^{-1}w)),
\end{array}
$$

where the left vertical arrow is the surjection induced by $C \rightarrow A\left[\frac{1}{p}\right]$; $T_i \mapsto \pi_i$. This map τ_A depends by sign on the ordering on the set of

prime ideals $\{(\pi_1), \ldots, (\pi_s)\}$, but is independent of the choices of the generators π_i of (π_i).

0.8. — The history of the explicit reciprocity law is long ([AH], [Iw], [Sa], [Sh], [W]..., I am sorry that I can not list many excellent works) and I have not yet well understood the relation between this work and others. For example, there should be a close relation between the method of COLEMAN [C] and that of this paper. Can one use the Lubin-Tate groups in our setting?

I am much obliged to Professors J.-M. FONTAINE and W. MESSING who explained their cohomology theory to me. I hope that this paper serves as a good propaganda of the importance of their theory. I express my sincere gratitude to Professor B. KAHN who introduced the paper [B1] to me with valuable remarks.

Conventions. — In this paper, p denotes a fixed prime number and $\mathbb{F}_p$ denotes $\mathbb{Z}/p\mathbb{Z}$. Rings are assumed to be commutative and with unit. For a ring S, S^x denotes the group of all invertible elements of S. For a scheme or a ring S, S_n denotes $S \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}$ and $\Omega_S^{\bullet}$ denotes the absolute differential module $\Omega_{S/\mathbb{Z}}^{\bullet}$.

1. p-bases and crystalline cohomology

A fundamental fact in the crystalline cohomology theory is that, if X is a scheme embedded in a *smooth* scheme Z, the crystalline cohomology of X is computed by using differentials on Z and the divided power envelope of X in Z ([BO], (7.2)). In later sections, we shall use schemes of the type (for example) $\mathrm{Spec}(\mathbb{Z}/p^n\mathbb{Z}[[T_1, \ldots, T_s]])$ as the ambient space Z, and so the smoothness of Z is a too strong assumption for us. The purpose of this section is to explain that the above fundamental fact in the crystalline cohomology theory holds if the ambient space has p-bases locally (see definition (1.3) for our definition of p-bases).

Definition 1.1. — Let $A \to B$ be a homomorphism of rings over $\mathbb{F}_p$. We say that B is *relatively perfect over* A if the map

$$A^{(p)} \otimes_A B \longrightarrow B^{(p)}; \quad x \otimes y \longmapsto xy^p$$

is an isomorphism, where $A^{(p)}$ denotes the ring over A with the underlying ring A and with the structural map $A \to A^{(p)}; x \mapsto x^p$, and $B^{(p)}$ denotes the ring over B defined similarly.

Definition 1.2. — Let $n \geq 1$ and let $A \to B$ a homomorphism of rings over $\mathbb{Z}/p^n\mathbb{Z}$. We say that B is *relatively perfect over* A if B_1 ($= B \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$,

cf. conventions) is relatively perfect over A_1 and B is formally etale over A (for the discrete topology) in the sense of [Gr], chap. 0, § 19.

Concerning homomorphisms between rings over $\mathbb{F}_p$, we have the following implications :

$$\text{(etale)} \implies \text{(relatively perfect)} \implies \text{(formally etale)}$$

(see [Gr], chap. 0, § 21 or [K4] (1.3)). So, for a ring over $\mathbb{F}_p$, definition (1.2) coincides with definition (1.1).

Definition 1.3. — Let $n \geq 1$ and let $A \to B$ be a homomorphism of rings over $\mathbb{Z}/p^n\mathbb{Z}$. A family $(b_\lambda)_{\lambda \in \Lambda}$ of elements of B is said to be a *p-basis of B over A* if the homomorphism from the polynomial ring

$$A[T_\lambda]_{\lambda \in \Lambda} \longrightarrow B ; \quad T_\lambda \longmapsto b_\lambda$$

is relatively perfect.

For rings over $\mathbb{F}_p$, this definition is slightly different from that of [Gr], chap. 0, § 21.1, when A and B are not assumed to be reduced.

The *p*-basis in the absolute sense is important for us. By our definition (1.3), for a ring B over $\mathbb{F}_p$, a family $(b_\lambda)_{\lambda \in \Lambda}$ of elements of B is a *p*-basis over $\mathbb{F}_p$ if and only if each element of B is expressed in the form

$$\sum_s x_s^p \cdot \prod_{\lambda \in \Lambda} b_\lambda^{s(\lambda)},$$

for a unique family (x_s) of elements of B, where s ranges over all functions $\Lambda \to \{0, 1, \ldots, p - 1\}$ with finite supports. If B is reduced, this coincides with the definition of [Gr], chap. 0, § 21.1. For a ring over $\mathbb{Z}/p^n\mathbb{Z}$, we have :

PROPOSITION 1.4. — *Let B be a ring over $\mathbb{Z}/p^n\mathbb{Z}$, and let $(b_\lambda)_{\lambda \in \Lambda}$ be a family of elements of B. Then the following two conditions are equivalent.*

(i) *$(b_\lambda)_{\lambda \in \Lambda}$ forms a p-basis of B over $\mathbb{Z}/p^n\mathbb{Z}$.*

(ii) *$(b_\lambda \mod pB)_{\lambda \in \Lambda}$ forms a p-basis of B/pB over $\mathbb{F}_p$, and B is flat over $\mathbb{Z}/p^n\mathbb{Z}$.*

Proof. — This proposition follows from the following two results (1.5) and (1.6).

PROPOSITION 1.5 (O. GABBER). — *Let A be a regular nœtherian ring over $\mathbb{F}_p$, and let B be a ring over A having a p-basis $(b_\lambda)_{\lambda \in \Lambda}$ over A. Then B is flat over A. Furthermore, the homomorphism $A[T_\lambda]_{\lambda \in \Lambda} \to B$; $T_\lambda \to b_\lambda$ is flat.*

Proof. — It suffices to see that B is flat over $A[T_\lambda]_{\lambda \in \Lambda'}$ for any finite subset Λ' of Λ. Since $A[T_\lambda]_{\lambda \in \Lambda'}$ is regular nœtherian, we are reduced to showing that B is flat over A. The proof of the flatness of B over A due to O. GABBER is introduced in [K4], (5.2) in the case $\Lambda = \emptyset$, and the same proof works in the general case.

LEMMA 1.6. — *Let $A \to B$ be a homomorphism of rings over $\mathbb{Z}/p^n\mathbb{Z}$, and assume that B_1 is flat over A_1. Then the following two conditions are equivalent.*

(i) *B is relatively perfect over A.*

(ii) *B_1 is relatively perfect over A_1 and B is flat over A.*

For the proof, see [K2], Lemma 1.

For a scheme Z over $\mathbb{Z}/p^n\mathbb{Z}$, we say that Z has *p-bases over $\mathbb{Z}/p^n\mathbb{Z}$ locally* if there is an affine open covering $\{U_\lambda\}$ of Z such that the affine ring $\mathcal{O}(U_\lambda)$ of U_λ has a p-basis over $\mathbb{Z}/p^n\mathbb{Z}$ for any λ.

In this paper, we shall consider only the crystalline cohomology with base $\mathbb{Z}/p^n\mathbb{Z}$ ($n \geq 1$) where $\mathbb{Z}/p^n\mathbb{Z}$ is endowed with the canonical PD (divided power) structure on the ideal $p(\mathbb{Z}/p^n\mathbb{Z})$. The PD envelopes are defined always with respect to this PD structure of the base $\mathbb{Z}/p^n\mathbb{Z}$.

The theme of this section is the following

THEOREM 1.7. — *Let Z be a scheme over $\mathbb{Z}/p^n\mathbb{Z}$ having p-bases over $\mathbb{Z}/p^n\mathbb{Z}$ locally, let $i : X \hookrightarrow Z$ be an immersion, and let $u : X/(\mathbb{Z}/p^n\mathbb{Z})_{\mathrm{crys}} \to X_{\mathrm{zar}}$ be the canonical morphism of sites. Let D be the PD enveloppe of X in Z, and let $J_D^{[r]}$ ($r \in \mathbb{Z}$) be the r-th divided power of the ideal $J_D = \mathrm{Ker}(\mathcal{O}_{D,\mathrm{zar}} \to \mathcal{O}_{X,\mathrm{Zar}})$. (For $r \leq 0$, $J_D^{[r]}$ denotes $\mathcal{O}_D$.) On the other hand, let $\mathcal{O}_{X/(\mathbb{Z}/p^n\mathbb{Z})}$ be the structural sheaf of the crystalline site $X/(\mathbb{Z}/p^n\mathbb{Z})_{\mathrm{crys}}$, and let $J_{X/(\mathbb{Z}/p^n\mathbb{Z})}^{[r]}$ be the r-th divided power of the ideal $J_{X/(\mathbb{Z}/p^n\mathbb{Z})} = \mathrm{Ker}(\mathcal{O}_{X/(\mathbb{Z}/p^n\mathbb{Z})} \to \mathcal{O}_{X,\mathrm{zar}})$. Then, $Ru_*(J_{X/(\mathbb{Z}/p^n\mathbb{Z})}^{[r]})$ is canonically isomorphic in the derived category to the complex*

$$J_D^{[r]} \xrightarrow{\ \mathrm{d}\ } J_D^{[r-1]} \otimes_{\mathcal{O}_Z} \Omega_Z^1 \xrightarrow{\ \mathrm{d}\ } \cdots \to J_D^{[r-q]} \otimes_{\mathcal{O}_Z} \Omega_Z^q \to \cdots$$

(deg. 0)   (deg. q)

This theorem is proved in textbooks of the crystalline cohomology theory in the case where Z is smooth over the case S (with $\Omega_{Z/S}^\bullet$ instead of $\Omega_Z^\bullet$), e.g. [BO], (7.2). However, one can check that all the points where the smoothness of Z is used in the proof of (1.7) are the followings : Z is locally formally smooth over the base, and smooth morphisms have

locally the properties in the LEMMA (1.8) below; but they are valid for any morphisms having p-bases.

For a ring S over $\mathbb{Z}/p^n\mathbb{Z}$, let $S\langle T_\lambda\rangle_{\lambda\in\Lambda}$ be the PD polynomial ring over S with variables $(T_\lambda)_{\lambda\in\Lambda}$. For an ideal I of S, let $D_S(I)$ be its PD envelope ([BO], § 3).

LEMME 1.8. — *Let $h : A \to B$ be a homomorphism of rings over $\mathbb{Z}/p^n\mathbb{Z}$ and assume that B has a p-basis $(b_\lambda)_{\lambda\in\Lambda}$ over A. Then :*

(1) *The differential module $\Omega^1_{B/A}$ is a free B-module with basis $(\mathrm{d}b_\lambda)_{\lambda\in\Lambda}$.*

(2) *Let I and J be ideals of A and B, respectively, such that $A/I \cong B/J$ via h. Then, if $b_\lambda \in J$ for all λ, we have an isomorphism*

$$D_A(I)\langle T_\lambda\rangle_{\lambda\in\Lambda} \xrightarrow{\cong} D_B(J) ; \quad T_\lambda^{[i]} \longmapsto b_\lambda^{[i]}.$$

In particular, in the case $I = (0)$, we have

$$A\langle T_\lambda\rangle_{\lambda\in\Lambda} \xrightarrow{\cong} D_B(J) ; \quad T_\lambda^{[i]} \longmapsto b_\lambda^{[i]}.$$

(3) *Let $C = B \otimes_A B$ and let J be the kernel of $C \to B ; x \otimes y \mapsto xy$. Then we have an isomorphism*

$$B\langle T_\lambda\rangle_{\lambda\in\Lambda} \xrightarrow{\cong} D_C(J) ; \quad T_\lambda^{[i]} \longrightarrow \left(b_\lambda\otimes 1 - 1\otimes b_\lambda\right)^{[i]}.$$

The condition $b_\lambda \in J$ for all $\lambda \in \Lambda$ in (2) is not so restrictive : if $(b_\lambda)_{\lambda\in\Lambda}$ is a p-basis over A and $(a_\lambda)_{\lambda\in\Lambda}$ are elements of A such that $a_\lambda \equiv b_\lambda \mod J$ for all λ, then $(b_\lambda - a_\lambda)_{\lambda\in\Lambda}$ is a p-basis over A satisfying $b_\lambda - a_\lambda \in J$ for all λ.

In the rest of this section, we prove (1.8). First, (1.8) (3) is a consequence of the case $I = (0)$ of (1.8) (2) applied to the homomorphism $B \to C$. Next, with C and J as in (1.8) (3), we have

$$\Omega^1_{B/A} = J/J^2 ; \quad x\,\mathrm{d}y \rightleftarrows y\otimes x - xy\otimes 1.$$

So, if we can prove

(*) $$A[T_\lambda]_{\lambda\in\Lambda}/(T_\lambda; \lambda \in \Lambda)^2 \xrightarrow{\cong} B/J^2 ; \quad T_\lambda \longmapsto b_\lambda$$

under the assumption of (1.8) (2) with $I = (0)$, instead of

(**) $$A\langle T_\lambda\rangle_{\lambda\in\Lambda} \xrightarrow{\cong} D_B(J) ; \quad T_\lambda^{[i]} \longmapsto b_\lambda^{[i]}$$

then (1.8)(1) will follow from (*) just as (1.8)(3) followed from (**). Since the proof of (*) is quite similar to that of (**), we leave the proof of (*) to the reader.

Now we prove (1.8) (2). We show that we may assume $\Lambda = \emptyset$ (i.e. that B is relatively perfect over A). Let I' be the ideal of $A[T_\lambda]_{\lambda \in \Lambda}$ generated by I and $(T_\lambda)_{\lambda \in \Lambda}$. Then, $D_A(I)\langle T_\lambda \rangle_{\lambda \in \Lambda}$ and $D_{A[T_\lambda]_{\lambda \in \Lambda}}(I')$ are canonically isomorphic, as is seen easily by using the universal property of the PD envelope. Since $A[T_\lambda]_{\lambda \in \Lambda}/I' \cong B/J$, we may replace $A[T_\lambda]_{\lambda \in \Lambda}$ by A, and hence we may assume $\Lambda = \emptyset$.

LEMMA 1.9. — *Let*

$$
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & B \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
R & \xrightarrow{\ c\ } & R/I
\end{array}
$$

be a commutative diagram of rings over $\mathbb{Z}/p^n\mathbb{Z}$ such that B is relatively perfect over A. Assume that there is a number $N \geq 1$ such that $x^N = 0$ for any $x \in I$. Then there is a unique homomorphism $t : B \to R$ such that $t \circ h = a$ and $c \circ t = b$.

Proof. — Note that the formally etale property is not directly applied, for we do not assume that I itself is nilpotent. However, by the formally etale property, we may assume that A is a ring over $\mathbb{F}_p$. Then, the proof of [Gr], chap. 0, (21.2.7) in fact proves the above lemma for rings over $\mathbb{F}_p$.

To prove the case $\Lambda = \emptyset$ of (1.8) (2), we apply (1.9) to the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ h\ } & B \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
D_A(I) & \xrightarrow{\ c\ } & A/I \cong B/J.
\end{array}
$$

Since $\mathrm{Ker}(c)$ is a PD ideal, any element x of $\mathrm{Ker}(c)$ satisfies

$$
x^{p^n} = p^n\,! \cdot x^{[p^n]} = 0.
$$

So by (1.9), there is a homomorphism $t : B \to D_A(I)$ such that $t \cdot h = a$ and $c \cdot t = b$. This t induces $D_B(J) \to D_A(I)$ compatible with the PD

structures. We see that the composite maps $D_A(I) \to D_B(J) \to D_A(I)$ and $D_B(J) \to D_A(I) \to D_B(J)$ are the identity maps. Indeed, the former is clear, and to see the latter, it suffices to show that the composite $B \to D_A(I) \to D_B(J)$ coincides with the canonical map. This follows from the uniqueness of the map t in (1.9) applied to the diagram

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
D_B(J) & \longrightarrow & B/J.
\end{array}
$$

2. Cohomology of Fontaine-Messing

In this section, we give reviews on the cohomology groups of Fontaine-Messing, which we denote by $H^\bullet(X, \mathcal{S}_n(r))$ (in this paper we shall assume $r < p$ and X is a scheme satisfying the condition (2.4) below), and on the symbol maps

$$
K_r^M\left(\Gamma(X, \mathcal{O}_{X_{n+1}})\right) \longrightarrow H^r\left(X, \mathcal{S}_n(r)\right) \qquad (0 \leq r < p).
$$

These cohomology groups were defined by Fontaine-Messing [FM] by using a certain Grothendieck topology called "syntomic topology". But in the following we adopt a different form of the definition, which is explained in [K5], chap. I, §§ 1–3, and which seems convenient for the use in this paper. Since the details are already given in [K5], we shall sometimes omit the proofs of lemmas. (In [K5], we worked on the etale site, but we work here on the Zariski site for we do not consider etale local problems in this paper. This does not make any essential change in the arguments for the definitions.)

First, we consider the following condition $F(X, Z, r)$ $(r \geq 0)$ on an immersion $X \hookrightarrow Z$ of schemes. For $n \geq 1$, let D_n be the PD enveloppe of X_n in Z_n.

$$
F(X, Z, r) \quad \left\{
\begin{array}{l}
\textit{The sequences :} \\[4pt]
J^{[i]}_{D_{m+n}} \xrightarrow{\ p^m\ } J^{[i]}_{D_{m+n}} \xrightarrow{\ p^n\ } J^{[i]}_{D_{m+n}} \longrightarrow J^{[i]}_{D_n} \to 0 \\[8pt]
\textit{are exact for all } m,\ n \geq 1 \textit{ and for all } 0 \leq i \leq r, \\
\textit{where the notations are as in (1.7).}
\end{array}
\right.
$$

LEMMA 2.1 (by FONTAINE-MESSING). — *Let X and Z be as above and assume Z is locally nœtherian. Assume that for any $x \in X_1$, p is a non-zero-divisor in $\mathcal{O}_{X,x}$ and $\mathrm{Ker}(\mathcal{O}_{Z,x} \to \mathcal{O}_{X,x})$ is generated by an $\mathcal{O}_{Z,x}$-regular sequence ([Gr], chap. 0, § 15). Then the conditions $F(X, Z, r)$ are satisfied for all r.*

For the proof, see [K5], chap. I, (1.3) (the assumption is slightly different there, but the same proof works).

LEMMA 2.2. — *Let $X \hookrightarrow Z$ and $X \hookrightarrow Z'$ be immersions with the same source X, and assume that Z_n and Z'_n have p-bases over $\mathbb{Z}/p^n\mathbb{Z}$ locally for any $n \geq 1$. Let $r \geq 0$. Then, the condition $F(X, Z, r)$ is satisfied if and only if $F(X, Z', r)$ is satisfied.*

Proof. — By considering the immersion $X \hookrightarrow Z \times_{\mathbf{Z}} Z'$, we may assume that there is a morphism $Z' \to Z$ compatible with the immersions such that Z'_n has p-bases over Z_n locally for any n. We may work locally, so let $Z = \mathrm{Spec}(A)$, $Z' = \mathrm{Spec}(B)$, let I (resp. J) be the ideal of A (resp. B) defining X, and let $I_n = IA_n$ (resp. $J_n = IB_n$). By (1.8) (2), if $(b_\lambda)_{\lambda \in \Lambda}$ is a p-basis of B_n over A_n such that $b_\lambda \in J_n$ for all $\lambda \in \Lambda$, we have

$$D_{A_n}(I_n)\langle T_\lambda \rangle_{\lambda \in \Lambda} \xrightarrow{\cong} D_{B_n}(J_n) \, ; \quad T_\lambda \longmapsto b_\lambda.$$

Let $\bar{I}_n = \mathrm{Ker}(D_{A_n}(I_n) \to A_n/I_n)$, and define $\bar{J}$ similarly. From the above isomorphism, we see that for each $i \geq 0$, there is an isomorphism of $\mathbb{Z}/p^n\mathbb{Z}$-modules

$$\bigoplus_s \bar{I}_n^{[i-|s|]} \xrightarrow{\cong} \bar{J}_n^{[i]} \, ; \quad (x_s)_s \longmapsto \sum_s x_s \cdot \prod_{\lambda \in \Lambda} b_\lambda^{[s(\lambda)]}$$

where s ranges over all functions $\Lambda \to \{0, 1, 2, \ldots\}$ with finite supports and $|s|$ denotes $\sum_{\lambda \in \Lambda} s(\lambda)$. This proves (2.2).

Definition 2.3. — Let Z be a scheme. A morphism $f : Z \to Z$ is called a *frobenius* of Z if $f \otimes \mathbb{Z}/p\mathbb{Z} : Z_1 \to Z_1$ is the absolute frobenius of Z_1 (induced by $\mathcal{O}_{Z_1} \to \mathcal{O}_{Z_1} ; x \longmapsto x^p$).

Now consider the following condition (2.4) on a scheme X.

2.4. — There is an immersion $X \hookrightarrow Z$ satisfying the following (i), (ii) and (iii).

   (i) For any $n \geq 1$, Z_n has p-bases over $\mathbb{Z}/p^n\mathbb{Z}$ locally.
   (ii) Z has a frobenius.
   (iii) The condition $F(X, Z, p-1)$ is satisfied.

In the rest of this section, X always denotes a scheme satisfying the condition (2.4). Let $X \hookrightarrow Z$ be any immersion such that Z satisfies (i) of (2.4) and Z is endowed with a fixed frobenius f. Note that (2.4) (iii) is then satisfied by (2.2). We denote the complex of sheaves on $(X_1)_{\mathrm{zar}}$

$$
J_{D_n}^{[r]} \xrightarrow{\ \mathrm{d}\ } J_{D_n}^{[r-1]} \otimes_{\mathcal{O}_{Z_n}} \Omega^1_{Z_n} \xrightarrow{\ \mathrm{d}\ } \cdots \rightarrow J_{D_n}^{[r-q]} \otimes_{\mathcal{O}_{Z_n}} \Omega^q_{Z_n} \rightarrow \cdots
$$

$\quad$ (deg. 0) $\hspace{6.5cm}$ (deg. q)

by $\mathsf{J}_{n,X,Z}^{[r]}$. Let $\mathsf{E}_{n,X,Z} = \mathsf{J}_{n,X,Z}^{[0]}$. Then, the frobenius f induces $\mathcal{O}_{D_n} \rightarrow \mathcal{O}_{D_n}$ and satisfies

$$
f\big(\mathsf{J}_{n,X,Z}^{[r]}\big) \subset p^r \, \mathsf{E}_{n,X,Z}
$$

for any $n \geq 1$ and for any $0 \leq r < p$ (cf. [K5], chap. I, (1.3)). By the condition $F(X,Z,p-1)$ and by (1.8) (1), the sequence

$$
\mathsf{J}_{n+r,X,Z}^{[r]} \xrightarrow{\ p^r\ } \mathsf{J}_{n+r,X,Z}^{[r]} \xrightarrow{\ p^n\ } \mathsf{J}_{n+r,X,Z}^{[r]} \rightarrow \mathsf{J}_{n,X,Z}^{[r]} \rightarrow 0
$$

is exact. So, for $0 \leq r < p$, we obtain a homomorphism of complexes $f_r : \mathsf{J}_{n,X,Z}^{[r]} \rightarrow \mathsf{E}_{n,X,Z}$ as the composite

$$
\mathsf{J}_{n,X,Z}^{[r]} \cong \mathsf{J}_{n+r,X,Z}^{[r]}/p^n \xrightarrow{\ f\ } p^r \, \mathsf{E}_{n+r,X,Z} \xrightarrow[\cong]{\ p^{-r}\ } \mathsf{E}_{n+r,X,Z}/p^n \cong \mathsf{E}_{n,X,Z}.
$$

Definition 2.5. — For $0 \leq r < p$, we define the complex $\mathcal{S}_n(r)_{X,Z}$ on $(X_1)_{\mathrm{zar}}$ to be the "mapping fiber" of

$$
f_r - 1 : \mathsf{J}_{n,X,Z}^{[r]} \longrightarrow \mathsf{E}_{n,X,Z}.
$$

Precisely, the degree q part of $\mathcal{S}_n(r)_{X,Z}$ is

$$
\big(J_{D_n}^{[r-q]} \otimes_{\mathcal{O}_{Z_n}} \Omega^q_{Z_n}\big) \oplus \big(\mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^{q-1}_{Z_n}\big)
$$

and the boundary operator of $\mathcal{S}_n(r)_{X,Z}$ is given by

$$
(x,y) \longmapsto \big(\mathrm{d}x, (f_r-1)(x) - \mathrm{d}y\big)
$$

$(x \in J_{D_n}^{[r-q]} \otimes_{\mathcal{O}_{Z_n}} \Omega^q_{Z_n}, \ y \in \mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^{q-1}_{Z_n})$.

Definition 2.6. — We denote by $H^q(X, \mathcal{S}_n(r))$ $(q \in \mathbb{Z})$ the q-th hyper-cohomology of the complex $\mathcal{S}_n(r)_{X,Z}$.

To justify the notation $H^*(X, \mathcal{S}_n(r))$, we show that in the derived category, $\mathcal{S}_n(r)_{X,Z}$ is independent of the choice of Z. Indeed, for two embeddings $X \hookrightarrow Z$ and $X \hookrightarrow Z'$ as above, we have the third embedding $X \hookrightarrow Z'' = Z \times_Z Z'$, where Z'' is endowed with the frobenius which is the product of the fixed frobeniuses of Z and Z'. We see that

$$\mathcal{S}_n(r)_{X,Z} \longrightarrow \mathcal{S}_n(r)_{X,Z''} \quad \text{and} \quad \mathcal{S}_n(r)_{X,Z'} \longrightarrow \mathcal{S}_n(r)_{X,Z''}$$

are quasi-isomorphisms by THEOREM (1.7).

Definition 2.7. — In the case where X and Z are affine schemes $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$, respectively, we denote by $\mathcal{S}_n(r)_{A,B}$ (where $r < p$) the complex of abelian groups obtained from $\mathcal{S}_n(r)_{X,Z}$ by applying the global section functor $\Gamma(X, \cdot)$ to each component of $\mathcal{S}_n(r)_{X,Z}$.

Since each component of $\mathcal{S}_n(r)_{X,Z}$ is a quasi-coherent sheaf, the cohomology group $H^q(\mathcal{S}_n(r)_{A,B})$ in (2.7) coincides with $H^q(X, \mathcal{S}_n(r))$ which we denote also by $H^q(A, \mathcal{S}_n(r))$.

We next define the symbol map. Define the product structure

$$\mathcal{S}_n(r)_{X,Z} \times \mathcal{S}_n(r')_{X,Z} \longrightarrow \mathcal{S}_n(r+r')_{X,Z}, \qquad (r + r' < p),$$

by $\qquad\qquad (x, y)(x', y') = \big(xx', (-1)^q f_r(x)y' + yx'\big),$

where (x, y) (resp. (x', y')) belongs to the degree q part (resp. q' part) of $\mathcal{S}_n(r)_{X,Z}$ (resp. $\mathcal{S}_n(r')_{X,Z}$). This product structure induces a product

$$H^q\big(X, \mathcal{S}_n(r)\big) \otimes H^{q'}\big(X, \mathcal{S}_n(r')\big) \longrightarrow H^{q+q'}\big(X, \mathcal{S}_n(r+r')\big)$$

$(r + r' < p)$ which satisfies

$$ab = (-1)^{qq'} ba, \quad \text{and} \quad (ab)c = a(bc), \qquad (a \in H^q, b \in H^{q'}).$$

(See [K5], chap. I, § 2.) On the other hand, we define a canonical homomorphism

$$\Gamma\big(X, \mathcal{O}_{X_{n+1}}^x\big) \longrightarrow H^1\big(X, \mathcal{S}_n(1)\big)$$

as follows. Denote by i the inclusion map $X \hookrightarrow Z$, let N be the kernel of $i^{-1}(\mathcal{O}_{Z_{n+1}}^x) \to \mathcal{O}_{X_{n+1}}^x$, and let C be the complex

$$N \qquad \longrightarrow \qquad i^{-1}\big(\mathcal{O}_{Z_{n+1}}^x\big)$$

$$\text{(deg . 0)} \qquad\qquad\qquad \text{(deg . 1)}$$

which is canonically quasi-isomorphic to $\mathcal{O}_{X_{n+1}}^x[-1]$. Let $s : C \to \mathcal{S}_n(r)_{X,Z}$ be the following homomorphism of complexes. For $x \in \mathcal{O}_{Z_{n+1}}$, let $\bar{x}$ be the image of x in $\mathcal{O}_{D_{n+1}}$. Then, the degree zero part of s is

$$N \longrightarrow J_{D_n} ; \quad x \longmapsto \log(\bar{x}),$$

and the degree one part of s is

$$i^{-1}\big(\mathcal{O}_{Z_{n+1}}^x\big) \longrightarrow \big(\mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^1_{Z_n}\big) \oplus \mathcal{O}_{D_n}$$

$$x \longmapsto \Big(1 \otimes \frac{\mathrm{d}x}{x}, \frac{1}{p} \log\Big(\overline{\frac{f(x)}{x^p}}\Big)\Big).$$

Here log is the logarithm

$$x \mapsto \sum_{i \geq 1} (-1)^i (i-1)!(x-1)^{[i]}$$

defined by the PD structure on $J_{D_{n+1}}$. Since $\log\overline{\big(f(x)/x^p\big)} \in p\mathcal{O}_{D_{n+1}}$, $p^{-1}\log\overline{\big(f(x)/x^p\big)}$ makes sense. This map s induces

$$\Gamma\big(X, \mathcal{O}_{X_{n+1}}^x\big) = H^1(X, C) \longrightarrow H^1\big(X, \mathcal{S}_n(1)\big).$$

By the product structure of $\mathcal{S}_n(r)_{X,Z}$ defined above, we obtain a homomorphism

$$\underbrace{\Gamma(X, \mathcal{O}_{X_{n+1}}^x) \otimes \cdots \otimes \Gamma(X, \mathcal{O}_{X_{n+1}}^x)}_{r \text{ times}} \longrightarrow H^r\big(X, \mathcal{S}_n(r)\big)$$

($r < p$). Of course, this map is independant of the choice of Z.

LEMMA 2.8. — *The above map factors through* $K_r^M\big(\Gamma(X, \mathcal{O}_{X_{n+1}})\big)$.

See [K5], chap. I (3.2) for the proof. The key point of the proof is that, by the functoriality of the above map, we may assume that

$$X = \mathrm{Spec}\Big(Z\Big[T, \frac{1}{T(T-1)}\Big]\Big), \quad Z = \mathrm{Spec}\big(Z[T]\big),$$

and that the frobenius of Z is $Z[T] \to Z[T]$; $T \mapsto T^p$.

If $X = Z = \mathrm{Spec}(B)$, the image of $\{a_1, \ldots, a_r\}$ under the symbol map $K_r^M(B) \to H^r(B, \mathcal{S}_n(r))$ coincides with the class of

$$\Big(\frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_r}{a_r}, \theta_f(a_1, \ldots, a_r)\Big) \in \Omega^r_{B_n} \oplus \Omega^{r-1}_{B_n},$$

where $\theta_f(a_1, \ldots, a_r)$ denotes

$$\sum_{i=1}^{r}(-1)^{i-1}\frac{1}{p}\log\left(\frac{f(a_i)}{a_i^p}\right)\frac{\mathrm{d}a_1}{a_1}\wedge\cdots\wedge\frac{\mathrm{d}a_{i-1}}{a_{i-1}}$$

$$\wedge f_1\left(\frac{\mathrm{d}a_{i+1}}{a_{i+1}}\right)\wedge\cdots\wedge f_1\left(\frac{\mathrm{d}a_r}{a_r}\right).$$

COROLLARY 2.9. — *For B as above, there is a unique homomorphism*

$$K_r^M(B) \longrightarrow \Omega_{B_n}^{r-1}/\mathrm{d}\Omega_{B_n}^{r-2}\,; \quad \{a_1,\ldots,a_r\} \longmapsto \theta_f(a_1,\ldots,a_r).$$

If the cardinal number of the p-basis of B_1 is equal to $(r-1)$, this homomorphism is indenpendent of the choice of the frobenius f.

The last statement follows from the fact that the symbol map and the isomorphism $\Omega_{B_n}^{r-1}/\mathrm{d}\Omega_{B_n}^{r-2} \cong H^r(B, \mathcal{S}_n(r))$ are independent of f.

3. Modification

In this section, B denotes a ring endowed with a fixed frobenius f and a fixed family $(b_\lambda)_{\lambda\in\Lambda}$ of elements satisfying the following conditions :

3.1.1. — In B, p is a non-zero divisor.

3.1.2. — B_1 has a p-basis over $\mathbb{F}_p$ and $(b_\lambda \mod pB_\lambda)_\lambda$ forms a part of a p-basis of B_1 over $\mathbb{F}_p$.

3.1.3. — $f(b_\lambda) \in b_\lambda^p \cdot B^x$ for any $\lambda \in \Lambda$.

Let J be an ideal of B, let $A = B/J$, and assume that $\mathrm{Spec}(A)$ satisfies the condition (2.4). (Note that by (1.4), $Z = \mathrm{Spec}(B)$ satisfies the condition (2.4) (i).)

In this section, we shall define complexes $\mathcal{S}_n'(r)_{A,B}$ (with $0 \le r < p$), and on certain further assumptions (3.2.1)–(3.2.3), symbol maps

$$K_r^M\big(A[\pi_\lambda^{-1}]_{\lambda\in\Lambda}\big) \longrightarrow H^r\big(\mathcal{S}_n'(r)_{A,B}\big),$$

where π_λ denotes the image of b_λ in A for each λ. There will be a map $\mathcal{S}_n(r)_{A,B} \to \mathcal{S}_n'(r)_{A,B}$ and a commutative diagram

$$
\begin{array}{ccc}
K_r^M(A) & \longrightarrow & H^r(\mathcal{S}_n(r)) \\
\downarrow & & \downarrow \\
K_r^M\big(A[\pi_\lambda^{-1}]_{\lambda\in\Lambda}\big) & \longrightarrow & H^r\big(\mathcal{S}_n'(r)_{A,B}\big).
\end{array}
$$

In the next section, we shall apply this construction to the case where $p \in \pi_1^{e_1} \cdots \pi_s^{e_s} A^\times$ for some $e_1, \ldots, e_s \geq 1$. The symbol map

$$K_r^M \left(A\left[\tfrac{1}{p}\right] \right) \to H^r \left(\mathcal{S}_n'(r)_{A,B} \right)$$

of this section will be important, for the cohomology of § 2 has no sense for schemes such as $\mathrm{Spec}\left(A\left[\tfrac{1}{p}\right] \right)$ on which p is invertible.

Contrary to the case of $\mathcal{S}_n(r)_{A,B}$, I can not relate $\mathcal{S}_n'(r)_{A,B}$ to the crystalline cohomology theory, and I do not know if the cohomology $H^*(\mathcal{S}'(r))_{A,B}$ and the above symbol map depend functorially only on A and the ideals $(\pi_1), \ldots, (\pi_s)$.

Let $B' = B[b_\lambda^{-1}; \lambda \in \Lambda]$, and $A' = A[\pi_\lambda^{-1}; \lambda \in \Lambda]$. We denote $(B')_n$ as B_n'. Let

$$(\Omega_{B_n}^\bullet)' = \bigoplus_{q \in \mathbf{Z}} (\Omega_{B_n}^q)'$$

be the B_n-subalgebra of $\Omega_{B_n'}^\bullet$ generated by $\Omega_{B_n}^1$ and db_λ/b_λ (with $\lambda \in \Lambda$). Then $(\Omega_{B_n}^\bullet)'$ is a subcomplex of $\Omega_{B_n'}^\bullet$.

Let $\mathbb{E}_{n,A,B}'$ be the complex $D_{B_n}(J_n) \otimes_{B_n} (\Omega_{B_n}^\bullet)'$ (J_n denotes $JB_n = J/p^n J$) with the boundary operator

$$D_{B_n}(J_n) \otimes_{B_n} (\Omega_{B_n}^q)' \longrightarrow D_{B_n}(J_n) \otimes_{B_n} (\Omega_{B_n}^{q+1})'$$

$$a \otimes w \longmapsto a \otimes dw + da \wedge w.$$

For $i \in \mathbf{Z}$, let $\bar{J}_n^{[i]}$ be the i-th divided power of the PD ideal

$$\bar{J}_n = \mathrm{Ker}\left(D_{B_n}(J_n) \to A_n \right).$$

We denote by $(\mathcal{J}_{n,A,B}^{[r]})'$ the subcomplex of $\mathbb{E}_{n,A,B}'$ whose degree q part is $\bar{J}_n^{[r-q]} \otimes_{B_n} (\Omega_{B_n}^q)'$. We define the map

$$f_r : \left(\mathcal{J}_{n,A,B}^{[r]} \right)' \longrightarrow \mathbb{E}_{n,A,B}'$$

for $0 \leq r < p$ by the same argument as in § 2, and define the complex $\mathcal{S}_n'(r)_{A,B}$ as the "mapping fiber" of $f_r - 1 : (\mathcal{J}_{n,A,B}^{[r]})' \to \mathbb{E}_{n,A,B}'$. We define the product structure of $\mathcal{S}_n'(r)_{A,B}$ by the same way as in § 2.

Now we make the following assumptions (3.2.1)–(3.2.3).

3.2.1. — The maps

$$A^x \oplus \mathbf{Z}^{(\Lambda)} \longrightarrow (A')^x ; \quad (x, (m_\lambda)) \longmapsto x \cdot \prod_\lambda \pi_\lambda^{m_\lambda} ;$$

$$B^x \oplus \mathbf{Z}^{(\Lambda)} \longrightarrow (B')^x ; \quad (x, (m_\lambda)) \longmapsto x \cdot \prod_\lambda b_\lambda^{m_\lambda}$$

are bijective. Here $\mathbb{Z}^{(\Lambda)}$ denotes the direct sum of copies of $\mathbb{Z}$ with the index set Λ.

3.2.2. — The elements b_λ and the ideal J are contained in the Jacobson radical of B.

3.2.3. — For any $r \leq 3$ and for any injection $j : \{1,\ldots,r\} \to \Lambda$, $\pi_{j(1)},\ldots,\pi_{j(r)}$ form an A-regular sequence.

On these assumptions, we define the homomorphism

$$h : (A')^x \to H^1(\mathcal{S}'_n(1)_{A,B})$$

as follows. For $a \in (A')^x$, take $b \in (B')^x$ having the image a in $(A')^x$, and let $h(a) \in H^1(\mathcal{S}'_n(1)_{A,B})$ be the class of

$$\left(1 \otimes \frac{db}{b}, \frac{1}{p}\log\left(\frac{f(b)}{b^p}\right)\right) \in D_{B_n}(J_n) \otimes_{B_n} (\Omega^1_{B_n})' \oplus D_{B_n}(J_n).$$

The fact that $h(a)$ does not depend on the choice of b is proved as follows. Since $\mathrm{Ker}(B^x \to A^x) = \mathrm{Ker}((B')^x \to (A')^x)$, it is sufficient to show that the class of $(1 \otimes db/b, p^{-1}\log(f(b)/b^p))$ in $H^1(\mathcal{S}'_n(1)_{A,B})$ is zero if $b \in \mathrm{Ker}(B^x \to A^x)$. But its class in $H^1(\mathcal{S}_n(1)_{A,B})$ is already zero as in §2.

For $r < p$, the product structure of $\mathcal{S}'_n(r)_{A,B}$ defines a map

$$\underbrace{(A')^x \otimes \cdots \otimes (A')^x}_{r\ \mathrm{times}} \longrightarrow H^r(\mathcal{S}'_n(r)_{A,B}) ;$$

$$a_1 \otimes \cdots \otimes a_r \longmapsto h(a_1)\ldots h(a_r).$$

PROPOSITION 3.3. — *The above map factors through* $K_r^M(A')$.

To prove this, we use :

LEMMA 3.4. — *Let* $x,y \in (A')^x$ *and assume* $x + y = 1$. *Then there are elements* $\tilde{x}$ *and* $\tilde{y}$ *of* $(B')^x$ *having the images* x *and* y *in* $(A')^x$, *respectively, such that* $\tilde{x} + \tilde{y} = 1$.

The proof is straightforward by using the assumptions (3.2.1)–(3.2.3), and so we omit it.

By the commutative diagram

$$
\begin{array}{ccccc}
K_r^M(B') & \longleftarrow & (B')^x \otimes \cdots \otimes (B')^x & \longrightarrow & (A')^x \otimes \cdots \otimes (A')^x \\
\downarrow & & \downarrow & & \downarrow \\
H^r\big(\mathcal{S}_n(r)_{B',B'}\big) & \longleftarrow & H^r\big(\mathcal{S}'_n(r)_{B,B}\big) & \longrightarrow & H^r\big(\mathcal{S}'_n(r)_{A,B}\big)
\end{array}
$$

and by (3.4), it is sufficient for the proof of (3.3) to show the injectivity of $H^r(\mathcal{S}'_n(r)_{B,B}) \to H^r(\mathcal{S}_n(r)_{B',B'})$. This is reduced to :

LEMMA 3.5. — *The map* $(\Omega_{B_n}^q)'/\mathrm{d}(\Omega_{B_n}^{q-1})' \to \Omega_{B'_n}^q/\mathrm{d}\Omega_{B'_n}^{q-1}$ *is injective for any* q.

I have only the following long proof. Take a p-basis $(b_\lambda)_{\lambda \in \Sigma}$ of B_n over $\mathbb{Z}/p^n\mathbb{Z}$ containing $(b_\lambda)_{\lambda \in \Lambda}$, let $S = \mathbb{Z}[T_\lambda]_{\lambda \in \Sigma}$ and let $S_n \to B_n$ be the map $T_\lambda \mapsto b_\lambda$ $(\lambda \in \Sigma)$. Define $S'_n = S_n[T_\lambda^{-1}]_{\lambda \in \Lambda}$ and let $(\Omega_{S_n}^\bullet)' \subset \Omega_{S'_n}^\bullet$ be the S_n-subalgebra generated by $\Omega_{S_n}^1$ and $\mathrm{d}T_\lambda/T_\lambda$, $\lambda \in \Lambda$. Then we have :

3.5.1. — The map $(\Omega_{S_n}^q)'/\mathrm{d}(\Omega_{S_n}^{q-1})' \to \Omega_{S'_n}^q/\mathrm{d}\Omega_{S'_n}^{q-1}$ is injective.

The proof of (3.5.1) is as follows. Let $M^\bullet$ be the $\mathbb{Z}/p^n\mathbb{Z}$-subalgebra of $\Omega_{S'_n}^\bullet$ generated by $\Omega_{\mathbb{Z}/p^n\mathbb{Z}[T_\lambda]_{\lambda \in \Sigma-\Lambda}}^1$ and $\mathrm{d}T_\lambda/T_\lambda$ $(\lambda \in \Lambda)$. For each function $s : \Lambda \to \mathbb{Z}$ with finite support, let, for $q \in \mathbb{Z}$,

$$
M_s^q = \Big(\prod_{\lambda \in \Lambda} T_\lambda^{s(\lambda)}\Big) \cdot M^q \subset \Omega_{S'_n}^q.
$$

Then $M_s^\bullet$ are subcomplexes of $\Omega_{S'_n}^\bullet$ and $\Omega_{S'_n}^\bullet$ is the direct sum of these subcomplexe. This proves (3.5.1), for $(\Omega_{S_n}^\bullet)' = \bigoplus_{s \geq 0} M_s^\bullet$.

Now we consider the homomorphisms

$$
W_n(S_1) \longrightarrow S_n, \; W_n(B_1) \longrightarrow B_n \,; \quad (x_0, \ldots, x_{n-1}) \longmapsto \sum_{i=0}^{n-1} p^i \, \tilde{x}_i^{p^{n-i}}
$$

where $\tilde{x}_i$ is any lifting of x_i. Then by [K2], §1, we have an isomorphism

$$
W_n(B_1) \otimes_{W_n(S_1)} S_n \xrightarrow{\cong} B_n.
$$

The action of $W_n(S_1)$ (resp. $W_n(B_1)$) commutes with d on $\Omega^\bullet_{S'_n}$ (resp. $\Omega^\bullet_{B'_n}$), and we obtain isomorphisms of complexes

$$(3.5.2) \qquad\qquad W_n(B_1) \otimes_{W_n(S_1)} (\Omega^\bullet_{S_n})' \cong (\Omega^\bullet_{B_n})' \; ;$$

$$(3.5.3) \qquad\qquad W_n(B_1) \otimes_{W_n(S_1)} \Omega^\bullet_{S'_n} \cong \Omega^\bullet_{B'_n} \, .$$

Furthermore, $W_n(S_1) \to W_n(B_1)$ is flat by [K2], Lemma 2, and by the flatness of $S_1 \to B_1$ (1.5). Hence by (3.5.2) and (3.5.3), (3.5) is reduced to (3.5.1).

4. The isomorphism theorem

In this section, we prove THEOREM (0.3) (1) and THEOREM (0.6).

4.1. — In (4.1)–(4.11), we work in the following situation. Let B be a nœtherian ring endowed with a frobenius f, ξ an element of B, and let q be an integer such that $0 \le q < p - 1$. Assume that $p > 2$ and the following (4.1.1)–(4.1.3) are satisfied.

4.1.1. — B_1 has a finite p-basis over F_p consisting just of q elements.

4.1.2. — In B, p and $\xi - 1$ are contained in the Jacobson radical of B and form a B-regular sequence.

4.1.3. — The map $f_q - 1 : \Omega^q_{B_1} \to \Omega^q_{B_1}$ is surjective.

A typical example is $B = \mathbb{Z}_p[[T_1, \ldots, T_q]]$, $f(T_i) = T_i^p$ $(1 \le i \le q)$, and $\xi = 1 + T_1^{i_1} \ldots T_q^{i_q}$ for some $i_1, \ldots, i_q \ge 1$.

Fix $n \ge 1$, and let $\Phi(T) \in \mathbb{Z}[T]$ be the cyclotomic polynomial $(T^{p^n} - 1)(T^{p^{n-1}} - 1)^{-1}$. We define $J = \Phi(\xi)B$ and $A = B/J$. By the assumption (4.1.2), $\Phi(\xi)$ is a non-zero-divisor of B and p is a non-zero-divisor in A. So by (2.1), the groups $H^*(A, \mathcal{S}_n(r))$ $(0 \le r < p)$ are defined. (In the case of the above example, A is the regular local ring $\mathbb{Z}_p[\zeta][[T_1, \ldots, T_q]]/(T_1^{i_1} \ldots T_q^{i_q} - (\zeta - 1))$, where ζ is a primitive p^n-th root of 1).

In this section, in (4.4)–(4.11) we prove the following THEOREM (4.3) which gives a presentation of $H^{q+1}(A, \mathcal{S}_n(q+1))$, and then in the latter half, we apply (4.3) to the proofs of (0.3) (1), (0.6) and a related result (4.14). These results show

$$H^{q+1}\big(A, \mathcal{S}_n(q+1)\big) \cong H^{q+1}\big(A, \mathcal{S}'_n(q+1)_{A,B}\big) \cong \mathbb{Z}/p^n\mathbb{Z}$$

in the case of the above example.

4.2. — We fix some notations. We fix an integer c such that $1 \leq c < p$ and define $\alpha, \beta, h \in B$ by

$$\alpha = \sum_{i=1}^{c} \frac{(-1)^{i-1}}{i} \left(\zeta^{p^{n-1}} - 1\right)^i, \quad \beta = \sum_{i=1}^{c} \frac{(-1)^{i-1}}{i} \left(\zeta^{p^n} - 1\right)^i,$$
$$h = \beta \alpha^{-1}.$$

Note that α and β are non-zero-divisors of B by (4.1.2),

$$\alpha \in (\xi^{p^{n-1}} - 1) \cdot B^x, \quad \beta \in (\xi^{p^n} - 1) \cdot B^x,$$

and hence h is a generator of J.

For the proof of THEOREM (0.8), it is sufficient to consider the case $c = 1$ (then $h = \Phi(\xi)$), but for the proof of THEOREM (0.4) (1), we will need the case $2 \leq c < p$.

THEOREM 4.3. — *Let the assumptions and the notations be as above. Then the canonical map $\Omega_{B_n}^q \to H^{q+1}(A, \mathcal{S}_n(q+1))$ induces an isomorphism*

$$H^{q+1}\left(A, \mathcal{S}_n(q+1)\right) \cong \Omega_{B_n}^q / \left(d\Omega_{B_n}^{q-1} + \{f_q(w) - hw \,;\, w \in \Omega_{B_n}^q\}\right).$$

The outline of the proof of (4.3) is as follows. The key point will be to define the "inverse map"

$$H^{q+1}\left(A, \mathcal{S}_n(q+1)\right) \longrightarrow \Omega_{B_n}^q / \left(d\Omega_{B_n}^{q-1} + \{f_q(w) - hw \,;\, w \in \Omega_{B_n}^q\}\right).$$

For this, we shall define a PD structure on the principal ideal (h) of $B_n / \beta^c B_n$. This will give

$$D_{B_n}(J_n) \longrightarrow B_n / \beta^c B_n$$

$(J_n =_{\text{def}} J B_n = J/p^n J)$ and hence

$$D_{B_n}(J_n) \otimes_{B_n} \Omega_{B_n}^q \longrightarrow B_n / \beta^c B_n \otimes_{B_n} \Omega_{B_n}^q$$

where the last map induces the above "inverse map".

The following lemma will be useful.

LEMMA 4.4.

(1) $h^p \equiv ph^{p-1} + \beta^{p-1} \mod p\beta^c B.$

(2) $f(\alpha) = \beta + p^n\gamma$ for some $\gamma \in B$, and for this γ we have

$$f(h) \equiv p + \beta^{p-1} - p^n\beta^{p-2}\gamma \mod (p\beta^c B + p^{n+1}B).$$

(3) $f(B) \in \beta B + p^{n+1}B.$

Proof. — We use the congruence in $\mathbb{Z}_{(p)}[T]$

$$\sum_{i=1}^{c} \frac{(-1)^{i-1}}{i}(T^p - 1)^i \equiv p \cdot \sum_{i=1}^{c} \frac{(-1)^{i-1}}{i}(T - 1)^i$$
$$+ \left(\sum_{i=1}^{c} \frac{(-1)^{i-1}}{i}(T - 1)^i\right)^p \mod p(T - 1)^{c+1}\mathbb{Z}_{(p)}[T].$$

This congruence is proved modulo p and modulo $((T - 1)^{c+1})$ easily, so it holds modulo $(p) \cap ((T - 1)^{c+1}) = (p(T - 1)^{c+1})$. By putting $T = \xi^{p^{n-1}}$, we have

(4.5.1) $\qquad\qquad \beta \equiv p\alpha + \alpha^p \mod p\alpha^{c+1}B.$

By multiplying (4.5.1) by $b^p\beta^{-1}$ and α^{-1}, respectively, we obtain (1) and

(4.5.2) $\qquad\qquad h \equiv p + \alpha^{p-1} \mod p\alpha^c B.$

By applying the frobenius f to (4.5.2), we have

(4.5.3) $\qquad\qquad f(h) \equiv p + f(\alpha)^{p-1} \mod pf(\alpha)^c B.$

The first assertion in (2) is verified easily and the second follows from the first and from (4.5.3). Finally, multiplying (4.5.3) by $f(\alpha)$ and by using the first assertion in (2), we obtain (3).

LEMMA 4.6. — *Let* $m \geq 1.$

(1) *The ideal* (h) *of* $B_m/\beta^c B_m$ *has a unique* PD *structure such that*

$$h^{[i]} = \begin{cases} \dfrac{h^i}{i\,!} & \text{if } 0 \leq i < p, \\[2mm] \dfrac{p^{i+1-p}}{i\,!}h^{p-1} & \text{if } i \geq p. \end{cases}$$

(2) *If $m \leq n+1$, $f(\beta^c B) \subset \beta^c B + p^m B$, and the induced homomorphism $f : B_m/\beta^c B_m \to B_m/\beta^c B_m$ preserves the* PD *structure in* (1).

Proof. — Let $S = B/\beta^c B$. Since p is a non-zero-divisor in S and since $h^p = ph^{p-1}$ in S by (4.4) (1), we see that for any $x \in hS$ and for any $i \geq 1$, there is a unique element y of hS such that $x^i = i!\,y$. The map

$$x \bmod p^m S \longmapsto y \bmod p^m S$$

defines a PD structure on the ideal hS_m of S_m having the property in (1). Now let $m \leq n+1$. Then, the first assertion in (2) follows from (4.4) (3). To show that the induced frobenius $f : S_m \to S_m$ preserves the PD structure, it suffices to prove $f(h^{[i]}) = f(h)^{[i]}$ for any i. The case $i < p$ is clear so assume $i \geq p$. By (4.4)(2), we have in S_m

$$f(h) = p + p^n x \quad \text{for some } x \in S_m.$$

Hence we have in S_m,

$$f\big(h^{[i]}\big) = f\Big(\frac{p^{i+1-p}}{i!}h^{p-1}\Big) = \frac{p^{i+1-p}}{i!}(p + p^n x)^{p-1} = p^{[i]},$$

$$f\big(h^{[i]}\big) = (p + p^n x)^{[i]} = p^{[i]}\big(1 + p^{n-1}x\big)^i = p^{[i]}.$$

4.7. — Now we analyse $H^{q+1}(A, \mathcal{S}_n(q+1))$. By using the complex $\mathcal{S}_n(q)_{A,B}$, we identify $H^{q+1}(A, \mathcal{S}_n(q+1))$ with the cokernel of

$$(4.7.1) \qquad \big(\bar{J}_n \otimes_{B_n} \Omega_{B_n}^q\big) \oplus \big(D_{B_n}(J_n) \otimes_{B_n} \Omega_{B_n}^{q-1}\big)$$

$$\xrightarrow{\ (f_{q+1}-1, \mathrm{d})\ } D_{B_n}(J_n) \otimes_{B_n} \Omega_{B_n}^q$$

where $\bar{J}_n = \mathrm{Ker}(D_{B_n}(J_n) \to A_n)$. Let $S = B/\beta^c B$. The homomorphisms

$$D_{B_{n+1}}(J_{n+1}) \longrightarrow S_{n+1}, \quad D_{B_n}(J_n) \longrightarrow S_n$$

defined by the PD structures in (4.6) preserve the frobenius, and hence induce a homomorphism from the diagram (4.7.1) to the diagram

$$(4.7.2) \qquad hS_n \otimes_{B_n} \Omega_{B_n}^q \oplus S_n \otimes_{B_n} \Omega_{B_n}^{q-1} \xrightarrow{\ (f_1 \otimes f_q - 1, \mathrm{d})\ } S_n \otimes_{B_n} \Omega_{B_n}^q$$

where $f_1 : hS_n \to S_n$ is the map

$$hx \longmapsto \big(1 - p^{n-1}\beta^{p-2}\gamma\big)f(x), \qquad (x \in S_n),$$

with γ as in (4.4) (2). Thus we obtained a homomorphism

(4.7.3) $H^{q+1}\big(A, \mathcal{S}_n(q+1)\big) \longrightarrow$

$\qquad \Omega^q_{B_n}/(\beta^c \Omega^q_{B_n} + \mathrm{d}\Omega^{q-1}_{B_n} + \{(1 - p^{n-1}\beta^{p-2}\gamma)f_q(w) - hw\,;\ w \in \Omega^q_{B_n}\}).$

On the other hand, we have

(4.7.4) $\qquad\qquad \beta\Omega^q_{B_n} \subset \{f_q(w) - hw\,;\ w \in \Omega^q_{B_n}\},$

which is proved as follows. Let $w \in \Omega^q_{B_n}$ and write $w = f_q(w') - w'$ by (4.1.3). Then $\beta w = f_q(\alpha w') - h\alpha w'$ (see the first assertion of (4.4) (2)). Thus the homomorphism (4.7.3) defines

(4.7.5) $H^{q+1}\big(A, \mathcal{S}_n(q+1)\big) \longrightarrow \Omega^q_{B_n}/(\mathrm{d}\Omega^{q-1}_{B_n} + \{f_q(w) - hw; w \in \Omega^q_{B_n}\}).$

The composite

$$\Omega^q_{B_n}/\mathrm{d}\Omega^{q-1}_{B_n} \xrightarrow{\ \text{can.}\ } H^{q+1}\big(A, \mathcal{S}_n(q+1)\big)$$

$$\xrightarrow{\ (4.7.5)\ } \Omega^q_{B_n}/\big(\mathrm{d}\Omega^{q-1}_{B_n} + \{f_q(w) - hw\,;\ w \in \Omega^q_{B_n}\}\big)$$

coincides with the natural projection, and hence for the proof of (4.3), it remains to show that "can." annihilates $\{f_q(w) - hw\,;\ w \in \Omega^q_{B_n}\}$ and that "can." is surjective. The former fact will be proved in (4.10), and the latter will be proved in (4.11).

We need the following lemmas. Note that since $\xi^{p^n} - 1 \in J$, we have an element $\log(\xi^{p^n}) \in \bar{J}_n$.

LEMMA 4.8.

(1) *For sufficiently large* $N \geq 1$, *the image of* $\bar{J}_n^{[N]} \otimes_{B_n} \Omega^q_{B_n}$ *in* $H^{q+1}(A, \mathcal{S}_n(q+1))$ *is zero.*

(2) *The image of* $D_{B_n}(J_n)(\log(\xi^{p^n}))^{[i]} \otimes_{B_n} \Omega^q_{B_n}$ *for* $i \geq 2$, *and the images of* $\log(\xi^{p^n}) \otimes \Omega^q_{B_n}$ *and* $\beta \otimes \Omega^q_{B_n}$ *in* $H^{q+1}(A, \mathcal{S}_n(q+1))$ *are zero.*

(3) *In* $H^{q+1}(A, \mathcal{S}_1(q+1))$, *the image of* $\bar{J}_1^{[2]} \otimes_{B_1} \Omega^q_{B_1}$ *is zero.*

Proof. — We see easily that $f_1(\bar{J}_n^{[N]}) = 0$ for sufficiently large N. For such N and for $x \in \bar{J}_n^{[N]}$ and $w \in \Omega^q_{B_n}$, the class of $x \otimes w$ in $H^{q+1}(A, \mathcal{S}_n(q+1))$ coincides with that of

$$f_{q+1}(x \otimes w) = f_1(x) \otimes f_q(w) = 0.$$

In the same way, we can deduce (3) from $f_1(\bar{J}_1^{[2]}) = 0$.

Now we prove (2). As is easily seen, we have in $D_{B_{n+1}}(J_{n+1})$, $f(\log(\xi^{p^n})) = p\log(\xi^{p^n})$ and hence $f((\log(\xi^{p^n})^{[i]}) = p^i(\log(\xi^{p^n}))^{[i]}$. For $x \in D_{B_n}(J_n)$ and $w \in \Omega_{B_n}^q$, and for $i \geq 1$, the class of $x \cdot (\log(\xi^{p^n}))^{[i]} \otimes w$ coincides with that of

$$f_{q+1}\big(x \cdot \big(\log(\xi^{p^n})\big)^{[i]} \otimes w\big) = p^{i-1}f(x)\big(\log(\xi^{p^n})\big)^{[i]} \otimes f_q(w).$$

In the case $i \geq 2$, by iterating this argument, we see that the class of $x \cdot (\log(\xi^{p^n}))^{[i]} \otimes w$ is zero. Next, for $w \in \Omega_{B_n}^q$, by taking $w' \in \Omega_{B_n}^q$ such that $w = f_q(w') - w'$ (4.1.3), we have

$$\log(\xi^{p^n}) \otimes w = f_{q+1}\big(\log(\xi^{p^n}) \otimes w'\big) - \log(\xi^{p^n}) \otimes w'.$$

Finally, to prove the assertion for $B \otimes \Omega_{B_n}^q$ in (2), it suffices to show for any $N \geq 1$, the element $\beta - \log(\xi^{p^n})$ of $D_{B_n}(J_n)$ is contained in the ideal of $D_{B_n}(J_n)$ generated by $\bar{J}_n^{[N]}$ and by $\log(\xi^{p^n})^{[i]}$, $i \geq 2$. But this fact follows from the relation between the logarithm and the exponential

$$\xi^{p^n} - 1 \equiv \sum_{i=1}^{N-1} \big(\log(\xi^{p^n})\big)^{[i]} \pmod{\bar{J}_n^{[N]}}.$$

LEMMA 4.9.
The image of $(f_1(h) - 1) \otimes \Omega_{B_n}^q$ is zero in $H^{q+1}(A, \mathcal{S}_n(q+1))$.

Proof. — Let x be an element of $\bar{J}_{n+2}$ such that $px = \beta^{p-1}$ whose existence follows from (4.4) (1) and from $h^p = p!h^{[p]}$. Then, by (4.4) (2) we have

(4.9.1) $f_1(h) - 1 = \bar{x} + \beta y$ for some $y \in B_n$,

where $\bar{x}$ denotes the image of x in $D_{B_n}(J_n)$. By (4.8) (2), it suffices to prove that the image of $\bar{x} \otimes \Omega_{B_n}^q$ in $H^{q+1}(A, \mathcal{S}_n(q+1))$ is zero. Since the class of $\bar{x} \otimes w$ ($w \in \Omega_{B_n}^q$) coincides with that of $f_1(\bar{x}) \otimes f_q(w)$, it suffices to prove that for any $N \geq 1$, $f_1(\bar{x})$ is contained in the ideal of $D_{B_n}(J_n)$ generated by $\bar{J}_n^{[N]}$ and $(\log(\xi^{p^n}))^{[i]}$ with $i \geq 2$. To see this, since $D_{B_{n+2}}(J_{n+2})/\bar{J}_{n+2}^{[N]}$ is a flat $\mathbb{Z}/p^{n+2}\mathbb{Z}$-module, it is sufficient to show that $f(\beta^{p-1})$ is contained in the ideal of $D_{B_{n+2}}(J_{n+2})$ generated by $\bar{J}_{n+2}^{[N]}$ and $p^2(\log(\xi^{p^n}))^{[i]}$ with $i \geq 2$. However this is deduced easily from the facts that β is contained in the ideal generated by $\bar{J}_{n+2}^{[N]}$ and $(\log(\xi^{p^n}))^{[i]}$ with $i \geq 1$ and that $f(\log(\xi^{p^n})) = p\log(\xi^{p^n})$ in $D_{B_{n+1}}(J_{n+1})$.

4.10. — Now we prove that for any $w \in \Omega^q_{B_n}$, the class of

$$1 \otimes (f_q(w) - hw) \quad \text{in} \quad H^{q+1}(A, \mathcal{S}_n(q+1))$$

is zero. The class of $h \otimes w$ coincides with that of $f_{q+1}(h \otimes w) = f_1(h) \otimes f_q(w)$, and hence we are reduced to (4.9).

4.11. — Finally we prove that the canonical map

$$\Omega^q_{B_n} \longrightarrow H^{q+1}(A, \mathcal{S}_n(q+1))$$

is surjective. This is reduced to the surjectivity of

$$\Omega^q_{B_1} \longrightarrow H^{q+1}(A, \mathcal{S}_1(q+1)),$$

and to (4.8) (3).

This completes the proof of THEOREM (4.3).

4.12. — From now on, and until the end of this section, A denotes a regular local ring satisfying the conditions (0.2.1)–(0.2.3). Let $\mathfrak{a}$, $R = A/\mathfrak{a}$, $\pi_1, \ldots, \pi_s$ and r be as in § 0. We assume further that A is complete with respect to the $\mathfrak{a}$-adic topology.

To apply the results in (4.1)–(4.11), we construct the ring B by using :

LEMMA 4.13. — *Let S be a ring over $\mathbb{F}_p$ having a p-basis over $\mathbb{F}_p$. Then there exists a ring P in which p is a non-zero-divisor such that $P/pP \cong S$ and $P \xrightarrow{\cong} \varprojlim_n P/p^n P$.*

Proof. — The assertion is clear in the case $S = \mathbb{F}_p[T_\lambda]_{\lambda \in \Lambda}$. The general case follows from this case by [K2], Lemma 1.

Let A and R be as above and let P be a ring having the properties in (4.13) with $S = R$. Since P_n is formally smooth over $\mathbb{Z}/p^n\mathbb{Z}$ for any $n \geq 1$ (1.4), we can lift the isomorphism $P/pP \xrightarrow{\cong} A/\mathfrak{a}$ (resp. the absolute frobenius $P/pP \to P/pP$; $x \mapsto x^p$) to a homomorphism $P \to A$ (resp. $f : P \to P$). By fixing such homomorphisms, let $B = P[[T_1, \ldots, T_s]]$, let $B \to A$ be the surjective homomorphism $T_i \mapsto \pi_i$, and extend f to $B \to B$ by $T_i \mapsto T_i^p$ $(1 \leq i \leq s)$. Then, A, B, f and T_i satisfy all the assumptions in § 3 with $\Lambda = \{1, \ldots, s\}$ and $b_\lambda = T_\lambda$. (For the fact $\mathrm{Spec}(A)$ satisfies (2.4), apply (2.1).) Let $J = \mathrm{Ker}(B \to A)$, $J_n = JB_n$ $(n \geq 1)$.

From now on, we assume that A contains a primitive p^n-th root ζ of 1. As in § 0, let

$$
\begin{aligned}
H &= \operatorname{Coker}\big(F - 1 : W_n\Omega_R^r \longrightarrow W_n\Omega_R^r / dW_n\Omega_R^{r-1}\big) \\
&= \operatorname{Coker}\big(f_r - 1 : \Omega_{P_n}^r \longrightarrow \Omega_{P_n}^r / d\Omega_{P_n}^{r-1}\big),
\end{aligned}
$$

where the last equality follows from the identification (0.7.2). We prove :

THEOREM 4.14. — *Let A and B be as above and assume $p > r + s + 1$.*

(1) *The canonical map*

$$
H^{r+s+1}(A, \mathcal{S}_n(r+s+1)) \to H^{r+s+1}(\mathcal{S}_n'(r+s+1)_{A,B})
$$

is an isomorphism.

(2) *The map*

$$
H = \Omega_{P_n}^r / \big(d\Omega_{P_n}^{r-1} + (f_r - 1)(\Omega_{P_n}^r)\big) \longrightarrow H^{r+s+1}\big(\mathcal{S}_n'(r+s+1)_{A,B}\big)
$$

$$
w \longmapsto \text{ the class of } \log(\xi^{p^n}) \otimes w \wedge \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_s}{T_s}
$$

is an isomomorphism, where ξ denotes a lifting of ζ to B.

As is easily seen, $\log(\xi^{p^n}) \in D_{B_n}(J_n)$ is independent of the choice of ξ. Since $f_1(\log(\xi^{p^n})) = \log(\xi^{p^n})$, the map in (4.14) (2) is well defined.

LEMMA 4.15. — *Let ξ be a lifting of ζ to B. Then, the couple (B, ξ) satisfies the assumptions (4.1.1)–(4.1.3) with $q = r + s$.*

Proof. — The condition (4.1.1) is clear and (4.1.2) follows from (4.16) (3) below. We consider (4.1.3). By using $F_1(dT_i) = T_i^{p-1}\,dT_i$, we see easily that for any $w \in \Omega_{B_n}^{r+s}$, the sequence $\{f_{r+s}^i(w)\}_{i \geq 0}$ converges to zero for the $(T_1, \ldots, T_s)$-adic topology on $\Omega_{B_n}^{r+s}$ where f_{r+s}^i denotes the i-fold ietration of f_{r+s}. The sum $w' = -\sum_{i \geq 0} f_{r+s}^i(w)$ satisfies $w = f_{r+s}(w') - w'$.

LEMMA 4.16. — *Let $p = u\pi_1^{e_1} \cdots \pi_s^{e_s}$ ($u \in A^x$, $e_i \geq 1$) be the prime factorization of p.*

(1) *In A, $\zeta - 1$ has a factorization of the form $v\pi_1^{e_1'} \cdots \pi_s^{e_s'}$ such that $v \in A^x$ and $e_i = p^{n-1}(p-1)e_i'$ for any i.*

(2) *The ideal $J_1 = J/pJ$ of B_1 is generated by $T_1^{e_1} \cdots T_s^{e_s} \mod p$.*

(3) *For any lifting ξ of ζ to B, the ideal $(\xi - 1)$ of B_1 is generated by $T_1^{e_1'} \cdots T_s^{e_s'} \mod p$.*

Proof. — First, (1) follows from $(\zeta - 1)^{p^{n-1}(p-1)} \in p \cdot (\mathbb{Z}[\zeta])^x$. Take a lifting $\tilde{u}$ (resp. $\tilde{v}$) of u (resp. v) to B. Then $p - \tilde{u}T_1^{e_1} \ldots T_s^{e_s} \in J$. Since the family $(T_1, \ldots, T_s, p - \tilde{u}T_1^{e_1} \ldots T_s^{e_s})$ is a part of a minimal system of generators of the maximal ideal of B, $p - \tilde{u}T_1^{e_1} \ldots T_s^{e_s}$ is a prime element. By $\dim(B) = \dim(A) + 1$, J is generated by this prime element. Finally, $\xi - 1 \equiv \tilde{v} T_1^{e_1'} \cdots T_s^{e_s'} \mod J$, and by (2), this proves (3).

LEMMA 4.17. — *Let $\xi \in B$ be a lifting of ζ to B and let*

$$\Phi(T) = (T^{p^n} - 1)(T^{p^{n-1}} - 1)^{-1} \in \mathbb{Z}[T].$$

Then, $\Phi(\xi)$ generates J.

Proof. — It suffices to prove that $\Phi(\xi) \mod pJ$ generates $J/pJ \subset B_1$. But this follows from

$$(\xi^{p^n} - 1)\left(\xi^{p^{n-1}} - 1\right)^{-1} \equiv (\xi - 1)^{p^{n-1}(p-1)} \mod pB$$

and from (4.16) (2) and (3).

By (4.15) and (4.17), we can apply all the results in (4.1)–(4.11) to the present couple (A, B) (with $q = r + s$).

In the following, we fix a lifting ξ of ζ to B such that $\xi - 1$ belongs to the ideal $(T_1, \ldots, T_s)$ of B. Such choice of ξ is possible for we can take for example, $1 + \tilde{v} T_1^{e_1'} \ldots T_s^{e_s'}$ as ξ. Fix an integer c such that $1 \le c < p$, and let $\alpha, \beta, h \in B$ be as in (4.2).

Then, by using the fact that the element γ in (4.4) (2) belongs to $T_1 \cdots T_s B$, which follows easily from $\xi - 1 \in (T_1, \ldots, T_s)$, we see :

LEMMA 4.18.

If $2 \le c < p$, all the arguments and results in (4.7)–(4.11) hold when we replace there

$$H^{q+1}(A, \mathcal{S}_n(q+1)) \quad \text{by} \quad H^{q+1}(\mathcal{S}_n'(q+1)_{A,B}), \quad \Omega_{B_n}^q \quad \text{by} \quad (\Omega_{B_n}^q)',$$
$$H^{q+1}(A, \mathcal{S}_1(q+1)) \quad \text{by} \quad H^{q+1}(\mathcal{S}_1'(q+1)_{A,B}) \quad \text{and} \quad \Omega_{B_1}^q \quad \text{by} \quad (\Omega_{B_1}^q)'$$

except that we replace (4.7.4) and its proof by

$$\beta T_1 \cdots T_s (\Omega_{B_n}^q)' \subset \left\{ f_q(w) - hw \, ; \, w \in (\Omega_{B_n}^q)' \right\}$$

(this inclusion follows from the original (4.7.4)), we do not change the part of (4.8) (2) concerning $\log(\xi^{p^n}) \otimes \Omega_{B_n}^q$ and $\beta \otimes \Omega_{B_n}^q$, and we replace (4.9.1) by

$$f_1(h) - 1 = \bar{x} + \beta T_1 \cdots T_s y \quad \text{for some} \quad y \in B_n.$$

Consequently, we have :

PROPOSITION 4.19. — *Let $2 \le c < p$. Then there is an isomorphism*

$$H^{r+s+1}\big(\mathcal{S}'_n(r+s+1)_{A,B}\big)$$
$$\cong (\Omega^{r+s}_{B_n})' / \big(\mathrm{d}(\Omega^{r+s+1}_{B_n})' + \{f_{r+s}(w) - hw\, ;\ w \in (\Omega^{r+s}_{B_n})'\}\big).$$

By using the relation

$$\beta^{-1}\big(f_{r+s}(w) - hw\big) = f_{r+s}(\alpha^{-1}w) - \alpha^{-1}w \qquad \big(w \in (\Omega^{r+s}_{B_n})'\big)$$

and $\mathrm{d}\beta = 0$, this isomorphisms (4.3) and (4.19) composed with

$$\Omega^{r+s}_{B_n} \xrightarrow{\cong} \beta^{-1}\Omega^{r+s}_{B_n}, \quad (\Omega^{r+s}_{B_n})' \xrightarrow{\cong} \beta^{-1}(\Omega^{r+s}_{B_n})'\, ; \qquad w \longmapsto \beta^{-1}w$$

gives the following commutative diagram.

(4.20)

$$
\begin{array}{ccc}
H^{r+s+1}\big(A, \mathcal{S}_n(r+s+1)\big) & \xrightarrow{\ \cong\ } & \beta^{-1}\Omega^{r+s}_{B_n}/\mathcal{M} \\[2mm]
\Big\downarrow & & \Big\downarrow \\[2mm]
H^{r+s+1}\big(\mathcal{S}'_n(r+s+1)_{A,B}\big) & \xrightarrow[\ \cong\ ]{} & \beta^{-1}(\Omega^{r+s}_{B_n})'/\mathcal{M}'
\end{array}
$$

with

$$\mathcal{M} = \beta^{-1}\,\mathrm{d}\Omega^{r+s-1}_{B_n} + (f_{r+s} - 1)(\alpha^{-1}\Omega^{r+s}_{B_n}),$$

$$\mathcal{M}' = \beta^{-1}\,\mathrm{d}(\Omega^{r+s-1}_{B_n})' + (f_{r+s} - 1)\big(\alpha^{-1}(\Omega^{r+s}_{B_n})\big)'.$$

LEMMA 4.21. — *Let*

$$i : \Omega^r_{P_n} \longrightarrow \beta^{-1}\Omega^{r+s}_{B_n} / \big(\beta^{-1}\,\mathrm{d}\Omega^{r+s-1}_{B_n} + (f_{r+s} - 1)(\alpha^{-1}\Omega^{r+s}_{B_n})\big)$$

be the homomorphism $w \mapsto w \wedge \mathrm{d}T_1/T_1 \wedge \cdots \wedge \mathrm{d}T_s/T_s$ (note that we have $(T_1 \cdots T_s)^{-1} \subset \beta^{-1}B$). Then :

  (1) *The map i factors through $H = \Omega^r_{P_n} / (\mathrm{d}\Omega^{r-1}_{P_n} + (f_r - 1)(\Omega^r_{P_n}))$.*
  (2) *The induced maps*

$$H \longrightarrow \beta^{-1}\Omega^{r+s}_{B_n} / \big(\beta^{-1}\,\mathrm{d}\Omega^{r+s-1}_{B_n} + (f_{r+s} - 1)(\alpha^{-1}\Omega^{r+s}_{B_n})\big)$$

$$H \longrightarrow \beta^{-1}(\Omega^{r+s}_{B_n})' / \big(\beta^{-1}\,\mathrm{d}(\Omega^{r+s-1}_{B_n})' + (f_{r+s} - 1)(\alpha^{-1}(\Omega^{r+s}_{B_n})')\big)$$

are surjective.

(3) *The composite*

$$H \xrightarrow{i} \beta^{-1}(\Omega_{B_n}^{r+s})' / (\beta^{-1} d(\Omega_{B_n}^{r+s-1})' + (f_{r+s} - 1)(\alpha^{-1}(\Omega_{B_n}^{r+s})'))$$

$$\xrightarrow[(4.20)]{\cong} H^{r+s+1}(\mathcal{S}_n'(r+s+1)_{A,B})$$

coincides with the homomorphism in (4.14) (2).

Proof. — The assertion (1) is clear. Since $\beta - \log(\xi^{p^n})$ is contained in the ideal of $D_{B_n}(J_n)$ generated by $\bar{J}_n^{[N]}$ and $(\log(\xi^{p^n}))^{[i]}$ ($i \geq 2$) for any N, (3) follows from (4.8) (2) modified as indicated in (4.18). Finally, to prove (2), it suffice to show that the induced maps :

$$\Omega_{P_1}^r \longrightarrow \beta^{-1}\Omega_{B_1}^{r+s} / (\beta^{-1} d\Omega_{B_1}^{r+s-1} + (f_{r+s} - 1)(\alpha^{-1}\Omega_{B_1}^{r+s})),$$

$$\Omega_{P_1}^r \longrightarrow \beta^{-1}(\Omega_{B_1}^{r+s})' / (\beta^{-1} d(\Omega_{B_1}^{r+s-1})' + (f_{r+s} - 1)(\alpha^{-1}(\Omega_{B_1}^{r+s})')),$$

are surjective. Since $\beta \in T_1^{p_{e_1'}^n} \cdots T_s^{p_{e_s'}^n} \cdot B_1^x$ in B_1, we are reduced to the case $m = 1$ of the following (4.22).

LEMMA 4.22. — *Fix* $m \geq 1$. *For* $t \in N$ *and* $j = (j_1, \ldots, j_s) \in \mathbb{Z}^s$, *let* F_j^t *be the subgroup* $(\prod_{i=1}^s T_i^{-j_i})(\Omega_{B_m}^t)'$ *of* $\Omega_{B_m'}^t$. *Then we have :*

(1) *If* $j > 0$ *and* $p \nmid j_i$ *for some* i ($1 \leq i \leq s$), *and if* j' *denotes* $(j_1, \ldots, j_{i-1}, j_i - 1, j_{i+1}, \ldots, j_s)$ *for such* i,

$$F_j^{r+s} = dF_j^{r+s-1} + F_{j'}^{r+s}.$$

(2) *If* $j > 0$ *and if* $p \mid j_i$ *for all* i ($i \leq i \leq s$), *and if* j' *denotes* $(p^{-1}j_1, \ldots, p^{-1}j_s)$, *then*

$$F_j^{r+s} = dF_j^{r+s-1} + f_{r+s}(F_{j'}^{r+s}).$$

(3) $F_0^{r+s} = dF_0^{r+s-1} + F_{-1}^{r+s}$ *where* 0 *and* -1 *denotes* $(0, \ldots, 0)$ *and* $(-1, \ldots, -1)$, *respectively.*

(4) $F_{-1}^{r+s} = \Omega_{B_m}^{r+s}$ *and* $F_{-1}^{r+s} \subset (f_{r+s} - 1)F_{-1}^{r+s}$.

The proofs of (1), (2), (3) are straightforwards, and assertion (4) follows from (4.1.3).

Find that this (4.22) also proves the surjectivity of

$$\Omega_{P_n}^r \longrightarrow \Omega_{B_n'}^{r+s} / (d\Omega_{B_n'}^{r+s-1} + (f_{r+s} - 1)(\Omega_{B_n'}^{r+s})).$$

Now we consider the residue homomorphism.

LEMMA 4.23. — *The residue homomorphism*

$$\mathrm{Res} : \Omega^{r+s}_{B'_n} \longrightarrow \Omega^r_{P_n}$$

in § 0 *is equal to the map*

$$\Omega^{r+s}_{B'_n} \cong \Omega^s_{B'_n/P_n} \otimes_{P_n} \Omega^r_{P_n} \longrightarrow \Omega^r_{P_n} \;;$$

$$T_1^{-j_1} \ldots T_s^{-j_s} w \otimes \eta \longmapsto \left(\mathrm{Res} \left[{}_{T_1^{j_1}, \ldots, T_s^{j_s}}^{w} \right] \right) \eta$$

$(w \in \Omega^s_{B_n/P_n}, \; j_1, \ldots, j_s \geq 0, \; \eta \in \Omega^r_{P_n})$ *where the notation is as in* [H], *chap.* III, § 9. *By passing to the quotients, it induces a map*

$$\Omega^{r+s}_{B'_n} / \left(\mathrm{d}\Omega^{r+s-1}_{B'_n} + (f_{r+s} - 1)\Omega^{r+s}_{B'_n} \right)$$
$$\longrightarrow \Omega^r_{P_n} / \left(\mathrm{d}\Omega^{r-1}_{P_n} + (f_r - 1)\Omega^r_{P_n} \right) = H.$$

The proof is straightforward and we omit it.

These results show that all arrows in

$$H \longrightarrow \beta^{-1}\Omega^{r+s}_{B_n} / \left(\beta^{-1}\,\mathrm{d}\Omega^{r+s-1}_{B_n} + (f_{r+s} - 1)(\alpha^{-1}\Omega^{r+s}_{B_n}) \right)$$

$$\longrightarrow \beta^{-1}(\Omega^{r+s}_{B_n})' / \left(\beta^{-1}\,\mathrm{d}\left(\Omega^{r+s-1}_{B_n}\right)' + (f_{r+s} - 1)(\alpha^{-1}(\Omega^{r+s}_{B_n})') \right)$$

$$\longrightarrow \Omega^{r+s}_{B'_n} / \left(\mathrm{d}\Omega^{r+s-1}_{B'_n} + (f_{r+s} - 1)(\Omega^{r+s}_{B'_n}) \right) \xrightarrow{\ \mathrm{Res}\ } H$$

are bijective. By (4.20) and (4.21) (3), this proves THEOREM (4.14).

Finally we prove THEOREM (0.3) (1) and THEOREM (0.6). The only remained problem is that the composite maps

$$K^M_{r+s+1}\left(A\left[\tfrac{1}{p}\right]\right) \longrightarrow H^{r+s+1}\left(\mathcal{S}'_n(r+s+1)_{A,B}\right) \xrightarrow[\substack{\cong \\ (4.14)}]{} H$$

$$H^{r+s+1}\left(A, \mathcal{S}_n(r+s+1)\right) \longrightarrow H^{r+s+1}\left(\mathcal{S}'_n(r+s+1)_{A,B}\right) \xrightarrow[\substack{\cong \\ (4.14)}]{} H$$

are independent of the choices of generators π_i of (π_i) $(1 \leq i \leq s)$ and of the ring homomorphism $P \to A$. If $(\pi'_i)_{1 \leq i \leq s}$ and $P' \to A$ are other choices, there exists an isomorphism

$$\varphi : P[[T_1, \ldots, T_s]] \xrightarrow{\cong} P'[[T_1, \ldots, T_s]]$$

such that $\varphi(P) \subset P'$, such that φ maps the ideal (T_i) onto (T_i) for each i, and such that the diagram

$$
\begin{array}{ccc}
P[[T_1,\ldots,T_s]] & \xrightarrow{\ \varphi\ } & P'[[T_1,\ldots,T_s]] \\
\end{array}
$$

$$
\begin{array}{ccccc}
 & {}^{T_i}\searrow & & {}^{T_i}\nearrow & \\
 & {}_{\pi_i}\searrow & A & {}_{\pi_i'}\nearrow &
\end{array}
$$

commutes. (Use the formally smooth property.) By this fact, we are reduced to proving the commutativity of the diagram

$$
\begin{array}{ccc}
\Omega^{r+s}_{P_n[[T_1,\ldots,T_s]][1/T_1\cdots T_s]} & \xrightarrow{\ \mathrm{Res}\ } & \Omega^{r}_{P_n} \\
\text{by }\varphi\ \downarrow & & \text{by }\varphi\ \downarrow \quad\searrow \\
\Omega^{r+s}_{P_n'[[T_1,\ldots,T_s]][1/T_1\cdots T_s]} & \xrightarrow{\ \mathrm{Res}\ } & \Omega^{r}_{P_n'} \quad\nearrow \quad H
\end{array}
$$

The commutativity of the left square follows from the properties of the residue homomorphism in [H], chap. III, §9. The commutativity of the right triangle follows from the functoriality of the identification $\Omega^{r}_{P_n}/d\Omega^{r-1}_{P_n} \cong W_n\Omega^{r}_R$.

5. Cohomological symbols

In this section we prove THEOREM (0.3)(2).

5.1. — Let A be a regular local ring satisfying the conditions (0.2.1), (0.2.2). Fix $n \geq 1$. The exact sequence of Kummer

$$
0 \longrightarrow \mathbb{Z}/p^n\mathbb{Z}(1) \longrightarrow \mathbb{G}_m \xrightarrow{\ p^n\ } \mathbb{G}_m \longrightarrow 0
$$

on $\mathrm{Spec}(A[\tfrac{1}{p}])_{\mathrm{et}}$ induces an isomorphism

$$
h_1 : A[\tfrac{1}{p}]^{x}/(A[\tfrac{1}{p}]^{x})^{p^n} \cong H^1\Big(\mathrm{Spec}(A[\tfrac{1}{p}])_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z}(1)\Big).
$$

The cohomological symbol map is defined by

$$h_q : K_q^M\big(A\big[\tfrac{1}{p}\big]\big)/p^n K_q^M\big(A\big[\tfrac{1}{p}\big]\big) \longrightarrow H^q\Big(\mathrm{Spec}\big(A\big[\tfrac{1}{p}\big]\big)_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z}(q)\Big)$$

$$\{a_1,\ldots,a_q\} \longmapsto h_1(a_1) \cup \cdots \cup h_1(a_q).$$

($\cup$ is the cup product.) On the other hand, for $q \geq 1$, let

$$H_{p^n}^q(R) =_{\mathrm{def}} \mathrm{Coker}\Big(F - 1 : W_n\Omega_R^{q-1} \longrightarrow W_n\Omega_R^{q-1}/dW_n\Omega_R^{q-2}\Big).$$

Let i_1 be the composite map

$$H_{p^n}^1(R) \cong H^1\big(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z}\big)$$

$$\cong H^1\Big(\mathrm{Spec}(A)_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z}\Big) \longrightarrow H^1\Big(\mathrm{Spec}(A\big[\tfrac{1}{p}\big])_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z}\Big)$$

where the first isomorphism comes from the exact sequence of Artin–Schreier–Witt

$$0 \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow W_n(\mathcal{O}) \xrightarrow{F_1} W_n(\mathcal{O}) \longrightarrow 0$$

on $\mathrm{Spec}(R)_{\mathrm{et}}$, and the second is by the henselian property of $(A, \mathfrak{a})$.

LEMMA 5.2. — *For $q \geq 1$, there exists a unique homomorphism*

$$i_q : H_{p^n}^q(R) \longrightarrow H^q\Big(\mathrm{Spec}\big(A\big[\tfrac{1}{p}\big]\big)_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z}(q-1)\Big)$$

such that

$$i_q\big(w \cdot d\log(b_1)\cdots d\log(b_{q-1})\big) = i_1(w) \cup h_{q-1}\big(\{\tilde{b}_1,\ldots,\tilde{b}_{q-1}\}\big)$$

for any $w \in W_n(R)$, $b_1,\ldots,b_{q-1} \in R^x$, and for any lifting $\tilde{b}_i$ of b_i to A^x.

Proof. — The presentation of $W_n\Omega_R^q$ given in [K1], §2.2, Corollary 3 shows that it is sufficient to prove the following :

Let $b \in R^x$, $\tilde{b}$ any lifting of b to A^x and let $w = (b,0,\ldots,0) \in W_n(R)$. Then $i_1(w) \cup h_1(\tilde{b}) = 0$.

Let R_w (resp. A_w) be the cyclic etale extension of R (resp. A) corresponding to the image of w in $H^1(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z})$. Consider the diagram

$$
\begin{array}{ccc}
R^x/N\big((R_w)^x\big) & \xleftarrow{\ \cong\ } & A^x/N\big((A_w)^x\big) \\
\Big\downarrow{\scriptstyle\cap} & & \Big\downarrow \\
H_{p^n}^2(R) & & H^2\big(\mathrm{Spec}(A\big[\tfrac{1}{p}\big])_{\mathrm{et}}, \mathbb{Z}/p^n(1)\big)
\end{array}
$$

where the left and right vertical maps are respectively

$$x \longmapsto \mathrm{class}\big(w\,\mathrm{d}\log(x)\big), \quad x \longmapsto i_1(w) \cup h_1(x),$$

and where N denotes the norm maps. The left vertical arrow is injective since it is the composite of

$$R^x/N\big((R_w)^x\big) = H^2\big(G,(R_w)^x\big) \hookrightarrow {}_{p^n}H^2\big(\mathrm{Spec}(R)_{\mathrm{et}},\mathbb{G}_m\big)$$
$$\cong H^1\big(\mathrm{Spec}(R)_{\mathrm{et}},\mathbb{G}_m/(\mathbb{G}_m)^{p^n}\big) \cong H^2_{p^n}(R)$$

$(G = \mathrm{Gal}(R_w/R),\ {}_{p^n}(\ )$ means $\mathrm{Ker}(p^n))$ where the last isomorphism follows from the exact sequence of etale sheaves

$$0 \longrightarrow \mathbb{G}_m/(\mathbb{G}_m)^{p^n} \longrightarrow W_n\Omega^1_{\mathrm{Spec}(R)}$$
$$\xrightarrow{\ F-1\ } W_n\Omega^1_{\mathrm{Spec}(R)}/\mathrm{d}V^{n-1}\Omega^1_{\mathrm{Spec}(R)} \longrightarrow 0$$

([CSS], 1.4) and from $\mathrm{d} = F^{n-1}\mathrm{d}V^{n-1}$. By the diagram, this proves $i_1(w) \cup h_1(b) = 0$.

COROLLARY 5.3. — *Let $\pi_1,\ldots,\pi_s$ and r be as in (0.2). Then there exists a homomorphism*

$$i_A : H = H^{r+1}_{p^n}(R) \longrightarrow H^{r+s+1}\Big(\mathrm{Spec}\big(A[\tfrac{1}{p}]\big)_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z}(r+s)\Big)$$

such that

$$i_A(w\,\mathrm{d}\log(b_1)\cdots\mathrm{d}\log(b_r)) = i_1(w)h_{r+s}\big(\{\tilde{b}_1,\ldots,\tilde{b}_r,\pi_1,\ldots,\pi_s\}\big)$$

for $w \in W_n(R)$, $b_1,\ldots,b_r \in R^x$ and for any liftings $\tilde{b}_i$ of b_i. This map depends by sign on the ordering on the set of prime ideals $\{(\pi_1),\ldots,(\pi_s)\}$, but is independent of the choices of generators π_i of (π_i).

Proof. — This follows from (5.2) and the fact $W_n\Omega^q_R = 0$ for $q > r$.

This homomorphism i_A is bijective in the case A is a discrete valuation ring (see [K3]). We conjecture that it is bijective in general.

In the following, A is as in (0.2.1)–(0.2.3). We fix prime elements $\pi_1,\ldots,\pi_s$ of A in (0.2.2) and a primitive p^n-th root ζ of 1. We identify the twists $\mathbb{Z}/p^n\mathbb{Z}(*)$ with $\mathbb{Z}/p^n\mathbb{Z}$ via $\mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}/p^n\mathbb{Z}(1)\,;\,1 \mapsto \zeta$.

LEMMA 5.4. — *For $w \in W_n(R)$, let y_w be any element of $A\left[\frac{1}{p}\right]^x$ such that*

$$y_w \mod \left(A\left[\tfrac{1}{p}\right]^x\right)^{p^n} = h_1^{-1} \circ i_1(w).$$

Then for $u_1, \ldots, u_r \in A^x$, the image of $\{y_w, u_1, \ldots, u_r, \pi_1, \ldots, \pi_s\}$ in

$$H^{r+s+1}\left(\mathrm{Spec}\left(A\left[\tfrac{1}{p}\right]\right)_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z}(r+s+1)\right)$$

under h_{r+s+1} and that under $-i_A \cdot \tau_A$ coincide.

Proof. — It is sufficient to show that $-\tau_A(\{y_w, u_1, \ldots, u_r, \pi_1, \ldots, \pi_s\})$ coincides with the class of $w\, d\log(\bar{u}_1) \ldots d\log(\bar{u}_r)$ in H where $\bar{u}_i = u_i \mod \mathfrak{a}$. Take $B = P[[T_1, \ldots, T_s]] \to A;\ T_i \mapsto \pi_i\ (1 \geq i \geq s)$ and a lifting $\xi \in B$ of ζ as before. Write $w = (a_0, \ldots, a_{n-1})$ and $\widehat{w} = \sum_{i=0}^{n-1} p^i \tilde{a}_i^{p^{n-i}} \in P_n$, where $\tilde{a}_i$ is a lifting of a_i to P. Then, the problem is reduced to showing that the image of y_w under the symbol map

$$\mathrm{class}\left(\frac{1}{p}\log\left(\frac{f(\tilde{y}_w)}{\tilde{y}_w^p}\right), \frac{\mathrm{d}\tilde{y}_w}{\tilde{y}_w}\right) \in H^1\left(\mathcal{S}_n(1)_{A,B}\right)$$

coincides with the class of

$$-\left(\log(\xi^{p^n})\widetilde{w}, 0\right) \in D_{B_n}(J_n) \oplus D_{B_n}(J_n) \otimes_{B_n} \Omega^1_{B_n}.$$

The image of w in $H^1(\mathrm{Spec}(R)_{\mathrm{et}}, \mathbb{Z}/p^n\mathbb{Z})$ defines cyclic etale extensions R_w, A_w, A'_w, etc. of R, A, $A' = A\left[\frac{1}{p}\right]$, etc. respectively, and an injection $\chi : G \to \mathbb{Z}/p^n\mathbb{Z}$ of $G = \mathrm{Gal}(R_w/R) = \mathrm{Gal}(A_w/A)$. There exists $x \in (A'_w)^x$ such that $x^{p^n} = y_w$, and by the definition of the correspondence $w \rightleftarrows y_{w'}$, x satisfies $\sigma(x)(x)^{-1} = \zeta^{\chi(\sigma)}$ for $\sigma \in G$. Let $\tilde{x}$ be a lifting of x to $(B'_w)^x$. Then $z = \tilde{x}^{p^n}(\tilde{y}_w)^{-1}$ belong to $\mathrm{Ker}((B'_w)^x \to (A'_w)^x)$. We have in $(D_{B_n}(J_n))_w$:

(5.4.1) $\sigma\big(\log(z)\big) - \log(z) = \log\left(\left(\frac{\sigma(\tilde{x})}{\tilde{x}}\right)^{p^n}\right) = \chi(\sigma)\log(\xi^{p^n}),$

(5.4.2) $f_1\big(\log(z)\big) - \log(z) = -\frac{1}{p}\log\left(\frac{f(\tilde{y}_w)}{\tilde{y}_w^p}\right).$

On the other hand, there exists $v \in W_n(R_w)$ such that $F(v) - v = w$ and $\sigma(v) - v = \chi(\sigma)$ for $\sigma \in G$. Define $\hat{v} \in P_n$ just as in the definition of $\widehat{w}$. Then $f(\hat{v}) - \hat{v} = \widehat{w}$ and $\sigma(\hat{v}) - \hat{v} = \chi(\sigma)$ for $\sigma \in G$. By (5.4.1), we see that the element

$$c =_{\mathrm{def}} \log(\xi^{p^n})\hat{v} - \log(z) \in \mathrm{Ker}\left((D_{B_n}(J_n)_w \to (A_n)_w\right)$$

is G-invariant and hence belongs to $\mathrm{Ker}(D_{B_n}(J_n) \to A_n)$. Furthermore

$$dc = -\frac{\mathrm{d}z}{z} = \frac{\mathrm{d}\tilde{y}_w}{\tilde{y}_w}$$

$$f_1(c) - c = \log(\xi^{p^n})\widehat{w} + \frac{1}{p}\log\Big(\frac{f(\tilde{y}_w)}{\tilde{y}_w^p}\Big)$$

by (5.4.2). This shows that the classes of

$$(1/p\log\big(f(\tilde{y}_w)/\tilde{y}_w^p\big), \quad \mathrm{d}\tilde{y}_w/\tilde{y}_w \quad \text{and} \quad -(\log(\xi^{p^n})\widetilde{w}, 0)$$

coincide in $H^1(\mathcal{S}_n(1)_{A,B})$. $\square$

In the case of a discrete valuation ring, the elements

$$\{y_w, u_1, \ldots, u_r, \pi_1, \ldots, \pi_s\}$$

in (5.4) generates $K_{r+s+1}^M(K)/p^n K_{r+s+1}^M(K)$ as is seen from the bijectivities of h_{r+s+1} and i_A. So by (5.4), we have already completed the proof of COROLLARY (0.4).

In the general case, THEOREM (0.3) (2) follows from (5.4) and the following (5.5), by a standard norm argument for the reduction to the infinite residue field case.

LEMMA 5.5. — *If the residue field of A is an infinite field, the elements $\{y_w, u_1, \ldots, u_r, \pi_1, \ldots, \pi_s\}$ in (5.4) generate*

$$K_{r+s+1}^M\big(A\big[\tfrac{1}{p}\big]\big)/p^n K_{r+s+1}^M\big(A\big[\tfrac{1}{p}\big]\big).$$

For the proof of (5.5), we need :

LEMMA 5.6. — *Let S be a regular local ring over $\mathbb{F}_p$ with infinite residue field, and assume that S has a finite p-basis consisting of q elements. Then $K_i^M(S)/pK_i^M(S) = (0)$ if $q < i \leq p$.*

Proof. — A proof is as follows. O. GABBER proved that the homomorphism

$$K_i^M(S)/pK_i^M(S) \longrightarrow \Omega_S^i ; \quad \{x_1, \ldots, x_i\} \longmapsto \frac{\mathrm{d}x_1}{x_1} \wedge \cdots \wedge \frac{\mathrm{d}x_i}{x_i}$$

is injective for any i. The lemma follows since $\Omega_S^i = (0)$ for $i > q$. (In the case S is a field, the proof of this injectivity due to S. BLOCH is given

in [BK], § 2.) The proof of O. GABBER is not yet available to the author, so we give another proof of (5.6) by using Quillen's K-group $K_*^Q(S)$.

We may assume that S contains an infinite field. Indeed, if $\bar{S}$ denotes the local ring of $\bigcup_{i \geq 0} S[T^{p^{-i}}]$ at the prime ideal generated by the maximal ideal of S, $\bar{S}$ satisfies the assumption of (5.6) with the same q, and the map $K_*^M(S)/pK_*^M(S) \to K_*^M(\bar{S})/pK_*^M(\bar{S})$ is injective as is seen by the specialization argument. Furthermore, $\bar{S}$ contains the infinite field $\bigcup_i \mathbb{F}_p(T^{p^{-i}})$.

Now we use the fact that for any local ring S containing an infinite field, there is a homomorphism $K_i^Q(S) \to K_i^M(S)$ such that the composite $K_i^M(S) \to K_i^Q(S) \to K_i^M(S)$ is the multiplication by $(i-1)!$. This was proved by SUSLIN [S1], [S2] in the case where S is a field, and as in SOULÉ [So] and in GUIN [Gu], the result of Suslin is extended to such local rings. By this fact, it is sufficient to show that the map $K_i^M(S)/pK_i^M(S) \to K_i^Q(S)/pK_i^Q(S)$ is zero for S in (5.6) and for $i > q$.

Let k be the residue field of S and let $[k : k^p] = p^t$ (so $t \leq q$). By (5.7) below, it is sufficient to prove that for $i > q$ and for elements $a_1, \ldots, a_i$ of S^x such that the images of $a_1, \ldots, a_t$ in k form a p-basis of k and such that $a_{t+1}, \ldots, a_q$ form a system of regular parameters of S, the image of $\{a_1, \ldots, a_i\}$ in $K_i^Q(S)/pK_i^Q(S)$ is zero. For $0 \leq j < q$, let $S^{(j)} = S[a_1^{1/p}, \ldots, a_j^{1/p}]$. Then, $S^{(q)} = S^{1/p}$ (cf. (5.7) (2)). By using the norm homomorphism

$$N_{S^{1/p}/S} = N_{S^{(1)}/S^{(0)}} \circ \cdots \circ N_{S^{(q)}/S^{(q-1)}} : K_*^Q(S^{1/p}) \longrightarrow K_*^S(S),$$

we have in $K_i^Q(S)$

$$\begin{aligned} \{a_1, \ldots, a_i\} &= N_{S^{1/p}/S}\big(\{a_1^{1/p}, \ldots, a_q^{1/p}, a_{q+1}, \ldots, a_i\}\big) \\ &= p^{i-q} N_{S^{1/p}/S}\big(\{a_1^{1/p}, \ldots, a_i^{1/p}\}\big) \in pK_i^Q(S). \quad \square \end{aligned}$$

In the above proof, we used :

LEMMA 5.7. — *Let S be a regular local ring of characteristic p having a finite p-basis over $\mathbb{F}_p$ consisting of q elements. Let $\mathfrak{m}_S$ be the maximal ideal of S, $k = S/\mathfrak{m}_S$, and let t be the integer such that $[k : k^p] = p^t$. Then :*

(1) $q = \dim(S) + t$.

(2) *If $a_1, \ldots, a_t$ are elements of S whose images in k form a p-basis of k over $\mathbb{F}_p$, and if $(a_i)_{t+1 \leq i \leq q}$ is a regular system of parameters of S, then $a_1, \ldots, a_q$ form a p-basis of S over $\mathbb{F}_p$.*

(3) $K_q^M(S)/pK_q^M(S)$ *is generated by elements of the form* $\{a_1, \ldots, a_q\}$ *such that* $a_1, \ldots, a_q$ *satisfy the hypothesis of* (2).

We omit the proofs of (1) and (2). In $K_i^M(k)/pK_i^M(k)$ $(i \geq 0)$, a symbol $\{x_1, \ldots, x_i\}$ (with $x_1, \ldots, x_i \in k^x$) vanishes unless

$$\left[k(x_1^{1/p}, \ldots, x_i^{1/p}) : k\right] = p^i$$

([K1], §1, Lemma 7). By this and by the fact that the kernel of the surjection $K_{t+1}^M(S) \to K_{t+1}^M(k)$ is generated by elements of the form $\{1+x, y_1, \ldots, y_t\}$ $(x \in \mathfrak{m}_S, y_1, \ldots, y_t \in S^x)$, we see that $K_q^M(S)/pK_q^M(S)$ is generated by elements of the form $\{a_1, \ldots, a_q\}$ such that $a_i \mod \mathfrak{m}_S$ $(1 \leq i \leq t)$ form a p-basis of k and $a_i - 1 \in \mathfrak{m}_S$ for $t+1 \leq i \leq q$. It is easy to express an element of this form as the sum of elements of the form in the hypothesis of (2).

Proof of (5.5). — We denote $K_{r+s+1}^M(A[\frac{1}{p}])$ by V. For $j = (j_1, \ldots, j_s) \in \mathbb{N}^s$ and for a subset E of $\{1, \ldots, s\}$, let V_E^j be the subgroup of V generated by elements of the form $\{a, b_1, \ldots, b_{r+s}\}$ such that

$$a \in \text{Ker}\left\{A^x \to \left(A \,/\, (\prod_{i=1}^{s} \pi_i^{j_i}) A\right)^x\right\}$$

and $b_1, \ldots, b_{r+s} \in (A[\pi_i^{-1} ; i \in E])^x$. In the case $E = \{1, \ldots, s\}$, we denote V_E^j by V^j. Since a subset of V/p^nV generates V/p^nV if its image generates V/pV, the proof of (5.5) is reduced to the case $n = 1$. In the following we assume $n = 1$.

LEMMA 5.8. — *If* $a \equiv 1 \mod (\zeta - 1)^p A$, $a \equiv y_w \mod \left(A[\frac{1}{p}]^x\right)^p$ *for some* w. *If* $a \equiv 1 \mod (\zeta - 1)^p \mathfrak{a} A$, *then* $a \in \left(A[\frac{1}{p}]^x\right)^p$.

Proof. — The equation $(1 + (\zeta - 1)T)^p = a$ can be rewritten as

$$T^p - T + g(T) = (a - 1)(\zeta - 1)^{-p} \quad \text{for some } g(T) \in R[T]\mathfrak{a}.$$

By the henselian property of $(A, \mathfrak{a})$, this equation (and hence the equation $T^p = a$) has a solution in A (resp. in a cyclic etale finite extension of A of degree p) if the class of $(a - 1)(\zeta - 1)^{-p}$ in R is contained (resp. not contained) in $\{x^p - x ; x \in R\}$.

LEMMA 5.9. — *Let* $i \geq 0$, S *a local ring,* $I \neq S$ *an ideal of* S, $x \in I$, *and let* N *be the subgroup of* $K_{i+1}^M(S[\frac{1}{x}])$ *generated by elements of the*

form $\{1 + ax, b_1, \ldots, b_i\}$ such that $a \in I$, $b_1, \ldots, b_i \in S\left[\frac{1}{x}\right]^x$. Then there is a homomorphism

$$\rho : \Omega^i_{S/I} \longrightarrow K^M_{i+1}\left(S\left[\tfrac{1}{x}\right]\right)/N$$

such that $\rho(a_0 \, \mathrm{d}a_1/a_1 \wedge \cdots \wedge \mathrm{d}a_i/a_i) = \{1 + \tilde{a}_0 x, \tilde{a}_1, \ldots, \tilde{a}_i\}$ mod N for any $a_0 \in S$, $a_1, \ldots, a_i \in S^x$ and their liftings $\tilde{a}_0, \ldots, \tilde{a}_i$ to A.

Cf. [BK], § 4 for the proof.

By applying (5.9) to the case $S = A$, $I = \mathfrak{a}$, $x = (\zeta - 1)^p$ and by using the fact that $\Omega^i_R = (0)$ for $i > r$, we see $V^j \subset pV$ if $j > ep/(p-1)$ $(e = (e_1, \ldots, e_s) \in \mathbb{N}^s)$ and that $V^{ep/(p-1)}/\sum_{j > ep/(p-1)} V^j$ is generated by elements of the form $\{1 + a(\zeta - 1)^p, u_1, \ldots, u_r, \pi_1, \ldots, \pi_s\}$ such that $a \in A$ and $u_1, \ldots, u_r \in A^x$. Hence by (5.8), for the proof of (5.5), it is sufficient to show $V \subset V^{ep/(p-1)} + pV$. We prove this in the following three steps. Denote $(0, \ldots, 0)$, $(1, \ldots, 1) \in \mathbb{N}^s$ simply by 0 and 1, respectively.

(5.10.1) $V \subset \sum_{j > 0} V^j_\phi + pV$.

(5.10.2) $\sum_{j > 0} V^j_\phi \subset V^1 + pV$.

(5.10.3) Let $j \in \mathbb{N}^s$, $j \geq 1$, $i \in \{1, \ldots, s\}$, and let

$$j' = (j_1, \ldots, j_{i-1}, j_i + 1, j_{i+1}, \ldots, j_s) \in \mathbb{N}^s.$$

Then if $p \nmid j_i$ (resp. if $j_1 \equiv \cdots \equiv j_s \equiv 0$ mod p and if $j \leq ep/(p-1)$ and $j_i < e_i p/(p-1)$), $V^j \subset V^{j'} + pV$.

LEMMA 5.11. — *For any distinct integers $c_1, \ldots, c_t$ $(t \geq 1)$ taken from $\{1, \ldots, s\}$, $A/(\pi_{c_1}, \ldots, \pi_{c_t})$ has a p-basis over $\mathbb{F}_p$ consisting of $r + s - t$ elements.*

This follows from (5.7) (2) applied to $A/(\pi_i)$ $(1 \leq i \leq s)$.

Proof of (5.10.1). — Note $V = V^0_{\{1, \ldots, s\}}$. Let $E = \{c_1, \ldots, c_t\}$ (with $t = \mathrm{Card}(E)$) be any subset of $\{1, \ldots, s\}$. Then, as is checked easily, the map

$$K^M_{r+s+1-t}(A) \longrightarrow V^0_E \Big/ \Big(\sum_{E' \subsetneq E} V^0_{E'} + \sum_{j > 0} V^j_\phi \Big) ;$$

$$x \longmapsto \{x, \pi_{c_1}, \ldots, \pi_{c_t}\}$$

is surjective and factors through $K^M_{r+s+1-t}(A/(\pi_{c_1}, \ldots, \pi_{c_t}))$. Since the last group is p-divisible by (5.6) and (5.11), we have

$$V^0_E \subset \sum_{E' \subsetneq E} V^0_{E'} + \sum_{j>0} V^j_\phi + pV.$$

This prove (5.10.1).

Proof of (5.10.2). — Let $j \in \mathbb{N}^s$, $j > 0$, $0 \le i \le s$, and assume $j_i = 0$. Let $E = \{c_1, \ldots, c_t\}$ ($t = \mathrm{Card}(E)$) be any subset of the support of j, and let $j' = (j_1, \ldots, j_{i-1}, 1, j_{i+1}, \ldots, j_s)$. We can check easily that for $a \in A$, the homomorphism

$$K^M_{r+s-t}(A) \longrightarrow \left(V^j_E + V^{j'}_{E\cup\{i\}} \right) \Big/ \left(\sum_{E' \subsetneq E} V^j_{E'} + V^{j'}_{E\cup\{i\}} \right);$$

$$x \longmapsto \left\{ 1 + a\pi_1^{j_1} \cdots \pi_s^{j_s}, x, \pi_{c_1}, \ldots, \pi_{c_t} \right\}$$

factors through the p-divisible group $K^M_{r+s-t}(A/(\pi_{c_1}, \ldots, \pi_{c_t}, \pi_i))$, and the images of these homomorphisms for varying a generate the target group. Hence we have

$$V^j_E \subset \sum_{E' \subsetneq E} V^j_{E'} + V^{j'}_{E\cup\{i\}} + pV$$

which proves (5.10.2).

Proof of (5.10.3). — The proofs for the two cases in (5.10.3) are similar, so we give here only the proof for the case where $j_1 \equiv \cdots \equiv j_s \equiv 0 \mod p$, $j \le ep/(p-1)$ and $j_i < e_ip/(p-1)$. Let $E = \{c_1, \ldots, c_t\}$ ($t = \mathrm{Card}(E)$) be any subset of $\{1, \ldots, s\}$. Then we can check easily that the homomorphism

$$\Omega^{r+s-t}_A \longrightarrow \left(V^j_E + V^{j'} \right) \Big/ \left(\sum_{E' \subsetneq E} V^j_{E'} + V^{j'} \right)$$

$$a \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{r+s-t}}{b_{r+s-t}} \longmapsto \left\{ 1 + a\pi_1^{j_1} \ldots \pi_s^{j_s}, b_1, \ldots, b_{r+s-t}, \pi_{c_1}, \ldots, \pi_{c_t} \right\}$$

($a \in A$, $b_1, \ldots, b_{r+s-t} \in A^x$) is surjective and factors through $\Omega^{r+s-t}_{A/(\pi_{c_1}, \ldots, \pi_{c_t})}$ (cf. (5.9)). If $i \notin E$, it factors through $\Omega^{r+s-t}_{A/(\pi_{c_1}, \ldots, \pi_{c_t}, \pi_i)} = (0)$. Assume $i \in E$ and let $S = A/(\pi_{c_1}, \ldots, \pi_{c_t})$. We show that the image of Ω^{r+s-t}_S in $(V^j_E + V^{j'} + pV)/(\sum_{E' \subsetneq E} V^j_{E'} + V^{j'} + pV)$ is zero. Since S has a p-basis over $\mathbb{F}_p$ consisting of $r + s - t$ elements, Ω^{r+s-t}_S is generated by elements of the form $a^p\, db_1/b_1 \wedge \cdots \wedge db_{r+s-t}/b_{r+s-t}$ ($a \in S$,

$b_1, \ldots, b_{r+s-t} \in S^x$) and by $\mathrm{d}\Omega_S^{r+s-t-1}$. The images of the former elements vanish by

$$1 + a^p \pi_1^{j_1} \cdots \pi_s^{j_s} \equiv \left(1 + a\pi_1^{j_1/p} \cdots \pi_s^{j_s/p}\right)^p \mod \pi_i\left(\pi_1^{j_1} \cdots \pi_s^{j_s}\right) A$$

($a \in A$). The image of $\mathrm{d}\Omega_S^{r+s-t-1}$ is zero by

$$\left\{1 - a\pi_1^{j_1} \cdots \pi_s^{j_s}, a\right\} = -\left\{1 - a\pi_1^{j_1} \cdots \pi_s^{j_s}, \pi_1^{j_1} \cdots \pi_s^{j_s}\right\} \in pK_2^M\left(A\left[\tfrac{1}{p}\right]\right)$$

($a \in A$). This completes the proof of (5.5) and hence of (0.3) (2).

Remark 5.12. — In our explicit reciprocity law (0.3), we assumed $p > r + s + 1$. After the author wrote up this paper, he found that this assumption can be weakened to $p \neq 2$ by the following method. Let $\mathcal{S}_n(q)_{A,B}^{\geq q-1}$ be the degree $\geq q-1$ part of $\mathcal{S}_n(q)_{A,B}$. Then this complex $\mathcal{S}_n(q)_{A,B}^{\geq q-1}$ is defined even in the case $p \leq q$. If $p \neq 2$, we obtain symbol maps, independence from the choice of B for $H^q(\mathcal{S}_n(q)_{A,B}^{\geq q-1})$, and this group (instead of $H^q(\mathcal{S}_n(q)_{A,B})$ is used to extend our results to the case $p \neq 2$. In the case $p = 2$, we have an essential difficulty. In fact, the explicit reciprocity law for $p = 2$ given in [B1] is essentially different from the case $p \neq 2$.

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Note. — The author appologizes to the editors for the delay in his revising of this paper, and he thanks Prof. C. SOULÉ for the help in revising. During this delay, the following new mathematical aspects appeared concerning the contents of this paper.

1) The contents of § 1 are treated in the paper :

BERTHELOT (P.) and MESSING (W.). — Théorie de Dieudonné cristalline III, *The Grothendieck Festschrift*, vol. III, Birkhäuser 1990, p. 173–247.

2) The modified complex $S'_n(r)_{A,B}$ in § 3 is related to the crystalline cohomology theory with logarithmic poles developped in the following papers :

FALTINGS (G.). — F-isocrystals on open varieties : Results and Conjectures, *The Grothendieck Festschrift*, vol. II, Birkhäuser 1990, p. 219–248.

KATO (K.). — Logarithmic structures of Fontaine-Illusie, *Proc. of first JAMI conference*, John Hopkins Univ. Press, 1990, p. 80–99.