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## ON A PROBLEM OF TAMAS VARGA

BY

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RÉSUMÉ. — Dans la première partie, on considère des propriétés quantitatives des nombres  $q$  où un développement en base  $q$  de 1 possède un nombre non borné de chiffres 0 consécutifs. Dans la seconde partie, on étudie la distribution des sommes finies  $\sum \varepsilon_i q^i$ , où  $\varepsilon_i = 0$  ou 1 pour des valeurs spéciales de  $q$ . La troisième partie est consacrée à l'étude de la distribution des chiffres dans les développements gloutons des nombres  $x$  presque partout dans  $[0, 1]$ . Finalement, on pose des problèmes ouverts.

ABSTRACT. — In the first part we investigate the quantitative properties of the numbers  $q$  for which there exists an expansion of 1 in base  $q$  where the length of consecutive 0-digits is not bounded. In the second part we study the distribution of the finite sums  $\sum \varepsilon_i q^i$ ,  $\varepsilon_i = 0$  or 1 for special values  $q$ . The third part is devoted to the study of the digit distribution of the greedy expansion of a.e.  $x \in [0, 1]$ . Finally we give some open problems.

*Dedicated to academician Vera T. Sós  
on the occasion of her birthday*

During his marvellous mathematical teaching activity Tamás VARGA found a lot of deep new problems. We mention the following one : in a heads or tails game repeated  $n$  times how long sequences of consecutive heads can be found? In other words, if we consider the dyadic expansion

$$x = \sum_1^{\infty} \frac{\varepsilon_k(x)}{2^k}$$

of a randomly chosen number  $0 \leq x \leq 1$ , what can be asserted about the

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longest sequence of consecutive 0-digits (resp. 1-digits) between the first  $n$  digits? This problem has thoroughly been investigated by many authors, see e.g. [1] and [2].

1.—In the paper [4], one of the authors modified the problem as follows. Let  $1 < q < 2$  be an arbitrary number and consider the expansions of the number 1 of the following type :

$$(1) \quad 1 = \sum_{i=1}^{\infty} \frac{1}{q^{n_i}}, \quad n_i \in \mathbb{N} \text{ are different (natural numbers).}$$

For some values  $q$  this expansion is not unique so the uniqueness problem can be investigated as well (see [5], [6]). Recently the second and third author proved in [7] that in the case  $q = \sqrt{2}$  there exists an expansion 1 with the property :

$$(2) \quad \sup_i (n_{i+1} - n_i) = \infty.$$

The authors of the present paper, V. KOMORNIK and M. HORVÁTH solved the uniqueness problem of the expansion (1) in [5] and [6], further P. ERDŐS and I. JOÓ studied in [8] the properties of the sequence  $(n_{i+1} - n_i)$ .

In this paper consider the following properties of the expansion (1)

$$(2') \quad \sup_i \frac{n_i}{i} = \infty,$$

$$(3) \quad \lim_i (n_{i+1} - n_i) = \infty,$$

$$(3') \quad \lim_i \frac{n_i}{i} = \infty.$$

Obviously  $(2') \Rightarrow (2)$  and  $(3) \Rightarrow (3')$ , further the reverse statements do not hold in general.

THEOREM 1 (cf. [8]). — *The set*

$$A := \left\{ q \in ]1, 2[ : \text{there exists an expansion (1) satisfying (2)} \right\}$$

*is residual and of full measure in ]1, 2[.*

*Problem 1.* — Does the statement of the THEOREM 1 remain true after substituting  $(2')$  in place of  $(2)$  in the definition of  $A$ ?

THEOREM 2. — *The set*

$$B := \left\{ q \in ]1, 2[ : \text{there exists an expansion (1) satisfying (3')} \right\}$$

is of first category and of measure zero.

*Proof.* — It is enough to make the proof for the sets

$$B \cap ]1 + \delta, 2[, \quad \delta > 0.$$

Consider arbitrary numbers  $1 + \delta < q_1 < q_2 < 2$ , and sufficiently large  $t \in \mathbb{N}$  :

$$(4) \quad 1 = \sum_{i=1}^n \frac{\varepsilon_i}{q_1^i} = \sum_{i=1}^n \frac{\varepsilon_i}{q_2^i} + \frac{1}{q_2^{n+t}}.$$

It follows from  $1 + \delta < q_1$  that there exists  $k \leq c(\delta)$  with  $\varepsilon_k = 1$ . Consequently

$$\begin{aligned} \frac{1}{q_2^{n+t}} &= \sum_{i=1}^n \varepsilon_i \left( \frac{1}{q_1^i} - \frac{1}{q_2^i} \right) \geq \frac{1}{q_1^k} - \frac{1}{q_2^k} \\ &= \frac{[q_1 + (q_2 - q_1)]^k - q_1^k}{q_1^k q_2^k} \geq \frac{k(q_2 - q_1)}{q_1 q_2^k} \geq \frac{q_2 - q_1}{q_2^{k+1}} \end{aligned}$$

i.e.

$$(5) \quad q_2 - q_1 \leq \frac{1}{q_2^{n+t-k-1}} \leq \frac{1}{q_2^{n+t-c(\delta)}} \leq \frac{1}{(1 + \delta)^{n+t-c(\delta)}}.$$

Denote  $B_n$  the set of those  $1 + \delta < q < 2$  for which there exists an expansion of 1 satisfying  $n_i/i \geq t$  for  $n_i > n$ . Take a number  $N > 2n$ . We see that between  $\frac{1}{2}N$  and  $N$  there exist  $\geq \frac{1}{3}t$  consecutive zeros for any  $q \in B_n$ . Indeed, assume the contrary. Then between  $\frac{1}{2}N$  and  $N$  there exist

$$\geq \frac{\frac{1}{2}N}{\frac{1}{3}t} = \frac{3}{2} \frac{N}{t} \text{ 1-digits}$$

and then  $i \geq \frac{3}{2}N/t$  and  $n_i \leq N$  would imply that  $n_i/i \leq \frac{2}{3}t$ . The contradiction shows that there exists  $\geq \frac{1}{3}t$  consecutive zeros between  $\frac{1}{2}N$  and  $N$ ; hence  $q$  can be covered by an interval of length

$$\leq (1 + \delta)^{-N/2-t/3+c(\delta)},$$

see (5) above. For any  $q \in B_n$  and  $N$  we get an interval; the number of such intervals is not greater than  $N$  times the number of the sequences  $\varepsilon_1, \dots, \varepsilon_N$  with  $n_i \geq it$  and  $n_i > n$ . In particular  $\varepsilon_1 + \dots + \varepsilon_N \leq N/t$  (if  $N > nt$ ). The number of choices of  $N/t$  digits from  $\varepsilon_1, \dots, \varepsilon_N$  is

$$\begin{aligned} \binom{N}{[N/t]} &= \frac{N(N-1)\cdots(n - [N/t] + 1)}{[N/t]!} \\ &\leq c \frac{N^{[N/t]}}{([N/t]e^{-1})^{[N/t]} \sqrt{[N/t]}} \\ &\leq c \frac{N^{[N/t]}}{\sqrt{[N/t]} (N/(2te))^{[N/t]}} \\ &\leq c \sqrt{t/N} (2te)^{N/t} \\ &\leq c\sqrt{t} 2^{c(1+\ln t)N/t} \\ &\leq c\sqrt{t} 2^{N\varepsilon(t)} \quad (\varepsilon(t) \rightarrow 0 \text{ when } t \rightarrow \infty). \end{aligned}$$

Hence the sum of the length of the above intervals covering  $B_n$  is not greater than  $c\sqrt{t}(1 + \delta)^{c(\delta)-t/3} \cdot N(2^{\varepsilon(t)}/\sqrt{1 + \delta})^N$ . Given  $\delta > 0$  we can choose  $t \geq t(\delta)$  satisfying  $2^{\varepsilon(t)}/\sqrt{1 + \delta} < 1$ . If we fix  $t$  and let  $N$  tend to infinity, we see that the set  $B_n$  can be covered by finite systems of intervals of arbitrary small length sum. Consequently  $B_n$  is nowhere dense and of measure zero.

Since  $B \subset \bigcup_1^\infty B_n$  the proof of THEOREM 2 is complete.  $\square$

*Remark.* — By (3)  $\Rightarrow$  (3') the same statement holds with (3) instead of (3').

**THEOREM 3.** — *Define the set*

$$C := \left\{ q \in ]1, 2[ : \text{there exists an expansion (3) satisfying (1)} \right\},$$

*For any interval  $I \subset ]1, 2[$  the intersection  $I \cap C$  has  $2^{\aleph_0}$  many points.*

*Proof.* — Take any value  $1 < q_0 < 2$  and any expansion  $1 = \sum \varepsilon_i/q_0^i$ . The set of  $q$  for which there exists an expansion of 1 starting with  $\varepsilon_1, \dots, \varepsilon_n$ , forms an interval whose length tends to zero when  $n \rightarrow \infty$ . This can be verified just as in (5). Consequently it is enough to prove that given  $q_0$  and  $n$  arbitrary we have  $2^{\aleph_0}$  many  $q \in C$  whose “good” expansion

starts with  $\varepsilon_1, \dots, \varepsilon_n$ . Fix a sequence  $n < n_1 < n_2 \dots$  satisfying (3) and construct a set  $\mathcal{P}$  of infinite subsets of the set  $\{n_1, n_2, \dots\}$  such that  $P_1, P_2 \in \mathcal{P}$  implies  $P_1 \subset P_2$  or  $P_2 \subset P_1$  and there are  $2^{\aleph_0}$  elements of  $\mathcal{P}$ . This can be done in the usual way mapping the set  $\{n_k\}$  onto the set of rational numbers  $\mathbb{Q}$  in a one-to-one way and then consider the sets  $\mathbb{Q} \cap ]-\infty, x[$ ,  $x \in \mathbb{R}$ . Now for every  $P \in \mathcal{P}$  it corresponds to a  $q = q_P$  by the rule

$$1 = \sum_{i=1}^n \frac{\varepsilon_i}{q^i} + \sum_{n_i \in P} \frac{1}{q^{n_i}}.$$

Then  $q_P \in C$  and for different  $P$  the value  $q_P$  is also different (in case  $P_1 \subset P_2$  we have  $q_{P_1} < q_{P_2}$ ).

THEOREM 3 is proved.  $\square$

2. — Now consider the following problem. For given  $1 < q < 2$  define the sets

$$A_n := A_n(q) := \left\{ \sum_{i=0}^{n-1} \varepsilon_i q^i : \varepsilon_i = 0 \text{ or } 1 \right\}, \quad n = 1, 2, \dots$$

If we arrange the sums  $A_n$  in a sequence  $y_1^{(n)} \leq \dots \leq y_{2^n}^{(n)}$ , we can write

$$A_n = \left\{ y_k^{(n)} : 1 \leq k \leq 2^n \right\}.$$

LEMMA 1. — We have  $y_{k+1}^{(n)} - y_k^{(n)} \leq 1$  for all  $k$  and  $n$ .

*Proof.* — Almost the same is proved in [6]. It runs as follows. We apply induction on  $n$ . If  $n = 1$  then  $A_n = \{0, 1\}$  so the statement is true. Suppose it for  $A_n$  and prove for  $A_{n+1}$ . Obviously we have

$$(6) \quad A_{n+1} = A_n \cup (q^n + A_n).$$

Now if in  $A_n$  there is an element larger than  $q^n$ , the smallest element of  $q^n + A_n$ , then the inductual hypothesis gives the statement by (6). If not, we have to check that the distance between the largest element of  $A_n$  and  $q^n$  is not larger than 1, i.e.

$$(1 + q + q^2 + \dots + q^{n-1}) + 1 \geq q^n \quad \text{i.e.} \quad \frac{q^n - 1}{q - 1} \geq q^n - 1.$$

But this is true since  $1 < q < 2$ .

LEMMA 2. — *The polynomial*

$$P_r(x) := x^{r+1} - \sum_{k=0}^r x^k, \quad r \geq 1$$

has exactly one zero  $\xi_r$  in  $]1, 2[$  and  $\xi_r \rightarrow 2$  monotone increasingly as  $r \rightarrow \infty$ .

*Proof.* — Define the polynomial  $Q_r$  by

$$P_r(x) = x^{r+1} - \frac{x^{r+1} - 1}{x - 1} = \frac{x^{r+2} - 2x^{r+1} + 1}{x - 1} := \frac{Q_r(x)}{x - 1}.$$

We see that the polynomial  $Q_r$  decreases for  $1 \leq x \leq x_0$ , increases for  $x_0 \leq x \leq 2$ , where  $x_0 = 2(r+1)/(r+2)$ , further  $Q_r(1) = 0, Q_r(2) = 1$ . It shows that  $Q_r(x)$  has exactly one zero  $\xi_r$  in  $]1, 2[$  and  $x_0 < \xi_r < 2$ . It implies at once that  $\xi_r \rightarrow 2$  as  $r \rightarrow \infty$ . On the other hand

$$Q_r(\xi_{r-1}) = \xi_{r-1}^{r+2} - 2\xi_{r-1}^{r+1} + 1 = 1 - \xi_{r-1} < 0$$

which shows that  $\xi_{r-1} < \xi_r$ .

LEMMA 3. — *Let  $n \geq 1, q = \xi_r$  for some  $r \geq 1$  and  $A_n = A_n(q)$ . Then we have  $A_n \cap ]q^n, 1 + q^n[ = \emptyset$ .*

*Proof.* — By  $q = \xi_r$  we have  $P_r(q) = 0$ , i.e.

$$(7) \quad 1 = \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^{r+1}}.$$

Iterating this we get the other representation

$$(8) \quad 1 = \sum_{\substack{k=1 \\ k \nmid r+1}}^{\infty} \frac{1}{q^k}.$$

Next we show that

$$(9) \quad 1 \geq \sum_{\substack{k=1 \\ k \neq r}}^{\infty} \frac{1}{q^k},$$

and equality holds for  $r = 1$ . Indeed, we can transform the numbers  $q^{-r}, q^{-2r-1}, q^{-3r-2}$ , etc. of (8) by the aid of (7) to obtain

$$1 = \frac{1}{q} + \frac{1}{q^2} + \dots + \frac{1}{q^{r-1}} + \frac{1}{q^{r+1}} + \left( \frac{2}{q^{r+2}} + \dots + \frac{2}{q^{2r}} \right) \\ + \frac{1}{q^{2r+1}} + \frac{1}{q^{2r+2}} + \left( \frac{2}{q^{2r+3}} + \dots + \frac{2}{q^{3r+1}} \right) \\ + \frac{1}{q^{3r+2}} + \frac{1}{q^{3r+3}} + \left( \frac{2}{q^{3r+4}} + \dots + \frac{2}{q^{4r+2}} \right) + \dots$$

which implies (9). This shows that if  $y \in A_n$  and  $y > q^n$  then the first  $r$  digits  $\varepsilon_{n-1}, \varepsilon_{n-2}, \dots, \varepsilon_{n-r}$  of

$$y = \sum_{i=1}^{n-1} \varepsilon_i q^i$$

must be 1 (otherwise  $y < q^n$ ). If  $\varepsilon_{n-r-1} = 1$  then  $q^n = \sum_{i=n-r-1}^{n-1} \varepsilon_i q^i$  and hence  $y > q^n$  clearly implies  $y \geq q^n + 1$ . If  $\varepsilon_{n-r-1} = 0$  then define

$$y_1 := y - \sum_{i=n-r}^{n-1} q^i \in A_{n-r-1}.$$

Since  $q^{n-1} + \dots + q^{n-r} = q^n - q^{n-r-1}$ , this implies

$$q^{n-r-1} < y_1 < q^{n-r-1} + 1.$$

Iterating this process we finally find a value  $1 \leq n \leq r + 1$  and  $y \in A_n$ ,  $q^n < y < 1 + q^n$ . But this is impossible since  $n \leq r + 1$  implies that  $q^n \geq 1 + q + \dots + q^{n-1}$ . The contradiction proves the LEMMA 3.  $\square$

Now introduce the Fibonacci-type sequence  $F_n^{(k)}$  by the recursion :

$$(10) \quad F_n^{(k)} = \begin{cases} 0 & \text{for } n < 0, \\ \sum_{i=1}^k F_{n-i}^{(k)} + 1 & \text{for } n \geq 0. \end{cases}$$

We see that

$$F_n^{(k)} = 2^n \quad \text{for } 0 \leq n \leq k, \quad F_{k+1}^{(k)} = 2^{k+1} - 1, \\ F_{k+2}^{(k)} = 2^{k+2} - 3, \quad F_{k+3}^{(k)} = 2^{k+3} - 11.$$



THEOREM 4. — Let  $n, r = 1, 2 \dots$  and  $q = \xi_r$ . Then

$$(a) \quad |A_n(q)| = F_n^{(r+1)},$$

$$(b) \quad \min_{\substack{1 \leq k \leq 2^n \\ y_{k+1}^{(n)} \neq y_k^{(n)}}} (y_{k+1}^{(n)} - y_k^{(n)}) \geq \frac{1}{q},$$

and equality holds for  $n \geq r + 1$ .

*Proof.* — Consider first the case  $n \leq r$ . Then

$$q^n - (1 + q + \dots + q^{n-1}) \geq \frac{1}{q} > 0$$

hence

$$A_n \cap (q^n + A_n) = \emptyset$$

and then from the obvious relation

$$A_{n+1} = A_n \cup (q^n + A_n)$$

we see at once that

$$|A_{n+1}| = 2|A_n| = \dots = 2^n |A_1| = 2^{n+1} = F_{n+1}^{r+1}$$

further (b) is also true and equality holds only for  $n = r + 1$ .

Now let  $n \geq r + 1$ . Then  $A_n$  and  $q^n + A_n$  has nonempty intersection since  $q^n = q^{n-1} + \dots + q^{n-r-1}$ . We show that in this case the sets  $A_n$  and  $q^n + A_n$  has overlapping maximal possible. Namely every  $y \in A_n$ ,  $y \geq q^n$  belongs to  $q^n + A_n$ . More precisely :

$$(*) \quad \begin{cases} \text{Every } y \in A_n, y \geq q^n \text{ has an expansion} \\ y = q^{n-1} + \dots + q^{n-r-1} + \sum_{k=0}^{n-r-2} \varepsilon_k q^k. \end{cases}$$

To prove (\*) we apply induction on  $n$ . For  $n = r + 1$  the only element  $y \in A_n$  with  $y \geq q^n$  is  $q^n = q^{n-1} + \dots + q + 1$ . Suppose (\*) for  $n$  and prove

it for  $n + 1$ . Let  $y \in A_{n+1}$  and  $y \geq q^{n+1}$ . If in the expansion  $y = \sum_{k=0}^n \varepsilon_k q^k$

we have  $\varepsilon_0 = 0$ , then applying the induction hypothesis to  $y/q \in A_n$ ,  $y/q \geq q^n$  we are ready. If  $y = q^{n+1}$  then  $y = q^n + q^{n-1} + \dots + q^{n-r}$  which is also a good representation. Finally if  $y > q^{n+1}$  and  $\varepsilon_0 = 1$  then by Lemma 3,  $y - 1 \geq q^{n+1}$ , hence we can apply the induction

hypothesis for  $y-1/q \in A_n, y-1/q \geq q^n$  so (\*) holds indeed. Consequently  $A_{n+1} = 2A_n - A_{n-r-1}$ , if  $n \geq r+1$ . So we can prove (a) by induction as follows :

$$\begin{aligned} |A_{n+1}| &= 2F_n^{(r+1)} - F_{n-r-1}^{(r+1)} \\ &= F_n^{(r+1)} + \left(1 + \sum_{i=1}^{r+1} F_{n-i}^{(r+1)}\right) - F_{n-r-1}^{(r+1)} \\ &= 1 + \sum_{i=1}^{r+1} F_{n+1-i}^{(r+1)} = F_{n+1}^{(r+1)}. \end{aligned}$$

The proof of (b) for  $n \geq r+1$  is obvious : in  $A_n$  and in  $A_n + q^n$  the minimal distance is  $1/q$  and they overlap maximally hence in  $A_{n+1}$  the minimal distance is also  $1/q$ . THEOREM 4 is proved.  $\square$

**3.** — In the following part of this paper we consider two other problems related to the papers of ERDŐS, RÉNYI [3] and ERDŐS, RÉVÉSZ [1]. To formulate the first one, fix a number  $1 < q < 2$  and expand any number  $0 \leq x \leq 1$  by the so-called *greedy expansion*

$$x = \sum_1^{\infty} \frac{\varepsilon_n(x)}{q^n}, \quad \varepsilon_n(x) = \begin{cases} 0 \\ 1. \end{cases}$$

We assert that

THEOREM 5. — *There exists a constant  $c > 0$  with the following properties. Consider the set of those  $x \in [0, 1]$  for which the greedy expansion of  $x$  contains a sequence of  $\geq c \log n$  consecutive 0-digits between the first  $n$  digits  $\varepsilon_1(x), \dots, \varepsilon_n(x)$  for all indices  $n \geq n_0(x)$ . This set has full measure in  $[0, 1]$ .*

The second problem arises in a heads or tails game with an asymmetric piece of money. We represent it as a random variable whose value is zero with probability  $p, 0 < p < 1$  and 1 with probability  $q = 1 - p$ . Consider a sequence  $x_1, x_2, \dots$  of independent random variables with such distributions. Introduce the quantities

$$\alpha_n := \log n - \log \log \log n + K$$

with some constant  $K < 0$  to be specified later. We prove the

THEOREM 6. — *The following event has probability 1 : between the first  $n$  digits  $x_1, \dots, x_n$  there exist  $\alpha_n$  consecutive 0 digits for sufficiently large  $n > n_0$ ; here  $n_0 = n_0(\omega)$  may depend on the concrete value of the sequence  $(x_n(\omega))_1^{\infty}$ .*

We mention the following open

*Problem 2.* — THEOREM 5 does not remain true for large  $c > 0$  (this is the case if  $q = 2$ , see ERDÖS, RÉNYI [3]).

For the proof of THEOREMS 5 and 6 we need some lemmas. Denote

$$P(\varepsilon_1, \dots, \varepsilon_n) = |\{x \in [0, 1] : \varepsilon_1 = \varepsilon_1(x), \dots, \varepsilon_n = \varepsilon_n(x)\}|$$

the probability of the event that the greedy expansion of  $x$  begins with the digits  $\varepsilon_1, \dots, \varepsilon_n$ .

LEMMA 4.

$$(a) \quad P(\varepsilon_1, \dots, \varepsilon_n, 1) \leq \frac{1}{q-1} P(\varepsilon_1, \dots, \varepsilon_n, 0),$$

$$(b) \quad P(\varepsilon_1, \dots, \varepsilon_n) \leq \frac{q}{q-1} P(\varepsilon_1, \dots, \varepsilon_n, 0).$$

*Proof.* — (b) follows from (a) since

$$\begin{aligned} P(\varepsilon_1, \dots, \varepsilon_n) &= P(\varepsilon_1, \dots, \varepsilon_n, 0) + P(\varepsilon_1, \dots, \varepsilon_n, 1) \\ &\leq \frac{q}{q-1} P(\varepsilon_1, \dots, \varepsilon_n, 0). \end{aligned}$$

To see (a) denote  $I_n$  the length of the segment

$$\left\{x \in [0, 1] : \varepsilon_1 = \varepsilon_1(x), \dots, \varepsilon_n = \varepsilon_n(x)\right\}.$$

The left endpoint is  $x = \sum_{k=1}^n \frac{\varepsilon_k}{q^k}$ . Hence :

i) If  $I_n < \frac{1}{q^{n+1}}$ , then

$$P(\varepsilon_1, \dots, \varepsilon_n, 0) = P(\varepsilon_1, \dots, \varepsilon_n), \quad P(\varepsilon_1, \dots, \varepsilon_n, 1) = 0.$$

ii) If  $I_n \in \left[\frac{1}{q^{n+1}}, \frac{1}{q^{n+1}} \frac{q}{q-1}\right]$ , then

$$P(\varepsilon_1, \dots, \varepsilon_n, 0) = \frac{1}{q^{n+1}}, \quad P(\varepsilon_1, \dots, \varepsilon_n, 1) = P(\varepsilon_1, \dots, \varepsilon_n) - \frac{1}{q^{n+1}}$$

and hence (a) follows.  $\square$

From LEMMA 4 we obtain immediately the

LEMMA 5.

$$P(\varepsilon_1, \dots, \varepsilon_n, \frac{1}{0}, \dots, \frac{\alpha_n}{0}) \geq \left(\frac{q-1}{q}\right)^{\alpha_n} P(\varepsilon_1, \dots, \varepsilon_n).$$

Now denote  $S_k(x) := \sum_{j=1}^k \frac{\varepsilon_j(x)}{q^j}$ . We obtain from LEMMA 5 by induction

(in  $[n/\alpha_n]$ ) the

LEMMA 6.

$$\left| \left\{ x : S_{\ell\alpha_n}(x) \neq S_{(\ell+1)\alpha_n}(x), \ell = 0, 1, \dots, \left[ \frac{n}{\alpha_n} \right] \right\} \right| \leq \left( 1 - \left( \frac{q-1}{q} \right)^{\alpha_n} \right)^{[n/\alpha_n]+1}$$

*Proof of the Theorem 5.* — Let

$$\alpha_n := \log n - \log \log n - \log \log \log n - K$$

where  $\log$  denotes the logarithm of base  $q/(q-1)$  and  $K = K(q) > 0$  is a constant “large enough”. Then

$$\left( 1 - \left( \frac{q-1}{q} \right)^{\alpha_n} \right)^{(q/(q-1))^{\alpha_n}} \rightarrow \frac{1}{e} \quad (\text{as } n \rightarrow \infty),$$

hence

$$\left( 1 - \left( \frac{q-1}{q} \right)^{\alpha_n} \right)^{(q/(q-1))^{\alpha_n}} \leq \frac{1}{e}.$$

(We know that  $\sqrt[k+1]{1 - (1 - k^{-1})^k} \leq \frac{k}{k+1}$  i.e.

$$(1 - k^{-1})^k \leq \left( 1 - \frac{1}{k+1} \right)^{k+1}.)$$

We have

$$\left( 1 - \left( \frac{q-1}{q} \right)^{\alpha_n} \right)^{[n/\alpha_n]+1} \leq e^{-((q-1)/q)^{\alpha_n} ([n/\alpha_n]+1)}.$$

The exponent can be estimated as follows if  $n > n_0$  :

$$\begin{aligned} -\left(\frac{q-1}{q}\right)^{\alpha_n} \left(\left[\frac{n}{\alpha_n}\right] + 1\right) &\leq -\frac{1}{2} \left(\frac{q-1}{q}\right)^{\alpha_n} \frac{n}{\alpha_n} \\ &= -\frac{1}{2} \left(\frac{q}{q-1}\right)^K \frac{\log n \log \log n}{n} \frac{n}{\alpha_n} \\ &\leq -\frac{1}{3} \left(\frac{q}{q-1}\right)^K \log \log n \\ &\leq -R \log \log n; \end{aligned}$$

but if  $K = K(q)$  is large enough, then the condition  $n > n_0$  can be omitted. We obtained that

$$|P_n| = \left| \left\{ x : S_{\ell\alpha_n}(x) \neq S_{(\ell+1)\alpha_n}(x), \ell = 0, 1, \dots, [n/\alpha_n] \right\} \right| \leq \frac{1}{(\log n)^2}$$

is  $R$  is large enough, i.e.  $K$  is large enough. This means that

$$\sum |P_{[(\frac{q}{q-1})^m]}| < \infty$$

and according to the Borel-Cantelli lemma almost every  $x \in [0, 1]$  belongs to finitely many set  $P_{[(q/(q-1))^m]}$ , i.e. for a.e.  $x \in [0, 1]$  the first  $[(q/(q-1))^m]$  digit contains 0-sequence of length

$$(*) \quad cm \quad \text{if} \quad m > m_0(x).$$

Now if

$$\left[ \left( \frac{q}{q-1} \right)^m \right] \leq n < \left[ \left( \frac{q}{q-1} \right)^{m+1} \right],$$

then  $m \asymp \log n$ , i.e. it follows from (\*) that for a.e.  $x \in [0, 1]$  among the first  $n$  digits there exists 0-sequence of length  $\geq c \log n$ . Theorem 5 is proved.  $\square$

*Proof of the Theorem 6.* — We need some lemmas.

LEMMA 7. — *The probability of the event that a sequence of length  $2n$  contains a sequence of zeros of length  $n$  is  $p^n(1 + np)$ .*

*Proof.* — Consider the sequence of  $n$  consecutive zeros with minimal first index. The probability of the event that this minimal index is the first is  $p^n$ , the probability of the event that it is  $k$  ( $2 \leq k \leq n+1$ ) is  $qp^n$  because the  $(k-1)$ th digit must be equal to 1. Hence LEMMA 7 follows.  $\square$

LEMMA 8. — *Let  $0 < \alpha_n < n$  be arbitrary and consider the sets*

$$\begin{aligned} B_k &:= (S_{k+\alpha_n} = S_k), & (k = 0, 1, \dots, n - \alpha_n), \\ C_\ell &:= \bigcup_{k=\ell\alpha_n}^{(\ell+1)\alpha_n} B_k, & (\ell = 0, 1, \dots, 2([n/\alpha_n] - 1)), \\ D_n &:= D := \bigcup_{\ell=0}^{[n/\alpha_n]-1} C_{2\ell}. \end{aligned}$$

Then the probability of  $\bar{D}$  is

$$[1 - p^{\alpha_n}(1 + \alpha_n q)]^{[n/(2\alpha_n)]}.$$

*Proof.* — The events  $C_{2\ell}$  are independent and one of them has probability  $p^{\alpha_n}(1 + \alpha_n q)$  by LEMMA 7.  $\square$

Now we give upper estimate for the probability of  $\bar{D}$ . Because

$$p^{\alpha_n}(1 + \alpha_n q) \rightarrow 0 \quad \text{as } \alpha_n \rightarrow \infty,$$

hence

$$\begin{aligned} & \left[ (1 - p^{\alpha_n}(1 + \alpha_n q))^{(p^{\alpha_n}(1 + \alpha_n q))^{-1}} \right]^{p^{\alpha_n}(1 + \alpha_n q)[n/\alpha_n]} \\ & \leq \left( \frac{1}{e} \right)^{p^{\alpha_n}(1 + \alpha_n q)[n/\alpha_n]} =: W. \end{aligned}$$

Let  $\alpha_n := \log n - \log \log \log n + K$  with some constant  $K$ , where  $\log$  is of base  $1/p$ . Then we have

$$p^{\alpha_n} = p^K \frac{\log \log n}{n}$$

hence the exponent of  $1/e$  in  $W$  is

$$\geq p^K \frac{\log \log n}{n} q \log n \frac{n}{4 \log n} = \frac{1}{4} \log \log n p^K q,$$

consequently

$$W \leq \left[ \left( \frac{1}{e} \right)^{\log \log n} \right]^{p^K q/4}.$$

Choose  $-K$  to be large enough, then the probability of  $\bar{D}_{1/p^n}$  can be estimated as follows :

$$|\bar{D}_{1/p^n}| \leq \left[ \left( \frac{1}{e} \right)^{\log \log(1/p^n)} \right]^{p^K q/4} \leq \frac{c_1}{n^{c_2 p^K}},$$

hence

$$\sum |\bar{D}_{1/p^n}| \leq c_1 \sum \frac{1}{n^{c_2 p^K}} < \infty$$

so according to Borel-Cantelli lemma, almost every  $x$  belongs to finitely many  $\bar{D}_{1/p^n}$  only i.e. for every  $n > n_0(x)$  the first  $n$  digits contain consecutive 0-s of length  $\alpha_n$ . THEOREM 6 is proved.  $\square$

At last we state the following questions.

*Problem 3.* — Investigate the behaviour of  $n_{i+1}/n_i$  for the greedy, lazy or arbitrary expansions (see [8]).

*Problem 4.* — Is the set investigated in THEOREM 5 residual in  $[0, 1]$ ?

*Problem 5.* — If  $q$  is not the root of the equations

$$1 = \frac{1}{q} + \frac{1}{q^2} + \cdots + \frac{1}{q^{r+1}}, \quad r = 1, 2, \dots$$

then

$$\inf(y_{n+1} - y_n) = 0,$$

where  $y_n$  is the strictly increasing list of the values

$$\sum_{i=1}^k \varepsilon_i q^i \quad k = 1, 2, \dots; \quad \varepsilon_i = \begin{cases} 0 \\ 1. \end{cases}$$

*Problem 6.* — The statement analogous to THEOREM 5 with lazy expansions and consecutive 1 digits.

*Problem 7.* — THEOREM 3 for greedy expansion.

*Problem 8.* — By [6], THEOREM 2 the set of  $q$  for which the greedy expansion of 1 contains consecutive 0-sequences of length  $\geq \log_2 m$  between the first  $m$  digits for infinitely many  $m$ , is residual and of full measure in  $]1, 2[$ . Does it remain true if we require  $\geq c \log m$  consecutive 0 between the first  $m$  digits for every  $m \geq m(q)$  (the constant  $c > 0$  can be chosen appropriately small)?

*Problem 9.* — In [14] we showed, among others, that the value  $q$  defined by

$$1 = \sum_{i=1}^9 \frac{1}{q^i} + \sum_{j=1}^n \frac{1}{q^{9+10j}} + \sum_{k=1}^{\infty} \frac{1}{q^{9+10n+5k}} \quad (n \geq 1)$$

has the property that 1 has exactly  $n + 1$  expansions. Describe the set of all  $q$ 's having this property.

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