

# BULLETIN DE LA S. M. F.

CHARLES FAVRE

## **Note on pull-back and Lelong number of currents**

*Bulletin de la S. M. F.*, tome 127, n° 3 (1999), p. 445-458

[http://www.numdam.org/item?id=BSMF\\_1999\\_\\_127\\_3\\_445\\_0](http://www.numdam.org/item?id=BSMF_1999__127_3_445_0)

© Bulletin de la S. M. F., 1999, tous droits réservés.

L'accès aux archives de la revue « Bulletin de la S. M. F. » (<http://smf.emath.fr/Publications/Bulletin/Presentation.html>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

**NOTE ON PULL-BACK AND  
LELONG NUMBER OF CURRENTS**

BY CHARLES FAVRE (\*)

---

ABSTRACT. — We prove a uniform estimate of the Lelong number of the pull-back of a plurisubharmonic function by a holomorphic map in term of the original Lelong number of this function.

RÉSUMÉ. — NOTE SUR LE NOMBRE DE LELONG DES PULL-BACK DE COURANTS. Cet article est consacré à l'étude du nombre de Lelong  $\nu(f^*u, 0)$  du pull-back d'une fonction plurisousharmonique  $u$  par une application holomorphe  $f: (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$  génériquement de rang maximal. Nous prouvons l'estimée  $\nu(f^*u, 0) \leq C_f \times \nu(u, 0)$  avec une constante  $C_f$  uniforme en  $u$ .

**1. Statement of the main result**

Fix  $f: (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$  a holomorphic germ, and  $T$  a positive closed current of bidegree  $(1, 1)$  defined in a neighborhood of the origin in  $(\mathbb{C}^n, 0)$ . Let  $u \in \text{PSH}(\mathbb{C}^n, 0)$  be a plurisubharmonic (psh) potential for  $T$  such that  $T = dd^c u$ . One can set

$$f^*T := dd^c(u \circ f)$$

as soon as the psh function  $u \circ f$  is not identically  $-\infty$ .

DEFINITION 1 (Lelong number, see [LG86]). — Let  $u \in \text{PSH}(\mathbb{C}^n, 0)$ . The function  $r \mapsto \sup_{|z|=r} u(z)$  is an increasing convex function of  $\log r$ .

---

(\*) Texte reçu le 19 septembre 1998, révisé le 17 février 1999, accepté le 10 mars 1999.  
C. FAVRE, Department of Mathematics, Royal Institute of Technology, S-100 44 Stockholm (Sweden). Email: favre@math.kth.se.  
Département de Mathématiques, Bât. 425, Université Paris-Sud, 91405 Orsay CEDEX (France). Charles.Favre@anh.math.u-psud.fr.

AMS classification: 32F05.

Keywords: Lelong number, pull-back, plurisubharmonic function.

- We can hence define the *Lelong number* of  $u$  at 0 by setting

$$\nu(u, 0) := \max\{c \geq 0; \text{ such that } u(z) \leq c \log |z| + O(1)\}$$

which is a finite non-negative real number.

- For a positive closed  $(1,1)$  current  $T$  in  $(\mathbb{C}^n, 0)$ , the *Lelong number* of  $T$  at 0 is

$$\nu(T, 0) := \nu(u, 0)$$

for any psh potential  $T = dd^c u$ .

For a given positive closed current  $T$  of bidegree  $(1,1)$  so that  $f^*T$  exists, we are interested in estimating the Lelong number of the pull-back  $\nu(f^*T, 0)$  in terms of  $\nu(T, 0)$ . Our theorem can be stated as follows.

**THEOREM 2.** — *Let  $f: (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$  be a holomorphic map. Then the following conditions are equivalent:*

- 1) *the map  $f$  has generic (maximal) rank equal to  $n$ ;*
- 2) *for any positive closed current  $T$  of bidegree  $(1,1)$   $f^*T$  is well defined, and the operator  $f^*$  is continuous for the weak topology of currents;*
- 3) *the range of  $f$  is not pluripolar;*
- 4) *for any positive closed current  $T$  of bidegree  $(1,1)$   $f^*T$  is well defined, and there exists a constant  $C > 0$  (depending only on  $f$ ) such that one has the inequality*

$$\nu(T, 0) \leq \nu(f^*T, 0) \leq C \cdot \nu(T, 0)$$

*between Lelong numbers of  $T$  and  $f^*T$  at the origin.*

**REMARK 3.** — The proof gives an estimate for the constant  $C$  above. Assume  $n = m$  and 1) is satisfied. Then 4) holds with

$$C = 1 + 2(\mu(Jf, 0) + n - 1),$$

where  $\mu(Jf, 0)$  is the order of vanishing of the Jacobian determinant of  $f$  at 0.

Using this remark, we also have a semi local version of Theorem 2.

**COROLLARY 4.** — *Let  $X$  and  $Y$  be two connected complex manifolds, and  $f: X \rightarrow Y$  be a holomorphic map whose generic rank is maximal equal to  $\dim(Y)$ . Then for any compact set  $K \subset X$ , there exists a constant  $C_K > 0$  such that for all positive closed current  $T$  of bidegree  $(1,1)$  and all  $p \in K$ , one has the inequality*

$$\nu(T, p) \leq \nu(f^*T, p) \leq C_K \cdot \nu(T, f(p))$$

*between Lelong numbers.*

Before giving a proof of this theorem and of its corollary, we will make some remarks about the stated results.

The main result of Theorem 2 is contained in the implication 1)  $\Rightarrow$  4). All the others are either obvious, or were known before.

The second assertion is contained in [M96]. We also refer the reader to this article for more general problems concerning pull-back of positive closed currents by holomorphic mappings.

The upper estimate given in 4) was already known in several different cases (the other inequality is easy to prove).

PROPOSITION 5 (see [De93]). — *Let  $f$  be a finite holomorphic germ  $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$  of local degree  $d$  and  $T$  a positive closed current (of any bidegree). Then*

$$\nu(f^*T, 0) \leq d \times \nu(T, 0).$$

C. Kiselman also proved 1)  $\Rightarrow$  4) for monomial morphisms.

PROPOSITION 6 (see [K87]). — *Let  $M = [a_{ij}] \in M(n, \mathbb{N})$  be an  $n \times n$  matrix with non-negative integer coefficients. We assume that  $\det M \neq 0$ . If*

$$f(z) = \left( \prod_{j=1}^n z_j^{a_{1j}}, \dots, \prod_{j=1}^n z_j^{a_{nj}} \right)$$

then for any positive closed (1,1) current  $T$

$$\nu(f^*T, 0) \leq \max_i \left\{ \sum_j a_{ij} \right\} \cdot \nu(T, 0).$$

Diller in [D98] also proved the main estimate 4) for birational mappings of  $\mathbb{P}^2$ .

A warning concerning the implication 1)  $\Rightarrow$  3). When  $f$  does not have generic maximal rank, it is not true in general that the image of  $f$  is contained in a countable union of hypersurfaces. It is contained in a countable union of polydisks of dimension strictly less than  $n$ .

EXAMPLE 7 (see [H73, 4.2]). — *Define  $f: (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^3, 0)$  by*

$$f(z, w, t) = (z, ze^w, ze^{e^w}).$$

*Note that  $f$  is independent of the last variable  $t$ . Then the set  $f(\mathbb{C}^3, 0)$  is pluripolar, but it is not included in a countable union of hypersurfaces.*

*Proof.* — We give a short proof of these facts. We begin proving that  $f(\mathbb{C}^3, 0)$  is pluripolar. Decompose the mapping  $f = \pi \circ g \circ p$  with

$$\begin{aligned} p(z, w, t) &= (z, w), \\ g(x, y) &= (y, e^x, e^{e^x}), \\ \pi(z, w, t) &= (z, zw, zt). \end{aligned}$$

The range of  $g$  is included in the hypersurface  $g(\mathbb{C}^2, 0) \subset \{e^w = t\}$ , hence is pluripolar. The morphism  $\pi$  is an isomorphism outside  $\{z = 0\}$ . As countable union of pluripolar sets remains pluripolar, we see that the image

$$\begin{aligned} f(\mathbb{C}^3, 0) &= \pi \circ g \circ p(\mathbb{C}^3, 0) = \pi(g(\mathbb{C}^2, 0)) \\ &= \{0\} \bigcup_{k \geq 0} \pi(g(\mathbb{C}^2, 0) \cap \{|z| > 1/k\}) \end{aligned}$$

is also pluripolar.

For the second fact, we proceed as follows. Assume first that  $f(\mathbb{C}^3, 0)$  is included in an hypersurface defined by a non identically zero holomorphic map  $h$ . We thus have the identity

$$h(z, ze^w, ze^{e^w}) = 0$$

for every  $z, w$  in a neighborhood of  $0 \in \mathbb{C}$ . Expand  $h$  in power series  $h = \sum_{k \geq 0} h_k$  where  $h_k$  is a homogeneous polynomial of degree  $k$  in three variables. Take an index  $k_0 \in \mathbb{N}$  such that  $h_{k_0} \not\equiv 0$ . Then

$$h_{k_0}(z, ze^w, ze^{e^w}) = z^{k_0} h_{k_0}(1, e^w, e^{e^w}) \equiv 0.$$

This would contradict the fact that the three functions  $(1, e^w, e^{e^w})$  are algebraically independent.

Now assume  $f(\mathbb{C}^3, 0) \subset \bigcup_{n \in \mathbb{N}} H_n$  is included in a countable union of hypersurfaces. For each  $n \in \mathbb{N}$ , the complex space  $f^{-1}H_n$  is also an hypersurface by what precedes. But we have

$$(\mathbb{C}^3, 0) \subset f^{-1}f(\mathbb{C}^3, 0) \subset \bigcup_{n \in \mathbb{N}} f^{-1}H_n,$$

which can not contain any open subset of  $(\mathbb{C}^3, 0)$ .  $\square$

Finally, a word about the motivations of this article. The author came to the problem of estimating Lelong numbers of pull-back of positive closed  $(1, 1)$  current while working on dynamics of rational maps of the projective space  $f: \mathbb{P}^k \rightarrow \mathbb{P}^k$  with maximal generic rank. Let us give a simple application of Theorem 2 in this context. We first recall some well-known facts which can be found for instance in [Si99].

We let  $\pi: \mathbb{C}^{k+1} - \{0\} \rightarrow \mathbb{P}^k$  be the natural projection onto  $\mathbb{P}^k$ , and take  $F = (F_0, \dots, F_k): \mathbb{C}^{k+1} \rightarrow \mathbb{C}^{k+1}$  a polynomial lift of  $f$  so that

$$F \circ \pi = \pi \circ f.$$

We assume that the  $k + 1$  polynomials  $\{F_i\}_{0 \leq i \leq k}$  do not contain any common factors. The indeterminacy set of  $f$  is equal to

$$I(f) := \pi \left( \bigcap_{i=0}^k F_i^{-1} \{0\} \right).$$

Given any positive closed current  $T$  of bidegree  $(1, 1)$  on  $\mathbb{P}^k$ , one can find a psh function  $G$  on  $\mathbb{C}^{k+1}$ , called its *potential*, such that

1) there exists a constant  $c > 0$  for which for all  $Z \in \mathbb{C}^{k+1}$  and for all  $\lambda \in \mathbb{C}$ ,

$$G(\lambda Z) = c \log |\lambda| + G(Z);$$

2)  $\pi^* T = dd^c G$ .

Conversely, given a psh function  $G$  on  $\mathbb{C}^{k+1}$  satisfying the homogeneity condition 1), one can find a unique positive closed current  $T$  of bidegree  $(1, 1)$  on  $\mathbb{P}^k$  such that 2) holds.

DEFINITION 8. — Let  $f: \mathbb{P}^k \rightarrow \mathbb{P}^k$  be a rational map of maximal generic rank  $k$ , and  $T$  be a positive current of bidegree  $(1, 1)$  with potential  $G$ . We define  $f^* T$  to be the positive closed current of bidegree  $(1, 1)$  whose potential is  $G \circ F$ .

The study of the operator  $f^*$  turns out to give many interesting informations on  $f$  and on its dynamics (see [Si99]). When  $f$  is not holomorphic, for any positive closed current  $T$  of bidegree  $(1, 1)$ , the current  $f^* T$  admits singularity points even if  $T$  has a smooth potential. The computation of Lelong numbers of  $f^* T$  can be viewed as a quantitative measure of how bad the singularities of this current are. The estimate 4) allows us to extend a result of [D98].

PROPOSITION 9. — *Let  $f: \mathbb{P}^k \rightarrow \mathbb{P}^k$  be a rational map with maximal generic rank and  $T$  be a positive closed current of bidegree  $(1,1)$ . Then  $\nu(f^*T, p) > 0$  if and only if either  $p \in I(f)$  or  $\nu(T, f(p)) > 0$ .*

*Proof.* — Assume that  $p \notin I(f)$ . As  $f$  has generic maximal rank, we can apply Theorem 2. This yields a constant  $C_f > 0$  such that

$$\nu(T, f(p)) \leq \nu(f^*T, p) \leq C_f \cdot \nu(T, f(p)).$$

And it follows that  $\nu(f^*T, p) > 0$  if and only if  $\nu(T, f(p)) > 0$ . It remains to check that if  $p$  belongs to  $I(f)$ , then  $\nu(f^*T, p) > 0$ . Choose  $\sigma$  a local section of  $\pi$  around  $p$ , and  $G \in \text{PSH}(\mathbb{C}^{k+1})$  a potential for  $T$ . One can find a constant  $A > 0$  so that

$$|F(\sigma(z))| \leq A|z - p|$$

for points  $z$  near  $p$ . As the function  $G$  satisfies an homogeneity relation, one can bound it by

$$G(Z) \leq B \log |Z| + O(1),$$

with  $B > 0$ . We thus have

$$G(F(\sigma(z))) \leq B \log |z - p| + O(1) \quad \text{and} \quad \nu(f^*T, p) \geq B > 0,$$

which concludes the proof.  $\square$

NOTE. — The main theorem has been proved independently by C. Kiselman (see [K99]) with a different method. His proof relies on volume estimates of sublevel sets of psh functions.

ACKNOWLEDGEMENTS. — I would like to thank first J. Merker for simplifying substantially the original proof, and C. Kiselman, N. Sibony and B. Teissier for many valuable discussions I had with them. Many thanks also for the referee who gave very constructive remarks on the first version.

**2. Proof of the main theorem**

We shall first prove the equivalence between the first three assertions. We conclude by proving  $4) \Rightarrow 3)$ , and  $1) \Rightarrow 4)$ .

$1) \Rightarrow 2)$ . — We assume that  $f$  has generic maximal rank equal to  $n$ . If  $u \in \text{PSH}(\mathbb{C}^n, 0)$  is non degenerate, the psh function  $u \circ f$  can not be identically  $-\infty$  as the range of  $f$  contains some open ball. Hence  $f^*T$  is well-defined for any closed positive current  $T$  of bidegree  $(1, 1)$ . For a sequence of positive closed  $(1, 1)$  current  $T_j \rightarrow T$  converging weakly towards  $T$ , one can find a sequence  $u_j$  of psh potential of  $T_j$  converging in  $L^1_{\text{loc}}$  to  $u$  a psh potential for  $T$ . It remains to check that  $u_j \circ f \rightarrow u \circ f$  in  $L^1_{\text{loc}}$ .

As  $f$  has maximal generic rank,  $u_j \circ f \rightarrow u \circ f$  almost everywhere. Now one can extract a subsequence  $u_{j_k} \circ f$  converging in  $L^1_{\text{loc}}$  to a psh function (see [Ho83] p.94). As any such limit should be equal to  $u \circ f$ , we infer  $u_j \circ f \rightarrow u \circ f$  in  $L^1_{\text{loc}}$ , thus  $f^*T_j \rightarrow f^*T$  in the weak topology.

$2) \Rightarrow 3)$ . — If the range  $f(\mathbb{C}^n, 0)$  is pluripolar, one can find  $u \in \text{PSH}(\mathbb{C}^n, 0)$  non-degenerate such that  $u \circ f \equiv -\infty$ . In that case, if  $T := \text{dd}^c u$ ,  $f^*T$  is not defined.

We also give an example of a sequence of positive closed currents of bidegree  $(1, 1)$  so that  $T_n \rightarrow T$ ,  $f^*T_n$  and  $f^*T$  are all well-defined, but for which the sequence  $f^*T_n$  fails to converge to  $f^*T$ . For this, work in the unit ball, and take  $f(z, w) = (0, w)$ ,  $T_n = \text{dd}^c u_n$ , with

$$u_n(z, w) = \max\{n^{-1} \log |z|, -2 + |w|^2\}.$$

Then  $T_n \rightarrow 0$  but  $f^*T_n = \text{dd}^c |w|^2$ .

$3) \Rightarrow 1)$ . — We only sketch the proof. We proceed by induction on  $m$ . Assume  $f: (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$  is a holomorphic germ such that  $\text{rk } Df_z$  the rank of  $Df_z$  is smaller than  $n - 1$  for any  $z \in (\mathbb{C}^m, 0)$ . Set

$$N := \max\{\text{rk } Df_z\} \leq n - 1,$$

and define for each  $k \leq N$ ,

$$V_k := \{z \in (\mathbb{C}^m, 0); \text{rk } Df_z \leq k\}.$$

By assumption,  $V_N$  contains an open neighborhood of the origin. Define

$$W := V_N - V_{N-1}.$$

The set  $V_k$  is the set where all minors of  $Df_z$  of size  $k + 1$  have zero determinant, and hence defines a closed analytic subspace of  $(\mathbb{C}^m, 0)$ .



Hence  $W$  is a Zariski open set of  $V_N$ . Now, on  $W$  the rank of the differential of  $f$  is constant equal to  $N$ . We can thus apply locally the constant rank theorem. Take any countable covering  $\{U_i\}_{i \in I}$  of  $W$  by open subsets such that for each  $i \in I$ , the set  $f(U_i)$  is a (non-closed) analytic subset of  $(\mathbb{C}^n, 0)$  of dimension  $N < n$ . For any  $i \in I$   $f(U_i)$  is pluripolar. A countable union of pluripolar sets remains pluripolar, hence  $f(W) = \bigcup_{i \in I} f(U_i)$  is pluripolar.

As  $\dim(V_{N-1}) < m$ , we can apply the induction hypothesis to conclude that

$$f(\mathbb{C}^m, 0) = f(W) \cup f(V_{N-1})$$

is pluripolar too.

The implication 4)  $\Rightarrow$  3) follows from 2)  $\Rightarrow$  3).

In fact, we even have that when the range of  $f$  is pluripolar, the supremum of  $(\nu(T, 0))^{-1} \nu(f^*T, 0)$  over all positive closed current  $T$  of bidegree  $(1, 1)$  for which  $f^*T$  is well-defined, is not finite.

Take  $u \in \text{PSH}(\mathbb{C}^n, 0)$  non-degenerate such that  $u \circ f \equiv -\infty$ . For any  $\alpha > 0$ , define

$$v_\alpha(z) := \max\{\alpha \log |z|, u(z) + \log |z|\}.$$

Then

$$\nu(f^*v_\alpha, 0) = \alpha \cdot \nu(\log |f|, 0), \quad \nu(v_\alpha, 0) = \min\{\alpha, \nu(u, 0) + 1\}.$$

Hence for  $\alpha \geq \nu(u, 0) + 1$ ,

$$(\nu(\text{dd}^c v_\alpha, 0))^{-1} \nu(f^* \text{dd}^c v_\alpha, 0) = C\alpha$$

with  $C = (\nu(u, 0) + 1)^{-1} \nu(\log |f|, 0)$ .

1)  $\Rightarrow$  4). — Let us first prove the following general result.

LEMMA 10. — *If  $f: (\mathbb{C}^m, 0) \rightarrow (\mathbb{C}^n, 0)$  is an arbitrary holomorphic germ, and  $T$  is a positive closed current of bidegree  $(1, 1)$  so that  $f^*T$  is well-defined, one has the inequality*

$$\nu(f^*T, 0) \geq \nu(T, 0)$$

between Lelong numbers.

*Proof.* — We fix  $u \in \text{PSH}(\mathbb{C}^n, 0)$  a local potential for  $T$ . We always have  $|f(Z)| \leq A|Z|$  for some constant, so that the estimate

$$u(Z) \leq \nu(T, 0) \log |Z| + O(1)$$

implies

$$u(f(Z)) \leq \nu(T, 0) \log |Z| + O(1)$$

which gives us the stated inequality.  $\square$

We now proceed with the proof of the upper bound for  $\nu(f^*T, 0)$  given in 4). As before,  $u$  will denote a local potential for  $T$ .

Let us show how to reduce the proof of this estimate to the equidimensional case *i.e.* when  $n = m$ .

We assume the estimate has already been proved for  $n = m$ . By assumption, the rank of the Jacobian derivative of  $f$  is generically equal to  $n$ . We can therefore find a closed embedding

$$i: L = (\mathbb{C}^n, 0) \hookrightarrow (\mathbb{C}^m, 0)$$

of a piece of  $n$ -plane into  $(\mathbb{C}^m, 0)$  such that the rank of the Jacobian derivative of the restriction

$$\bar{f} := f \circ i$$

to  $(\mathbb{C}^n, 0)$  is also generically equal to  $n$ . We can now apply the estimate to  $\bar{f}$  and use Lemma 10. We get

$$\nu(f^*T, 0) \leq \nu(i^* \circ f^*T, 0) \leq \nu(\bar{f}^*T, 0) \leq C_{\bar{f}} \cdot \nu(T, 0).$$

Let us deal now with the equidimensional case. The assumption on  $f$  can be rewritten as its Jacobian derivative does not vanish identically on a neighborhood of the origin.

Take a line  $L$  passing through 0 intersecting  $\text{Crit}(f)$  the critical set of  $f$  only at 0, and not tangent to any irreducible component of  $\text{Crit}(f)$ . We can assume it is given in coordinates  $z = (z_1, \dots, z_n)$  by

$$L := \{z_2 = \dots = z_n = 0\}.$$

We can find an open cone around this line  $L$

$$\mathcal{C} := \{z \in U; \text{dist}(z, L) < \varepsilon|z|\}$$

such that  $\mathcal{C} \cap \text{Crit}(f) = \emptyset$ .

Instead of working in this cone, it is more convenient to work on an open set. We thus consider the blow-up  $\pi$  of the origin 0, and replace the germ  $f$  by the composition  $g := f \circ \pi$ . In coordinates,

$$\pi(z) = (z_1, z_1z_2, \dots, z_1z_n).$$

We look at  $g$  in the open set  $\overline{\pi^{-1}\{\mathcal{C}\}}$ . Define

$$E = \pi^{-1}\{0\} = \{z_1 = 0\}.$$

Let us point out some special properties of the map  $g$ .

- 1)  $\text{Crit}(g) = E$ .
- 2)  $g^{-1}\{0\} = E$ .

We can thus write the Jacobian determinant of  $g$  under the form

$$Jg(z) = z_1^N \psi(z)$$

for some integer  $N \in \mathbb{N}$  and some holomorphic function  $\psi$  which does not vanish at any point of  $E$ . In a sufficiently small neighborhood  $V$  of the origin, we can find a constant  $C > 0$  such that for all  $z \in V$

$$(1) \quad |Jg(z)| \geq C|z_1|^N.$$

For the proof of Remark 3 and Corollary 4, we will need the following estimation on the integer  $N$ . It gives precisely a control on the constant  $C$  of assertion 4) of the theorem.

LEMMA 11. — *The integer  $N$  introduced above can be chosen as*

$$N = \mu(Jf, 0) + n - 1,$$

where  $\mu(Jf, 0)$  is the order of vanishing of the holomorphic function  $Jf$  at the point 0.

*Proof.* — Set  $N_0 := \mu(Jf, 0)$ . We first check that for a (generic) suitable choice of line  $L$ , one has in a small cone  $\mathcal{C}$  around  $L$  as above

$$(2) \quad |Jf(z)| \geq C|z|^{N_0}.$$

Expand the holomorphic jacobian determinant  $Jf$  in power series

$$Jf = \sum_{k \geq N_0} h_k$$

where  $h_k$  is a homogeneous polynomial of degree  $k$  and  $h_{N_0}$  is not identically zero. Let  $\mathbb{P}^{n-1}$  be the set of complex lines in  $\mathbb{C}^n$  passing through the origin, and for a point  $z \in \mathbb{C}^n$  set  $L_z = \mathbb{C}z$ . By homogeneity of  $h_{N_0}$ , one can define the continuous function  $H: \mathbb{P}^{n-1} \rightarrow \mathbb{R}_+$  by

$$H(L_z) = |z|^{-N_0} |h_{N_0}(z)|.$$

Take a generic line  $L$  such that  $H(L) > 0$ . Then for all lines  $L'$  close to  $L$ , one has  $H(L') \geq \frac{1}{2}H(L)$ . Hence in a small cone  $\mathcal{C}$  around  $L$ , one has  $H(L_z) \geq \frac{1}{2}H(L)$ .

We infer for all  $z \in \mathcal{C}$ ,

$$\begin{aligned} |f(z)| &\geq \left| h_{N_0}(z) - \sum_{k \geq N_0+1} h_k(z) \right| \\ &\geq |h_{N_0}(z)| - \left| \sum_{k \geq N_0+1} h_k(z) \right| \\ &\geq 2^{-1}H(L)|z|^{N_0} - C'|z|^{N_0+1} \geq C|z|^{N_0}, \end{aligned}$$

for some constants  $C, C' > 0$ .

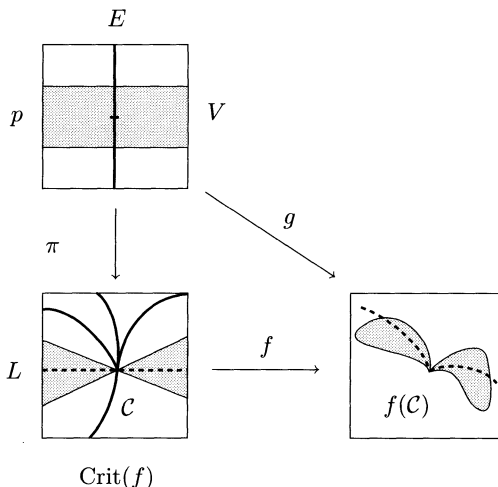
Now a direct computation yields

$$\det(D\pi_z) = z_1^{n-1}.$$

Therefore, if we have chosen a line  $L$  so that equation (2) applies, we get for all  $z \in \mathcal{C}$ ,

$$\begin{aligned} |\det(Dg_z)| &= |\det(D\pi_z) \cdot \det(Df_{\pi(z)})| \\ &\geq |z_1|^{n-1} \cdot C|z_1|^{\mu(Jf,0)}, \end{aligned}$$

which concludes the proof of Lemma 11.  $\square$



In the sequel, we will assume that  $V$  is a small ball in  $\mathbb{C}^n$  endowed with the usual euclidean metric. If  $r > 0$  and  $K$  is a compact set, we set

$$B(K, r) := \{z; \text{dist}(z, K) < r\}.$$

The key lemma is:

LEMMA 12. — *There exists two integers  $N_0, N_1 \in \mathbb{N}^*$ , and two positive constants  $C_0, C_1 > 0$  such that for all  $z \in V$ ,*

$$g(B(z, C_0|z_1|^{N_0})) \supset B(g(z), C_1|z_1|^{N_1}).$$

*Moreover, we can choose  $N_0 = N + 1$ , and  $N_1 = 2N + 1$  (with the above notations).*

*Proof.* — The idea is to approximate the range of  $g(B(z, r))$  by  $Dg_z(B(z, r))$  and estimate the size of the latter.

We have  $|Jg(z)| \geq C|z_1|^N$  for all  $z \in V$ . In  $V$ , all eigenvalues of  $Dg_z$  are uniformly bounded by some constant  $D > 0$ . Therefore for all  $z \in V - E$ ,

$$|Dg_z^{-1}|^{-1} \geq \inf\{|\lambda|; \lambda \in \text{Spec}(Dg_z)\} \geq \frac{C}{D^{n-1}}|z_1|^N.$$

And for all  $z \in V$ , for all  $r > 0$ ,

$$Dg_z(B(z, r)) \supset B(g(z), C'|z_1|^N r),$$

for some constant  $C' > 0$ . Now by Taylor's formula, there exists another constant  $C'' > 0$  such that for all  $z, w \in V$ ,

$$|g(w) - g(z) - Dg_z \cdot (w - z)| \leq C''|w - z|^2.$$

If we choose  $M > N$  and take  $r = |z_1|^M$ , we infer for  $z$  sufficiently small

$$g(B(z, |z_1|^M)) \supset B(g(z), C'|z_1|^{N+M} - C''|z_1|^{2M}),$$

which gives the desired result with  $N_1 = N + M$ .  $\square$

To conclude, we follow Diller [D98]. Define

$$\Delta_r := L \cap \{|z| \leq r\}.$$

We first apply Lemma 12 to each point of the set  $\partial\Delta_r$ . We obtain

$$(3) \quad g(B(\partial\Delta_r, C_0 r^{N_0})) \supset B(\partial g(\Delta_r), C_1 r^{N_1}).$$

We consider now translated of  $g(\Delta_r)$  by vectors  $z$  of norm  $|z| < C_1 r^{N_1}$ . The estimate (3) tells us that  $\partial(z + g(\Delta_r))$  is still included in the range of  $g$ . We have more precisely for all  $|z| \leq C_1 r^{N_1}$ ,

- 1)  $z \in z + g(\Delta_r)$ ,
- 2)  $\partial(z + g(\Delta_r)) \subset g(B(\partial\Delta_r, C_0 r^{N_0}))$ .

We are now in position to prove the desired inequality. We start with

$$u(g(z)) \leq \nu(g^*u, 0) \log |z| + D$$

for some constant  $D \in \mathbb{R}$ . We want to prove an analog estimate for  $u$ . Fix  $z \in V$  and  $r > 0$  such that  $|z| < C_1 r^{N_1}$ . Then the maximum principle applied to  $u$  on the analytic disk  $z + g(\Delta_r)$  yields

$$\begin{aligned} u(z) &\leq \max_{z+g(\Delta_r)} u \leq \max_{\partial(z+g(\Delta_r))} u \\ &\leq \max_{g(B(\partial\Delta_r, C_0 r^{N_0}))} u \\ &\leq \max_{w \in B(\partial\Delta_r, C_0 r^{N_0})} u(g(w)) \\ &\leq \max_{w \in B(\partial\Delta_r, C_0 r^{N_0})} \nu(g^*u, 0) \log |w| + D \\ &\leq \nu(g^*u, 0) \log r + D' \end{aligned}$$

for  $D' := D + \nu(g^*u, 0) \log(\frac{3}{2})$  (by possibly reducing  $C_0$  we can assume that  $C_0 r^{N_0-1} \leq \frac{1}{2}$ ). As this is true for any  $r$  satisfying  $|z| \leq C_1 r^{N_1}$ , we obtain

$$u(z) \leq \frac{1}{N_1} \nu(g^*u, 0) \log |z| + D''.$$

Thus  $\nu(u, 0) \geq N_1^{-1} \nu(g^*u, 0)$ . To conclude the proof we use the general inequality in Lemma 10

$$\nu(u, 0) \geq \frac{1}{N_1} \nu(g^*u, 0) \geq \frac{1}{N_1} \nu((f \circ \pi)^*u, 0) \geq \frac{1}{N_1} \nu(f^*u, 0).$$

The proof combined with Lemmas 11 and 12 gives more precisely (see Remark 3):

LEMMA 13. — *If  $f: (\mathbb{C}^n, z) \rightarrow (\mathbb{C}^n, f(z))$  is a germ of holomorphic map of maximal generic rank, then for any positive closed current  $T$  of bidegree  $(1,1)$ , one has the inequality*

$$\nu(f^*T, z) \leq (2(n - 1 + \mu(Jf, 0)) + 1) \cdot \nu(T, f(z))$$

between Lelong numbers.

*Proof of Corollary 4.* — We localize first the problem and assume that  $X = B^m(0, 1)$ ,  $Y = B^n(0, 1)$  are unit balls respectively in  $\mathbb{C}^m$  and  $\mathbb{C}^n$ . As before, it is sufficient to prove it in the equidimensional case *i.e.*  $X = Y = B^n(0, 1)$ .

As  $f$  has maximal generic rank, we can apply Lemma 13 at each point  $z \in K$ . Now on the compact set  $K$ , the function  $z \mapsto \mu(Jf, z)$  is upper semi continuous, hence bounded above by a constant  $C_K$ . This yields Corollary 4.  $\square$

## BIBLIOGRAPHY

- [D98] DILLER (J.). — *Birational maps, positive currents and dynamics*, preprint, 1998.
- [De93] DEMAILLY (J.-P.). — *Monge-Ampère operators, Lelong numbers and intersection theory, complex analysis and geometry*. — Univ. Ser. Math., Plenum Press, p. 115–193, 1993.
- [H73] HIRONAKA (H.). — *Introduction to real analytic sets and real analytic maps*. — Instituto Matematico “L. Tonelli” dell’ Univ. di Pisa, 1973.
- [Ho83] HÖRMANDER (L.). — *The analysis of linear partial differential operators*, I. — Springer Verlag, 1983.
- [K87] KISELMAN (C.). — *Un nombre de Lelong raffiné*. — Séminaire d’Analyse complexe et Géométrie 1985-1987, Faculté des sciences de Tunis, 1987.
- [K99] KISELMAN (C.). — *Le nombre de Lelong des images inverses des fonctions plurisousharmoniques*, to appear in Bull. Sci. Math., 1999.
- [LG86] LELONG (P.), GRUMAN (L.). — *Entire functions of several complex variables*, Grundlehren der math. Wiss., n° 282, Springer Verlag, 1986.
- [M96] MEO (M.). — *Image inverse d’un courant positif fermé par une application analytique surjective*, C.R. Acad. Sci. Paris, t. **322**, série I, 1996, p. 1141–1144.
- [Si99] SIBONY (N.). — *Dynamique des applications rationnelles de  $\mathbb{P}^k$* , to appear in Panoramas et Synthèses **8**, Soc. math. France, 1999.