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## FORMALITY OF THE FUNCTION SPACE OF FREE MAPS INTO AN ELLIPTIC SPACE

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ABSTRACT. — Let  $X$  be an  $n$ -connected elliptic space and  $Y$  a non rationally contractible, finite-type,  $q$ -dimensional CW complex, where  $q \leq n$ . We show that the function space  $X^Y$  of free maps from  $Y$  into  $X$  is formal if and only if the rational cohomology algebra  $H^*(X; \mathbb{Q})$  is free, that is,  $X$  has the rational homotopy type of a product of odd dimensional spheres.

RÉSUMÉ. — FORMALITÉ DES ESPACES DE FONCTIONS LIBRES DANS UN ESPACE ELLIPTIQUE. — Soient  $X$  un espace elliptique  $n$ -connexe et  $Y$  un CW complexe non rationnellement contractile, de type fini et de dimension  $q \leq n$ . Nous montrons que l'espace  $X^Y$  des fonctions libres de  $Y$  dans  $X$  est formel si et seulement si l'algèbre  $H^*(X, \mathbb{Q})$  est libre, *i.e.*  $X$  a le type d'homotopie rationnelle d'un produit de sphères de dimensions impaires.

### 1. Introduction

D. Sullivan's minimal model  $(\wedge V, d)$  satisfies a nilpotence condition on  $d$ , *i.e.*, there is a well ordered basis  $\{v_i\}_{i \in I}$  of  $V$  such that,  $i < j$  if  $\deg v_i < \deg v_j$  for each  $i, j \in I$  and  $d(v_i) \in \wedge V_{< i}$ . Here  $V_{< i}$  denotes the subspace of  $V$  generated by basis elements  $\{v_j; j \in I, j < i\}$ . According to [9, Def. 1.2],  $(\wedge V, d)$  is called *normal* if  $\text{Ker}[d|_V] = \text{Ker}(d|_V)$  where

$$\text{Ker}[d|_V] := \{v_i \in V; i \in I, d(v_i) \text{ is cohomologous to zero in } (\wedge V_{< i}, d)\}.$$

Let  $F$ ,  $E$  and  $B$  be connected nilpotent spaces and let  $\mathcal{M}(B)$  be a normal minimal model. In this paper, we say that a rational fibration [7, p. 200]

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$F \xrightarrow{i} E \xrightarrow{\pi} B$  is *M.N* if there is a KS-extension:

$$(1.1) \quad \begin{array}{ccccc} \mathcal{M}(B) & \xrightarrow{\text{inclusion}} & (\mathcal{M}(B) \otimes \wedge V, D) & \xrightarrow{\text{projection}} & (\wedge V, \bar{D}) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ A^*(B) & \xrightarrow{\pi^*} & A^*(E) & \xrightarrow{i^*} & A^*(F) \end{array}$$

in which  $(\mathcal{M}(B) \otimes \wedge V, D)$  is minimal (*i.e.*,  $D$  is decomposable) and normal by a suitable change of KS-basis. Here  $A^*(X)$  denotes the rational de-Rham complex of a space  $X$ ,  $\mathcal{M}(F) \cong (\wedge V, \bar{D})$  and “ $\simeq$ ” means quasi-isomorphic, *i.e.*, the map induces an isomorphism in cohomology. We remark that “M.N” is a characteristic of the rational fibration but not of the total space.

Many rational fibrations are M.N. For example, the rational fibration given by a KS-extension:

$$(\wedge(x, y), 0) \longrightarrow (\wedge(x, y, z), D) \longrightarrow (\wedge z, 0)$$

with  $|x| = 3$  (where  $|v|$  means  $\text{deg}(v)$  for  $v \in V$ ),  $|y| = 3$ ,  $|z| = 5$  and  $D(z) = xy$  is M.N. Of course, any rationally trivial fibration is M.N. On the other hand, many rational fibrations are not M.N. For example, in the KS-model of the Hopf fibration  $S^3 \rightarrow S^7 \rightarrow S^4$ , the model of the total space  $(\mathcal{M}(S^4) \otimes \wedge(x_3), D)$  with  $|x_3| = 3$  is not even minimal. The rational fibration given by a KS-extension:

$$(\wedge(x, y), d) \longrightarrow (\wedge(x, y, z), D) \longrightarrow (\wedge z, 0)$$

with  $|x| = 2$ ,  $|y| = 5$ ,  $|z| = 3$ ,  $D(x) = d(x) = 0$ ,  $D(y) = d(y) = x^3$  and  $D(z) = x^2$  is minimal but can not be normal by any change of KS-basis.

In the following, a fibration means a rational fibration. A nilpotent space  $X$  or the minimal model  $\mathcal{M}(X)$  is called (rationally) *formal* if there is a quasi-isomorphism from  $\mathcal{M}(X)$  to  $(H^*(X; \mathbb{Q}), 0)$  (see [3]). The reason we consider M.N-type fibrations is that we can then state a necessary (but perhaps not sufficient) condition for the formality of the total space as in [3, Thm 4.1] when the base space is formal (see Lemma 2.3).

A fibration  $F \rightarrow E \rightarrow B$  is called:

- $\sigma \cdot F$  if it has a rational section;
- *W.H.T* if  $\pi_*(E) \otimes Q = (\pi_*(B) \otimes Q) \oplus (\pi_*(F) \otimes Q)$  for the rational number field  $Q$  and
- *H.T* if it is rationally trivial (see [11]).

The following lemma expresses the relations among these different types of fibrations.

LEMMA 1.1.

1) “ $M.N$ ” is embedded in the sequence of implications:

$$\sigma \cdot F \implies M.N \implies W.H.T,$$

where the reversed implications are false in general.

2) If a fibration  $F \rightarrow E \rightarrow B$  is  $\sigma \cdot F$  and  $E$  is formal, then  $B$  is formal (compare [4, Lemme 2])

Our object of interest is the function space  $X^Y$  of free, continuous maps from a connected space  $Y$  into a connected space  $X$ , endowed with the compact-open topology. Observe that  $X^Y$  is infinite dimensional and is connected if  $X$  is  $n$ -connected and  $Y$  is a  $q$ -dimensional CW-complex, where  $q \leq n$ . Furthermore,  $X^Y$  is the total space of the fibration:

$$(*) \quad (X, *)^{(Y, *)} \longrightarrow X^Y \xrightarrow{\pi} X,$$

where  $(X, *)^{(Y, *)}$  is the function space of pointed maps, and  $\pi$  is the evaluation at the base point. We know that  $(*)$  has a section  $s$ , where  $s(x)$  is the constant map at  $x$ . Therefore  $(*)$  is  $\sigma \cdot F$ . When  $Y = S^1$ , N. Dupont and M. Vigué-Poirrier proved the following formality result.

THEOREM (see [4, Théorème]). — *Let  $X$  be a simply connected space where  $H^*(X; Q)$  is finitely generated. Then  $X^{S^1}$  is formal iff  $H^*(X; Q)$  is free, i.e.,  $X$  has the rational homotopy type of a product of Eilenberg Maclane spaces.*

Our goal in this article is to generalize the theorem of Dupont and Vigué-Poirrier to  $X^Y$ , when  $Y$  is of finite-type, i.e.,  $\pi_i(Y) \otimes Q$  is finite-dimensional for all  $i$ , provided that  $X$  is elliptic, i.e., the total dimensions of  $H^*(X; Q)$  and  $\pi_*(X) \otimes Q$  are finite. More precisely, we prove the following theorem.

THEOREM 1.2. — *Let  $X$  be an  $n$ -connected elliptic space, and let  $Y$  be a non rationally contractible, finite-type,  $q$ -dimensional CW complex, where  $q \leq n$ . Then  $X^Y$  is formal iff  $H^*(X; Q)$  is free, i.e.,  $X$  has the rational homotopy type of a product of odd dimensional spheres.*

In proving Theorem 1.2, we use a model due to Brown and Szczarba [2] for the connected component in  $X^Y$  of a map  $f: Y \rightarrow X$ , which is constructed from minimal models of  $X$ ,  $Y$  and  $f$ . We remark that, under the hypotheses of Theorem 1.2, this *non-formalizing tendency* of  $X^Y$  does not depend on the rational homotopy type of  $Y$ . We cannot easily relax the connectivity hypothesis.

For example, when  $X = \mathbb{C}P^2$  and  $Y = S^3$ , we can see  $X_{(0)}^Y \simeq (\mathbb{C}P^2 \times K(Q, 2))_{(0)}$  by the calculation in [2]. In particular,  $X_{(0)}^Y$  is formal even though  $X$  does not have the rational homotopy type of a product of odd dimensional spheres. Also we must consider each connected component of  $X^Y$  in the general case.

In the following sections, our category is CDGA, that is, the objects are commutative differential graded algebras (cdga) over  $Q$ , and the morphisms are maps of differential graded algebra. Also,  $H^*(\ )$  means  $H^*(\ ; Q)$  and  $I(S)$  denotes the ideal in the algebra  $A$  generated by a basis of a subspace  $S$  in  $A$ . When  $B$  is a subalgebra of  $A$  and both  $A$  and  $B$  contain  $S$ , then  $I(S)$  denotes the ideal in the algebra  $A$  and  $I_B(S)$  the ideal in the algebra  $B$ , unless otherwise noted.

**2. Two changes of KS-basis**

When a cdga  $\mathcal{A}$  is formal, we can choose a minimal model  $\mathcal{M} = (\wedge V, d)$  of  $\mathcal{A}$  such that  $V = \text{Ker}(d|_V) \oplus \text{Ker}(\psi|_V)$  for a quasi-isomorphism  $\psi: \mathcal{M} \rightarrow (H^*(\mathcal{A}), 0)$ . Therefore, according to [3, Thm 4.1],  $\mathcal{A}$  is formal iff there is a complement  $N$  to  $\text{Ker}(d|_V)$ ,  $V = \text{Ker}(d|_V) \oplus N$ , such that any  $d$ -cocycle of  $I(N)$  is  $d$ -exact. We remark this ‘ $\mathcal{M}$ ’ must be a normal minimal model. Conversely, if  $\mathcal{M} = (\wedge V, d)$  is a normal minimal model and formal,  $H^*(\mathcal{M})$  is generated by  $\text{Ker}(d|_V)$  as an algebra (see [9, Lemma 1.8]). Therefore for any quasi-isomorphism  $\psi: \mathcal{M} \rightarrow (H^*(\mathcal{A}), 0)$ , we have  $V = \text{Ker}(d|_V) \oplus \text{Ker}(\psi|_V)$ .

Following [8, p. 5], we use the term “change of KS-basis” in this paper as follows. Suppose that

$$(B^*, d_B) \longrightarrow (B^* \otimes \wedge V, \delta) \longrightarrow (\wedge V, \bar{\delta})$$

is a KS-extension with KS-basis  $\{v_i\}_{i \in I}$ , *i.e.*, a well-ordered basis of  $V$  such that  $i < j$  if  $|v_i| < |v_j|$  for each  $i, j \in I$  and  $\delta(v_i) \in B^* \otimes \wedge V_{<i}$ . Define a map of algebras  $\phi: B^* \otimes \wedge V \rightarrow B^* \otimes \wedge V$  by setting

$$\phi|_B = \text{id}_B \quad \text{and} \quad \phi(v_i) = v_i + \chi_i$$

on basis elements of  $V$ , where  $\chi_i \in B^* \otimes \wedge V_{<i}$  (To be exact, this is different from the definition of “KS-change of basis” of [8, p. 5] since  $\chi_i$  may not be contained in  $B^+ \otimes \wedge V$ .) Finally, define a new differential  $D$  on  $B^* \otimes \wedge V$  by

$$D = \phi^{-1} \circ \delta \circ \phi.$$

Then we have an isomorphism of KS-extensions

$$(2.1) \quad \begin{array}{ccccc} (B^*, d_B) & \xrightarrow{\text{incl.}} & (B^* \otimes \wedge V, D) & \xrightarrow{\text{proj.}} & (\wedge V, \bar{D}) \\ \downarrow = & & \phi \downarrow \cong & & \bar{\phi} \downarrow \cong \\ (B^*, d_B) & \xrightarrow{\text{incl.}} & (B^* \otimes \wedge V, \delta) & \xrightarrow{\text{proj.}} & (\wedge V, \bar{\delta}), \end{array}$$

where  $D|_{B^*} = \delta|_{B^*} = d_B$ .

In this section we introduce two changes of KS-basis. If the fibration (1.1) is  $M.N$ , the normal minimal model  $\mathcal{M}(E) \cong (\mathcal{M}(B) \otimes \wedge V, D)$  is given by a change of KS-basis that we denote  $\phi_1$ , one of two basis changes studied in this section.

*Proof of Lemma 1.1.*

1) The implication  $(\sigma \cdot F \Rightarrow M.N)$  is given in terminology of (1.1) with KS-basis  $\{v_i\}_{i \in I}$  as follows. We know that (1.1) is  $\sigma \cdot F$  iff  $Dv - \bar{D}v \in \mathcal{M}^+(B) \otimes \wedge^+ V$  for  $v \in V$  (see [10, VI.6.(1)]). Therefore the minimality follows. Suppose there are  $\{v_i\}_{i \in J}$  with  $J \subset I$  such that  $Dv_i$  is cohomologous to 0. For  $i \in J$ , we can change KS-basis inductively, as  $\phi_1(v_i) = v_i - \chi_i$  if  $Dv_i = D(\chi_i)$  where  $\chi_i \in B^* \otimes \wedge V_{<i}$  and  $\phi_1(v_i) = v_i$  for  $i \in I - J$ . Put  $\tilde{D} = \phi_1^{-1} \circ D \circ \phi_1$  and then we have  $\tilde{D}(v_i) = 0$  for  $i \in J$ . Thus we have  $\text{Ker}[\tilde{D}|_V] = \text{Ker}(\tilde{D}|_V)$ . We put again  $D = \tilde{D}$ . Since again  $Dv - \bar{D}v \in \mathcal{M}^+(B) \otimes \wedge^+ V$  for  $v \in V$ , we have for  $\mathcal{M}(B) = (\wedge V_B, d_B)$

$$\text{Ker}[D|_{V_B}] = \text{Ker}[d_B|_{V_B}] = \text{Ker}(d_B|_{V_B}) = \text{Ker}(D|_{V_B}).$$

The implication  $(M.N \Rightarrow W.H.T)$  is clear from the decomposability of  $D$ . On the other hand, the first and last examples in Section 1 provide counter-examples to the first and second converses, respectively.

2) From 1), we can assume  $(\wedge(V_B \oplus V), D)$  of (1.1) is minimal and normal. Since  $E$  is formal, there is a complement  $N$  to  $\text{Ker}(D|_{V_B \oplus V})$  in  $V_B \oplus V$  such that any  $D$ -cocycle of  $I(N)$  is  $D$ -exact since  $H^*(E)$  is generated by  $\text{Ker}(D|_{V_B \oplus V})$  (see [3], [9]). Then  $N \cap V_B$  is a complement to  $\text{Ker}(D|_{V_B}) = \text{Ker}(d_B|_{V_B})$  in  $V_B$ . From  $(D - \bar{D})(V) \subset \wedge^+ V_B \otimes \wedge^+ V$  and  $D|_{V_B} = d_B$ , we see any  $d_B$ -cocycle of  $I_B(N \cap V_B)$  is  $d_B$ -exact, where  $I_B(S)$  is the ideal of  $\wedge V_B$  generated by a subset  $S$  of  $V_B$ . The formality of  $B$  follows again from [3, Thm 4.1].  $\square$

As in the proof of Lemma 1.1, part 1), given any KS-extension (1.1), we can change KS-basis

$$\phi_1: (\wedge(V_B \oplus V), \tilde{D}) \cong (\wedge(V_B \oplus V), D)$$

so that  $\text{Ker}[\tilde{D}|_V] = \text{Ker}(D|_V)$ . We put again  $D = \tilde{D}$ .

Next we introduce the second type change of KS-basis, which we denote  $\phi_2$ .

LEMMA 2.1. — *Let  $E$  be a formal space and  $F \rightarrow E \rightarrow B$  a  $W.H.T$  fibration, with KS-model  $(\wedge V_B, d_B) \rightarrow (\wedge(V_B \oplus V), D) \rightarrow (\wedge V, \bar{D})$ . Then, for any complement  $N$  to  $\text{Ker}(D|_V)$  in  $V$ , there is a change KS-basis*

$$\phi_2: (\wedge(V_B \oplus V), \tilde{D}) \cong (\wedge(V_B \oplus V), D)$$

*such that any  $\tilde{D}$ -cocycle of  $I(N)$  is  $\tilde{D}$ -exact.*

*Proof.* — Since the fibration is W.H.T,  $(\Lambda(V_B \oplus V), D)$  is a minimal model of  $E$  with a KS-basis  $\{v_i\}_{i \in I}$  of  $V$ . Let  $\psi: (\Lambda(V_B \oplus V), D) \rightarrow (H^*(E), 0)$  be a quasi-isomorphism and  $K = \text{Ker}(\psi|_V)$  with the sub-KS-basis  $\{v_i\}_{i \in I_1}$ . Then  $K$  is a complement of  $\text{Ker}(D|_V)$  in  $V$  such that any  $D$ -cocycle of  $I(K)$  is  $D$ -exact (see [3], [9]). Let  $\{v_j\}_{j \in I_2}$  be the sub-KS-basis of  $\text{Ker}(D|_V)$  where we assume that  $I$  is indexed by  $i > j$  if  $|v_i| = |v_j|$  for  $i \in I_1$  and  $j \in I_2$ . Here  $I = I_1 \cup I_2$  and  $I_1 \cap I_2 = \emptyset$ . Then we can choose a basis of the given complement  $N$  to  $\text{Ker}(D|_V)$  as  $\{v_i + \sum_{j \in J_i} a_{ij} v_j\}_{i \in I_1}$  with some  $a_{ij} \in Q$ . Here  $J_i = \{j \in I_2; |v_j| = |v_i|\}$ . There is a regular linear transformation  $\phi_2: V \rightarrow V$  given by

$$\phi_2(v_i) = v_i - \sum_{j \in J_i} a_{ij} v_j \text{ for } i \in I_1 \quad \text{and} \quad \phi_2(v_j) = v_j \text{ for } j \in I_2.$$

Extend it to an algebra map

$$\phi_2: \Lambda(V_B \oplus V) \longrightarrow \Lambda(V_B \oplus V)$$

by  $\phi_2|_{V_B} = \text{id}_{V_B}$  and define  $\tilde{D} = \phi_2^{-1} D \phi_2$ . Then  $N$  is also a complement to  $\text{Ker}(\tilde{D}|_V)$  in  $V$  and equals  $\text{Ker}(\psi \phi_2|_V)$ . If an element  $w$  of  $I(N)$  is a  $\tilde{D}$ -cocycle, then  $[w] = \psi \phi_2(w) = 0$  and  $w$  is  $\tilde{D}$ -exact since  $\psi^* \phi_2^*$  is an isomorphism on cohomology and  $\text{Ker}[\tilde{D}|_V] = \text{Ker}(D|_V)$ .  $\square$

**COROLLARY 2.2.** — *Let  $\mathcal{M} = (\Lambda V, d)$  be a normal minimal model. If  $\mathcal{M}$  is formal, for any complement  $N$  to  $\text{Ker}(d|_V)$  there is a change of basis  $(\Lambda V, \tilde{d}) \cong (\Lambda V, d)$  of  $V$  so that any  $\tilde{d}$ -cocycle of  $I(N)$  is  $\tilde{d}$ -exact.*

*Proof.* — It follows by applying Lemma 2.1 when the base space is the one-point space.  $\square$

Let a fibration  $F \rightarrow E \rightarrow B$  be M.N. Let  $\mathcal{M}(B) = (\Lambda V_B, d_B)$  be normal and  $(\Lambda V_B, d_B) \xrightarrow{i} (\Lambda(V_B \oplus V), D)$  a KS-extension, which is normal by a suitable KS-basis change  $\phi_1$ . Let  $B$  be formal. Then there is a quasi-isomorphism  $\rho_B: (\Lambda V_B, d_B) \rightarrow (H^*(B), 0)$  embedded in the commutative diagram:

$$(2.2) \quad \begin{array}{ccc} (\Lambda V_B, d_B) & \xrightarrow{i} & (\Lambda(V_B \oplus V), D) \\ \rho_B \downarrow & & \downarrow \rho \\ (H^*(B), 0) & \xrightarrow{i'} & (H^*(B) \otimes \Lambda V, D'), \end{array}$$

which is push out in CDGA, *i.e.*,  $D'|_V := (\rho_B \otimes 1) \circ D|_V$ . Here  $i$  and  $i'$  are inclusions and  $\rho$  is a quasi-isomorphism since  $\rho_B$  is [1]. There is a complement  $N_B$  to  $\text{Ker}(d_B|_{V_B})$  in  $V_B$  such that any  $d_B$ -cocycle of  $I_B(N_B)$  is  $d_B$ -exact (see [3], [9]). We remark that

$$N_B = \text{Ker}(\rho_B|_{V_B}) = \text{Ker}(\rho|_{V_B \oplus V}).$$

LEMMA 2.3. — *Let  $E$  and  $B$  be formal spaces and  $F \rightarrow E \rightarrow B$  be an  $M.N$  fibration, with  $KS$ -model  $(\wedge V_B, d_B) \rightarrow (\wedge(V_B \oplus V), D) \rightarrow (\wedge V, \bar{D})$ . Then, for any complement  $N$  to  $\text{Ker}(D|_V)$  in  $V$ , there is a change of  $KS$ -basis*

$$\phi_2: (\wedge(V_B \oplus V), \tilde{D}) \cong (\wedge(V_B \oplus V), D)$$

such that any  $\tilde{D}$ -cocycle of  $I(N_B \oplus N)$  is  $\tilde{D}$ -exact.

*Proof.* — Let  $\psi: (\wedge(V_B \oplus V), D) \rightarrow (H^*(E), 0)$  be a quasi-isomorphism and  $\rho: (\wedge(V_B \oplus V), D) \rightarrow (H^*(B) \otimes \wedge V, D')$  a quasi-isomorphism as in (2.2). If  $b \in N_B$ ,  $\rho(b) = 0$  since  $\text{Ker}(\rho_B|_{V_B}) = \text{Ker}(\rho|_{V_B})$ . Then

$$0 = \psi^* \rho^{*-1}[\rho(b)] = [\psi(b)] = \psi(b)$$

in  $H^*(E)$ . Hence  $N_B \subset \text{Ker}(\psi|_{V_B})$ . We can change  $KS$ -basis by some  $\phi_2$  for a given complement  $N$  to  $\text{Ker}(D|_V)$  in  $V$  as in Lemma 2.1, so that  $N = \text{Ker}(\psi\phi_2|_V)$ . Then  $\phi_2|_{V_B} = \text{id}_{V_B}$  and therefore we have

$$\text{Ker}(\psi\phi_2|_{V_B \oplus V}) = \text{Ker}(\psi|_{V_B}) \oplus \text{Ker}(\psi\phi_2|_V) \supset N_B \oplus \text{Ker}(\psi\phi_2|_V) = N_B \oplus N.$$

Thus we have that any  $\tilde{D}$ -cocycle of  $I(N_B \oplus N)$  is  $\tilde{D}$ -exact since  $\psi\phi_2$  is a quasi-isomorphism and

$$\begin{aligned} \text{Ker}[\tilde{D}|_{V_B \oplus V}] &= \text{Ker}[D\phi_2|_{V_B \oplus V}] = \text{Ker}[D|_{V_B \oplus V}] \\ &= \text{Ker}(D|_{V_B \oplus V}) = \text{Ker}(D\phi_2|_{V_B \oplus V}) = \text{Ker}(\tilde{D}|_{V_B \oplus V}). \quad \square \end{aligned}$$

### 3. Proof of Theorem 1.2

We begin this section by recalling the construction of the model of  $X^Y$  due to Brown and Szczarba [2]. Let  $(\wedge V, d)$  a free cdga and  $(B, d_B)$  a finite-type cdga. Let  $(B_*, d_*)$  be the differential graded coalgebra with  $B_q = \text{Hom}(B^{-q}, Q)$ . The differential  $d_*$  on  $B_*$  is the dual of  $d_B$  and the coproduct  $\partial: B_* \rightarrow B_* \otimes B_*$  is the dual of multiplication. Let  $\wedge(\wedge V \otimes B_*)$  be the free cdga generated by the vector space  $\wedge V \otimes B_*$  with the differential induced by the tensor product differential  $\tilde{d}$  on  $\wedge V \otimes B_*$ , and let  $I$  be the ideal in  $\wedge(\wedge V \otimes B_*)$  generated by  $1 \otimes 1 - 1$  and by the all elements of the form

$$v_1 v_2 \otimes \beta - \sum_i (-1)^{|v_2| \cdot |\beta_i|} (v_1 \otimes \beta_i)(v_2 \otimes \beta'_i)$$

with  $v_1, v_2 \in \wedge V$ ,  $\beta_i, \beta'_i \in B_*$  and  $\partial\beta = \sum_i \beta_i \otimes \beta'_i$ . Then there is a natural isomorphism

$$\kappa: \wedge(V \otimes B_*) \cong \wedge(\wedge V \otimes B_*)/I$$



as graded algebras, induced by the inclusion  $V \otimes B_* \rightarrow \Lambda V \otimes B_*$  (see [2, Thm 3.5]). Note that  $\tilde{d}(I) \subset I$ . Define  $\delta$  on  $\Lambda(V \otimes B_*)$  by  $\delta = \kappa^{-1} \tilde{d} \kappa$ . For example, if  $dv = v_1 v_2$  where  $v_1, v_2 \in V$  and  $\partial\beta = \sum_i \beta_i \otimes \beta'_i$  (see [2, p. 6]),

$$\delta(v \otimes \beta) = \sum_i (-1)^{|v_2| |\beta_i|} (v_1 \otimes \beta_i) \cdot (v_2 \otimes \beta'_i) + (-1)^{|v|} v \otimes d_*(\beta).$$

In the following, we suppose that  $X$  is  $n$ -connected and finite-type, with  $\mathcal{M}(X) = (\Lambda V, d)$  (so that  $V = \bigoplus_{i>n} V^i$ ), and  $Y$  is a non rationally contractible, finite-type,  $q$ -dimensional CW complex, where  $q \leq n$ . Let  $\mathcal{M}(Y) = (\Lambda V_Y, d_Y)$  and  $\mathcal{M}_*(Y) = \text{Hom}(\mathcal{M}(Y), Q)$ . Then

$$(\Lambda(\Lambda V \otimes \mathcal{M}_*(Y)), \tilde{d})/I \cong (\Lambda(V \otimes \mathcal{M}_*(Y)), \delta) \cong (\Lambda W, \delta) \otimes \mathcal{C} \simeq (\Lambda W, \delta),$$

where  $W \subset V \otimes \mathcal{M}_*(Y)$  with

$$W \equiv V \otimes \{\text{the cohomology classes of } d_Y\text{-cocycles}\}_* \cong V \otimes H_*(Y)$$

as vector spaces and  $\mathcal{C}$  a contractible cdga. Here a basis of  $W$  is inductively constructed so that  $V \otimes y_* \subset W$  for  $y \in V_Y$  with  $d_Y(y) = 0$  and  $\delta(W) \subset \Lambda W$  (see [2]). According to [2, Thm 1.5], the minimal model of  $X^Y$  is given by  $\mathcal{M}(X^Y) \cong (\Lambda W, \delta)$ .

Write  $W = V \oplus W_+$ , where  $W_+ := W \cap (V \otimes \mathcal{M}_+(Y))$ . Then a KS-model of the fibration (\*) (see Section 1) for a normal minimal model  $\mathcal{M}(X) = (\Lambda V, d)$  is given as

$$(**) \quad (\Lambda V, d) \longrightarrow (\Lambda V \otimes \Lambda W_+, \delta) \longrightarrow (\Lambda W_+, \bar{\delta}),$$

where  $(\Lambda W, \delta) = (\Lambda V \otimes \Lambda W_+, \delta)$  is minimal but may not be normal. Since (\*) has a section, it is M.N by Lemma 1.1, part 1). Then there is a KS-basis change of (\*\*) that can be given as follows:

$$(3.1) \quad \begin{array}{ccccc} (\Lambda V, d) & \xrightarrow{i} & (\Lambda V \otimes \Lambda W_+, D) & \longrightarrow & (\Lambda W_+, \bar{D}) \\ & \downarrow = & \phi_1 \downarrow \cong & & \bar{\phi}_1 \downarrow \cong \\ (\Lambda V, d) & \longrightarrow & (\Lambda V \otimes \Lambda W_+, \delta) & \longrightarrow & (\Lambda W_+, \bar{\delta}), \end{array}$$

where  $(\Lambda W, D) = (\Lambda V \otimes \Lambda W_+, D)$  is normal with  $D = \phi_1^{-1} \delta \phi_1$ .

In the following, we suppose that  $X$  is elliptic (i.e.,  $\dim_Q V < \infty$ ) and  $X^Y$  is formal, which implies that  $X$  is formal by Lemma 1.1, part 2). If  $X$  is elliptic, it is known that  $H^*(X)$  is a Poincaré algebra (see [6]). Furthermore, if  $X$  is elliptic

and formal, it is known that  $(\wedge V, d)$  is two stage, i.e.,  $V = V_0 \oplus V_1$  with  $dV_0 = 0$  and  $dV_1 \subset \wedge V_0$  (see [5]). Therefore we can put

$$W_{(0)} = \{V_0 \otimes \mathcal{M}_*(Y)\} \cap W \quad \text{and} \quad W_{(1)} = \{V_1 \otimes \mathcal{M}_*(Y)\} \cap W.$$

Then  $W = W_{(0)} \oplus W_{(1)}$ ,  $V_0 \subset W_{(0)}$ ,  $V_1 \subset W_{(1)}$ ,  $\delta W_{(0)} = 0$ , and  $\delta W_{(1)} \subset \wedge W_{(0)}$ . Then  $\phi_1|_{\text{Ker}(\delta|_W)} = \text{id}_{\text{Ker}(\delta|_W)}$  and especially  $\phi_1|_{W_{(0)}} = \text{id}_{W_{(0)}}$ .

For the quasi-isomorphism  $\rho_X: (\wedge V, d) \rightarrow (H^*(X), 0)$  with  $\text{Ker}(\rho_X|_V) = V_1$ , there are the push outs:

$$(3.2) \quad \begin{array}{ccc} (\wedge V, d) & \longrightarrow & (\wedge V \otimes \wedge W_+, \delta) \\ \downarrow \rho_X & & \downarrow \eta \\ (H^*(X), 0) & \longrightarrow & (H^*(X) \otimes \wedge W_+, \delta') \end{array}$$

and for the KS-basis change  $\phi_2: (\wedge V \otimes \wedge W_+, \tilde{D}) \cong (\wedge V \otimes \wedge W_+, D)$  corresponding to a certain complement  $N$  to  $\text{Ker}(D|_V)$  in  $V$  as in Lemma 2.1,

$$(3.3) \quad \begin{array}{ccc} (\wedge V, d) & \xrightarrow{i} & (\wedge V \otimes \wedge W_+, \tilde{D}) \\ \downarrow \rho_X & & \downarrow \tilde{\rho} \\ (H^*(X), 0) & \xrightarrow{i'} & (H^*(X) \otimes \wedge W_+, \tilde{D}'), \end{array}$$

where  $\rho_X$ ,  $\eta$  and  $\tilde{\rho}$  are quasi-isomorphisms.

CLAIM. —  $(H^*(X) \otimes \wedge W_+, \tilde{D}') \cong (H^*(X) \otimes \wedge W_+, \delta')$  as *cdgas*.

*Proof of Claim.* — Since (3.3) is a push out, there is a map  $(\phi_1\phi_2)'$  such that the following commutes:

$$(3.4) \quad \begin{array}{ccc} (\wedge V \otimes \wedge W_+, \tilde{D}) & \xrightarrow{\phi_1\phi_2} & (\wedge V \otimes \wedge W_+, \delta) \\ \downarrow \tilde{\rho} & & \downarrow \eta \\ (H^*(X) \otimes \wedge W_+, \tilde{D}') & \xrightarrow{(\phi_1\phi_2)'} & (H^*(X) \otimes \wedge W_+, \delta'). \end{array}$$

On the other hand, since (3.2) is a push out, there is a map  $(\phi_2^{-1}\phi_1^{-1})'$  such that the following commutes:

$$(3.5) \quad \begin{array}{ccc} (\wedge V \otimes \wedge W_+, \delta) & \xrightarrow{\phi_2^{-1}\phi_1^{-1}} & (\wedge V \otimes \wedge W_+, \tilde{D}) \\ \downarrow \eta & & \downarrow \tilde{\rho} \\ (H^*(X) \otimes \wedge W_+, \delta') & \xrightarrow{(\phi_2^{-1}\phi_1^{-1})'} & (H^*(X) \otimes \wedge W_+, \tilde{D}'). \end{array}$$

Then  $(\phi_2^{-1}\phi_1^{-1})' \circ (\phi_1\phi_2)' = \text{id}$  and  $(\phi_1\phi_2)' \circ (\phi_2^{-1}\phi_1^{-1})' = \text{id}$  by universality. Hence  $(\phi_1\phi_2)'$  is an isomorphism in (3.4).  $\square$

*Proof of Theorem 1.2.* — The *if* part is obvious since  $\delta = 0$  if  $d = 0$ . The *only if* part is shown as follows. Suppose  $V_1 \neq 0$ . Let  $v = v_i$  be a non-zero basis element in a basis  $\{v_j\}_{j \in I}$  of  $V_1$ , where  $i = \max\{j \in I; |v_j| = |v|\}$ , and let  $y$  be a non-zero basis element of  $V_Y$  with  $d_Y(y) = 0$  for  $\mathcal{M}(Y) = (\wedge V_Y, d_Y)$ . Such an element  $y$  surely exists since  $Y$  is not rationally contractible and since  $V_Y$  has a well-ordered basis  $\{y_i\}_i$  such that  $d_Y(y_i) \in \wedge(V_{Y < i})$ . Then we can regard  $v \otimes y_*$  as a basis element of  $W_{(1)}$  with the index  $k$  of the basis for some  $k$  and  $\{v_j \otimes y_*\}_{j < i} \subset W_{< k}$  from the construction of  $W$ .

Suppose  $\delta(v \otimes y_*) = \delta(\chi)$  for some  $\chi \in \wedge W_{< k}$ . We can uniquely write

$$\chi = \sum_{j < i} \theta_j(v_j \otimes y_*) + \mu$$

for  $\theta_j \in \wedge V_{< i}$  and  $\mu \notin \wedge V \otimes (V \otimes y_*)$ . Then

$$0 = \delta(v \otimes y_*) - \delta\chi = \kappa^{-1} \tilde{d}\kappa((v - \theta) \otimes y_*) = \kappa^{-1}(d(v - \theta) \otimes y_*)$$

for  $\theta = \sum_{j < i} \theta_j v_j$  and  $\delta(\mu) = 0$  since

(a)  $\tilde{d}(V \otimes z_*) \subset \wedge V \cdot (V \otimes z_*)$  for any  $z \in V_Y$ , since  $d_*(z_*) = z_* \circ d_Y = 0$  due to the decomposability of  $d_Y$ , and

(b)  $\tilde{d}(V \otimes z_*) \subset \wedge V \cdot (V \otimes (\wedge^{>1} V_Y)_*) \oplus \wedge V \cdot \wedge^{>1}(V \otimes \mathcal{M}_+(Y))$  for any  $z \in \wedge^{>1} V_Y$ .

Since the derivation  $( ) \otimes y_* : \wedge V \rightarrow \wedge W$  is injective,  $d(v) = d(\theta)$  for  $\theta \in \wedge V_{< i}$ , which contradicts the normality of  $(\wedge V, d)$ . Thus  $\delta(v \otimes y_*)$  is not cohomologous to zero.

We see therefore

$$\phi_1(v \otimes y_*) = v \otimes y_*$$

in (3.1) from the definition of change of KS-basis in the proof of Lemma 1.1 (1). Also

$$D(v \otimes y_*) = \phi_1^{-1} \delta \phi_1(v \otimes y_*) = \phi_1^{-1} \delta(v \otimes y_*) = \delta(v \otimes y_*) \neq 0$$

since  $\delta(W_{(1)}) \subset \wedge W_{(0)}$  and  $\phi_1|_{W_{(0)}} = \text{id}_{W_{(0)}}$ . Hence  $v \otimes y_* \notin \text{Ker}(D|_W)$ . Then, from Lemma 2.1, we can change KS-basis  $\phi_2 : (\wedge W, \tilde{D}) \cong (\wedge W, D)$ , so that any  $\tilde{D}$ -cocycle of  $I(N)$  is  $\tilde{D}$ -exact, for some subspace  $N$  of  $W_{(1)} \cap W_+$  with  $v \otimes y_* \in N$ . We fix a particular  $N$ .

Let  $[w]$  be the fundamental class of  $H^*(X)$ . Then  $[w] \cdot (v \otimes y_*)$  is a  $\delta'$ -cocycle of  $H^*(X) \otimes \wedge W_+$ . In fact, if  $dv = \sum_i a_i v_{i_1} \cdots v_{i_{n_i}}$  for  $v_i \in V_0$  and  $a_i \in Q$ ,

$$\begin{aligned} \delta'([w] \cdot (v \otimes y_*)) &= [w] \cdot (\rho_X \otimes 1) \delta(v \otimes y_*) \\ &= \sum_i \sum_{1 \leq j \leq n_i} \pm a_i [wv_{i_1} \cdots \hat{v}_{i_j} \cdots v_{i_{n_i}}] \cdot (v_{i_j} \otimes y_*) \end{aligned}$$

must be zero since the degree of  $wv_{i_1} \cdots \hat{v}_{i_j} \cdots v_{i_{n_i}}$  is always greater than the formal dimension of  $X$ .

Let  $\phi_2(v \otimes y_*) = v \otimes y_* + c$  with  $c$  a  $D$ -cocycle. Since  $\phi_1|_{W_{(0)}} = \text{id}_{W_{(0)}}$ , we obtain

$$0 = Dc = \phi_1^{-1} \delta \phi_1(c) = \delta \phi_1(c),$$

i.e.,  $\phi_1(c)$  is a  $\delta$ -cocycle. Then  $[w](v \otimes y_* + \phi_1(c))$  is a  $\delta'$ -cocycle but cannot be  $\delta'$ -exact since  $v \otimes y_* + \phi_1(c)$  contains a non-zero element of  $W_{(1)} \cap W_+$  due to the definition of change of KS-basis and since

- (a)  $\delta'(W_+) \subset H^*(X) \otimes (W_{(0)} \cap W_+)$ , and
- (b)  $\delta'(\Lambda^{>1}W_+) \subset H^*(X) \otimes \Lambda^{>1}W_+$ .

Then, according to the Claim above,  $[w](v \otimes y_*)$  is a non-exact  $\tilde{D}'$ -cocycle, since

$$(\phi_1 \phi_2)'([w](v \otimes y_*)) = [w](\phi_1(v \otimes y_*) + \phi_1(c)) = [w]((v \otimes y_*) + \phi_1(c))$$

in (3.4). Since  $\tilde{\rho}$  is a quasi-isomorphism in (3.3), there exists a non-exact  $\tilde{D}$ -cocycle  $w \cdot (v \otimes y_*) + \xi$  in  $\Lambda W$ , such that  $\tilde{\rho}(\xi) = 0$ . Since

$$\text{Ker}(\tilde{\rho}|_{V \oplus W_+}) = \text{Ker}(\rho_X|_{V \oplus W_+}) = V_1,$$

we obtain that  $\xi \in I(V_1)$  and thus  $w \cdot (v \otimes y_*) + \xi \in I(V_1 \oplus N)$ . This contradicts Lemma 2.3. Hence  $V_1 = 0$ .

Since  $\dim_Q H^*(X) < \infty$ , this means  $V_0 = V_0^{\text{odd}}$  and  $H^*(X) = \Lambda(V_0^{\text{odd}})$ , i.e.,  $X$  has the rational homotopy type of a product of odd dimensional spheres if  $V_0 \neq 0$  and is rationally contractible if  $V_0 = 0$ .  $\square$

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