ÉTALE COHOMOLOGY AND REDUCTION OF ABELIAN VARIETIES

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Abstract. — In this paper we study the étale cohomology groups associated to abelian varieties. We obtain necessary and sufficient conditions for an abelian variety to have semistable reduction (or purely additive reduction which becomes semistable over a quadratic extension) in terms of the action of the absolute inertia group on the étale cohomology groups with finite coefficients.

Résumé (*Cohomologie étale et réduction des variétés abéliennes*)
   Nous étudions les groupes de cohomologies étales associés aux variétés abéliennes. Nous obtenons des conditions nécessaires et suffisantes pour qu'une variété abeliénne ait une réduction semistable (ou une réduction purement additive qui devient semistable sur une extension quadratique), en termes de l'action du groupe d'inertie absolu sur les groupes de cohomologies étales à coefficients finis.

1. Introduction

Suppose X is a smooth projective variety over a field F, v is a discrete valuation on F, and ℓ is a prime number not equal to the residue characteristic of v. Let F^s denote a separable closure of F, let $\bar{v}$ be an extension of v to F^s,

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let I denote the inertia subgroup at $\bar{v}$ of $\mathrm{Gal}(F^s/F)$, and let $\overline{X} = X \times_F F^s$. For every positive integer k, the group I acts naturally on the k-th ℓ-adic étale cohomology group $H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$. Grothendieck proved the Monodromy Theorem (see the Appendix to [10], and 1.2 and 1.3 of [4]), which says that I acts on $H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ via quasi-unipotent operators, $i.e.$, for every $\sigma \in I$ we have $(\sigma^m - 1)^r H^k_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) = 0$ for some positive integers m and r. It is known (see 3.7 of [4], and 3.5 and 3.6 of [5]) that if $k = 1$, then one may take $r = 2$. It easily follows (see Theorem 5.6 (i), (ii) below) that if X is an abelian variety, then one may take $r = k + 1$. It is shown in [6] (see 3.4 and 3.8 of [4], and p. vi of [3]) that one may take $r = k + 1$ whenever one knows the Purity Conjecture (3.1 of [4]) and resolution of singularities.

From now on, suppose X is a d-dimensional abelian variety. The Néron model $\mathcal{X}$ of X at v is a smooth separated model of X over the valuation ring R such that for every smooth scheme $\mathcal{Y}$ over R and morphism $\varphi : \mathcal{Y} \otimes_R F \to X$ over F there is a unique morphism $\mathcal{Y} \to \mathcal{X}$ over R which extends φ. The generic fiber of $\mathcal{X}$ can be canonically identified with X, and $\mathcal{X}$ is a commutative group scheme over R whose group structure extends that of X. Let X_v^0 denote the identity component of the special fiber of $\mathcal{X}$ at v. Over an algebraic closure of the residue field, there is an exact sequence of algebraic groups

$$0 \to U \times \mathcal{T} \longrightarrow X_v^0 \longrightarrow B \to 0,$$

where B is an abelian variety, $\mathcal{T}$ is the maximal algebraic torus in X_v^0, and U is a unipotent group. By definition, X is $semistable$ at v if and only if $U = 0$. As I-modules, $H^1_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell) \cong \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(X), \mathbb{Z}_\ell)$, where $T_\ell(X)$ is the ℓ-adic Tate module. Grothendieck's Galois Criterion for Semistability says that X is semistable at v if and only if every $\sigma \in I$ acts on $T_\ell(X)$ as a unipotent operator of echelon ≤ 2, $i.e.$, if and only if $(\sigma - 1)^2 H^1_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell) = 0$ for every $\sigma \in I$.

Suppose n is a positive integer relatively prime to the residue characteristic of v. Raynaud's criterion says that if I acts trivially on the n-torsion X_n, and $n \geq 3$, then X is semistable at v. The authors (see [13] and [15]) proved that if $n \geq 5$, then X is semistable at v if and only if $(\sigma - 1)^2 X_n = 0$ for every $\sigma \in I$. In other words, necessary and sufficient conditions for semistability can be read off not only from the ℓ-adic representation, as shown by Grothendieck, but also from the mod n representation (for $n \geq 5$). The aim of this paper (see Theorem 5.10) is to generalize this result to the case of the higher étale cohomology groups H^k. Note that $H^1_{\text{ét}}(\overline{X}, \mathbb{Z}/n\mathbb{Z})$ and X_n are dual. Assume that $0 < k < 2d$, that $k < r \in \mathbb{Z}$, and that n does not belong to a certain finite set $N(r)$ of prime powers, defined explicitly in terms of r in § 2. (For example, $N(2) = \{1, 2, 3, 4\}$.) We show that if k is odd, then X is semistable at v if and only if $(\sigma - 1)^r H^k_{\text{ét}}(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = 0$ for every $\sigma \in I$. If k is even, we show (under an additional assumption; see § 7) that $(\sigma - 1)^r H^k_{\text{ét}}(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = 0$ for every $\sigma \in I$ if and only if either X is semistable at v or X has purely additive

reduction at v but is semistable over a (ramified) quadratic extension of F. We call the latter abelian varieties briefly unstable (see Definitions 5.2 and 5.3).

We gave such necessary and sufficient conditions for semistability in the case $k = 1$ and $r = 2$ in [13] and [15]. Poincaré duality then gives such criteria in the case $k = 2d - 1$. One can check that in the cases $k = 1$ or $2d - 1$, one cannot replace $N(r)$ by a smaller set. However, when $2 \leq k \leq 2d - 2$ one may replace $N(r)$ by an explicitly defined subset $N'(r)$ for which the result is sharp (see § 6).

In § 2 we introduce basic definitions and notation. Section 3 deals with multilinear algebra in characteristics 0 and ℓ and over $\mathbb{Z}_\ell$. We use the Jordan decompositions of exterior powers of linear operators to obtain a Minkowski-Serre type result. Section 4 contains some abelian variety results that will be used later. In § § 5–6 we state and prove our main results. We give necessary and sufficient conditions for semistability, and also necessary and sufficient conditions for an abelian variety to either be semistable or have purely additive reduction which becomes semistable over a quadratic extension. In § 6 we shrink the exceptional set in the criteria when $2 \leq k \leq 2d - 2$. We prove that this exceptional set is minimal.

We hope that our results and/or methods will be useful in the study of semistability for the more general class of motives [9].

2. Notation and definitions

If F is a field, let F^s denote a separable closure. Throughout this paper, X is a d-dimensional abelian variety defined over F, and v is a discrete valuation on F of residue characteristic $p \geq 0$. Let $\bar{v}$ denote an extension of v to F^s, and let I denote the inertia subgroup at $\bar{v}$ of $\mathrm{Gal}(F^s/F)$. If ℓ is a prime not equal to the characteristic of F, let

$$\rho_{\ell,X} : \mathrm{Gal}(F^s/F) \longrightarrow \mathrm{Aut}\big(T_\ell(X)\big) \cong \mathrm{GL}_{2d}(\mathbb{Z}_\ell)$$

denote the ℓ-adic representation on the Tate module $T_\ell(X) = \varprojlim X_{\ell^n}$. Let $V_\ell(X) = T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, let $\overline{X} = X \times_F F^s$, and let

$$H_\ell^k = H_{\text{ét}}^k(\overline{X}, \mathbb{Z}_\ell) = \mathrm{Hom}_{\mathbb{Z}_\ell}\big(\wedge^k(T_\ell(X)), \mathbb{Z}_\ell\big).$$

Then H_ℓ^k is a free $\mathbb{Z}_\ell$-module of rank $\binom{2d}{k} =: b$, and $H_\ell^k \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = \wedge_{\mathbb{Q}_\ell}^k(V^*)$, where $V^* = \mathrm{Hom}_{\mathbb{Q}_\ell}(V, \mathbb{Q}_\ell)$ is the dual vector space. Let

$$\rho_{\ell,k} : I \longrightarrow \mathrm{Aut}(H_\ell^k) \cong \mathrm{GL}_b(\mathbb{Z}_\ell)$$

be the representation giving the action of I on H_ℓ^k.

DEFINITION 2.1. — If r is a positive integer, define a finite set of prime powers $N(r)$ by

$$N(r) = \{\text{prime powers } \ell^m \mid 0 \le m(\ell - 1) \le r\}.$$

For example, $N(1) = \{1, 2\}$, $N(2) = \{1, 2, 3, 4\}$, $N(3) = \{1, 2, 3, 4, 8\}$, $N(4) = \{1, 2, 3, 4, 5, 8, 9, 16\}$.

3. Some linear algebra

LEMMA 3.1. — *Suppose g is a linear operator on a finite-dimensional vector space over a field of characteristic zero. Suppose $(g^m - 1)^2 = 0$ for some positive integer m. If g is unipotent, then $(g - 1)^2 = 0$. If $-g$ is unipotent, then $(g + 1)^2 = 0$.*

Proof. — If g is unipotent then all the eigenvalues of $g^{m-1} + \cdots + g + 1$ are $m \neq 0$. Therefore $g^{m-1} + \cdots + g + 1$ is an invertible operator. Since

$$0 = (g^m - 1)^2 = (g - 1)^2(g^{m-1} + \cdots + g + 1)^2$$

we have $(g - 1)^2 = 0$.

If $-g$ is unipotent then all the eigenvalues of g^m are $(-1)^m$. Since g^m is unipotent, $m = 2r$ is even. We have

$$0 = (g^m - 1)^2 = (g^{2r} - 1)^2 = (g + 1)^2(g - 1)^2(g^{2(r-1)} + \cdots + g^2 + 1)^2.$$

All the eigenvalues of $g - 1$ are -2 and all the eigenvalues of $g^{2(r-1)} + \cdots + g^2 + 1$ are $r \neq 0$. Therefore $g - 1$ and $g^{2(r-1)} + \cdots + g^2 + 1$ are invertible, and $(g + 1)^2 = 0$. $\square$

THEOREM 3.2. — *Suppose m, r and g are positive integers, ℓ is a prime number, and $A \in \mathrm{M}_g(\mathbb{Z}_\ell)$ satisfies $(A - 1)^r \in \ell^m \mathrm{M}_g(\mathbb{Z}_\ell)$. Suppose λ is an eigenvalue of A which is a root of unity. If $r = m(\ell - 1)$ then $\lambda^\ell = 1$. If $r < m(\ell - 1)$ then $\lambda = 1$.*

Proof. — This is a special case of Theorem 6.7 of [12]. $\square$

THEOREM 3.3. — *Suppose $\bar{\mathbb{Z}}$ is the ring of algebraic integers in $\bar{\mathbb{Q}}$, n and r are positive integers, λ is a root of unity in $\bar{\mathbb{Z}}$, and $(\lambda - 1)^r \in n\bar{\mathbb{Z}}$. Then $\lambda = 1$ if $n \notin N(r)$.*

Proof. — This is a special case of Corollary 3.3 of [12]. $\square$

LEMMA 3.4 (see Lemma 4.3 of [13]). — *Suppose $\mathcal{O}$ is an integral domain of characteristic zero, and ℓ is a prime number. Suppose r, s, and m are positive integers such that $r \geq m\ell^{s-1}(\ell - 1)$. Suppose $\alpha \in \mathcal{O}$ and $\alpha^{\ell^s} = 1$. Then $(\alpha - 1)^r \in \ell^m \mathbb{Z}[\alpha]$.*

Suppose V is a finite-dimensional vector space over $\mathbb{Q}_\ell$. Suppose k is an integer and $0 < k < \dim(V)$. Let

$$f_k : \mathrm{GL}(V) \longrightarrow \mathrm{GL}\big(\wedge^k(V)\big)$$

denote the natural representation defined by $f_k(g) = \wedge^k(g)$. For $g \in \mathrm{GL}(V)$, let $g = s_g u_g = u_g s_g$ be the Jordan decomposition, where $u_g, s_g \in \mathrm{GL}(V)$, u_g is unipotent, and s_g is semisimple. Then $f_k(g) = f_k(s_g)f_k(u_g) = f_k(u_g)f_k(s_g)$, $f_k(u_g)$ is unipotent, and $f_k(s_g)$ is semisimple.

REMARK 3.5. — In the notation of §2, with "$*$" denoting the dual map, for every $\sigma \in I$ we have

$$\rho_{\ell,k}(\sigma) = \big(f_k(\rho_{\ell,X}(\sigma^{-1}))\big)^*.$$

REMARK 3.6. — The kernel of f_k is $\{\gamma \in \mathbb{Z}_\ell \mid \gamma^k = 1\}$. In particular, if $(k, \ell - 1) = 1$ then f_k is injective.

LEMMA 3.7. — *The element $\wedge^k(g)$ is unipotent if and only if there exists a k-th root of unity $\gamma \in \mathbb{Z}_\ell$ such that γg is unipotent.*

Proof. — If $\wedge^k(g)$ is unipotent, then $\wedge^k(s_g) = 1$. By Remark 3.6, s_g is a k-th root of unity in $\mathbb{Z}_\ell$. Let $\gamma = s_g^{-1}$. □

Now assume that V is even-dimensional, choose a non-degenerate alternating bilinear form on V, and let $\mathrm{Sp}(V) \subset \mathrm{GL}(V)$ be the corresponding symplectic group. Write

$$\rho_k : \mathrm{Sp}(V) \longrightarrow \mathrm{GL}(\wedge^k(V))$$

for the restriction of f_k to $\mathrm{Sp}(V)$. It is well-known that $u_g, s_g \in \mathrm{Sp}(V)$ for every $g \in \mathrm{Sp}(V)$.

REMARK 3.8. — Remark 3.6 easily implies that the kernel of ρ_k is $\{1\}$ if k is odd and is $\{1, -1\}$ if k is even.

LEMMA 3.9. — *Suppose that $g \in \mathrm{Sp}(V)$.*

  (i) *If $(g - 1)^2 = 0$, then $(\rho_k(g) - 1)^{k+1} = 0$.*
  (ii) *If k is even and $(g + 1)^2 = 0$, then $(\rho_k(g) - 1)^{k+1} = 0$.*
  (iii) *If γg is unipotent for some $\gamma \in \mathbb{Q}_\ell$, then $\gamma \in \{\pm 1\}$.*
  (iv) *Suppose $\wedge^k(g)$ is unipotent. If k is odd then g is unipotent. If k is even then either g or $-g$ is unipotent.*

Proof. — Note that

$$(\rho_k(g) - 1)(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = gv_1 \wedge \cdots \wedge gv_k - v_1 \wedge v_2 \cdots \wedge v_k.$$

Part (i) follows by substituting $g = 1 + \eta$, and (ii) follows from (i) applied to $-g$. Suppose now that g is not unipotent. Then g has an eigenvalue $\lambda \neq 1$. Since $g \in \mathrm{Sp}(V)$, λ^{-1} is also an eigenvalue of g. Suppose also that γg is unipotent for some $\gamma \in \mathbb{Q}_\ell$. Then $\gamma\lambda$ and γ/λ are eigenvalues of the unipotent element γg and thus are equal to 1. Therefore $\gamma^2 = 1$, *i.e.*, $\gamma \in \{\pm 1\}$, so either g or $-g$ is unipotent. If $\wedge^k(g)$ is unipotent, then γg is unipotent for some $\gamma \in \mathbb{Q}_\ell$, by Lemma 3.7. If k is odd then $\wedge^k(-g) = -\wedge^k(g)$ is not unipotent, so $-g$ is not unipotent and thus g is unipotent.   □

Theorem 3.12 below will be used in § 6. To prove it, we first prove a lemma and a theorem.

LEMMA 3.10. — *Suppose $\mathcal{W}$ is an $(\ell-1)$-dimensional vector space over a field of prime characteristic $\ell \geq 5$, and A is a unipotent linear operator on $\mathcal{W}$ whose Jordan form consists of one Jordan block of size $\ell-1$. If $r \in \mathbb{Z}$ and $2 \leq r \leq \ell-3$, then $(\wedge^r(A) - 1)^{\ell-1} \neq 0$.*

Proof. — It follows immediately from Corollary III.2.7(a) on p. 43 and Proposition III.2.10(b) on p. 45 of [1] that all but one of the Jordan blocks of $\wedge^r(A)$ have size ℓ, and the size of the remaining block is 1 or $\ell - 1$ (since the size is less than ℓ and is congruent mod ℓ to $\dim(\wedge^r(\mathcal{W})) = \binom{\ell-1}{r} \equiv (-1)^r$). Since $\ell \geq 5$ and $2 \leq r \leq \ell - 3$, we have $\dim(\wedge^r(\mathcal{W})) \geq \ell$. Therefore, $\wedge^r(A)$ must have a Jordan block of size ℓ, so $(\wedge^r(A) - 1)^{\ell-1} \neq 0$.   □

THEOREM 3.11. — *Suppose $\mathcal{V}$ is a finite-dimensional vector space over a field of characteristic $\ell \geq 5$, and A is a unipotent linear operator on $\mathcal{V}$. Suppose $k \in \mathbb{Z}$, $2 \leq k \leq \dim(\mathcal{V}) - 2$, and $(A - 1)^{\ell-2} \neq 0$. Then $(\wedge^k(A) - 1)^{\ell-1} \neq 0$.*

Proof. — Let $\mathcal{W}$ be an A-invariant $(\ell-1)$-dimensional subspace of $\mathcal{V}$ such that the Jordan form of the restriction of A to $\mathcal{W}$ is a Jordan block of size $\ell - 1$. Then $\wedge^j(\mathcal{W})$ is a $\wedge^j(A)$-invariant subspace of $\wedge^j(\mathcal{V})$, for all j. By Lemma 3.10,

$$\left(\wedge^j(A) - 1\right)^{\ell-1}\left(\wedge^j(\mathcal{W})\right) \neq 0$$

if $2 \leq j \leq \ell - 3$. We are therefore done if $k \leq \ell - 3$. Suppose $k > \ell - 3$. Then $\wedge^{k-(\ell-3)}(\mathcal{V}/\mathcal{W}) \neq 0$. Since A is unipotent, $\wedge^{k-(\ell-3)}(A)$ is also unipotent, and there exists a non-zero $\wedge^{k-(\ell-3)}(A)$-invariant element $u \in \wedge^{k-(\ell-3)}(\mathcal{V}/\mathcal{W})$. Then

$$0 \neq \left(\wedge^{\ell-3}(A) - 1\right)^{\ell-1}\left(\wedge^{\ell-3}(\mathcal{W})\right) \cong \left(\wedge^{\ell-3}(A) - 1\right)^{\ell-1}(\wedge^{\ell-3}(\mathcal{W})) \otimes u$$
$$= (\wedge^k(A) - 1)^{\ell-1}(\wedge^{\ell-3}(\mathcal{W}) \otimes u)$$
$$\subset \left(\wedge^k(A) - 1\right)^{\ell-1}(\wedge^{\ell-3}(\mathcal{W}) \otimes \wedge^{k-(\ell-3)}(\mathcal{V}/\mathcal{W})).$$

There is a $\wedge^k(A)$-equivariant projection from the image in $\wedge^k(\mathcal{V})$ of

$$\wedge^{\ell-3}(\mathcal{W}) \otimes \wedge^{k-(\ell-3)}(\mathcal{V})$$

onto $\wedge^{\ell-3}(\mathcal{W}) \otimes \wedge^{k-(\ell-3)}(\mathcal{V}/\mathcal{W})$. Therefore, $0 \neq (\wedge^k(A) - 1)^{\ell-1}(\wedge^k(\mathcal{V}))$. $\qquad\square$

THEOREM 3.12. — *Suppose ℓ is a prime number, $\ell \geq 5$, V is a finite-dimensional $\mathbb{Q}_\ell$-vector space, T is a $\mathbb{Z}_\ell$-lattice in V, and g is a quasi-unipotent linear operator on V such that $g(T) = T$. Suppose k and m are positive integers and $2 \leq k \leq \dim(V) - 2$. If $(\wedge^k(g) - 1)^{m(\ell-1)} \in \ell^m \mathrm{End}(\wedge^k(T))$, then $\wedge^k(g)$ is unipotent.*

Proof. — By Theorem 3.2, all the eigenvalues of $\wedge^k(g)$ are ℓ-th roots of unity. In particular, $\wedge^k(g)^\ell$ is unipotent. By Lemma 3.7 there exists a k-th root of unity $\gamma \in \mathbb{Z}_\ell$ such that $(\gamma g)^\ell$ is unipotent. Replacing g by γg, we may assume that g^ℓ is unipotent.

First, suppose that g is semisimple. Then $g^\ell = 1$ and $\wedge^k(g)^\ell = 1$. By Theorem 6.8 of [12] there is a $\wedge^k(g)$-invariant splitting of the free $\mathbb{Z}_\ell$-module $\wedge^k(T)$ into a direct sum of free $\mathbb{Z}_\ell$-modules $\wedge^k(T) = P_1 \oplus P_2$ such that $\wedge^k(g)$ acts as the identity on P_1, and

$$\wedge^k(g)^{\ell-1} + \cdots + \wedge^k(g) + 1 = 0 \quad \text{on} \quad P_2.$$

This implies easily that

$$\left(\wedge^k(g) - 1\right)^{\ell-1} \in \ell \, \mathrm{End}\left(\wedge^k(T)\right).$$

If $g \neq 1$ then g has an eigenvalue which is a primitive ℓ-th root of unity, and therefore by Theorem 3.2, $(g - 1)^{\ell-2} \notin \ell \mathrm{End}(T)$. If we let $\mathcal{V} = T/\ell T$, and let $A : \mathcal{V} \to \mathcal{V}$ be the linear operator induced by g, then $(A - 1)^{\ell-2} \neq 0$, but $(\wedge^k(A) - 1)^{\ell-1} = 0$. This contradicts Theorem 3.11, and proves that $g = 1$ when g is semisimple.

Next we will induct on the maximum of the multiplicities of the roots of the minimal polynomial $P(t)$ of g (*i.e.*, on the maximal size of the Jordan blocks for g). Let $P_1(t) \in \mathbb{Z}_\ell[t]$ be the monic polynomial whose roots are the same as those of $P(t)$, but all with multiplicity one. Then P_1 divides P, and $P_1 = P$ if and only if g is semisimple. Let

$$T_0 = \left\{ x \in T \mid P_1(g)(x) = 0 \right\}.$$

Then T_0 is a pure free $\mathbb{Z}_\ell$-submodule of T which is g-invariant, and the restriction $g_0 : T_0 \to T_0$ is semisimple. Let $T_1 = T/T_0$ and let g_1 denote the induced automorphism $g_1 : T_1 \to T_1$. Then T_1 is a free $\mathbb{Z}_\ell$-module of finite rank, and the maximal multiplicity of a root of the minimal polynomial of

$$g_0 \oplus g_1 : T_0 \oplus T_1 \longrightarrow T_0 \oplus T_1$$

is strictly less than that of $P(t)$, if g is not semisimple. Note that $(g_0 \oplus g_1)^\ell$ is unipotent, since g^ℓ is unipotent. Further, $g_0 \oplus g_1$ is unipotent if and only if g

is unipotent. To apply induction and finish the proof, it suffices to check that

$$(1) \qquad (\wedge^k(g_0 \oplus g_1) - 1)^{m(\ell-1)} \in \ell^m \mathrm{End}\big(\wedge^k(T_0 \oplus T_1)\big).$$

For $0 \le i \le k$, let H_i be the image of the natural homomorphism

$$\wedge^i(T_0) \otimes \wedge^{k-i}(T) \longrightarrow \wedge^k(T).$$

Then

$$H_i/H_{i+1} \cong \wedge^i(T_0) \otimes \wedge^{k-i}(T_1),$$

and

$$\wedge^k(T) = H_0 \supset H_1 \supset \cdots \supset H_k = \wedge^k(T_0) \supset H_{k+1} := 0$$

is a natural filtration of $\wedge^k(g)$-stable pure $\mathbb{Z}_\ell$-submodules of $\wedge^k(T)$. Since H_i is pure in $\wedge^k(T)$, we have $H_i \cap \ell^m \wedge^k (T) = \ell^m H_i$. Therefore,

$$\big(\wedge^k(g) - 1\big)^{m(\ell-1)}(H_i) \subseteq \ell^m H_i.$$

Since

$$\bigoplus_{i=0}^k (H_i/H_{i+1}) = \bigoplus_{i=0}^k \big(\wedge^i(T_0) \otimes \wedge^{k-i}(T_1)\big) = \wedge^k(T_0 \oplus T_1),$$

we have (1). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Abelian variety lemmas

As stated earlier, we suppose X is an abelian variety over a field F, v is a discrete valuation on F of residue characteristic $p \ge 0$, and ℓ is a prime different from p. Recall that I is the inertia subgroup at $\bar{v}$ of $\mathrm{Gal}(F^s/F)$.

THEOREM 4.1 (Galois Criterion for Semistability). — *The   following   are equivalent:*

(i) *X is semistable at v;*
(ii) *I acts unipotently on $T_\ell(X)$, i.e. all the eigenvalues of $\rho_{\ell,X}(\sigma)$ are 1 for every $\sigma \in I$;*
(iii) *for every $\sigma \in I$, $(\rho_{\ell,X}(\sigma) - 1)^2 = 0$.*

Proof. — See 3.5 and 3.8 of [5], and Theorem 6 on p. 184 of [2]. $\qquad\square$

LEMMA 4.2. — *Suppose $\sigma \in I$. Then:*

(i) *$\rho_{\ell,X}(\sigma)$ is unipotent if and only if $(\rho_{\ell,X}(\sigma) - 1)^2 = 0$;*
(ii) *$-\rho_{\ell,X}(\sigma)$ is unipotent if and only if $(\rho_{\ell,X}(\sigma) + 1)^2 = 0$.*

Proof. — There exists a finite Galois extension $L \subset F^s$ of F such that if w is the restriction of $\bar{v}$ to L then X is semistable at w (see Prop. 3.6 of [5]). Let $I_w = I \cap \mathrm{Gal}(F^s/L)$ be the corresponding inertia group, let $m = [L : F]$, and let $g = \rho_{\ell,X}(\sigma)$. Then $\sigma^m \in I_w$. By Theorem 4.1, $(g^m - 1)^2 = 0$. Now apply Lemma 3.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following result follows immediately from Lemmas 3.9 (iii) and 4.2.

PROPOSITION 4.3. — *Suppose $\sigma \in I$. The following are equivalent:*

(i) *there exists $\gamma_\sigma \in \mathbb{Q}_\ell$ such that $\gamma_\sigma \rho_{\ell,X}(\sigma)$ is unipotent;*

(ii) *either $\rho_{\ell,X}(\sigma)$ or $-\rho_{\ell,X}(\sigma)$ is unipotent;*

(iii) *either $(\rho_{\ell,X}(\sigma) - 1)^2 = 0$ or $(\rho_{\ell,X}(\sigma) + 1)^2 = 0$.*

PROPOSITION 4.4. — *Let $J \subset I$ denote the first ramification group, and let τ be a lift to I of a topological generator of the procyclic group I/J. The following are equivalent:*

(i) *X has purely additive reduction at v;*

(ii) *1 is not an eigenvalue for the action of τ on $V_\ell(X)^J$;*

(iii) *$V_\ell(X)^I = 0$.*

Proof. — The equivalence of (ii) and (iii) is obvious. For the equivalence of (i) and (ii) see Corollary 1.10 of [7]. $\qquad\square$

5. Higher cohomology groups of abelian varieties

Write $V = V_\ell(X)$ and $T = T_\ell(X)$, and recall that $H_\ell^k = H_{\text{ét}}^k(\overline{X}, \mathbb{Z}_\ell)$. The image of $\rho_{\ell,X}$ lies in the symplectic group $\mathrm{Sp}(V)$, by the Galois-equivariance of the Weil pairing, and the fact that the inertia group acts as the identity on the ℓ-power roots of unity.

ASSUMPTION 5.1. — For the remainder of this paper (except for Remark 7.1) we will assume that if $p = 2$ then the valuation ring is henselian.

DEFINITION 5.2. — If $p \neq 2$ then we say that X is *briefly unstable* at v if X is purely additive at v and becomes semistable above v over a quadratic separable extension of F.

DEFINITION 5.3. — If $p = 2$ then we say that X is *briefly unstable* at v if X is purely additive at v and there exists a finite unramified extension M of F such that X is semistable above v over a quadratic separable extension of M.

REMARK 5.4. — By Theorem 4.1, the quadratic extension in Definitions 5.2 and 5.3 is ramified over v.

THEOREM 5.5. — *Suppose X is an abelian variety over a field F, v is a discrete valuation on F of residue characteristic $p \geq 0$, and ℓ is a prime different from p. Then the following are equivalent:*

(a) *X is either semistable or briefly unstable at v;*

(b) *for each $\sigma \in I$, either $\rho_{\ell,X}(\sigma)$ or $-\rho_{\ell,X}(\sigma)$ is unipotent.*

Proof. — Assume (a) holds. By Theorem 4.1, we may reduce to the case where X has purely additive reduction at v, M is a finite unramified extension of F, and L is a quadratic separable extension of M over which X is semistable above v. Then by Theorem 4.1, $\rho_{\ell,X}(\sigma)^2$ is unipotent for all $\sigma \in I$. Let $J \subset I$ be the first ramification subgroup. Then J, and therefore $\rho_{\ell,X}(J)$, is either trivial (if $p = 0$) or a pro-p-group. By [7] (see pp. 282–283), since $\ell \neq p$, $\rho_{\ell,X}(J)$ is either trivial or a finite p-group. If $s \in \rho_{\ell,X}(J) \subset \rho_{\ell,X}(I)$, then s^2 is unipotent and has finite order, and thus $s^2 = 1$. It follows that either $\rho_{\ell,X}(J) = \{1\}$, or $p = 2$ and $\rho_{\ell,X}(J)$ is a finite commutative group of exponent 2.

Suppose that $\rho_{\ell,X}(J) = \{1\}$. Then $V^J = V$. Let τ be a lift to I of a topological generator of the procyclic group I/J. Then $g := \rho_{\ell,X}(\tau)$ generates the procyclic group $\rho_{\ell,X}(I)$. By Prop. 4.4, 1 is not an eigenvalue of g. Since g^2 is unipotent, the only eigenvalue of g is -1, *i.e.*, $-g$ is unipotent. For each integer i either g^i or $-g^i$ is unipotent. Since in the ℓ-adic topology the set of integral powers of g is dense in $\rho_{\ell,X}(I)$ and the set of unipotent operators in $\mathrm{Aut}(T)$ is closed, therefore for each $\sigma \in I$ either $\rho_{\ell,X}(\sigma)$ or $-\rho_{\ell,X}(\sigma)$ is unipotent.

We may thus assume that $\rho_{\ell,X}(J) \neq \{1\}$, $p = 2$, and $\rho_{\ell,X}(J)$ is a finite commutative group of exponent 2. We may assume that $L \subset F^s$. Let w be the restriction of $\bar{v}$ to L. Let I_w denote the inertia subgroup at $\bar{v}$ of $\mathrm{Gal}(F^s/L)$. Clearly, $J_w := J \cap I_w$ is the first ramification subgroup of I_w, and J_w has index 2 in J. Since X is semistable at w, $\rho_{\ell,X}(\sigma)$ is unipotent for all $\sigma \in I_w$. Since $\rho_{\ell,X}(J)$ is finite, $\rho_{\ell,X}(\sigma) = 1$ for all $\sigma \in J_w$. Since J_w has index 2 in J, therefore $\rho_{\ell,X}(J)$ has order 2.

Since L/F is wild quadratic, therefore the inclusion $I_w \subset I$ induces a natural isomorphism $I_w/J_w = I/J$. Let τ_w be a lift to I_w of a topological generator of $I_w/J_w = I/J$. Since $\rho_{\ell,X}(\tau_w)$ is unipotent, and X has purely additive reduction at v, therefore $V^J = 0$ by Proposition 4.4.

Clearly J_w is normal in I, since it is the intersection of normal subgroups. We can view $\rho_{\ell,X}$ as a homomorphism from I/J_w to $\mathrm{Aut}(T)$. The image of

$$I_w \longrightarrow I_w/J_w \subset I/J_w = I/J \times I/I_w$$

is $I/J \times \{1\}$. Therefore $\rho_{\ell,X}(I/J \times \{1\})$ consists of unipotent operators. Let s be the non-trivial element of I/I_w and let $h = \rho_{\ell,X}(1 \times s)$. Then $h^2 = 1$. If $h = 1$ then $\rho_{\ell,X}(I/J_w)$ consists of unipotent operators, so X is semistable at v, which is not the case. So $h \neq 1$. If $h = -1$ then $\rho_{\ell,X}(I/J_w)$ is the union of $\rho_{\ell,X}(I/J \times \{1\})$ and $-\rho_{\ell,X}(I/J \times \{1\})$. Therefore for each $g \in \rho_{\ell,X}(I/J_w)$, either g or $-g$ is unipotent. So we have reduced to the case where $h \neq \pm 1$. But then V^J, the eigenspace of h corresponding to the eigenvalue 1, is non-zero. This contradiction proves that (a) implies (b).

To prove that (b) implies (a), suppose that X is not semistable at v, and suppose that for each $\sigma \in I$ either $\rho_{\ell,X}(\sigma)$ or $-\rho_{\ell,X}(\sigma)$ is unipotent. By Theorem 4.1, $\rho_{\ell,X}(\sigma)$ is not unipotent for some $\sigma \in I$. For such a σ, the

eigenvalues of $\rho_{\ell,X}(\sigma)$ are all -1. Thus $V^I = 0$. By Prop. 4.4, X has purely additive reduction at v. Let

$$(2) \qquad I_{v,X} = \big\{ \sigma \in I \mid \rho_{\ell,X}(\sigma) \text{ is unipotent} \big\}.$$

Then $I_{v,X} \neq I$. It is known (see pp. 354–355 of [5] and § 4 of [14]) that $I_{v,X}$ is an open normal subgroup of finite index in I. Since $\rho_{\ell,X}(I_{v,X})$ consists of unipotent operators, $V^{I_{v,X}} \neq 0$ by a theorem of Kolchin (p. 35 of [8]). The restriction map $\rho' : I \to \operatorname{Aut}(V^{I_{v,X}})$ factors through the finite group $I/I_{v,X}$. Therefore the image of ρ' is finite. If $\sigma \in I - I_{v,X}$, then $-\rho'(\sigma)$ is unipotent and of finite order, and thus $\rho'(\sigma) = -1$ on $V^{I_{v,X}}$. Therefore ρ' has kernel $I_{v,X}$ and image $\{\pm 1\}$, so $[I : I_{v,X}] = 2$.

First assume $p \neq 2$. Then I has exactly one subgroup of index 2 and thus this subgroup must be $I_{v,X}$. Let L/F be a ramified separable quadratic extension. We may assume that $L \subset F^s$. Let w be the restriction of $\bar{v}$ to L. The corresponding inertia group $I_w \subset \operatorname{Gal}(F^s/L)$ has index 2 in I and therefore is $I_{v,X}$. By Theorem 4.1, X is semistable at w.

Now assume that $p = 2$. Let $F^{\mathrm{ur}} = (F^s)^I$, the maximal extension of F unramified above v. The valuation ring of F^{ur} is henselian and the residue field is separably closed. Let L be the quadratic extension of F^{ur} corresponding to $I_{v,X} \subset I \cong \operatorname{Gal}(F^s/F^{\mathrm{ur}})$. Then $L = F^{\mathrm{ur}}(\sqrt{c})$ for some $c \in F^{\mathrm{ur}}$, $F(c)$ is unramified above v over F, and X is semistable over the ramified quadratic extension $F(\sqrt{c})$ of $F(c)$. □

THEOREM 5.6. — *Suppose X is an abelian variety over a field F, suppose v is a discrete valuation on F of residue characteristic $p \geq 0$, suppose k is a positive integer, suppose $k < 2\dim(X)$, and suppose ℓ is a prime number not equal to p.*

(i) *If k is odd then the following are equivalent:*
   (a) *X is semistable at v,*
   (b) *for each $\sigma \in I$, $\rho_{\ell,k}(\sigma)$ is unipotent,*
   (c) *for each $\sigma \in I$, $(\rho_{\ell,k}(\sigma) - 1)^{k+1} = 0$.*

(ii) *If k is even then the following are equivalent:*
   (a) *X is either semistable or briefly unstable at v,*
   (b) *for each $\sigma \in I$, $\rho_{\ell,k}(\sigma)$ is unipotent,*
   (c) *for each $\sigma \in I$, $(\rho_{\ell,k}(\sigma) - 1)^{k+1} = 0$.*

(iii) *If k is odd then the following are equivalent:*
   (a) *X is either semistable or briefly unstable at v,*
   (b) *for each $\sigma \in I$, either $\rho_{\ell,k}(\sigma)$ or $-\rho_{\ell,k}(\sigma)$ is unipotent,*
   (c) *for each $\sigma \in I$, either $(\rho_{\ell,k}(\sigma) - 1)^{k+1} = 0$ or $(\rho_{\ell,k}(\sigma) + 1)^{k+1} = 0$.*

Proof. — Clearly, (c) implies (b). That (a) implies (c) follows from Remark 3.5, combined with Theorem 4.1 and Lemmas 4.2 (i) and 3.9 (i) for (i), with Theorem 5.5 and Lemma 3.9 (ii) for (ii), and with Theorem 5.5 and Lemmas 4.2 and

3.9 (i) for (iii). Suppose we have (b). To conclude (a), apply Lemma 3.9 (iv) and Remark 3.5, combined with Theorem 5.5 for (ii) and (iii) and with Theorem 4.1 for (i). $\qquad\square$

COROLLARY 5.7. — *Suppose X is an abelian variety over a field F, and v is a discrete valuation on F of residue characteristic $p \geq 0$. Suppose k and r are positive integers, $k < 2\dim(X)$, and $k < r$. Suppose $\sigma \in I$.*

(i) *If either X is semistable at v, or k is even and X is briefly unstable at v, then*

$$(\sigma - 1)^r H_{\text{ét}}^k(\bar{X}, \mathbb{Z}_\ell) = 0$$

for every prime $\ell \neq p$, and

$$(\sigma - 1)^r H_{\text{ét}}^k(\bar{X}, \mathbb{Z}/n\mathbb{Z}) = 0$$

for every positive integer n not divisible by p.

(ii) *If k is odd and X is briefly unstable at v, then for every prime $\ell \neq p$, either*

$$(\sigma - 1)^r H_{\text{ét}}^k(\bar{X}, \mathbb{Z}_\ell) = 0 \quad or \quad (\sigma + 1)^r H_{\text{ét}}^k(\bar{X}, \mathbb{Z}_\ell) = 0,$$

and for every positive integer n not divisible by p, either

$$(\sigma - 1)^r H_{\text{ét}}^k(\bar{X}, \mathbb{Z}/n\mathbb{Z}) = 0 \quad or \quad (\sigma + 1)^r H_{\text{ét}}^k(\bar{X}, \mathbb{Z}/n\mathbb{Z}) = 0.$$

Proof. — The first parts follow from Theorem 5.6, since $r \geq k + 1$. The second parts follow from the first parts for all prime divisors ℓ of n, since for all i,

$$H_{\text{ét}}^k(\overline{X}, \mathbb{Z}/\ell^i\mathbb{Z}) = H_{\text{ét}}^k(\overline{X}, \mathbb{Z}_\ell) \otimes \mathbb{Z}/\ell^i\mathbb{Z}. \qquad\square$$

THEOREM 5.8. — *Suppose X is an abelian variety over a field F, and v is a discrete valuation on F of residue characteristic $p \geq 0$. Suppose k, n, and r are positive integers, $k < 2\dim(X)$, n is not divisible by p, and $n \notin N(r)$.*

(i) *Suppose that $(\sigma - 1)^r H_{\text{ét}}^k(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = 0$ for all $\sigma \in I$. Then either X is semistable at v, or k is even and X is briefly unstable at v.*

(ii) *Suppose k is odd, and suppose that for each $\sigma \in I$ either*

$$(\sigma - 1)^r H_{\text{ét}}^k(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = 0 \quad or \quad (\sigma + 1)^r H_{\text{ét}}^k(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = 0.$$

Then either X is semistable at v, or X is briefly unstable at v.

Proof. — Recall ([5], Thm. 4.3) that the characteristic polynomial of $\rho_{\ell,X}(\sigma)$ has integer coefficients and does not depend on the choice of $\ell \neq p$. Therefore the characteristic polynomial P_σ of $(\rho_{\ell,k}(\sigma) - 1)^r/n$ has coefficients in $\mathbb{Z}[1/n]$ and does not depend on the choice of $\ell \neq p$. Suppose that

$$(\sigma - 1)^r H_{\text{ét}}^k(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = 0.$$

Then for all prime divisors ℓ of n, $(\rho_{\ell,k}(\sigma) - 1)^r H_\ell^k \subseteq n H_\ell^k$, so P_σ has coefficients in $\mathbb{Z}_\ell$. Thus P_σ has integer coefficients. Since X is semistable over a finite separable extension of F, by Theorem 4.1, Lemma 3.9 (i), (ii), and Theorem 5.6

there is a positive integer m such that $(\rho_{\ell,k}(\sigma^m) - 1)^{k+1} = 0$. Let α be an eigenvalue of $\rho_{\ell,k}(\sigma)$. Then $(\alpha^m - 1)^{k+1} = 0$, so $\alpha^m = 1$. Since $(\alpha - 1)^r/n$ is an eigenvalue of $(\rho_{\ell,k}(\sigma) - 1)^r/n$ and therefore is a root of P_σ, it is an algebraic integer. Since $n \notin N(r)$, we have $\alpha = 1$ by Theorem 3.3. Thus $\rho_{\ell,k}(\sigma)$ is unipotent. Applying Theorem 5.6, we have (i). To obtain (ii), replace -1 by $+1$ in the above argument. $\qquad\square$

Next we specialize Theorem 5.8i to the case $r = 1$. Note that when $k = 1$ we recover Raynaud's criterion for semistability (Prop 4.7 of [5]).

THEOREM 5.9. — *Suppose X is an abelian variety over a field F, v is a discrete valuation on F, k and n are integers, $0 < k < 2\dim(X)$, $n \geq 3$, n is not divisible by the residue characteristic, and I acts as the identity on $H^k_{\text{ét}}(\overline{X}, \mathbb{Z}/n\mathbb{Z})$. Then either X is semistable at v, or k is even and X is briefly unstable at v.*

The next result is an immediate corollary of Theorems 5.8 and 5.6 and Corollary 5.7.

THEOREM 5.10. — *Suppose X is an abelian variety over a field F, suppose v is a discrete valuation on F of residue characteristic $p \geq 0$, suppose k, n, and r are positive integers, and suppose ℓ is a prime number. Suppose $k < 2\dim(X)$, suppose $k < r$, suppose $n\ell$ is not divisible by p, and suppose $n \notin N(r)$.*

(i) *If k is odd, then the following are equivalent:*
   (a) *X is semistable at v,*
   (b) *for each $\sigma \in I$, $(\sigma - 1)^r H^k_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell) = 0$,*
   (c) *for each $\sigma \in I$, $(\sigma - 1)^r H^k_{\text{ét}}(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = 0$.*

(ii) *If k is even then the following are equivalent:*
   (a) *X is either semistable or briefly unstable at v,*
   (b) *for each $\sigma \in I$, $(\sigma - 1)^r H^k_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell) = 0$,*
   (c) *for each $\sigma \in I$, $(\sigma - 1)^r H^k_{\text{ét}}(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = 0$.*

(iii) *If k is odd then the following are equivalent:*
   (a) *X is either semistable or briefly unstable at v,*
   (b) *for each $\sigma \in I$ either $(\sigma - 1)^r$ or $(\sigma + 1)^r$ kills $H^k_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$.*
   (c) *for each $\sigma \in I$ either $(\sigma - 1)^r$ or $(\sigma + 1)^r$ kills $H^k_{\text{ét}}(\overline{X}, \mathbb{Z}/n\mathbb{Z})$.*

6. Non-extremal cohomology groups

The results of §5 are sharp when $k = 1$ or $2d - 1$. In this section we obtain sharp results when $1 < k < 2d - 1$. Let

$$N'(r) = \{\ell^m \mid 0 \leq m(\ell - 1) < r, \text{ or } \ell = 2 \text{ or } 3 \text{ and } r = m(\ell - 1)\}.$$

Clearly, $N'(r) \subseteq N(r)$, and a prime power ℓ^m lies in $N(r) - N'(r)$ if and only if $\ell \geq 5$ and $r = m(\ell - 1)$.

THEOREM 6.1. — *Suppose X is an abelian variety over a field F, suppose v is a discrete valuation on F of residue characteristic $p \geq 0$, suppose k and r are positive integers, and suppose n is a positive integer which is not divisible by p. Suppose $2 \leq k \leq 2\dim(X) - 2$, suppose $k < r$, and suppose $n \notin N'(r)$.*

(i) *If k is odd, then the following are equivalent:*
    (a) *X is semistable at v,*
    (b) *for every $\sigma \in I$, $(\sigma - 1)^r H_{\text{ét}}^k(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = 0$.*

(ii) *If k is even then the following are equivalent:*
    (a) *X is either semistable or briefly unstable at v,*
    (b) *for every $\sigma \in I$, $(\sigma - 1)^r H_{\text{ét}}^k(\overline{X}, \mathbb{Z}/n\mathbb{Z}) = 0$.*

(iii) *If k is odd then the following are equivalent:*
    (a) *X is either semistable or briefly unstable at v,*
    (b) *for each $\sigma \in I$ either $(\sigma - 1)^r$ or $(\sigma + 1)^r$ kills $H_{\text{ét}}^k(\overline{X}, \mathbb{Z}/n\mathbb{Z})$.*

Proof. — If $n \notin N(r)$ then the assertion is contained in Theorem 5.10. Thus we may assume that $n = \ell^m$ with $\ell \geq 5$ and $r = m(\ell - 1)$. If (a) holds, then (b) follows from Corollary 5.7. If (b) holds, then (a) follows from Theorems 3.12 and 5.6. $\square$

The following example shows that the condition $n \notin N'(r)$ is sharp.

EXAMPLE 6.2. — Suppose that F is a discrete valuation field, m and r are positive integers, ℓ is a prime number, and $n = \ell^m$. Suppose that either $m(\ell - 1) < r$, or $\ell = 2$ and $r = m = m(\ell - 1)$, or $\ell = 3$ and $r = 2m = m(\ell - 1)$. Let X_1, Y be abelian varieties of positive dimension with good reduction over F, and such that Y has an automorphism of exact order ℓ. If $\ell \leq 3$, assume $\dim(Y) = 1$. Let L be a totally ramified degree ℓ extension of F, and let X_2 be the L/F-form of Y corresponding to a character $\text{Gal}(L/F) \hookrightarrow \text{Aut}(Y)$. Then $X = X_1 \times X_2$ has neither semistable nor purely additive reduction. However, by Lemma 3.4 we have $(\rho_{\ell,k}(\sigma) - 1)^r \in n\text{End}(H_\ell^k)$ for every $\sigma \in I$.

7. Semistability over quadratic extensions

REMARK 7.1. — When the valuation ring is henselian, then v extends uniquely to F^s, I is normal in $\text{Gal}(F^s/F)$, and the field $(F^s)^I$ is the (unique) maximal extension of F unramified over v. Note that Assumption 5.1 can be dropped in Theorems 5.6 (i), 5.10 (i), and 6.1 (i), in Corollary 5.7 (i) if X is semistable, and in Theorems 5.8 (i) and 5.9 when k is odd, since it is not used. (Assumption 5.1 is only used when we deal with brief instability.)

LEMMA 7.2. — *Suppose X is an abelian variety over a field F, v is a discrete valuation on F of residue characteristic $p \geq 0$, and the valuation ring is henselian. Suppose there exists a finite unramified extension M of F such that X is semistable above v over a quadratic separable extension L of M. Suppose*

that if $p = 2$, then the residue field either is separably closed or is algebraic over $\mathbb{F}_2$ (e.g., is finite). Then X is semistable above v over a quadratic separable extension of F.

Proof. — If K is a subfield of F^s, let I_K denote the inertia subgroup of I corresponding to K. Note that I_L is an open subgroup of I of index 2.

When $p \neq 2$, then F has a tamely ramified separable quadratic extension L'. Since $p \neq 2$, I has exactly one open subgroup of index 2, so $I_L = I_{L'}$. By Theorem 4.1 over L and over L', X is semistable above v over L'.

Now suppose $p = 2$.

If the residue field is separably closed, then F has no non-trivial unramified extensions, so $M = F$.

Suppose the residue field k is algebraic over $\mathbb{F}_2$, and let $G_k := \mathrm{Gal}(k^s/k)$. Then G_k is a torsion-free procyclic group, since it is a closed subgroup of $\hat{\mathbb{Z}}$. Let F^{ur} be the maximal unramified extension of F. Then $I = \mathrm{Gal}(F^s/F^{\mathrm{ur}})$. Since the valuation ring is henselian, $G_k = \mathrm{Gal}(F^{\mathrm{ur}}/F)$. We may assume that X is not semistable at v. Let $I_{v,X}$ be the group defined by formula (2) (proof of Theorem 5.5) with $\ell = 3$. Applying Theorem 4.1 over F and over L (for $\ell = 3$) shows that $I_{v,X}$ is a proper subgroup of I and $I_L \subseteq I_{v,X} \subset I$. Since $[I : I_L] = 2$, we have $I_{v,X} = I_L$ and $[I : I_{v,X}] = 2$. Let F' be the quadratic extension of F^{ur} cut out by $I_{v,X}$. Then F'/F is Galois, since the group $I_{v,X}$ is the intersection of I and

$$\{\sigma \in \mathrm{Gal}(F^s/F) \mid \rho_{3,X}(\sigma) \text{ is unipotent}\},$$

and both are stable under conjugation by $\mathrm{Gal}(F^s/F)$. Further, X is semistable (above v) over F' by Theorem 4.1, and $G := \mathrm{Gal}(F'/F)$ is an extension of G_k by $C := \mathrm{Gal}(F'/F^{\mathrm{ur}})$. This extension is central, since C has order 2 and thus has no non-trivial automorphisms. Since every group whose quotient by its center is (pro)cyclic must be commutative, G is commutative. Let Δ be the subset of squares in G. Then Δ is a closed subgroup, $\Delta \cap C = 1$, and $[G : \Delta] = 2$ or 4 (G_k is procyclic, so $2G_k$ has index 1 or 2 in G_k). Thus Δ is open and G/Δ either has order 2 (in which case let $H = \Delta$) or is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and we may choose the isomorphism so that the first factor is the image of C (and then let H be the preimage in G of the second factor). Then H is an open subgroup of G of index 2, so the corresponding subfield $L' := (F')^H$ is quadratic over F. Since $H \cap C = 1$, therefore F'/L' is unramified and so X is semistable over L'. □

COROLLARY 7.3. — *Suppose that if the residue characteristic is 2, then the residue field either is separably closed or is algebraic over $\mathbb{F}_2$. Then we may replace "briefly unstable at v" by "purely additive at v and becomes semistable above v over a quadratic separable extension of F" in the results of §§ 5–6.*

REMARK 7.4. — The proof of Lemma 7.2 shows that the condition that the residue field k be algebraic over $\mathbb{F}_2$ can be replaced by the condition that $\mathrm{Gal}(k^s/k)$ be a torsion-free procyclic group.

REMARK 7.5. — The group $I_{v,X}$ defined in (2) is independent of ℓ (see p. 355 of [5] and Theorem 4.2 of [14]). It follows that for each fixed $\sigma \in I$, whether or not $(\sigma - 1)^r$ kills $H^k_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)$ (or $H^k_{\text{ét}}(\overline{X}, \mathbb{Z}/n\mathbb{Z})$ for n not a power of 2) is independent of ℓ (and n), and depends only on whether or not $\sigma \in I_{v,X}$.

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