

## ON SYSTEMS OF LINEAR INEQUALITIES

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*In celebration of the 70th birthday of Professor Genjiro Fujisaki*

ABSTRACT. — We show in detail that the category of general Roth systems or the category of semi-stable systems of linear inequalities of slope zero is a neutral Tannakian category. On the way, we present a new proof of the semi-stability of the tensor product of semi-stable systems. The proof is based on a numerical criterion for a system of linear inequalities to be semi-stable.

RÉSUMÉ (*Sur certains systèmes d'inégalités linéaires*). — On démontre en détail que la catégorie des systèmes de Roth généraux ou la catégorie des systèmes semi-stables d'inégalités linéaires de pente zéro est une catégorie tannakienne neutre. En chemin, on présente une nouvelle preuve de la semi-stabilité du produit tensoriel de systèmes semi-stables. La preuve découle d'un critère numérique pour qu'un système d'inégalités linéaires soit semi-stable.

### Introduction

Let  $f_1, \dots, f_n$  be absolutely linearly independent linear forms in  $n$  variables  $T_1, \dots, T_n$  with real algebraic coefficients;  $c(1), \dots, c(n)$  real numbers such that

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$c(1)+\dots+c(n)=0$ ; and  $Q, \delta$  positive real numbers. We are primarily interested in properties of the rational integral solutions to the system of inequalities

$$|f_i(T_1, \dots, T_n)| < Q^{-c(i)-\delta} \quad (Q > 1; i = 1, \dots, n)$$

when  $\delta$  is fixed. For example, finiteness of the number of solutions.

Let  $L$  be the subfield of the field of real numbers generated by all the coefficients of  $f_1, \dots, f_n$ . If we do not seek sharp estimates, then it seems that the nature of the system comes from a descending filtration on the  $L$ -vector space  $LT_1 \oplus \dots \oplus LT_n$ : the family  $f_1, \dots, f_n$  is a basis which induces a basis of the associated graded vector space. The number  $c(i)$  is the weight of  $f_i$  with respect to the filtration. In fact, one sees easily that finiteness of the number of solutions is independent of choices of such a basis (modulo replacement of  $\delta$  by a slightly larger exponent). A system with finitely many solutions has been called a general Roth system.

From the viewpoint of filtrations, Faltings and Wüstholz [4] gave a projective geometric picture of the set of (rational) solutions to a (related) system of inequalities. In particular, it is coordinate-free. Faltings [3] has found a resemblance between filtered vector spaces and filtered isocrystals and he called semi-stable (of slope zero) a filtered vector space which gives rise to a general Roth system.

In the present article, we aim at proving that the category of general Roth systems, namely, the category of semi-stable filtered vector spaces of slope zero forms a neutral Tannakian category. It means that the category is equivalent to the category of finite dimensional representations of an affine group scheme over the base field. A key lemma is the one stating that a tensor product of semi-stable filtered vector spaces is again semi-stable. The lemma was used for a second proof of the subspace theorem of Schmidt and Schlickewei by Faltings and Wüstholz [4].

Reversing the order of reasoning, we obtain a new proof of the key lemma which depends on the subspace theorem. *Note that it is not a tautology*, because the original proof of Schmidt and Schlickewei does not require the key lemma. The subspace theorem provides us with a simple numerical criterion (Theorem 2.8) for a filtered vector space to be semi-stable. The key lemma is then a consequence (Corollary 2.9) of the criterion.

Our proof is elementary. The difficult parts are hiding in the subspace theorem and in Minkowski's theorem on the geometry of numbers. In Section 1, we make precise various definitions. The section is expository. In Section 2, we give the new proof of the key lemma.

NOTATION. — Let  $\mathbb{R}$  be the field of real numbers. By  $i \gg 0$ , we mean a real number  $i$  is large enough according to the context. The symbol ‘ $\circ$ ’ indicates composition of morphisms.

### 1. Category of linear inequalities

Let  $k$  be a finite extension field of the rational number field and let  $L$  be an algebraic extension field of  $k$ .

DEFINITION 1.1 (filtration, slope, and weight [9, p. 82])

For a finite dimensional  $k$ -vector space  $V$ , a family  $F^\bullet$  of  $L$ -vector spaces

$$F^i V \subset L \otimes_k V \quad (i \in \mathbb{R})$$

is called an  $L$ -filtration on  $V$  if and only if the conditions

$$\begin{aligned} F^i V &\supset F^j V \quad (i \leq j), \\ F^{-i} V &= L \otimes_k V, \quad F^i V = 0 \quad (i \gg 0) \quad \text{and} \\ F^i V &= \bigcap_{j < i} F^j V \end{aligned}$$

are satisfied. We denote the associated graduation by

$$\text{gr}^w(V, F^\bullet) = F^w V / F^{w+0} V \quad (w \in \mathbb{R}),$$

where

$$F^{w+0} V = \bigcup_{j > w} F^j V.$$

The *slope*  $M$  of the filtration is a real number

$$M(V, F^\bullet) = \frac{1}{\dim_k V} \sum_{w \in \mathbb{R}} w \dim_L \text{gr}^w(V, F^\bullet).$$

The slope of the zero-dimensional vector space is not defined. The real numbers  $w$  such that  $\text{gr}^w(V, F^\bullet) \neq 0$  are called the *weights* of the filtration. We often say  $V$  is an  $L$ -filtered  $k$ -vector space, instead of saying that  $(V, F^\bullet)$  is a  $k$ -vector space with an  $L$ -filtration. Similarly, we omit  $F^\bullet$  from  $M(V, F^\bullet)$  or  $\text{gr}^w(V, F^\bullet)$  and abbreviate  $F^i V$  to  $V^i$ .

DEFINITION 1.2 (subfiltration and quotient filtration). — Let  $W$  be a subspace over  $k$  of  $V$ . The  $L$ -filtration on  $W$  given by

$$W^i = (L \otimes_k W) \cap V^i \quad (i \in \mathbb{R})$$

is the *sub- $L$ -filtration* on  $W$  of  $V$ . The  $L$ -filtration on  $V/W$  defined as

$$(V/W)^i = (V^i + L \otimes_k W) / L \otimes_k W \quad (i \in \mathbb{R})$$

is the *quotient  $L$ -filtration* on  $V/W$  of  $V$ .

LEMMA 1.3 (see [4, p. 116]). — *Let  $W$  be a proper subspace over  $k$  of  $V$ , and  $F^\bullet$  an  $L$ -filtration on  $V$ . If we endow  $W$  with the subfiltration and  $V/W$  with the quotient filtration, then we have*

$$M(V) \dim_k V = M(W) \dim_k W + M(V/W) \dim_k (V/W).$$

*Proof.* — By definition, the sequences

$$0 \rightarrow W^w \rightarrow V^w \rightarrow (V/W)^w \rightarrow 0,$$

$$0 \rightarrow W^{w+0} \rightarrow V^{w+0} \rightarrow (V/W)^{w+0} \rightarrow 0$$

are both exact. By diagram chase,

$$0 \rightarrow \text{gr}^w W \rightarrow \text{gr}^w V \rightarrow \text{gr}^w(V/W) \rightarrow 0$$

is exact, too. The above equality follows at once.  $\square$

EXAMPLE 1.4. — Let  $v$  and  $u$  be non-zero elements of a  $k$ -vector space  $V$  such that

$$V = kv \oplus ku.$$

We attach to  $V$  the following filtration:

$$V^i = \begin{cases} L \otimes_k V & \text{for } i \leq 0, \\ L(v+u) & \text{for } 0 < i \leq 1, \\ 0 & \text{for } i > 1. \end{cases}$$

The subfiltration on  $W = kv$  is

$$W^i = \begin{cases} L \otimes_k W & \text{for } i \leq 0, \\ 0 & \text{for } i > 0. \end{cases}$$

For the subspace  $U = ku$ , the quotient filtration on  $V/U$  becomes

$$(V/U)^i = \begin{cases} L \otimes_k (V/U) & \text{for } i \leq 1, \\ 0 & \text{for } i > 1. \end{cases}$$

This is especially telling that although there is a canonical isomorphism of  $W = W/W \cap U$  onto  $V/U = (W+U)/U$  as vector spaces, they are not isomorphic as filtered vector spaces. In general, a subquotient filtration is not necessarily defined.

REMARK 1.5. — As is easily seen, in the case  $U \subset W \subset V$ , the *subquotient  $L$ -filtration* on  $W/U$  of  $V$  is well-defined.

DEFINITION 1.6 (filtered homomorphism [4, p. 117]). — For  $L$ -filtered  $k$ -vector spaces  $V$  and  $W$ , a *filtered homomorphism*  $f: V \rightarrow W$  is a  $k$ -linear map such that

$$f(V^i) \subset W^i \quad (i \in \mathbb{R})$$

when extended over  $L$ . It is said to be *strict* if

$$f(V^i) = [L \otimes_k f(V)] \cap W^i \quad (i \in \mathbb{R}).$$

The strictness of  $f$  means that the  $k$ -vector space  $V/\text{Ker } f$  with the quotient filtration of  $V$  (the *coimage*  $\text{Coim } f$  of  $f$ ) is isomorphic to the  $k$ -vector space  $f(V)$  with the subfiltration of  $W$  (the *image*  $\text{Im } f$  of  $f$ ).

REMARK 1.7. — For a subspace  $W$  over  $k$  of  $V$ , the canonical maps  $W \rightarrow V$  and  $V \rightarrow V/W$  are strictly filtered with respect to the induced filtrations. A composition of filtered homomorphisms is filtered. In Example 1.4, the canonical map  $W \rightarrow V/U$  is filtered but not strict. In Remark 1.5, the canonical map  $W \rightarrow V/U$  is strict.

LEMMA 1.8. — *If a filtered homomorphism  $f: V \rightarrow W$  is bijective as a  $k$ -linear map, then*

$$M(V) \leq M(W).$$

*Moreover, the equality is valid if and only if it is an isomorphism of filtered vector spaces.*

*Proof.* — Induction on the number of weights of  $V$ . First note that for the proof, the case  $L = k$  is sufficient.

When  $V$  has only one weight, the whole claim is almost trivial.

Suppose  $V$  has plural weights and  $w$  is the largest among them. Let the inclusion map be  $g: V^w \rightarrow V$ . We endow  $V/\text{Im } g$  and  $W/\text{Im } f \circ g$  with the respective quotient filtrations (the *cokernels*  $\text{Coker } g$  and  $\text{Coker } f \circ g$ ). The number of weights of  $\text{Im } g$  is one, and the number of weights of  $\text{Coker } g$  is fewer than the number of weights of  $V$ . The inductive assumption yields

$$M(\text{Im } g) \leq M(\text{Im } f \circ g) \quad \text{and} \quad M(\text{Coker } g) \leq M(\text{Coker } f \circ g).$$

From Lemma 1.3 we get the inequality we wanted. Furthermore, when we have  $M(V) = M(W)$ , the above inequalities must be equalities. By the inductive hypothesis,

$$\text{Im } g \simeq \text{Im } f \circ g \quad \text{and} \quad \text{Coker } g \simeq \text{Coker } f \circ g.$$

In particular,

$$\text{gr}^i \text{Im } g \simeq \text{gr}^i \text{Im } f \circ g \quad \text{and} \quad \text{gr}^i \text{Coker } g \simeq \text{gr}^i \text{Coker } f \circ g \quad (i \in \mathbb{R}).$$

By the third exact sequence in the proof of Lemma 1.3, we obtain

$$\text{gr}^i V \simeq \text{gr}^i W \quad (i \in \mathbb{R}),$$

hence  $V \simeq W$ . □

LEMMA 1.9. — *If a filtered homomorphism  $f: V \rightarrow W$  is injective as a  $k$ -linear map, then*

$$M(V) \dim_k V \leq M(W) \dim_k W.$$

*Proof.* — By definition, the induced morphism

$$V \rightarrow \text{Im } f$$

is filtered. Since it is also an isomorphism of  $k$ -vector spaces, we obtain by Lemma 1.8  $M(V) \leq M(\text{Im } f)$ . From Lemma 1.3, we get the desired inequality. □

DEFINITION 1.10 (direct sum). — Let  $V$  and  $W$  be  $L$ -filtered  $k$ -vector spaces. The  $k$ -vector space  $V \oplus W$  with an  $L$ -filtration

$$(V \oplus W)^i = V^i \oplus W^i \quad (i \in \mathbb{R})$$

is the  $L$ -filtered direct sum of  $V$  and  $W$ . We simply call it *direct sum* and write it  $V \oplus W$ . It is obvious that the inclusion maps and the projection maps are strictly filtered. By Lemma 1.3,

$$M(V \oplus W) \dim_k(V \oplus W) = M(V) \dim_k V + M(W) \dim_k W$$

when  $V$  and  $W$  are not 0-dimensional.

DEFINITION 1.11 (tensor product). — Let  $V$  and  $W$  be  $L$ -filtered  $k$ -vector spaces. The  $k$ -vector space  $V \otimes_k W$  with an  $L$ -filtration

$$(V \otimes_k W)^i = \sum_{j \in \mathbb{R}} V^j \otimes_L W^{i-j} \quad (i \in \mathbb{R})$$

is the  $L$ -filtered tensor product of  $V$  and  $W$ . We simply call it *tensor product* and write it  $V \otimes W$ .

LEMMA 1.12. — *One has*

$$M(V \otimes W) = M(V) + M(W)$$

when  $V$  and  $W$  are not 0-dimensional.

*Proof.* — Induction on the number of weights of  $V$ . We may suppose that  $L = k$ . When  $V$  has only one weight  $m$ , we have

$$(V \otimes W)^i = \sum_{j \leq m} V \otimes W^{i-j} = V \otimes W^{i-m} \quad (i \in \mathbb{R}).$$

Hence

$$\mathrm{gr}^i(V \otimes W) \simeq V \otimes \mathrm{gr}^{i-m} W \quad (i \in \mathbb{R}),$$

in particular

$$\dim \mathrm{gr}^i(V \otimes W) = \dim V \cdot \dim \mathrm{gr}^{i-m} W \quad (i \in \mathbb{R}).$$

Multiplying  $i$  on both sides,

$$\begin{aligned} i \dim \mathrm{gr}^i(V \otimes W) &= m \dim V \cdot \dim \mathrm{gr}^{i-m} W \\ &\quad + \dim V \cdot (i - m) \dim \mathrm{gr}^{i-m} W \quad (i \in \mathbb{R}). \end{aligned}$$

Summing up and dividing by  $\dim(V \otimes W) = \dim V \cdot \dim W$  each side,

$$\begin{aligned} M(V \otimes W) &= m \frac{1}{\dim W} \sum_{i \in \mathbb{R}} \dim \mathrm{gr}^{i-m} W + \frac{1}{\dim W} \sum_{i \in \mathbb{R}} (i - m) \dim \mathrm{gr}^{i-m} W \\ &= m + M(W) = M(V) + M(W). \end{aligned}$$

When  $V$  has plural weights, let  $w$  be the biggest. We have a (non-canonical) isomorphism of filtered vector spaces

$$V \simeq V^w \oplus (V/V^w).$$

By definition, this induces a filtered isomorphism

$$V \otimes W \simeq [V^w \otimes W] \oplus [(V/V^w) \otimes W].$$

Since the number of weights of  $V^w$  or  $V/V^w$  is smaller than the number of weights of  $V$ , we obtain

$$\begin{aligned} M(V \otimes W) &= M(V^w \otimes W) \frac{\dim V^w}{\dim V} + M((V/V^w) \otimes W) \frac{\dim(V/V^w)}{\dim V} \\ &= \frac{M(V^w) \dim V^w + M(V/V^w) \dim(V/V^w)}{\dim V} + M(W). \end{aligned}$$

Lemma 1.3 completes the proof.  $\square$

Now we try to define the category of linear inequalities. We take into account several filtrations simultaneously. The relation between filtered vector spaces and systems of linear inequalities is made clear at the top of the next section.

Let  $\mathfrak{M}(k)$  be the set of places of  $k$ .

DEFINITION 1.13 (category of linear inequalities). — Let  $\mathcal{C}$  be the following category: an object is a finite dimensional  $k$ -vector space  $V$  equipped with an  $L$ -filtration  $V_v^\bullet$  at each place  $v \in \mathfrak{M}(k)$  such that for except a finite number of  $v \in \mathfrak{M}(k)$ ,

$$V_v^0 = L \otimes_k V \quad \text{and} \quad V_v^{0+0} = 0.$$

For such  $V$  and  $W$ , the set of morphisms  $\text{Hom}(V, W)$  is the set of  $k$ -linear maps filtered for every  $v \in \mathfrak{M}(k)$  (call them *filtered*, simply):

$$\text{Hom}(V, W) \subset \text{Hom}_k(V, W)$$

It is a linear subspace over  $k$ . The composition of morphisms is well-defined and associative. It is also bi-linear. Identity maps are filtered and become identity morphisms in  $\mathcal{C}$ . The zero dimensional vector space  $0$  is the unique zero object in  $\mathcal{C}$ . The filtrations on a direct sum as  $k$ -vector spaces of objects are the direct sums of  $L$ -filtrations for all  $v \in \mathfrak{M}(k)$ . The inclusion maps and the projection maps are (strictly) filtered and it defines a direct sum and a direct product in  $\mathcal{C}$ . For a morphism  $f$ , we have the kernel  $\text{Ker } f$ , the coimage  $\text{Coim } f$ , the image  $\text{Im } f$ , and the cokernel  $\text{Coker } f$ .

We define the tensor product of objects as tensor product of  $k$ -vector spaces with the tensor product of  $L$ -filtrations for each  $v \in \mathfrak{M}(k)$ . The tensor product gives a bi-linear functor

$$\otimes: \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}.$$

We denote by  $I$  a one-dimensional vector space endowed with filtrations such that  $M_v(I) = 0$ , where  $M_v$  is the slope of filtration for  $v \in \mathfrak{M}(k)$ . There exists an isomorphism  $I \rightarrow I \otimes I$  in  $\mathcal{C}$ . The object  $I$  is an identity object of  $(\mathcal{C}, \otimes)$ :

$$\text{End}(I) = k \text{id}_I \simeq k$$

The action of  $\text{End}(I)$  is canonically identified with the original  $k$ -action.

Thus the category  $\mathcal{C}$  is a  $k$ -linear additive tensor category.

Let  $\mathcal{V}_k$  be the category of finite dimensional vector spaces over  $k$ . The forgetful functor

$$\omega: \mathcal{C} \longrightarrow \mathcal{V}_k$$

is a tensor functor.

REMARK 1.14. — It is not abelian as exemplified in Example 1.4. Namely, the coimage and the image of a morphism are not always isomorphic.

DEFINITION 1.15 (semi-stability [4, p. 116]). — The *slope* of an object  $V$  in  $\mathcal{C}$  is

$$M(V) = \sum_{v \in \mathfrak{M}(k)} M_v(V).$$

An object  $V$  is *semi-stable* if and only if it is not 0 and satisfies the inequality

$$M(W) \leq M(V)$$

for any non-zero subspace  $W$  over  $k$  of  $V$  with the sub- $L$ -filtrations (*i.e.*, for any *subobject*  $W$  of  $V$ ). Equivalently, thanks to Lemma 1.3, it is semi-stable if and only if it is not 0 and

$$M(V) \leq M(Q)$$

for any non-zero quotient space  $Q$  over  $k$  of  $V$  with the quotient  $L$ -filtrations (*i.e.*, for any *quotient object*  $Q$  of  $V$ ).

LEMMA 1.16. — *If  $V$  and  $W \in \mathcal{C}$  are both semi-stable, then so is  $V \otimes W$ .*

*Proof.* — Several proofs exist. See [4], [8], [3], [9], and [5]. Another proof is presented in the next section (Corollary 2.9).  $\square$

DEFINITION 1.17 (category of general Roth systems). — Let  $\mathcal{C}_0^{\text{ss}}$  be the (strictly) full subcategory of  $\mathcal{C}$  whose objects are the semi-stable objects with slope zero (and with integral or rational weights) and the zero-dimensional vector space. It is immediate to check that  $\mathcal{C}_0^{\text{ss}}$  is a  $k$ -linear tensor subcategory.

LEMMA 1.18. — *The category  $\mathcal{C}_0^{\text{ss}}$  is additive.*

*Proof.* — What has to be proved is that the direct sum of objects in  $\mathcal{C}_0^{\text{ss}}$  is semi-stable. Let  $V$  and  $W$  be any objects in  $\mathcal{C}$  and let  $S$  be any subobject of  $V \oplus W$ . From Lemma 1.9, we see

$$M(S) \dim_k S \leq M(V \oplus W) \dim_k (V \oplus W).$$

As we saw earlier in Definition 1.10, the right hand side equals

$$M(V) \dim_k V + M(W) \dim_k W.$$

If  $M(V) = M(W) = 0$ , then

$$M(S) \leq 0 = M(V \oplus W),$$

which means  $V \oplus W \in \mathcal{C}_0^{\text{ss}}$ . Note that when the weights of  $V$  and  $W$  are all integral or rational, the weights of  $V \oplus W$  are respectively also integral or rational by definition.  $\square$

LEMMA 1.19 (see [3, p. 649]). — *The category  $\mathcal{C}_0^{\text{ss}}$  is abelian.*

*Proof.* — Let  $V$  and  $W$  be objects in  $\mathcal{C}_0^{\text{ss}}$  and let  $f \in \text{Hom}(V, W)$ . We have

$$M(\text{Coim } f) \leq M(\text{Im } f),$$

for the induced morphism  $\text{Coim } f \rightarrow \text{Im } f$  is filtered. In addition, the semi-stability of  $V$  and  $W$  implies

$$0 = M(V) \leq M(\text{Coim } f) \quad \text{and} \quad M(\text{Im } f) \leq M(W) = 0.$$

We get

$$M(\text{Coim } f) = M(\text{Im } f) = 0.$$

From Lemma 1.8, we see that  $\text{Coim } f$  and  $\text{Im } f$  are isomorphic. Since a quotient object  $Q$  of  $\text{Coim } f$  is a quotient object of  $V$  and since a subobject  $S$  of  $\text{Im } f$  is a subobject of  $W$ ,

$$M(Q) \geq M(V) = M(\text{Coim } f) \quad \text{and} \quad M(S) \leq M(W) = M(\text{Im } f),$$

which shows  $\text{Coim } f$  and  $\text{Im } f$  are semi-stable. Furthermore, Lemma 1.3 implies

$$M(\text{Ker } f) = M(\text{Coker } f) = 0.$$

We observe similarly that  $\text{Ker } f$  and  $\text{Coker } f$  are semi-stable.  $\square$

REMARK 1.20. — In particular, every element of  $\text{Hom}(V, W)$  is strict (for all  $v \in \mathfrak{M}(k)$ ).

PROPOSITION 1.21. — *The category  $\mathcal{C}_0^{\text{ss}}$  is a neutral Tannakian category.*

*Proof.* — According to the paper [2, Prop. 1.20], the forgetful functor

$$\omega_0^{\text{ss}} : \mathcal{C}_0^{\text{ss}} \longrightarrow \mathcal{V}_k$$

is a fiber functor.  $\square$

## 2. Semi-stability and successive minima

Throughout this section, the field  $k$  will be assumed to be a finite extension of the rational number field.

Let  $V$  be an  $n$ -dimensional vector space over  $k$ ;  $T_1, \dots, T_n$  a basis of  $V$  over  $k$ ; and  $L$  an algebraic extension field of  $k$ . We denote by  $\mathfrak{M}(k)$  the set of places of  $k$ . We fix an extension of each  $v \in \mathfrak{M}(k)$  to  $L$  once for all and denote it by the same letter  $v$ .

Suppose that for each  $v \in \mathfrak{M}(k)$ , we are given  $L$ -linearly independent vectors  $f_{1v}, \dots, f_{nv}$  of  $L \otimes_k V$  and real numbers  $c(1; v), \dots, c(n; v)$ , almost all of which satisfy

$$f_{iv} = T_i \quad \text{and} \quad c(i; v) = 0.$$

Define  $k_{\mathbb{A}}$  as the adèle ring of  $k$  and  $V^*$  as the dual space  $\text{Hom}_k(V, k)$  to  $V$  over  $k$ . For a positive real number  $Q$ , let  $\Pi$  be the parallelotope in  $k_{\mathbb{A}} \otimes_k V^*$  given by

$$|\langle f_{iv}, t_v \rangle|_v \leq Q^{-c(i; v)} \quad (t_v \in k_v \otimes_k V^*).$$

Here  $k_v$  is the completion of  $k$  at  $v \in \mathfrak{M}(k)$ ;  $|\cdot|_v$  is the valuation on the completion  $L_v$  of  $L$  at  $v$  normalized so that

$$\text{if } v \text{ is real, } |2|_v = 2, \quad \text{if } v \text{ is complex, } |2|_v = 2^2 = 4,$$

so that  $\prod_{v \in \mathfrak{M}(k)} |\cdot|_v$  satisfies the product formula on  $k$ ; and  $\langle \cdot, \cdot \rangle$  is the canonical pairing between the elements of  $L_v \otimes_k V$  and of  $L_v \otimes_k V^*$ . We denote by  $\lambda_1, \dots, \lambda_n$  the successive minima for  $\Pi$  with respect to  $V^*$  [1, p. 18]. We regard  $\lambda_i = \lambda_i(Q)$  as a function of the parameter  $Q$ .

For each  $v \in \mathfrak{M}(k)$  and  $j \in \mathbb{R}$ , let  $V_v^j$  be the subspace over  $L$  of  $L \otimes_k V$  spanned by

$$f_{iv} \quad \text{with} \quad c(i; v) \geq j,$$

which leads to a family of  $L$ -filtrations on  $V$  (Definition 1.1) indexed by  $\mathfrak{M}(k)$ . The slope of filtration for  $v \in \mathfrak{M}(k)$  becomes

$$M_v(V) = \frac{1}{n} \sum_{i=1}^n c(i; v)$$

and the slope as an object of the category  $\mathcal{C}$  of linear inequalities (Definition 1.13 and Definition 1.15) is

$$M(V) = \sum_{v \in \mathfrak{M}(k)} M_v(V) = \frac{1}{n} \sum_{v \in \mathfrak{M}(k)} \sum_{i=1}^n c(i; v).$$

REMARK 2.1. — In the paper [3] of Faltings and in the paper [5] of Ferretti, the weight ‘of  $f_{iv}$ ’ is

$$1 + c(i; v).$$

Hence our  $M(V)$  and the invariant  $\mu(V)$  in [3] or [5] are related by the equation

$$\mu(V) = 1 + M(V).$$

REMARK 2.2. — When  $V$  is semi-stable (Definition 1.15), the system  $(f_{i,v}; c(i; v))_{i,v}$  is called a *general ( $v$ -adic) Roth system* (cf. e.g. [6, Satz 1.2]).

THEOREM 2.3 (Minkowski). — *There exist positive real numbers  $A(k, n)$  and  $B(k, n)$  depending on the number field  $k$  and on the dimension  $n$  such that*

$$A(k, n) \leq (\lambda_1 \cdots \lambda_n)^{[k:\mathbb{Q}]} \text{vol}(\Pi) \leq B(k, n).$$

*Proof.* — See [1, Theorems 3 and 6]. □

THEOREM 2.4 (Schmidt, Schlickewei). — *Suppose there exist a natural number  $d$  with  $1 \leq d < n$ , a positive number  $\delta$ , and an unbounded set  $\mathfrak{N}$  of positive real numbers such that*

$$\lambda_d < \lambda_{d+1} Q^{-\delta} \quad (Q \in \mathfrak{N}).$$

*Then there exist a  $d$ -dimensional subspace  $S^*$  of  $V^*$  over  $k$  and an unbounded subset  $\mathfrak{N}'$  of  $\mathfrak{N}$  such that the first  $d$  successive minima for  $\Pi(Q \in \mathfrak{N}')$  are attained in  $S^*$ .*

*Proof.* — See [7, Theorem 3], [6, Satz 2.1], and other papers of Schmidt and Schlickewei. □

LEMMA 2.5. — *The following conditions are equivalent:*

- 1)  $V$  is not semi-stable (Definition 1.15).
- 2) *There exist a natural number  $d$  with  $1 \leq d < n$ , a positive number  $\delta$ , and an unbounded set  $\mathfrak{N}$  of real numbers such that*

$$\lambda_d < \lambda_{d+1} Q^{-\delta} \quad (Q \in \mathfrak{N}).$$

*Proof.* — Suppose  $V$  is not semi-stable. Then there exists an  $m$ -dimensional subspace  $U$  of  $V$  over  $k$  with  $1 \leq m < n$  such that

$$M(V/U) < M(V).$$

By the definition of the quotient filtration (Definition 1.2), there exist for each  $v$ , indices  $j(1; v), \dots, j(n-m; v)$  such that  $f_{j(1;v)v}, \dots, f_{j(n-m;v)v}$  modulo  $U$  form a basis of  $L \otimes_k (V/U)$  and

$$M_v(V/U) = \frac{1}{n-m} \sum_{r=1}^{n-m} c(j(r; v); v).$$

Let

$$(V/U)^* = \{t \in V^* \mid \langle f, t \rangle = 0 \text{ for all } f \in U\}$$

be the dual space to  $V/U$  in  $V^*$  and let  $\delta$  be a positive number such that

$$M(V/U) + n\delta \leq M(V) - n\delta.$$

We have a family of positive constants  $K(v)$  ( $v \in \mathfrak{M}(k)$ ) depending only on  $U$  and  $f_{1v}, \dots, f_{nv}$ , which coincide with 1 except for at most finitely many places  $v$ , such that the  $(n-m)$ -dimensional parallelotope  $\tilde{\Pi}$  in  $k_{\mathbb{A}} \otimes_k (V/U)^*$  given by

$$|\langle f_{j(r;v)v}, t_v \rangle|_v \leq K(v) Q^{-c(j(r;v);v)} \quad (t_v \in k_v \otimes_k (V/U)^*; r = 1, \dots, n-m)$$

is contained in the intersection of  $k_{\mathbb{A}} \otimes_k (V/U)^*$  across  $\Pi$ . Let  $\tilde{\lambda}_1$  be the first minimum for  $\tilde{\Pi}$  with respect to  $(V/U)^*$ . Then

$$\lambda_1 \leq \tilde{\lambda}_1.$$

Since the  $(n-m)$ -dimensional volume of  $Q^{M(V/U)+\frac{1}{2}n\delta}\tilde{\Pi}$  is big for sufficiently large  $Q$ ,

$$\tilde{\lambda}_1 \leq Q^{M(V/U)+\frac{1}{2}n\delta} \quad (Q \gg 0)$$

(Minkowski's theorem). On the other hand, the volume of  $Q^{M(V)-\frac{1}{2}n\delta}\Pi$  is small for sufficiently large  $Q$ . We obtain

$$Q^{M(V)-\frac{1}{2}n\delta} < \lambda_n \quad (Q \gg 0).$$

Consequently

$$\lambda_1 Q^{\frac{1}{2}n\delta} \leq Q^{M(V/U)+n\delta} \leq Q^{M(V)-n\delta} < \lambda_n Q^{-\frac{1}{2}n\delta} \quad (Q \gg 0),$$

that is,

$$\lambda_1 < \lambda_n Q^{-n\delta} \quad (Q \gg 0).$$

For  $Q$  large enough, there is thus an integer  $d(Q) < n$  such that

$$\lambda_{d(Q)} < \lambda_{d(Q)+1} Q^{-\delta};$$

by the box principle, there certainly exists an integer  $d < n$  and an unbounded set  $\mathfrak{N}$  such that for every  $Q \in \mathfrak{N}$ , the above inequality holds.

Conversely, on the latter assumption, Theorem 2.4 tells us that there exist a  $d$ -dimensional subspace  $S^*$  of  $V^*$  over  $k$  and an unbounded subset  $\mathfrak{N}'$  of  $\mathfrak{N}$  such that the first  $d$  successive minima for  $\Pi(Q \in \mathfrak{N}')$  are attained in  $S^*$ . Let  $\check{\Pi}$  be the  $d$ -dimensional parallelotope  $k_{\mathbb{A}} \otimes_k S^* \cap \Pi$  and  $\check{\lambda}_1, \dots, \check{\lambda}_d$  the successive minima for  $\check{\Pi}$  with respect to  $S^*$ . What is stated above amounts to the equalities

$$\check{\lambda}_1 = \lambda_1, \dots, \check{\lambda}_d = \lambda_d \quad (Q \in \mathfrak{N}').$$

By the definition of quotient filtrations (Definition 1.2), there exist indices  $\ell(1;v), \dots, \ell(d;v)$  such that  $f_{\ell(1;v)v}, \dots, f_{\ell(d;v)v}$  are linearly independent when restricted on  $L \otimes_k S^*$  and

$$M_v(V/W) = \frac{1}{d} \sum_{r=1}^d c(\ell(r;v);v),$$

where  $W = \{f \in V \mid \langle f, S^* \rangle = 0\}$ . We have

$$\text{vol}(\check{\Pi}) \leq \check{C} \cdot Q^{-\sum_{v \in \mathfrak{M}(k)} \sum_{r=1}^d c(\ell(r;v);v)}.$$

Here  $\check{C}$  is a constant depending only on  $S^*$  and  $f_{1v}, \dots, f_{nv}$ . In addition, Theorem 2.3 implies

$$A(k, d) \leq (\check{\lambda}_1 \cdots \check{\lambda}_d)^{[k:\mathbb{Q}]} \text{vol}(\check{\Pi}).$$

Putting together, we obtain

$$\frac{A(k, d)}{\check{C}} Q^{dM(V/W)} \leq (\check{\lambda}_1 \cdots \check{\lambda}_d)^{[k:\mathbb{Q}]} = (\lambda_1 \cdots \lambda_d)^{[k:\mathbb{Q}]} \quad (Q \in \mathfrak{N}').$$

On the other hand, applying Theorem 2.3 to  $\Pi$ , we deduce

$$\begin{aligned} B(k, n) &\geq (\lambda_1 \cdots \lambda_d \lambda_{d+1} \cdots \lambda_n)^{[k:\mathbb{Q}]} \text{vol}(\Pi) \\ &\geq (\lambda_1 \cdots \lambda_d [\lambda_d Q^\delta]^{n-d})^{[k:\mathbb{Q}]} \text{vol}(\Pi) \quad (Q \in \mathfrak{N}) \\ &\geq ([\lambda_1 \cdots \lambda_d]^{n/d} Q^{(n-d)\delta})^{[k:\mathbb{Q}]} \text{vol}(\Pi) \\ &\geq \frac{A(k, d)^{n/d}}{\check{C}^{n/d}} Q^{nM(V/W) + [k:\mathbb{Q}](n-d)\delta} C Q^{-nM(V)} \quad (Q \in \mathfrak{N}'), \end{aligned}$$

where  $C$  is another constant which depends only on  $f_{1v}, \dots, f_{nv}$ . Rewriting the inequality,

$$Q^{M(V/W)} \leq \frac{B(k, n)^{1/n} \check{C}^{1/d}}{C^{1/n} A(k, d)^{1/d}} Q^{-[k:\mathbb{Q}](1-d/n)\delta + M(V)} \quad (Q \in \mathfrak{N}').$$

We get

$$Q^{M(V/W)} < Q^{M(V)} \quad (Q \in \mathfrak{N}', Q \gg 0).$$

Therefore  $V$  is not semi-stable. □

COROLLARY 2.6. — *The following statements are equivalent:*

- 1)  $V$  is semi-stable.
- 2) For any natural number  $d$  with  $1 \leq d < n$  and any  $\delta > 0$ , if the inequality

$$\lambda_d < \lambda_{d+1} Q^{-\delta}$$

is valid, then  $Q$  is bounded.

LEMMA 2.7. — *One has*

$$\lim_{Q \rightarrow \infty} \frac{1}{\dim_k V} \sum_{i=1}^n \frac{\log \lambda_i}{\log Q} = \frac{M(V)}{[k:\mathbb{Q}]}.$$

*Proof.* — We see

$$\text{vol}(\Pi) = C Q^{-\sum_{v \in \mathfrak{M}(k)} \sum_{i=1}^n c(i;v)}.$$

Here  $C$  is a constant depending only on  $f_{i,v}$  and fixed extensions of  $v$  to  $L$ . Substituting this expression for  $\text{vol}(\Pi)$  in the inequalities of Minkowski's theorem above and taking the logarithms, we obtain

$$\log A(k, n) \leq [k : \mathbb{Q}] \sum_{i=1}^n \log \lambda_i + \log C - \sum_{v \in \mathfrak{M}(k)} \sum_{i=1}^n c(i;v) \log Q \leq \log B(k, n).$$

Dividing by  $[k : \mathbb{Q}]n \log Q$ ,

$$\frac{\log A(k, n)}{[k : \mathbb{Q}]n \log Q} \leq \frac{1}{n} \sum_{i=1}^n \frac{\log \lambda_i}{\log Q} + \frac{\log C}{[k : \mathbb{Q}]n \log Q} - \frac{M(V)}{[k : \mathbb{Q}]} \leq \frac{\log B(k, n)}{[k : \mathbb{Q}]n \log Q}.$$

Let  $Q$  go to infinity! □

**THEOREM 2.8.** — *The following are equivalent:*

- 1)  $V$  is semi-stable.
- 2) One has

$$\lim_{Q \rightarrow \infty} \frac{\log \lambda_1}{\log Q} = \dots = \lim_{Q \rightarrow \infty} \frac{\log \lambda_n}{\log Q} = \frac{M(V)}{[k : \mathbb{Q}]}.$$

*Proof.* — Combine Corollary 2.6 with Lemma 2.7. □

**COROLLARY 2.9.** — *Let  $V'$  be another vector space over  $k$  with a family of  $L$ -filtrations indexed by  $\mathfrak{M}(k)$  as those on  $V$ . If  $V$  and  $V'$  are both semi-stable, then so is  $V \otimes V'$ .*

*Proof.* — We denote by  $m$  the dimension of  $V'$  and by  $V'^*$  the dual vector space to  $V'$  over  $k$ . Let  $g_{1,v}, \dots, g_{m,v}$  and  $d(1;v), \dots, d(m;v)$  be respectively bases of  $L \otimes_k V'$  and real numbers which define the given filtrations on  $V'$ . Let  $\Pi'$  be the parallelotope in  $k_{\mathbb{A}} \otimes_k V'^*$  given by the family  $(g_{j,v}, d(j;v))_{j,v}$  and  $\lambda'_m$  the last minimum for  $\Pi'$  with respect to  $V'^*$ . By definition, the filtrations on  $V \otimes V'$  are given by the tuple

$$(f_{i,v} \otimes g_{j,v}, c(i;v) + d(j;v))_{i,j,v},$$

so the parallelotope  $\Pi^{\otimes}$  in  $k_{\mathbb{A}} \otimes_k (V^* \otimes_k V'^*)$  we take into consideration is determined by the inequalities

$$\left| \left\langle f_{i,v} \otimes g_{j,v}, \sum_h t_v^{(h)} \otimes s_v^{(h)} \right\rangle \right|_v = \left| \sum_h \langle f_{i,v}, t_v^{(h)} \rangle \cdot \langle g_{j,v}, s_v^{(h)} \rangle \right|_v \leq Q^{-c(i;v) - d(j;v)},$$

where the superscript  $(h)$  runs through a finite set:

$$\sum_h t_v^{(h)} \otimes s_v^{(h)} \in (k_v \otimes_k V^*) \otimes_{k_v} (k_v \otimes_k V'^*) \simeq k_v \otimes_k (V^* \otimes_k V'^*).$$

We denote by  $\lambda_1^\otimes, \dots, \lambda_{mn}^\otimes$  its successive minima. Choose any set of  $k$ -linearly independent elements  $x^1, \dots, x^n$  of  $V^* \cap \lambda_n \Pi$  and any set of  $k$ -linearly independent elements  $y^1, \dots, y^m$  of  $V'^* \cap \lambda'_m \Pi'$ . The family  $(x^a \otimes y^b)_{a,b}$  is a basis of  $V^* \otimes_k V'^*$ . Because of the choice, for an Archimedean place  $v \in \mathfrak{M}(k)$

$$|\langle f_{i v}, x^a \rangle|_v \leq \lambda_n Q^{-c(i;v)}, \quad |\langle g_{j v}, y^b \rangle|_v \leq \lambda'_m Q^{-d(j;v)},$$

hence

$$|\langle f_{i v} \otimes g_{j v}, x^a \otimes y^b \rangle|_v \leq \lambda_n \lambda'_m Q^{-[c(i;v)+d(j;v)]}.$$

Similarly, for a non-Archimedean place  $v \in \mathfrak{M}(k)$

$$|\langle f_{i v}, x^a \rangle|_v \leq Q^{-c(i;v)}, \quad |\langle g_{j v}, y^b \rangle|_v \leq Q^{-d(j;v)},$$

hence

$$|\langle f_{i v} \otimes g_{j v}, x^a \otimes y^b \rangle|_v \leq Q^{-[c(i;v)+d(j;v)]}.$$

This means all  $x^a \otimes y^b$  are in  $\lambda_n \lambda'_m \Pi^\otimes$  and we obtain inequalities

$$\lambda_1^\otimes \leq \dots \leq \lambda_{mn}^\otimes \leq \lambda_n \lambda'_m.$$

Taking logarithms of each term and dividing by  $\log Q$ , we get

$$\frac{\log \lambda_1^\otimes}{\log Q} \leq \dots \leq \frac{\log \lambda_{mn}^\otimes}{\log Q} \leq \frac{\log \lambda_n}{\log Q} + \frac{\log \lambda'_m}{\log Q}.$$

On the assumption that  $V$  and  $V'$  are semi-stable, the expression at the right end converges when  $Q$  goes to infinity, that is,

$$\limsup_{Q \rightarrow \infty} \frac{\log \lambda_1^\otimes}{\log Q} \leq \dots \leq \limsup_{Q \rightarrow \infty} \frac{\log \lambda_{mn}^\otimes}{\log Q} \leq \frac{M(V) + M(V')}{[k : \mathbb{Q}]} = \frac{M(V \otimes V')}{[k : \mathbb{Q}]}.$$

On the other hand, in any case, we have by Lemma 2.7

$$\lim_{Q \rightarrow \infty} \frac{1}{mn} \sum_{w=1}^{mn} \frac{\log \lambda_w^\otimes}{\log Q} = \frac{M(V \otimes V')}{[k : \mathbb{Q}]}.$$

Therefore

$$\frac{M(V \otimes V')}{[k : \mathbb{Q}]} \leq \frac{1}{mn} \sum_{w=1}^{mn} \limsup_{Q \rightarrow \infty} \frac{\log \lambda_w^\otimes}{\log Q},$$

accordingly

$$\limsup_{Q \rightarrow \infty} \frac{\log \lambda_w^\otimes}{\log Q} = \frac{M(V \otimes V')}{[k : \mathbb{Q}]} \quad (w = 1, \dots, mn).$$

We now study the inferior limits. We see by Lemma 2.7, for any positive number  $\delta$ , if  $Q$  is sufficiently large, then

$$\begin{aligned} \frac{M(V \otimes V')}{[k : \mathbb{Q}]} - \frac{\delta}{(mn)^2} &< \frac{1}{mn} \sum_{w=1}^{mn} \frac{\log \lambda_w^\otimes}{\log Q} \\ &< \frac{1}{mn} \frac{\log \lambda_1^\otimes}{\log Q} + \frac{1}{mn} \sum_{w=2}^{mn} \left( \limsup_{Q \rightarrow \infty} \frac{\log \lambda_w^\otimes}{\log Q} + \frac{\delta}{mn} \right) \\ &= \frac{1}{mn} \frac{\log \lambda_1^\otimes}{\log Q} + \frac{mn-1}{mn} \left( \frac{M(V \otimes V')}{[k : \mathbb{Q}]} + \frac{\delta}{mn} \right). \end{aligned}$$

Rewriting the inequality, we obtain

$$\frac{M(V \otimes V')}{[k : \mathbb{Q}]} - \delta < \frac{\log \lambda_1^\otimes}{\log Q} \quad (Q \gg 0),$$

hence

$$\frac{M(V \otimes V')}{[k : \mathbb{Q}]} \leq \liminf_{Q \rightarrow \infty} \frac{\log \lambda_1^\otimes}{\log Q} \leq \dots \leq \liminf_{Q \rightarrow \infty} \frac{\log \lambda_{mn}^\otimes}{\log Q}.$$

The conclusion follows from the trivial inequality

$$\liminf_{Q \rightarrow \infty} \frac{\log \lambda_{mn}^\otimes}{\log Q} \leq \limsup_{Q \rightarrow \infty} \frac{\log \lambda_{mn}^\otimes}{\log Q} = \frac{M(V \otimes V')}{[k : \mathbb{Q}]} \quad \square$$

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