Bull. Soc. math. France
133 (3), 2005, p. 331–348

KLOOSTERMAN-FOURIER INVERSION FOR SYMMETRIC MATRICES

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ABSTRACT. — We formulate a Kloosterman transform on the space of generalized Kloosterman integrals on symmetric matrices, and obtain an inversion formula. The formula is a step towards a fundamental lemma of the Jacquet type. At the same time it hints towards a conjectural relative trace formula identity, associated with the metaplectic correspondence.

RÉSUMÉ (*Inversion de Kloosterman-Fourier pour les matrices symétriques*)
Nous définissons une transformation de Kloosterman sur l'espace des intégrales de Kloosterman généralisées sur les matrices symétriques et nous obtenons une formule d'inversion. Cette formule est une étape vers un lemme fondamental de type de Jacquet. En même temps, elle indique une identité conjecturale de la formule des traces relative associée à la correspondance métaplectique.

1. Introduction

Let F be a non-archimedean local field, $\mathcal{O}_F$ the ring of integers in F and $\wp$ the maximal ideal of $\mathcal{O}_F$. Let $|\cdot|$ denote the normalized absolute value on F so that for a uniformizer ϖ of F we have $|\varpi|^{-1} = \#(\mathcal{O}_F/\wp)$ is the size of the

Texte reçu le 23 avril 2003, révisé le 12 février 2004, accepté le 3 mars 2004

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2000 Mathematics Subject Classification. — 11F70, 11L05.

Key words and phrases. — Kloosterman integrals, Kloosterman transform, Jacquet's relative trace formula.

residual field. Let ψ be a non-trivial additive character of F. We recall the formula

$$(1)\qquad \int \widehat{f}(x)\psi(ax^2)\,\mathrm{d}x = |2a|^{-\frac{1}{2}}\gamma(a,\psi)\int f(x)\psi(-a^{-1}x^2)\,\mathrm{d}x$$

which we use to define the Weil constant γ. Here $a \in F^\times$, $f \in C_c^\infty(F)$ is a Schwartz function on F and $\widehat{f}$ is the Fourier transform of f defined by

$$\widehat{f}(x) = \int f(y)\psi(-2xy)\,\mathrm{d}y.$$

The measure $\mathrm{d}x$ is the self dual Haar measure on F with respect to ψ. Thus, it satisfies

$$(2)\qquad f(0) = |2|\int \widehat{f}(x)\,\mathrm{d}x.$$

Let $N = N_n$ be the subgroup of upper triangular unipotent matrices in $\mathrm{GL}_n(F)$. Define the non-degenerate character $\theta = \theta_n$ of N by

$$\theta\Big(u\Big) = \psi\Big(\sum_{i=1}^{n-1} x_{i,i+1}\Big)$$

where $u = (x_{i,j}) \in N$. The Haar measure $\mathrm{d}x$ on F determines a Haar measure on N and a self dual Haar measure on any finite dimensional F-vector space. We will use the measures determined by $\mathrm{d}x$ unless otherwise specified. Denote by $M_{m\times n}(F)$ the set of all $m \times n$ matrices with entries in F. Let

$$M_n(F) = M_{n\times n}(F)$$

and denote by $\mathcal{S} = \mathcal{S}_n$ the space of symmetric matrices

$$\mathcal{S} = \big\{X \in M_n(F);\ {}^tX = X\big\}.$$

We consider the action $(u,s) \mapsto {}^tusu$ of N on $\mathcal{S}$.

Definition 1.1. — An element $s \in \mathcal{S}$ is called *relevant* if θ is trivial on the stabilizer N_s of s in N.

Our objects of interest are the generalized Kloosterman integrals

$$(3)\qquad \omega[\Phi,\psi;s] = \int_{N_s\backslash N} \Phi({}^tusu)\theta(u^2)\,\mathrm{d}u$$

for a relevant $s \in \mathcal{S}, \Phi \in C_c^\infty(\mathcal{S})$. Let

$$S_n = \mathcal{S}_n \cap \mathrm{GL}_n(F).$$

The orbits in S_n are fully described in [6]. To describe a set of representatives for all orbits in S_n we view the elements of the Weyl group as permutation matrices in $\mathrm{GL}_n(F)$. Thus a complete set of representatives for the orbits in S_n is the set of all wa, where w is the longest element in the Weyl group of a standard Levi subgroup M of $\mathrm{GL}_n(F)$ and a is in the center of M. All relevant orbits in $\mathcal{S}_n$ with zero determinant contain an element of the form $\left(\begin{smallmatrix} s & \\ & 0\end{smallmatrix}\right)$, where $s \in S_{n-1}$. This is proved in [3] for Hermitian matrices. The proof for symmetric matrices is identical and we omit it here. Representatives for the relevant orbits of zero determinant are therefore given as above in terms of representatives of orbits in S_{n-1}. When $w = 1$ and a is a diagonal matrix, the stabilizer of a in N_n is trivial.

In this sense the diagonal matrices are representatives of the largest orbits. In a sense explained in [3] and [2], the Kloosterman integrals for smaller orbits, *i.e.* with $w \neq 1$ are determined by Kloosterman integrals of the largest orbits. For this reason, our main concern in this work is the space of Kloosterman integrals, restricted to the relevant diagonal matrices. These are of the form $a = \mathrm{diag}(a_1, \dots, a_n)$ where $a_1, \dots, a_{n-1} \in F^\times$ and $a_n \in F$. We will denote by $\omega[\Phi, \psi; a_1, \dots, a_n]$ the Kloosterman integral $\omega[\Phi, \psi; \mathrm{diag}(a_1, \dots, a_n)]$.

To state our main theorem it will be convenient to introduce a normalization. We introduce the normalizing factors

$$\sigma_n(a) = a_1^{n-1} a_2^{n-2} \cdots a_{n-1},\ \Gamma_n(a, \psi) = \gamma(a_1, \psi)^{n-1} \gamma(a_2, \psi)^{n-2} \cdots \gamma(a_{n-1}, \psi).$$

The normalized Kloosterman integral is

$$\widetilde{\omega}^\psi[\Phi, \psi; a_1, \dots, a_n] = \Gamma_n(-a, \psi) \big|\sigma_n(a)\big|^{\frac{1}{2}} \omega[\Phi, \psi; a_1, \dots, a_n]. \tag{4}$$

The purpose of the notation $\widetilde{\omega}^\psi$ is to emphasize the dependence of the normalization on the character ψ. Let $\Omega_{\psi,n}$ be the space of functions ω on $(F^\times)^{n-1} \times F$ of the form

$$\omega(a_1, \dots, a_n) = \widetilde{\omega}^\psi[\Phi, \psi; a_1, \dots, a_n]$$

for some $\Phi \in C_c^\infty(\mathcal{S})$. Denote by $[\cdot,\cdot] : F^\times \times F^\times \mapsto \{\pm 1\}$ the quadratic Hilbert symbol. It is defined by the condition $[a, b] = 1$ iff a is representable by the quadratic form $a = x^2 - by^2$. We define the Kloosterman transform $K_{\psi,n}$ on $\Omega_{\psi,n}$ by

$$\begin{aligned} K_{\psi,n}\omega(a_1, \dots, a_n) = \int \omega(p_1, \dots, p_n) \psi\Big(-\sum_{i=1}^{n} p_i a_{n+1-i} + \sum_{i=1}^{n-1} \frac{1}{p_i a_{n-i}} \Big) \\ \times \Big(\prod_{i=1}^{n-1} \prod_{j=1}^{n-i} [a_i, p_j] \Big) \mathrm{d}p_n \mathrm{d}p_{n-1} \cdots \mathrm{d}p_1 \end{aligned} \tag{5}$$

where the integral over $p_i \in F$, $i = 1, \dots, n$ is only iterated. Although the integrand is *a priori* only defined for $p_1, \dots, p_{n-1} \in F^\times$ and $p_n \in F$ we make sense of (5) in the proof of theorem 1.2. Denote by $w_n \in \mathrm{GL}_n(F)$ the permutation matrix with a unit anti-diagonal. For any matrix $X \in M_n(F)$ we will denote by $\mathrm{Tr}(X)$ the trace of X. For a function $\Phi \in C_c^\infty(\mathcal{S})$ let

$$\hat{\Phi}(s) = \int_{\mathcal{S}} \Phi(t) \psi\big(-\mathrm{Tr}(st) \big) \mathrm{d}t \tag{6}$$

be the standard Fourier transform of Φ. We will consider the Fourier transform

$$\check{\Phi}(s) = \hat{\Phi}(w_n s w_n) \tag{7}$$

of Φ. Our main theorem is

THEOREM 1.2. — *The integral* (5) *defining the Kloosterman transform on* $\Omega_{\psi,n}$ *is a convergent iterated integral. Moreover, let* $\Phi \in C_c^\infty(\mathcal{S})$ *then,*

$$\begin{aligned}(8)\qquad &\big(K_{\psi,n}\widetilde{\omega}^{\psi}[\Phi,\psi;.]\big)(a_1,\dots,a_n)\\ &\qquad = |2|^{\frac{1}{2}n(n-1)}\gamma(1,\bar{\psi})^{\frac{1}{2}n(n-1)}\widetilde{\omega}^{\bar{\psi}}[\check{\Phi},\bar{\psi};a_1,\dots,a_n].\end{aligned}$$

The theorem shows in particular that $K_{\psi,n}$ is a transform from $\Omega_{\psi,n}$ to $\Omega_{\bar{\psi},n}$. Combining the theorem with Fourier inversion on $\mathcal{S}$ we obtain the inversion of the Kloosterman transform.

COROLLARY 1.3. — *The Kloosterman transform satisfies*

$$K_{\bar{\psi},n} \circ K_{\psi,n} = |2|^{n(n-1)}\,\mathrm{Id}$$

where Id *is the identity map on* $\Omega_{\psi,n}$.

The motivation to the problem lies in a conjectural trace formula identity of the Jacquet type. The identity is concerned with the metaplectic correspondence of [1]. It is a lifting of genuine automorphic representations of the metaplectic double cover $\widetilde{\mathrm{GL}}_n$ of GL_n to automorphic representations of GL_n. Jacquet suggests the following characterization for the image of this lift: A cuspidal automorphic representation of GL_n with trivial central character is a lifting from $\widetilde{\mathrm{GL}}_n$ iff it is (H,χ)-distinguished for some subgroup H of orthogonal similitudes of GL_n and some idéle class quadratic character χ.

For more detail and definitions we refer to [6]. This characterization of the image of metaplectic correspondence will follow from the relative trace formula identity

$$\int K_\Phi(u)\theta(u^2)\mathrm{d}u = \int K_f(u_1,u_2)\theta(u_1u_2)\mathrm{d}u_1\mathrm{d}u_2. \tag{9}$$

Here k is a global field. The integration is over $u,u_1,u_2 \in N_n(k)\backslash N_n(\mathbb{A}_k)$, where on the right hand side N_n is viewed as its splitting in $\widetilde{\mathrm{GL}}_n$, K_f and K_Φ are kernel functions depending on the quadratic character χ, of operators corresponding to the smooth functions of compact support f on $\widetilde{\mathrm{GL}}_n(\mathbb{A}_k)$ and Φ on $S_n(\mathbb{A}_k)$. Again for more details we refer to [6]. The fundamental lemma for this situation is a matching of Kloosterman integrals. If Φ_0 is the characteristic function of $\mathcal{S}\cap K$ where $K = \mathrm{GL}_n(\mathcal{O}_F)$ is the standard maximal compact of $\mathrm{GL}_n(F)$, then $\omega[\Phi_0,\psi;a]$ matches in an appropriate way, an integral of the form

$$\int_{N\times N}\Psi_0({}^tu_1au_2)\theta(u_1u_2)\mathrm{d}u_1\mathrm{d}u_2 \tag{10}$$

where Ψ_0 is the unit element of the genuine spherical Hecke algebra of $\widetilde{\mathrm{GL}}_n(F)$. In [6], Mao proved the fundamental lemma for the case $n = 3$ by brute force computation. In [4], Jacquet developed a method to prove Kloosterman integral identities of the above type. The method requires in both sides an inversion formula for a Fourier-Kloosterman transform on the space of Kloosterman integrals. For the case of a quadratic extension the inversion formulas are obtained in [3] and the method is carried out in [4] to prove the identity of Kloosterman integrals which serves as a fundamental lemma for a relative trace formula. In this work we provide a step towards the fundamental lemma associated with the trace formula (9). At the same time, the formula (5) and mainly the oscillating factor $\prod_{i=1}^{n-1}\prod_{j=1}^{n-i}[a_i, p_j]$ in it, hint to a relation with the metaplectic group. If σ is the 2-cocycle that defines multiplication in $\widetilde{\mathrm{GL}}_n(F)$ as defined in [5], then for $a = \mathrm{diag}(a_1, \ldots, a_n)$ and $p = \mathrm{diag}(p_1, \ldots, p_n)$ we have

$$\sigma(a, w_n p w_n) = \prod_{i=1}^{n-1}\prod_{j=1}^{n-i}[a_i, p_j]. \tag{11}$$

The rest of this manuscript is organized as follows: The main tool we use to prove Theorem 1.2 is Weil's formula. In Chapter 2 we write it in a form convenient for our needs. We then prove the theorem by induction. Chapter 3 provides an inversion formula for some intermediate integrals designed to use an inductive argument. In Chapter 4 the inductive step is carried out to finish the proof of the inversion. Chapter 5 provides a much simpler formula associated with the smallest orbits. We present it here, since once the analogous results for the metaplectic case will be obtained, the method of Jacquet requires this formula in order to prove smooth matching. The proof of the inversion formula closely follows the guidelines of [3], the new ingredient is the occurrence of the Hilbert symbol in the Kloosterman transform. The problem was suggested to me by Jacquet. For the project and for much help and support, I am most thankful to him. Most of this work was written during my visit at IHÉS. I thank the IHÉS for a very pleasant and productive visit.

2. Weil's formula

Let

$$V_{n,m} = \left\{ \begin{pmatrix} 0_n & X \\ {}^tX & 0_m \end{pmatrix};\ X \in M_{n\times m}(F) \right\}.$$

We view $V_{n,m}$ as a self-dual space via the pairing

$$\left(\begin{pmatrix} 0_n & X \\ {}^tX & 0_m \end{pmatrix}, \begin{pmatrix} 0_n & {}^tY \\ Y & 0_m \end{pmatrix} \right) \longmapsto \mathrm{Tr}\left[-\begin{pmatrix} 0_n & X \\ {}^tX & 0_m \end{pmatrix}\begin{pmatrix} 0_n & {}^tY \\ Y & 0_m \end{pmatrix} \right] \tag{12}$$

where $X \in M_{n\times m}(F)$ and $Y \in M_{m\times n}(F)$. The Fourier transform of a function $\Phi \in C_c^\infty(V_{n,m})$ is defined by

$$\hat{\Phi}\begin{pmatrix} 0_n & {}^tX \\ X & 0_m \end{pmatrix} = \int_{M_{n\times m}} \Phi\begin{pmatrix} 0_n & Y \\ {}^tY & 0_m \end{pmatrix} \psi\big[-2\operatorname{Tr}(XY)\big]\,\mathrm{d}Y, \tag{13}$$

$X \in M_{m\times n}(F)$. We recall Weil's formula for the Fourier transform of a character of second order for the space $V_{n,m}$. Taking our definition of the Fourier transform into acount, from [7] we have that for all $A \in S_n, C \in S_m$ there is a constant $\gamma_m^n(A,C,\psi)$ such that for all $\Phi \in C_c^\infty(V_{n,m})$

$$\begin{aligned}\int_{M_{n\times m}(F)} &\Phi\begin{pmatrix} 0_n & X \\ {}^tX & 0_m \end{pmatrix} \psi\big[\operatorname{Tr}(C{}^tXAX)\big]\,\mathrm{d}X \\ &= |2|^{\frac12 mn} \cdot |\det A|^{-\frac12 m} \cdot |\det C|^{-\frac12 n} \\ &\quad \times \gamma_m^n(A,C,\psi) \int_{M_{m\times n}(F)} \hat{\Phi}\begin{pmatrix} 0_n & {}^tZ \\ Z & 0_m \end{pmatrix} \psi\big[-\operatorname{Tr}(C^{-1}ZA^{-1}{}^tZ)\big]\,\mathrm{d}Z.\end{aligned} \tag{14}$$

For $P \in M_{m\times n}(F)$ the Fourier transform of the function

$$\begin{pmatrix} 0_n & X \\ {}^tX & 0_m \end{pmatrix} \longmapsto \Phi\Big[\begin{pmatrix} 0_n & X \\ {}^tX & 0_m \end{pmatrix}\Big] \psi\big[\operatorname{Tr}(PX)\big] \tag{15}$$

$X \in M_{n\times m}(F)$, is the function

$$\begin{pmatrix} 0_n & {}^tZ \\ Z & 0_m \end{pmatrix} \longmapsto \hat{\Phi}\begin{pmatrix} 0_n & {}^t(Z - \frac12 P) \\ Z - \frac12 P & 0_m \end{pmatrix} \tag{16}$$

$Z \in M_{m\times n}(F)$. Applying Weil's formula (14) to this function, we get after the change of variables $Z \mapsto Z + \frac12 P$ that

$$\begin{aligned}\int_{M_{n\times m}(F)} &\Phi\Big[\begin{pmatrix} 0_n & X \\ {}^tX & 0_m \end{pmatrix}\Big] \psi\big[\operatorname{Tr}(PX) + \operatorname{Tr}(C{}^tXAX)\big]\,\mathrm{d}X \\ &= |2|^{\frac12 mn} |\det A|^{-\frac12 m} \cdot |\det C|^{-\frac12 n} \gamma_m^n(A,C,\psi) \int_{M_{m\times n}(F)} \hat{\Phi}\begin{pmatrix} 0_n & {}^tZ \\ Z & 0_m \end{pmatrix} \\ &\qquad \times \psi\big[-\operatorname{Tr}\big(C^{-1}(Z + \tfrac12 P)A^{-1}{}^t(Z + \tfrac12 P)\big)\big]\,\mathrm{d}Z.\end{aligned} \tag{17}$$

We remark that in the case $m = n = 1$ the Weil constant is the 1-dimensional Weil constant defined in (1)

$$\gamma_1^1(A,C,\psi) = \gamma(AC,\psi). \tag{18}$$

For general m and n we can describe the Weil constant in terms of the 1-dimensional case.

LEMMA 2.1. — *Let* $A = {}^tg_1 \operatorname{diag}(a_1,\ldots,a_n) g_1$ *and* $C = g_2 \operatorname{diag}(c_1,\ldots,c_m) {}^tg_2$, *where* $A \in S_n, C \in S_m, g_1 \in \mathrm{GL}_n(F)$ *and* $g_2 \in \mathrm{GL}_m(F)$, *then*

$$\gamma_m^n(A,C,\psi) = \prod_{\substack{1\le i\le n \\ 1\le j\le m}} \gamma(a_i c_j,\psi). \tag{19}$$

Proof. — We start with the case $m=1$. Since $C_c^\infty(V_{n,1}) \simeq C_c^\infty(F)^{\otimes n}$, it is enough in the definition of γ_1^n to consider functions Φ of the form $\Phi(v) = \prod_{i=1}^n f_i(x_i)$ where $v \in V_{n,1}$ is of the form $v = \begin{pmatrix} 0_n & X \\ {}^tX & 0 \end{pmatrix}$, $X = {}^t(x_1, \dots, x_n) \in F^n$ and $f_i \in C_c^\infty(F)$. Note that in this case $\hat{\Phi}(v) = \prod_{i=1}^n \hat{f}_i(x_i)$, therefore if $A = \mathrm{diag}(a_1, \dots, a_n)$ the factorization of $\gamma_1^n(A,1,\psi)$ is straightforward from (1) and (2). Indeed

$$\begin{aligned}\int \Phi\begin{pmatrix} 0_n & X \\ {}^tX & 0 \end{pmatrix}\psi\big[\mathrm{Tr}({}^tXAX\big]\,\mathrm{d}X &= \prod_{i=1}^n \int f_i(x_i)\psi(a_i x_i^2)\,\mathrm{d}x_i \\ &= |2|^{\frac12 n}\prod_{i=1}^n \Big(|a_i|^{-\frac12}\gamma(a_i,\psi)\int \hat{f}_i(x_i)\psi(-a_i^{-1}x_i^2)\,\mathrm{d}x_i\Big) \\ &= |2|^{\frac12 n}|\det A|^{-\frac12}\Big(\prod_{i=1}^n \gamma(a_i,\psi)\Big)\int \hat{\Phi}\begin{pmatrix} 0_n & {}^tZ \\ Z & 0 \end{pmatrix}\psi\big[-\mathrm{Tr}(ZA^{-1}{}^tZ)\big]\,\mathrm{d}Z,\end{aligned}$$

therefore in this case indeed we get

$$\gamma_1^n(A,1,\psi) = \prod_{i=1}^n \gamma(a_i,\psi). \tag{20}$$

Let $g \in \mathrm{GL}_n(F)$ be such that $A = {}^tgag$ with $a = \mathrm{diag}(a_1, \dots, a_n)$, for any $A \in S_n$. Note that the Fourier transform of the function

$$\begin{pmatrix} 0_n & X \\ {}^tX & 0_m \end{pmatrix} \longmapsto \Phi\begin{pmatrix} 0_n & g^{-1}X \\ {}^tX{}^tg^{-1} & 0 \end{pmatrix}$$

is the function

$$\begin{pmatrix} 0_n & {}^tZ \\ Z & 0_m \end{pmatrix} \longmapsto |\det g|\cdot\hat{\Phi}\begin{pmatrix} 0_n & {}^tg\,{}^tZ \\ Zg & 0 \end{pmatrix}$$

and therefore using the changes of variables $X \mapsto g^{-1}X$, $Z \mapsto Zg^{-1}$ and applying (20) for a we get

$$\begin{aligned}\int\Phi\begin{pmatrix} 0_n & X \\ {}^tX & 0 \end{pmatrix}\psi\big[\mathrm{Tr}({}^tXAX)\big]\,\mathrm{d}X &= |\det g|^{-1}\int\Phi\begin{pmatrix} 0_n & g^{-1}X \\ {}^tX\,{}^tg^{-1} & 0 \end{pmatrix}\psi\big[\mathrm{Tr}({}^tXaX)\big]\,\mathrm{d}X \\ &= |2|^{\frac12 n}|\det a|^{-\frac12}\Big(\prod_{i=1}^n \gamma(a_i,\psi)\Big)\int \hat{\Phi}\begin{pmatrix} 0_n & {}^tg\,{}^tZ \\ Zg & 0 \end{pmatrix}\psi\big[-\mathrm{Tr}(Za^{-1}{}^tZ)\big]\,\mathrm{d}Z \\ &= |2|^{\frac12 n}|\det g|^{-1}\cdot|\det a|^{-\frac12}\Big(\prod_{i=1}^n \gamma(a_i,\psi)\Big)\int \hat{\Phi}\begin{pmatrix} 0_n & {}^tZ \\ Z & 0 \end{pmatrix}\psi[-\mathrm{Tr}(ZA^{-1}{}^tZ)]\,\mathrm{d}Z.\end{aligned}$$

Since $\det A = \det g^2 \det a$ we get (20) for any $A \in S_n$. For a general m, if $C = \mathrm{diag}(c_1, \dots, c_m)$ is diagonal then

$$\mathrm{Tr}(C{}^tXAX) = \sum_{j=1}^m \mathrm{Tr}({}^tXc_jAX).$$

Therefore, as in the case $m=1$ the fact that $C_c^\infty(V_{n,m})\simeq C_c^\infty(V_{n,1})^{\otimes m}$ implies that

$$\gamma_m^n(A,C,\psi)=\prod_{j=1}^{m}\gamma_1^n(c_jA,1,\psi) \tag{21}$$

and (19) then follows from the case $m=1$. For any A and C as in the statement of the lemma, since

$$\mathrm{Tr}(C^tXAX)=\mathrm{Tr}(XC^tXA)$$

we may compute as before, denoting $c=\mathrm{diag}(c_1,\dots,c_m)$ and using the changes of variables $X\mapsto Xg_2^{-1}$ and $Z\mapsto g_2^{-1}Z$ that

$$\begin{aligned}\int \Phi\begin{pmatrix}0_n & X\\ {}^tX & 0\end{pmatrix}\psi\big[\mathrm{Tr}(C^tXAX)\big]\,\mathrm{d}X\\
&=|\det g_2|^{-n}\int \Phi\begin{pmatrix}0_n & Xg_2^{-1}\\ {}^tg_2^{-1}\,{}^tX & 0\end{pmatrix}\psi\big[\mathrm{Tr}(c^tXAX)\big]\,\mathrm{d}X\\
&=|2|^{\frac12 mn}\cdot|\det A|^{-\frac12 m}\cdot|\det c|^{-\frac12 n}\cdot\gamma_m^n(A,c,\psi)\\
&\qquad\int\hat\Phi\begin{pmatrix}0_n & {}^tZ{}^tg_2\\ g_2Z & 0\end{pmatrix}\psi[-\mathrm{Tr}(c^{-1}ZA^{-1}{}^tZ)\big]\,\mathrm{d}Z\\
&=|2|^{\frac12 mn}\cdot|\det g_2|^{-n}\cdot|\det A|^{-\frac12 m}\cdot|\det c|^{-\frac12 n}\cdot\gamma_m^n(A,c,\psi)\\
&\qquad\int\hat\Phi\begin{pmatrix}0_n & {}^tZ\\ Z & 0\end{pmatrix}\psi[-\mathrm{Tr}(C^{-1}ZA^{-1}{}^tZ)\big]\,\mathrm{d}Z.\end{aligned}$$

We then have

$$\gamma_m^n(A,C,\psi)=\gamma_m^n(A,c,\psi)$$

and the lemma follows from the case when C is diagonal. □

The symmetry on the right hand side of (19) implies that

$$\gamma_m^n(A,C,\psi)=\gamma_n^m(C,A,\psi). \tag{22}$$

From (1) it is easy to see that the one-dimensional Weil constant satisfies

$$\gamma(a,\psi)=\gamma(-a,\bar\psi). \tag{23}$$

Applying Weil's formula twice we get that

$$\gamma(a,\psi)\gamma(-a^{-1},\psi)=1. \tag{24}$$

We also recall that γ satisfies

$$\gamma(ac,\psi)=\gamma(1,\psi)^{-1}\gamma(a,\psi)\gamma(c,\psi)[a,c] \tag{25}$$

for $a,c\in F^\times$. Therefore, for A and C as in Lemma 2.1

$$\gamma_m^n(A,C,\psi)=\gamma(1,\psi)^{-mn}\Big(\prod_{i=1}^{n}\gamma(a_i,\psi)^m\Big)\Big(\prod_{j=1}^{m}\gamma(c_j,\psi)^n\Big)\Big(\prod_{i,j}[a_i,c_j]\Big). \tag{26}$$

In particular

$$\gamma_m^n(A, 1_m, \psi) = \prod_{i=1}^{n} \gamma(a_i, \psi)^m = \gamma_n^m(1_m, A, \psi) \tag{27}$$

and

$$\gamma_n^m(1_n, C, \psi) = \prod_{j=1}^{m} \gamma(c_j, \psi)^n = \gamma_m^n(C, 1_n, \psi). \tag{28}$$

The expression $\prod_{i,j}[a_i, c_j]$ is thus only dependent on A and C and we can and will denote

$$[A, C] = \prod_{i,j}[a_i, c_j]. \tag{29}$$

We finish this chapter, with an identity we will need later.

Lemma 2.2. — *One has*

$$\gamma_m^n(-A, 1_m, \psi)\gamma_m^n(A^{-1}, C, \psi) = \gamma(1, \bar\psi)^{mn}[A, C]\gamma_n^m(-C, 1_n, \bar\psi). \tag{30}$$

Proof. — From (26), (27) and (28) we get that

$$\begin{aligned} &\gamma_m^n(-A, 1_m, \psi)\gamma_m^n(A^{-1}, C, \psi) \\ &\quad = \gamma(1, \psi)^{-mn}\gamma_m^n(-A, 1_m, \psi)\gamma_m^n(A^{-1}, 1_m, \psi)\gamma_n^m(C, 1_n, \psi)[A, C]. \end{aligned} \tag{31}$$

Using (24) and (27) we get that

$$\gamma_m^n(-A, 1_m, \psi)\gamma_m^n(A^{-1}, 1_m, \psi) = 1.$$

From (23) and (28) we have $\gamma_n^m(C, 1_n, \psi) = \gamma_n^m(-C, 1_n, \bar\psi)$ and from (23) and (24) that $\gamma(1, \psi)^{-1} = \gamma(1, \bar\psi)$. Therefore, the right hand side of (31) is now equal to $\gamma(1, \bar\psi)^{mn}[A, C]\gamma_n^m(-C, 1_n, \bar\psi)$ and the lemma follows. □

3. Intermediate orbital integrals

For $n, m \geq 1$ and a function $\Phi \in C_c^\infty(\mathcal{S}_{m+n})$ we define the intermediate orbital integral

$$\begin{aligned} &\omega_m^n\Big[\Phi, \psi; \begin{pmatrix} A_n & \\ & B_m \end{pmatrix}\Big] \\ &\qquad = \int_{M_{n\times m}(F)} \Phi\Big[\begin{pmatrix} 1_n & \\ {}^tX & 1_m \end{pmatrix}\begin{pmatrix} A_n & \\ & B_m \end{pmatrix}\begin{pmatrix} 1_n & X \\ & 1_m \end{pmatrix}\Big]\theta_{m+n}\Big[\begin{pmatrix} 1_n & 2X \\ & 1_m \end{pmatrix}\Big]\,\mathrm{d}X \\ &\quad = \int_{M_{n\times m}(F)} \Phi\Big[\begin{pmatrix} A_n & A_nX \\ {}^tXA_n & B_m + {}^tXA_nX \end{pmatrix}\Big]\psi\big[2\,\mathrm{Tr}(\epsilon X)\big]\,\mathrm{d}X \end{aligned} \tag{32}$$

where $A_n \in S_n$, $B_m \in \mathcal{S}_m$ and $\epsilon = \epsilon_m^n = (\delta_{(i,j),(1,n)}) \in M_{m\times n}(F)$. We also define a normalized intermediate integral

$$(33)\qquad \widetilde{\omega}_m^{n,\psi}\Big[\Phi,\psi;\begin{pmatrix}A_n & \\ & B_m\end{pmatrix}\Big] = \gamma_m^n(-A_n,1_m,\psi)\cdot|\det A_n|^{\frac12 m}\cdot\omega_m^n\Big[\Phi,\psi;\begin{pmatrix}A_n & \\ & B_m\end{pmatrix}\Big].$$

Proposition 3.1. — *Let $C_m \in S_m, D_n \in \mathcal{S}_n$ and $\Phi \in C_c^\infty(\mathcal{S}_{m+n})$. The function mapping A_n to*

$$(34)\qquad [A_n,C_m]\psi\big[\mathrm{Tr}(w_mC_m^{-1}w_m\epsilon_m^nA_n^{-1\,t}\epsilon_m^n)\big] \times\int_{\mathcal{S}_m}\widetilde{\omega}_m^{n,\psi}\Big[\Phi,\psi;\begin{pmatrix}A_n & \\ & B_m\end{pmatrix}\Big]\psi\big[-\mathrm{Tr}(B_mw_mC_mw_m)\big]\,\mathrm{d}B_m$$

originally defined for $A_n \in S_n$, extends to a smooth function of compact support on $\mathcal{S}_n$. The iterated integral

$$(35)\qquad \int_{\mathcal{S}_n}\Big\{\int_{\mathcal{S}_m}\widetilde{\omega}_m^{n,\psi}\Big[\Phi,\psi;\begin{pmatrix}A_n & \\ & B_m\end{pmatrix}\Big]\psi\Big[-\mathrm{Tr}(B_mw_mC_mw_m)\Big]\mathrm{d}B_m\Big\}\cdot \times[A_n,C_m]\psi\big[\mathrm{Tr}(w_mC_m^{-1}w_m\epsilon_m^nA_n^{-1\,t}\epsilon_m^n)\big]\psi\big[-\mathrm{Tr}(A_nw_nD_nw_n)\big]\,\mathrm{d}A_n$$

is therefore convergent. It is equal to

$$(36)\qquad |2|^{\frac12 mn}\cdot\gamma(1,\bar\psi)^{mn}\cdot\widetilde{\omega}_n^{m,\bar\psi}\Big[\check\Phi,\bar\psi;\begin{pmatrix}C_m & \\ & D_n\end{pmatrix}\Big].$$

Proof. — First we remark that from the right hand side of (32) it is easily observed that for a fixed $A_n \in S_n$ the function

$$B_m \longmapsto \omega_m^n\Big[\Phi,\psi;\begin{pmatrix}A_n & \\ & B_m\end{pmatrix}\Big]$$

is smooth and of compact support on $\mathcal{S}_m$. Therefore, for a fixed $C_m \in S_m$ the function Θ defined by

$$(37)\qquad \Theta(D_n) = |2|^{\frac12 mn}\gamma(1,\bar\psi)^{mn}\cdot\widetilde{\omega}_n^{m,\bar\psi}\Big[\check\Phi,\bar\psi;\begin{pmatrix}w_mC_mw_m & \\ & w_nD_nw_n\end{pmatrix}\Big],$$

is in $C_c^\infty(\mathcal{S}_n)$. Consider the partial Fourier transform with respect to B_m

$$(38)\qquad \int_{\mathcal{S}_m}\widetilde{\omega}_m^n\Big[\Phi,\psi;\begin{pmatrix}A_n & \\ & B_m\end{pmatrix}\Big]\psi\big[\mathrm{Tr}(-C_mB_m)\big]\,\mathrm{d}B_m.$$

Expanding along (32) it becomes after a change of variables $B_m \mapsto B_m - {}^tXA_nX$

$$(39)\qquad \gamma_m^n(-A_n,1_m,\psi)\cdot|\det A_n|^{\frac12 m}\cdot \times\int\Phi\Big[\begin{pmatrix}A_n & A_nX\\ {}^tXA_n & B_m\end{pmatrix}\Big]\psi[\mathrm{Tr}(2\epsilon_m^nX)+\mathrm{Tr}(C_m{}^tXA_nX)]\mathrm{d}X\ \psi\big[\mathrm{Tr}(-C_mB_m)\big]\,\mathrm{d}B_m$$

and after another change of variables $X \mapsto A_n^{-1}X$ it becomes

$$(40)\quad \gamma_m^n(-A_n,1_m,\psi)|\det A_n|^{-\frac{1}{2}m} \int \Phi\Big[\begin{pmatrix} A_n & X \\ {}^tX & B_m\end{pmatrix}\Big]\psi\big[\mathrm{Tr}(2\epsilon_m^n A_n^{-1}X)+\mathrm{Tr}(C_m{}^tXA_n^{-1}X)\big]\,\mathrm{d}X \; \psi\big[-\mathrm{Tr}(B_mC_m)\big]\,\mathrm{d}B_m.$$

Applying Weil's formula (17) with $P = 2\,\epsilon_m^n A_n^{-1}$, we see that this is equal to

$$(41)\quad |2|^{\frac{1}{2}mn}\gamma_m^n(A_n^{-1},C_m,\psi)\gamma_m^n(-A_n,1_m,\psi)|\det C_m|^{-\frac{1}{2}n} \times\int \Phi\begin{pmatrix} A_n & Y \\ {}^tY & B_m\end{pmatrix}\psi\big[-2\,\mathrm{Tr}(XY)\big]\,\mathrm{d}Y \; \psi\big[-\mathrm{Tr}(C_m^{-1}(X+\epsilon_m^nA_n^{-1})A_n({}^tX+A_n^{-1t}\epsilon_m^n))\big]\,\mathrm{d}X \; \psi\big[-\mathrm{Tr}(B_mC_m)\big]\,\mathrm{d}B_m.$$

We now use Lemma 2.2. After expanding (41) it becomes:

$$(42)\quad |2|^{\frac{1}{2}mn}\cdot\gamma(1,\bar\psi)^{mn}\cdot\gamma_n^m(-C_m,1_n,\bar\psi)\cdot|\det C_m|^{-\frac{1}{2}n} \psi\big[-\mathrm{Tr}(C_m^{-1}\epsilon_m^nA_n^{-1t}\epsilon_m^n)\big][A_n,C_m] \times\int \Phi\begin{pmatrix} A_n & Y \\ {}^tY & B_m\end{pmatrix}\psi\big[-2\,\mathrm{Tr}(XY)-\mathrm{Tr}(C_m^{-1}XA_n{}^tX) -2\,\mathrm{Tr}(C_m^{-1}X{}^t\epsilon_m^n)\big]\,\mathrm{d}Y\,\mathrm{d}X\cdot\psi\big[-\mathrm{Tr}(B_mC_m)\big]\,\mathrm{d}B_m.$$

We showed so far that for a fixed $C_m \in S_m$, the function Ψ on S_n defined by

$$(43)\quad \Psi(A_n) = [A_n,C_m]\cdot\psi\big[\mathrm{Tr}(C_m^{-1}\epsilon_m^nA_n^{-1}\epsilon_n^m)\big]\cdot \times\int_{\mathcal{S}_m}\widetilde{\omega}_m^{n,\psi}\Big[\Phi,\psi;\begin{pmatrix} A_n & \\ & B_m\end{pmatrix}\Big]\psi\big[-\mathrm{Tr}(B_mC_m)\big]\,\mathrm{d}B_m$$

is equal to

$$(44)\quad |2|^{\frac{1}{2}mn}\cdot\gamma(1,\bar\psi)^{mn}\cdot\gamma_n^m(-C_m,1_n,\bar\psi)\cdot|\det C_m|^{-\frac{1}{2}n} \int \Phi\begin{pmatrix} A_n & Y \\ {}^tY & B_m\end{pmatrix}\psi\big[-2\,\mathrm{Tr}(XY)-\mathrm{Tr}(C_m^{-1}XA_n{}^tX) -2\,\mathrm{Tr}(C_m^{-1}X{}^t\epsilon_m^n)-\mathrm{Tr}(C_mB_m)\big]\,\mathrm{d}Y\,\mathrm{d}X\,\mathrm{d}B_m.$$

Changing variables $X \mapsto C_mX$ we get

$$(45)\quad \Psi(A_n) = |2|^{\frac{1}{2}mn}\cdot\gamma(1,\bar\psi)^{mn}\cdot\gamma_n^m(-C_m,1_n,\bar\psi)\cdot|\det C_m|^{\frac{1}{2}n} \times\int \Phi\begin{pmatrix} A_n & Y \\ {}^tY & B_m\end{pmatrix}\cdot\psi\big[-2\,\mathrm{Tr}(C_mXY)-\mathrm{Tr}(XA_n{}^tXC_m) -2\,\mathrm{Tr}(X{}^t\epsilon_m^n)-\mathrm{Tr}(B_mC_m)\big]\,\mathrm{d}Y\,\mathrm{d}X\,\mathrm{d}B_m.$$

Next we expand $\Theta(D_n)$:

$$(46)\quad \Theta(D_n) = |2|^{\frac{1}{2}mn} \cdot \gamma(1,\bar\psi)^{mn} \cdot \gamma_n^m(-C_m, 1_n, \bar\psi) \cdot |\det C_m|^{\frac{1}{2}n} \times \int_{M_{m\times n}(F)} \hat\Phi\Big[w_{m+n}\begin{pmatrix} w_m C_m w_m & w_m C_m w_m X \\ {}^tX w_m C_m w_m & w_n D_n w_n + {}^tX w_m C_m w_m X\end{pmatrix} w_{m+n}\Big] \psi\big[-2\operatorname{Tr}(\epsilon_n^m X)\big]\,\mathrm{d}X$$

$$= |2|^{\frac{1}{2}mn}\gamma(1,\bar\psi)^{mn} \cdot \gamma_n^m(-C_m, 1_n, \bar\psi) \cdot |\det C_m|^{\frac{1}{2}n} \times \int_{M_{m\times n}(F)} \hat\Phi\Big[\begin{pmatrix} D_n + w_n{}^tX w_m C_m w_m X w_n & w_n{}^tX w_m C_m \\ C_m w_m X w_n & C_m\end{pmatrix}\Big] \psi[-2\operatorname{Tr}(\epsilon_n^m X)]\,\mathrm{d}X.$$

After a change of variables $X \mapsto w_m X w_n, \Theta(D_n)$ becomes

$$(47)\qquad |2|^{\frac{1}{2}mn} \cdot \gamma(1,\bar\psi)^{mn} \cdot \gamma_n^m(-C_m, 1_n, \bar\psi) \cdot |\det C_m|^{\frac{1}{2}n} \cdot \times \int_{M_{m\times n}(F)} \hat\Phi\Big[\begin{pmatrix} D_n + {}^tX C_m X & {}^tX C_m \\ C_m X & C_m\end{pmatrix}\Big] \psi\big[-2\operatorname{Tr}(X^t\epsilon_m^n)\big]\,\mathrm{d}X$$

where we use the fact that

$$\operatorname{Tr}(\epsilon_n^m w_m X w_n) = X_{1,n} = \operatorname{Tr}(X^t\epsilon_m^n).$$

Expanding further we have

$$(48)\quad \Theta(D_n) = |2|^{\frac{1}{2}mn} \cdot \gamma(1,\bar\psi)^{mn} \cdot \gamma_n^m(-C_m, 1_n, \bar\psi) \cdot |\det C_m|^{\frac{1}{2}n} \cdot \times \int \Phi\Big[\begin{pmatrix} s_n & Y \\ {}^tY & B_m\end{pmatrix}\Big] \psi\Big[-\operatorname{Tr}\Big(\begin{pmatrix} s_n & Y \\ {}^tY & B_m\end{pmatrix}\begin{pmatrix} D_n + {}^tX C_m X & {}^tX C_m \\ C_m X & C_m\end{pmatrix}\Big)\Big] \mathrm{d}Y\,\mathrm{d}s_n\,\mathrm{d}B_m \psi\big[-2\operatorname{Tr}(X^t\epsilon_m^n)\big]\,\mathrm{d}X$$

$$= |2|^{\frac{1}{2}mn} \cdot \gamma(1,\bar\psi)^{mn} \cdot \gamma_n^m(-C_m, 1_n, \bar\psi) \cdot |\det C_m|^{\frac{1}{2}n} \times \int \Phi\Big[\begin{pmatrix} s_n & Y \\ {}^tY & B_m\end{pmatrix}\Big] \psi\big[-\operatorname{Tr}(s_n{}^tX C_m X) - \operatorname{Tr}(s_n D_n)\big]\,\mathrm{d}s_n\, \psi\big[-\operatorname{Tr}(B_m C_m) - 2\operatorname{Tr}(C_m X Y)\big]\,\mathrm{d}Y\,\mathrm{d}B_m\, \psi\big[-2\operatorname{Tr}(X^t\epsilon_m^n)\big]\,\mathrm{d}X.$$

Using Fourier inversion we then see that

$$(49)\qquad \widehat\Theta(-A_n) = \int_{\mathcal{S}_n} \Theta(D_n)\psi\big[\operatorname{Tr}(A_n D_n)\big]\,\mathrm{d}A_n$$

$$= |2|^{\frac{1}{2}mn}\gamma(1,\bar\psi)^{mn}\gamma_n^m(-C_m, 1_n, \bar\psi)|\det C_m|^{\frac{1}{2}n} \times \int \Phi\Big[\begin{pmatrix} A_n & Y \\ {}^tY & B_m\end{pmatrix}\Big] \psi\Big[-\operatorname{Tr}(A_n{}^tX C_m X)\Big] \psi\big[-\operatorname{Tr}(B_m C_m) - 2\operatorname{Tr}(C_m X Y)\big]\,\mathrm{d}Y\,\mathrm{d}B_m\, \psi[-2\operatorname{Tr}(X^t\epsilon_m^n)]\,\mathrm{d}X.$$

Comparing with (45) we get that

$$\widehat{\Theta}(-A_n) = \Psi(A_n) \tag{50}$$

for all $A_n \in S_n$. This proves the first part of the proposition. Furthermore, regarding Ψ now as its extension to $\mathcal{S}_n$ and writing explicitly the equality $\widehat{\Psi}(D_n) = \Theta(D_n)$ we have for all $C_m \in S_m, D_n \in \mathcal{S}_n$

$$\begin{aligned}\int_{\mathcal{S}_n} \Big[\int_{\mathcal{S}_m} \widetilde{\omega}_m^{n,\psi}\Big[\Phi,\psi;\begin{pmatrix}A_n & \\ & B_m\end{pmatrix}\Big]\psi\big[-\operatorname{Tr}(B_m C_m)\big]\,\mathrm{d}B_m\Big]\\ [A_n, C_m]\psi\big[\operatorname{Tr}(C_m^{-1}\epsilon_m^n A_n^{-1\,t}\epsilon_m^n)\big]\psi\big[-\operatorname{Tr}(A_n D_n)\big]\,\mathrm{d}A_n\\ = |2|^{\frac12 mn}\cdot\gamma(1,\bar\psi)^{mn}\cdot\widetilde{\omega}_n^{m,\bar\psi}\Big[\check\Phi,\bar\psi;\begin{pmatrix}w_m C_m w_m & \\ & w_n D_n w_n\end{pmatrix}\Big].\end{aligned} \tag{51}$$

Observing that $[A_n, C_m] = [A_n, w_m C_m w_m]$ we get the proposition by replacing (C_m, D_n) with $(w_m C_m w_m, w_n D_n w_n)$. □

We end this chapter with a reduction formula that we will need for the proof of the main theorem. Let $\Phi \in C_c^\infty(\mathcal{S}_{m+n})$ and define on $S_n \times \mathcal{S}_m$ the function

$$\Xi(A_n, B_m) = \omega_m^n\Big[\Phi,\psi;\begin{pmatrix}A_n & \\ & B_m\end{pmatrix}\Big]. \tag{52}$$

Associated with the action of $N_n \times N_m$ on $S_n \times \mathcal{S}_m$, we consider the generalized Kloosterman integral

$$\omega\Big[\Xi,\psi;(A_n,B_m)\Big] = \int \Xi({}^t u_1 A_n u_1, {}^t u_2 B_m u_2)\theta_n(u_1)\theta_m(u_2)\,\mathrm{d}u_1\,\mathrm{d}u_2. \tag{53}$$

For a relevant element $x \in \mathcal{S}_{m+n}$ of the form $x = \begin{pmatrix}x_n & \\ & x_m\end{pmatrix}$, where $x_n \in S_n$ and $x_m \in \mathcal{S}_m$ are relevant, we have

$$\omega[\Phi,\psi;x] = \omega\big[\Xi,\psi,(x_n,x_m)\big]. \tag{54}$$

4. Proof of the main theorem

We prove the functional equation by induction on n, the case $n = 1$ being simply the definition of the Fourier transform. For a fixed $a_1 \in F^\times$, let

$$\begin{aligned}\Psi(A_{n-1}) = [a_1, A_{n-1}]\psi\big[\operatorname{Tr}(a_1^{-1}\epsilon A_{n-1}^{-1}\,{}^t\epsilon)\big]\\ \times\int \widetilde{\omega}_1^{n-1,\psi}\Big[\Phi,\psi;\begin{pmatrix}A_{n-1} & \\ & p_n\end{pmatrix}\Big]\psi[-p_n a_1]\,\mathrm{d}p_n,\end{aligned} \tag{55}$$

$$\Theta(D_{n-1}) = |2|^{\frac12(n-1)}\cdot\gamma(1,\bar\psi)^{n-1}\cdot\widetilde{\omega}_{n-1}^{1,\bar\psi}\Big[\check\Phi,\bar\psi;\begin{pmatrix}a_1 & \\ & D_{n-1}\end{pmatrix}\Big]. \tag{56}$$

Applying Proposition 3.1, with $(1, n-1)$ in the role of (m,n), we get that Ψ extends to $\mathcal{S}_{n-1}$. In fact $\Psi, \Theta \in C_c^\infty(\mathcal{S}_{n-1})$ and $\Theta = \check\Psi$. Let $a^{(1)} =$

$\operatorname{diag}(a_2,\dots,a_n)$ be relevant in $\mathcal{S}_{n-1}$. By induction applied to Ψ we have

$$(57)\qquad \big(K_{\psi,n-1}\widetilde{\omega}^{\psi}[\Psi,\psi;\cdot\,]\big)(a^{(1)}) = |2|^{\frac12(n-1)(n-2)}\cdot\gamma(1,\bar\psi)^{\frac12(n-1)(n-2)}\cdot\widetilde{\omega}^{\bar\psi}\big[\Theta,\bar\psi;a^{(1)}\big].$$

By (54) and (19) we have

$$(58)\quad \omega[\Theta,\bar\psi;a^{(1)}] = |2|^{\frac12(n-1)}\cdot\gamma(1,\bar\psi)^{n-1}\cdot\gamma^1_{n-1}(-a_1,1_{n-1},\bar\psi)\cdot|a_1|^{\frac12(n-1)} \times\omega\big[\omega^1_{n-1}[\check\Phi,\bar\psi;\cdot\,],\bar\psi;(a_1,a^{(1)})\big]$$
$$= |2|^{\frac12(n-1)}\cdot\gamma(1,\bar\psi)^{n-1}\cdot\gamma(-a_1,\bar\psi)^{n-1}\cdot|a_1|^{\frac12(n-1)}\cdot\omega[\check\Phi,\bar\psi;a_1,\dots,a_n]$$

therefore the right hand side of (57) satisfies

$$(59)\quad |2|^{\frac12(n-1)(n-2)}\cdot\gamma(1,\bar\psi)^{\frac12(n-1)(n-2)}\cdot\widetilde{\omega}^{\bar\psi}[\Theta,\bar\psi;a^{(1)}]$$
$$= |2|^{\frac12 n(n-1)}\cdot\gamma(1,\bar\psi)^{\frac12 n(n-1)}\cdot\Gamma_{n-1}(-a^{(1)},\bar\psi)\cdot\big|\sigma_{n-1}(a^{(1)})\big|^{\frac12} \times\gamma(-a_1,\bar\psi)^{n-1}\cdot|a_1|^{\frac12(n-1)}\cdot\omega[\check\Phi,\bar\psi;a_1,\dots,a_n]$$
$$= |2|^{\frac12 n(n-1)}\cdot\gamma(1,\bar\psi)^{\frac12 n(n-1)}\cdot\widetilde{\omega}^{\bar\psi}[\check\Phi,\bar\psi;a_1,\dots,a_n].$$

Next we treat the left hand side of (57). We start with the computation of the Kloosterman integral of Ψ at $x=\operatorname{diag}(p_1,\dots,p_{n-1})$. Note that

$$[a_1,{}^tuxu]\cdot\psi\big[\operatorname{Tr}(a_1^{-1}\epsilon u^{-1}x^{-1\,t}u^{-1\,t}\epsilon)\big] = \Big(\prod_{j=1}^{n-1}[a_1,p_j]\Big)\psi\Big[\frac{1}{a_1p_{n-1}}\Big]$$

for all $u\in N_{n-1}$ is constant on the orbit of x. So

$$(60)\qquad \widetilde{\omega}^{\psi}[\Psi,\psi;p_1,\dots,p_{n-1}] = \Gamma_{n-1}(-p_1,\dots,-p_{n-1},\psi) \times\big|\sigma_{n-1}(p_1,\dots,p_{n-1})\big|^{\frac12}\Big(\prod_{j=1}^{n-1}[a_1,p_j]\Big)\psi\Big[\frac{1}{a_1p_{n-1}}\Big]$$
$$\times\iint\widetilde{\omega}_1^{n-1,\psi}\Big[\Phi,\psi;\begin{pmatrix}{}^tuxu & \\ & p_n\end{pmatrix}\Big]\psi[-p_na_1]\,\mathrm{d}p_n\theta(u)\,\mathrm{d}u.$$

Since Φ is smooth and of compact support, it is easy to see from the right hand side of (32), that for $\phi\in C_c^\infty(F^\times)$ the function

$$(61)\qquad (A_{n-1},p_n)\longmapsto\phi(\det A_{n-1})\cdot\widetilde{\omega}_1^{n-1,\psi}\Big[\Phi,\psi;\begin{pmatrix}A_{n-1} & \\ & p_n\end{pmatrix}\Big]$$

is smooth of compact support on $\mathrm{GL}(n-1,F)\times F$. Since the orbit ${}^tuxu$, $u\in N_{n-1}$ has fixed determinant, the double integral in (60) is absolutely convergent and we may switch order of integration. From (54) we get

$$(62)\qquad \begin{aligned}&\Gamma_{n-1}(-p^{(n)},\psi)\cdot\big|\sigma_{n-1}(p_1,\ldots,p_{n-1})\big|^{\frac12}\\ &\qquad\times\int\widetilde{\omega}_1^{n-1,\psi}\Big[\Phi,\psi;\begin{pmatrix}{}^tuxu & \\ & p_n\end{pmatrix}\Big]\theta(u)\,\mathrm{d}u\\ &=\Gamma_{n-1}(-p^{(n)},\psi)\cdot\big|\sigma_{n-1}(p_1,\ldots,p_{n-1})\big|^{\frac12}\cdot\gamma_1^{n-1}(-x,1,\psi)\\ &\qquad\times|\det x|^{\frac12}\int\omega_1^{n-1}\Big[\Phi,\psi;\begin{pmatrix}{}^tuxu & \\ & p_n\end{pmatrix}\Big]\theta(u)\,\mathrm{d}u\\ &=\widetilde{\omega}^{\psi}[\Phi,\psi;p_1,\ldots,p_n].\end{aligned}$$

By switching order of integration in the right hand side of (60) we get that

$$(63)\qquad \begin{aligned}&\widetilde{\omega}^{\psi}[\Psi,\psi;p_1,\ldots,p_{n-1}]\\ &=\Big(\prod_{j=1}^{n-1}[a_1,p_j]\Big)\psi\Big[\frac{1}{a_1p_{n-1}}\Big]\int\widetilde{\omega}^{\psi}\Big[\Phi,\psi;p_1,\ldots,p_n\Big]\psi[-p_na_1]\,\mathrm{d}p_n.\end{aligned}$$

Finally, we see that

$$(64)\qquad \begin{aligned}&\big(K_{\psi,n-1}\widetilde{\omega}^{\psi}[\Psi,\psi;\cdot\,]\big)(a^{(1)})\\ &=\int\Big(\prod_{j=1}^{n-1}[a_1,p_j]\Big)\psi\Big[\frac{1}{a_1p_{n-1}}\Big]\int\widetilde{\omega}^{\psi}[\Phi,\psi;p_1,\ldots,p_n]\psi[-p_na_1]\,\mathrm{d}p_n\\ &\qquad\times\psi\Big[-\sum_{i=1}^{n-1}p_ia_{n+1-i}+\sum_{i=1}^{n-2}\frac{1}{p_ia_{n-i}}\Big]\Big(\prod_{i=2}^{n-1}\prod_{j=1}^{n-i}[a_i,p_j]\Big)\mathrm{d}p_{n-1}\cdots\mathrm{d}p_1\\ &=\big(K_{\psi,n}\widetilde{\omega}^{\psi}[\Phi,\psi;\cdot\,]\big)(a_1,\ldots,a_n).\end{aligned}$$

The theorem now follows from (57), (59) and (64).

Corollary 1.3 is now immediate since $\gamma(1,\psi)\cdot\gamma(1,\bar{\psi})=1$. □

5. A formula for the smallest orbits

Let $\kappa(n)=\frac12 n(n+1)-1$.

Proposition 5.1. — *For $\Phi\in C_c^{\infty}(\mathcal{S}_n)$, the function $\phi(a)=\omega[\Phi,\psi,w_na]$ is a smooth function of compact support on $F^{\times}$. Furthermore, it satisfies the functional equation*

$$(65)\qquad |a|^{\kappa(n)}\cdot\omega[\Phi,\psi,w_na]=\int\omega\Big[\check{\Phi},\bar{\psi};\begin{pmatrix}-w_{n-1}a^{-1} & \\ & b\end{pmatrix}\Big]\,\mathrm{d}b.$$

Proof. — We can write ϕ as follows

$$(66)\qquad \omega[\Phi,\psi,w_na]=\int\Phi(m')\psi\Big(\sum_{i=2}^{n}x_{i,n+2-i}\Big)\underset{i+j\geq n+2}{\otimes}\mathrm{d}x_{i,j}$$

where

$$m'_{i,j} = \begin{cases} 0 & \text{if } i+j \le n, \\ a & \text{if } i+j = n+1, \\ ax_{i,j} & \text{if } i+j \ge n+2, \end{cases}$$

and $x_{i,j} = x_{j,i}$ if $i+j \ge n+2$. Let

$$\kappa_1(n) = \begin{cases} \frac{1}{4}n^2 & \text{if } n-\text{even}, \\ \frac{1}{4}(n^2-1) & \text{if } n-\text{odd}, \end{cases} \qquad \kappa_2(n) = \kappa_1(n-1) + n - 1.$$

Then $\kappa(n) = \kappa_1(n) + \kappa_2(n)$. After a change of variables (66) can be writen as

$$\omega[\Phi, \psi, w_n a] = |a|^{-\kappa_1(n)} \int \Phi(m) \psi\Big(a^{-1} \sum_{i=2}^{n} x_{i,n+2-i}\Big) \underset{i+j \ge n+2}{\otimes} \mathrm{d}x_{i,j} \tag{67}$$

where

$$m_{i,j} = \begin{cases} 0 & \text{if } i+j \le n, \\ a & \text{if } i+j = n+1, \\ x_{i,j} & \text{if } i+j \ge n+2, \end{cases}$$

and $x_{i,j} = x_{j,i}$ if $i+j \ge n+2$. The smoothness of ϕ follows from the fact that Φ is smooth. Also $\Phi(m) = 0$ for large enough $|a|$ and for all $x_{i,j}$ as above, since Φ is of compact support. Therefore $\phi(a)$ vanishes when $|a|$ is sufficiently large. Let X be the $n \times n$ symmetric matrix with 0 in the (i,j)-th entry whenever $i+j \le n+1$ and $x_{i,j}$ in the (i,j)-th entry whenever $i+j \ge n+2$. Let Z be a similar variable matrix with entries $z_{i,j}$, thus

$$m = \begin{pmatrix} & & & a \\ & & \cdot & x_{2,n} \\ & \cdot & & \vdots \\ a & x_{2,n} & \cdots & x_{n,n} \end{pmatrix}, \quad X = \begin{pmatrix} & & & 0 \\ & & \cdot & x_{2,n} \\ & \cdot & & \vdots \\ 0 & x_{2,n} & \cdots & x_{n,n} \end{pmatrix}, \quad Z = \begin{pmatrix} & & & 0 \\ & & \cdot & z_{2,n} \\ & \cdot & & \vdots \\ 0 & z_{2,n} & \cdots & z_{n,n} \end{pmatrix}. \tag{68}$$

Since Φ is smooth and of compact support, there are integers $\ell < k$ such that $\Phi(m) = 0$ unless $x_{i,j} \in \wp^\ell$ and $\Phi(m+Z) = \Phi(m)$ whenever $z_{i,j} \in \wp^k$ for all i,j. Therefore,

$$\phi(a) = \sum_{x_{i,j} \in \wp^\ell/\wp^k} \Phi(m) \int_{z_{i,j} \in \wp^k} \psi\Big(a^{-1} \sum_{i=2}^{n} (x_{i,n+2-i} + z_{i,n+2-i})\Big) \underset{i+j \ge n+2}{\otimes} \mathrm{d}z_{i,j}.$$

This integral factors, for example, through the integral $\int_{\wp^k} \psi(a^{-1} z_{2,n}) \mathrm{d}z_{2,n}$ which vanishes whenever $|a|$ is small enough. So we get also that $\phi(a) = 0$ for $|a|$ sufficiently small. We can write

$$\int \omega\Big[\check{\Phi}, \bar{\psi}; \begin{pmatrix} -w_{n-1}a^{-1} & \\ & b \end{pmatrix}\Big] \mathrm{d}b = \int \check{\Phi}(p') \psi\Big(-\sum_{i=1}^{n} y_{i,n+1-i}\Big) \underset{2n > i+j \ge n+1}{\otimes} \mathrm{d}y_{i,j} \, \mathrm{d}b$$

where

$$p'_{i,j} = \begin{cases} 0 & \text{if } i+j \le n-1, \\ -a^{-1} & \text{if } i+j = n, \\ -a^{-1}y_{i,j} & \text{if } 2n > i+j \ge n+1, \\ b & \text{if } i+j = 2n, \end{cases}$$

and $y_{i,j} = y_{j,i}$ if $i+j \ge n+1$. After a change of variables this becomes

$$\int \omega\Big[\check{\Phi}, \bar{\psi}; \begin{pmatrix} -w_{n-1}a^{-1} & \\ & b \end{pmatrix}\Big] \mathrm{d}b = |a|^{\kappa_2(n)} \int \check{\Phi}(p'')\psi\Big(a\sum_{i=1}^{n} y_{i,n+1-i}\Big) \bigotimes_{i+j \ge n+1} \mathrm{d}y_{i,j}$$

where

$$p''_{i,j} = \begin{cases} 0 & \text{if } i+j \le n-1, \\ -a^{-1} & \text{if } i+j = n, \\ y_{i,j} & \text{if } i+j \ge n+1, \end{cases}$$

and $y_{i,j} = y_{j,i}$ if $i+j \ge n+1$ which is the same as writing

$$\begin{aligned} (69) \qquad & \int \omega\Big[\check{\Phi}, \bar{\psi}; \begin{pmatrix} -w_{n-1}a^{-1} & \\ & b \end{pmatrix}\Big] \mathrm{d}b \\ & \qquad = |a|^{\kappa_2(n)} \int \hat{\Phi}(p)\psi\Big(a\sum_{i=1}^{n} y_{i,n+1-i}\Big) \bigotimes_{i+j \le n+1} \mathrm{d}y_{i,j} \end{aligned}$$

where

$$p_{i,j} = \begin{cases} y_{i,j} & \text{if } i+j \le n+1, \\ -a^{-1} & \text{if } i+j = n+2, \\ 0 & \text{if } i+j \ge n+3, \end{cases}$$

and $y_{i,j} = y_{j,i}$ if $i+j \le n+1$. Next let Y be the $n \times n$ symmetric matrix with

$$Y_{i,j} = \begin{cases} y_{i,j} & \text{if } i+j \le n+1, \\ 0 & \text{if } i+j \ge n+2. \end{cases}$$

Let A' be defined by $p = Y + A'$ and let X and Z be matrix variables as defined in (68). Thus

$$p = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & -a^{-1} \\ y_{1n} & -a^{-1} & \end{pmatrix}, \; Y = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & 0 \\ y_{1n} & 0 & \end{pmatrix}, \; A' = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & -a^{-1} \\ 0 & -a^{-1} & \end{pmatrix}.$$

Using Fourier inversion formula for the function $\hat{\Phi}(Y+(\cdot))$ in the (i,j)-th entries for all $i+j \geq n+2$, we obtain from (69) that

$$|a|^{-\kappa_2(n)}\int \omega\Big[\check{\Phi},\bar{\psi};\begin{pmatrix} -w_{n-1}a^{-1} & \\ & b\end{pmatrix}\Big]\,\mathrm{d}b$$

$$=\int \widehat{\Phi}(Y+Z)\psi\big[-\operatorname{Tr}(XZ)\big]\,\mathrm{d}Z\,\psi\big[\operatorname{Tr}(XA')\big]\,\mathrm{d}X\,\psi\Big(a\sum_{i=1}^{n} y_{i,n+1-i}\Big)\bigotimes_{i+j\leq n+1}\mathrm{d}y_{i,j}.$$

Let $A = w_n a$, one easily observes now that $\operatorname{Tr}(ZA) = \operatorname{Tr}(YX) = 0$ and therefore $\operatorname{Tr}((Y+Z)(A-X)) = \operatorname{Tr}(YA - ZX) = a\sum_{i=1}^{n} y_{i,n+1-i} - \operatorname{Tr}(XZ)$, so we get by the Fourier inversion formula applied to $\hat{\Phi}$ that

$$\text{(70)}\qquad |a|^{-\kappa_2(n)}\int \omega\Big[\check{\Phi},\bar{\psi};\begin{pmatrix} -w_{n-1}a^{-1} & \\ & b\end{pmatrix}\Big]\,\mathrm{d}b$$

$$=\int \widehat{\Phi}(Y+Z)\,\psi\big[\operatorname{Tr}((Y+Z)(A-X))\big]\mathrm{d}Y\,\mathrm{d}Z\,\psi\big[\operatorname{Tr}(XA')\big]\,\mathrm{d}X$$

$$=\int \Phi(A-X)\,\psi\big[\operatorname{Tr}(XA')\big]\,\mathrm{d}X = \int \Phi(A+X)\,\psi\big[\operatorname{Tr}(-XA')\big]\,\mathrm{d}X.$$

But $m = A + X$ and $\operatorname{Tr}(-XA') = a^{-1}\sum_{i=2}^{n} x_{i,n+2-i}$, so comparing with (67), the right hand side of (70) is equal to $|a|^{\kappa_1(n)}\omega[\Phi,\psi;w_n a]$. □

BIBLIOGRAPHY

[1] FLICKER (Y.Z.) & KAZHDAN (D.A.) – *Metaplectic correspondence*, Publ. Math. Inst. Hautes Études Sci., t. **64** (1986), pp. 53–110.

[2] JACQUET (H.) – *Facteurs de transfert pour les intégrales de Kloosterman*, C. R. Acad. Sci. Paris Sér. I Math., t. **336** (2003), no. 2, pp. 121–124.

[3] ______, *Smooth transfer of Kloosterman integrals*, Duke Math. J., t. **120** (2003), no. 1, pp. 121–152.

[4] ______, *Kloosterman identities over a quadratic extension*, Ann. Math. (2), t. **160** (2004), no. 2, pp. 755–779.

[5] KAZHDAN (D.A.) & PATTERSON (S.J.) – *Metaplectic forms*, Publ. Math. Inst. Hautes Études Sci., t. **59** (1982), pp. 35–142.

[6] MAO (Z.) – *A fundamental lemma for metaplectic correspondence*, J. reine angew. Math., t. **496** (1998), pp. 107–129.

[7] WEIL (A.) – *Sur certains groupes d'opérateurs unitaires*, Acta Math., t. **111** (1964), pp. 143–211.