SOME REMARKS
ON THE LOCAL CLASS FIELD THEORY
OF SERRE AND HAZEWINKEL

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Abstract. — We give a new approach for the local class field theory of Serre and Hazewinkel. We also discuss two-dimensional local class field theory in this framework.

Résumé (*Quelques remarques sur la théorie du corps de classes local de Serre et Hazewinkel*)

Nous donnons une nouvelle approche de la théorie du corps de classes local de Serre et Hazewinkel. Nous discutons également la théorie du corps de classes local de dimension deux dans ce cadre.

1. Introduction

The purpose of this paper is twofold. First, we give a geometric description of the local class field theory of Serre and Hazewinkel ([13], [7, Appendice]) in the equal characteristic case. The main result is Theorem A below. Second, we discuss its two-dimensional analog with the aim to seek an analog of Lubin-Tate theory for two-dimensional local fields. The discussion is taken place in Section 6 with a partial result.

We formulate Theorem A. The precise definitions of the terms used below are given in Section 2. Let k be a perfect field of characteristic $p > 0$ and set $K = k((T))$. The group of units of K can be viewed as a proalgebraic group

Takashi Suzuki, Department of Mathematics, University of Chicago, 5734 S University Ave, Chicago, IL 60637 • *E-mail :* suzuki@math.uchicago.edu

(more precisely, a pro-quasi-algebraic group) over k in the sense of Serre ([12]); we denote this proalgebraic group by $\mathbf{U}_K$. For each perfect k-algebra R (perfect means that the p-th power map is an isomorphism), we have the group of its R-rational points

$$\mathbf{U}_K(R) = \left\{ \sum_{i=0}^{\infty} a_i \mathbf{T}^i \;\middle|\; a_i \in R, \; a_0 \in R^\times \right\}.$$

Likewise, the multiplicative group $K^\times$ of K can be viewed as a perfect group scheme $\mathbf{K}^\times$, which is the direct product of $\mathbf{U}_K$ and the discrete infinite cyclic group generated by $\mathbf{T}$. Let K_p be the perfect closure of K. We consider the K_p-rational point $1 - T\mathbf{T}^{-1}$ of $\mathbf{K}^\times$ and the corresponding morphism

$$\varphi \colon \operatorname{Spec} K_p \to \mathbf{K}^\times.$$

This morphism induces a homomorphism

$$\eta \colon \operatorname{Gal}(K^{\mathrm{ab}}/K) = \pi_1^{\mathrm{et}}(\operatorname{Spec} K_p)^{\mathrm{ab}} \xrightarrow{\varphi} \pi_1^k(\mathbf{K}^\times)$$

on the fundamental groups. Here K^{ab} is the maximal abelian extension of K and $\pi_1^{\mathrm{et}}(\cdot)^{\mathrm{ab}}$ denotes the maximal abelian quotient of the étale fundamental group. The group $\pi_1^k(\mathbf{K}^\times)$ is the fundamental group of $\mathbf{K}^\times$ as a perfect group scheme over k, which classifies all surjective isogenies to $\mathbf{K}^\times$ with finite constant kernels. Now we state the main theorem of this paper:

THEOREM A. — 1. *The above defined map* $\eta \colon \operatorname{Gal}(K^{\mathrm{ab}}/K) \to \pi_1^k(\mathbf{K}^\times)$ *is an isomorphism.*
  2. *The inverse of η restricted to $\pi_1^k(\mathbf{U}_K)$ coincides with the reciprocity isomorphism* $\theta \colon \pi_1^k(\mathbf{U}_K) \xrightarrow{\sim} I(K^{\mathrm{ab}}/K)$ *of Serre and Hazewinkel ([13], [7]), where I denotes the inertia group.*

Note that Assertion 1 of the theorem says that we have an essentially one-to-one correspondence between surjective isogenies $A \twoheadrightarrow \mathbf{K}^\times$ with finite constant kernels and finite abelian extensions L of K by pullback by φ, as expressed as a cartesian diagram of the form

$$\begin{array}{ccc} \operatorname{Spec} L_p & \longrightarrow & A \\ \downarrow & & \downarrow \\ \operatorname{Spec} K_p & \xrightarrow{\;\varphi\;} & \mathbf{K}^\times. \end{array}$$

In other words, we have $\operatorname{Ext}_k^1(\mathbf{K}^\times, N) \xrightarrow{\sim} H^1(K, N)$ for any finite constant group N. Another remark is that the proof of Assertion 2 that we give in this paper does not use the fact that θ is an isomorphism. This means that Theorem A and our proof of it together give another proof of this fact, more specifically, the existence theorem of the local class field theory of Serre and

Hazewinkel ([13, §4], [7, 6.3]). For a generalization of Assertion 2 for the full groups $\pi_1^k(\mathbf{K}^\times)$ and $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, see Remark 3.5.

We outline the paper. After providing some preliminaries at Section 2, we give three different proofs of Theorem A. Each of them has its own advantages and interesting points. The first proof given in Section 3 may be the standard one. The method is purely local. In this proof, for Assertion 1, we explicitly calculate the groups $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ and $\pi_1^k(\mathbf{K}^\times)$ and the homomorphism η between them individually. For Assertion 2, we interpret the assertion to the compatibility of φ with norm maps and prove it by analyzing the diagonal divisors. The second proof given in Section 4 relies on Lubin-Tate theory. Hence it is applicable only for finite residue field cases. The third proof given in Section 5 is a geometric proof, which was suggested by the referee to the author. In this geometric proof, for Assertion 1, we need the Albanese property of the morphism $\varphi \colon \mathrm{Spec}\, K_p \to \mathbf{K}^\times$, which was established by Contou-Carrere in [6] (see also [5]). For Assertion 2, we need the local-global compatibility and the global version of Assertion 2, which was obtained by Serre in [13, §5].

In Section 6, we define a morphism analogous to $\varphi \colon \mathrm{Spec}\, K_p \to \mathbf{K}^\times$ for a field of the form $k((S))((T))$ using the second algebraic K-group instead of the multiplicative group. If k is a finite field, the field $k((S))((T))$ is called a two-dimensional local field ([8]) of positive characteristic. In analogy with the second proof of Theorem A by Lubin-Tate theory, we attempt to formulate an analogous theory to Lubin-Tate theory for the two-dimensional local field $k((S))((T))$. Although we have not yet obtained an analog of a Lubin-Tate formal group in this paper, we did obtain a result that may be thought of as a partial result for abelian extensions having p-torsion Galois groups (Proposition 6.1). In Section 7, we give an analog of the result of [2, §2.6] on $\mathcal{D}$-modules. Their result is for fields of the form $k((T))$ with k characteristic zero while ours $k((S))((T))$ with k characteristic zero. This is also regarded as an analog of Proposition 6.1.

We give a couple of comments on literature. The morphism $\varphi \colon \mathrm{Spec}\, K_p \to \mathbf{K}^\times$ had been introduced by Grothendieck ([4, August 9, 1960]) long before our paper. Besides Contou-Carrere as mentioned above, the morphism φ has been studied by many people. The $\mathcal{D}$-module version of Theorem A in the zero-characteristic case had been established in [2, §2.6]. Deligne had given a sketch of proof of results stronger than Assertion 1. This is written in Section e of his letter to Serre contained in [3]. His method is different from our method. The author did not realize these works at the time of writing this paper.

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2. Preliminaries

In this section, we give precise definitions of the terms that we used in
Introduction and will use in the following sections, fix notation, prove the in-
dependence of the choice of the prime element T (Proposition 2.3), and reduce
the proof of Theorem A to the case of algebraically closed residue fields.

2.1. Definitions and notation. — We work on the site $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$ of per-
fect schemes over a perfect field k of characteristic $p > 0$ with the fpqc
topology (compare with [11, Chapter III, §0, "Duality for unipotent perfect
group schemes"]; see also [17]). The category of sheaves of abelian groups
on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$ contains the category of commutative affine pro(-quasi)-
algebraic groups over k in the sense of Serre ([12]) as an thick abelian full
subcategory. We denote by Ext^i_k the i-th Ext functor for the category of
sheaves of abelian groups on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$. For a sheaf of abelian groups
A on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$ and a non-negative integer i, we define the fundamental
group $\pi_1^k(A)$ of A to be the Pontryagin dual of the torsion abelian group
$\mathrm{inj}\lim_{n \geq 1} \mathrm{Ext}^1_k(A, n^{-1}\mathbb{Z}/\mathbb{Z})$. If A is an extension of an étale group whose
group of geometric points is finitely generated as an abelian group by an
affine proalgebraic group, then $\mathrm{inj}\lim_{n \geq 1} \mathrm{Ext}^1_k(A, n^{-1}\mathbb{Z}/\mathbb{Z}) = \mathrm{Ext}^1_k(A, \mathbb{Q}/\mathbb{Z})$
and hence the Pontryagin dual of $\mathrm{Ext}^1_k(A, \mathbb{Q}/\mathbb{Z})$ coincides with $\pi_1^k(A)$. For a
k-algebra R, we denote by R_p the perfect k-algebra given by the injective limit
of p-th power maps $R \to R \to \cdots$, where the i-th copy of R in this system is
given the map $k \to R$, $a \mapsto a^{p^i}$ as its k-algebra structure map.

Let K be a complete discrete valuation field of equal characteristic with
residue field k. We denote by $\mathcal{O}_K$ the ring of integers of K and by $\mathfrak{p}_K$ the
maximal ideal of $\mathcal{O}_K$. We set $U_K = U_K^0 = \mathcal{O}_K^\times$ and $U_K^n = 1 + \mathfrak{p}_K^n$ for $n \geq 1$. We
fix an algebraic closure $\overline{K}$ of K. All algebraic extensions of K are taken inside
$\overline{K}$. We denote by K^{sep} ($\subset \overline{K}$) the separable closure of K, by K^{ur} the maximal
unramified extension of K and by K^{ab} the maximal abelian extension of K,
respectively. Since K has equal characteristic $p > 0$, the rings $\mathcal{O}_K$ and K have
canonical structures of k-algebras by the Teichmüller section $k \hookrightarrow \mathcal{O}_K$. Hence,
for a sheaf of abelian groups A on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$, we can consider the groups
$A((\mathcal{O}_K)_p)$ (resp. $A(K_p)$) of $(\mathcal{O}_K)_p$-rational (resp. K_p-rational) points of A. We
denote by $A((\mathfrak{p}_K)_p)$ the kernel of the natural map $A((\mathcal{O}_K)_p) \to A(k)$. The
reduction map $\mathcal{O}_K \twoheadrightarrow k$ and the Teichmüller section $k \hookrightarrow \mathcal{O}_K$ give a canonical
splitting $A((\mathcal{O}_K)_p) = A(k) \oplus A((\mathfrak{p}_K)_p)$.

We define a sheaf of rings $\mathbf{O}_K$ on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$ by setting $\mathbf{O}_K(R) = R \hat{\otimes}_k \mathcal{O}_K$ for each perfect k-algebra R,[(1)] where $\hat{\otimes}$ denotes the completed tensor product. Let $\mathbf{K}$ be the sheaf of rings on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$ with $\mathbf{K}(R) = \mathbf{O}_K(R) \otimes_{\mathcal{O}_K} K$. We set $\mathbf{U}_K = \mathbf{O}_K^\times$. For each $n \geq 0$, the sheaf of rings $\mathbf{O}_K$ has a subsheaf of ideals $\mathbf{p}_K^n$ with $\mathbf{p}_K^n(R) = \mathbf{O}_K(R) \otimes_{\mathcal{O}_K} \mathfrak{p}_K^n$. The presentation $\mathbf{O}_K = \mathrm{proj} \lim_{n \to \infty} \mathbf{O}_K/\mathbf{p}_K^n$ gives an affine proalgebraic ring structure for $\mathbf{O}_K$. Likewise, $\mathbf{U}_K$ has a subsheaf of groups $\mathbf{U}_K^n = 1 + \mathbf{p}_K^n$ for each $n \geq 1$ (for $n = 0$, we set $\mathbf{U}_K^0 = \mathbf{U}_K$). The presentation $\mathbf{U}_K = \mathrm{proj} \lim_{n \to \infty} \mathbf{U}_K/\mathbf{U}_K^n$ gives an affine proalgebraic group structure for $\mathbf{U}_K$. We have a split exact sequence $0 \to \mathbf{U}_K \to \mathbf{K}^\times \to \mathbb{Z} \to 0$. For a prime element T of $\mathcal{O}_K$, let $\mathbf{T}$ be the k-rational point of $\mathbf{O}_K$ defined by $\mathbf{T} = 1 \otimes T \in k \hat{\otimes}_k \mathcal{O}_K = \mathbf{O}_K(k)$. If we fix a prime element T, we may write $\mathbf{U}_K(R) = R[[\mathbf{T}]]^\times$ and $\mathbf{K}^\times(R) = R[[\mathbf{T}]][\mathbf{T}^{-1}]^\times$, which are the descriptions of $\mathbf{U}_K$ and $\mathbf{K}^\times$ given in Introduction. The rational point $1 - T\mathbf{T}^{-1} \in \mathbf{K}^\times(K_p)$ taken in Introduction should therefore be understood to be $1 - (T \otimes 1) \otimes T^{-1} \in ((K_p \hat{\otimes}_k \mathcal{O}_K) \otimes_{\mathcal{O}_K} K)^\times = \mathbf{K}^\times(K_p)$.

For a rational point $f \in \mathbf{K}^\times(K_p)$, we denote by φ_f the corresponding morphism of k-schemes $\mathrm{Spec}\, K_p \to \mathbf{K}^\times$. An extension $0 \to N \to A \to \mathbf{K}^\times \to 0$ with N finite constant pulls back by $\varphi_f \colon \mathrm{Spec}\, K_p \to \mathbf{K}^\times$ to an N-torsor on $\mathrm{Spec}\, K_p$. This defines a homomorphism $\mathrm{Ext}_k^1(\mathbf{K}^\times, N) \to H^1(K, N)$. By Pontryagin duality, we have a homomorphism $\mathrm{Gal}(K^{\mathrm{ab}}/K) \to \pi_1^k(\mathbf{K}^\times)$, which we denote by η_f.

2.2. A subset of $\mathbf{K}^\times(K_p)$. — In this subsection, we define a subset Ξ_K of $\mathbf{K}^\times(K_p)$ and prove some properties of it. This subset contains $1 - T\mathbf{T}^{-1}$ for all prime elements T and is convenient for proving both the independence of the prime element (Proposition 2.3) and the compatibility with norm maps (Proposition 3.4).

DEFINITION 2.1. — We define Ξ_K to be the set of elements $f \in \mathbf{K}^\times(K_p)$ satisfying the following conditions:

1. f is in $\mathbf{K}((\mathcal{O}_K)_p)$ and generates as an ideal of $\mathbf{K}((\mathcal{O}_K)_p)$ the kernel of the multiplication map $\mathbf{K}((\mathcal{O}_K)_p) = ((\mathcal{O}_K)_p \hat{\otimes}_k \mathcal{O}_K) \otimes_{\mathcal{O}_K} K \to K_p$ and
2. the natural map $\mathbf{K}((\mathcal{O}_K)_p) \to \mathbf{K}(k)$ sends f to 1.

PROPOSITION 2.2. —   1. *The set Ξ_K contains $1 - T\mathbf{T}^{-1}$ for any prime T.*
   2. *The group $\mathbf{U}_K((\mathfrak{p}_K)_p) = \mathrm{Ker}\big(\mathbf{U}_K((\mathcal{O}_K)_p) \to \mathbf{U}_K(k)\big)$ acts by multiplication on Ξ_K transitively without fixed points.*
   3. *The valuation map $\mathbf{K}^\times \to \mathbb{Z}$ sends every element of Ξ_K to -1.*

[(1)] This defines a sheaf on the site of perfect k-algebras with the fpqc topology. We actually need to take its Zariski sheafification to have a sheaf on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$. The two sites here have equivalent categories of sheaves. The same process is applied to all sheaves in this paper.

4. *For every $f \in \Xi_K$, we have the following commutative diagram with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I(K^{\mathrm{ab}}/K) & \longrightarrow & \mathrm{Gal}(K^{\mathrm{ab}}/K) & \longrightarrow & \mathrm{Gal}(k^{\mathrm{ab}}/k) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\eta_f} & & \downarrow{\scriptstyle\eta_f} & & \downarrow{\scriptstyle\wr} & & \\
0 & \longrightarrow & \pi_1^k(\mathbf{U}_K) & \longrightarrow & \pi_1^k(\mathbf{K}^\times) & \longrightarrow & \pi_1^k(\mathbb{Z}) & \longrightarrow & 0,
\end{array}$$

where the right vertical isomorphism $\mathrm{Gal}(k^{\mathrm{ab}}/k) \xrightarrow{\sim} \pi_1^k(\mathbb{Z})$ is the Pontryagin dual map of the natural isomorphism $H^1(k, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Ext}_k^1(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ times -1.

5. *For each prime element T, the restricted homomorphism $\eta_f \colon I(K^{\mathrm{ab}}/K) \to \pi_1^k(\mathbf{U}_K)$ for $f = 1 - T\mathbf{T}^{-1} \in \Xi_K$ factors as the composite of the inclusion $I(K^{\mathrm{ab}}/K) \hookrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$ and the homomorphism $\mathrm{Gal}(K^{\mathrm{ab}}/K) \to \pi_1^k(\mathbf{U}_K)$ induced by the morphism $\mathrm{Spec}\,K_p \to \mathbf{U}_K$ corresponding to the rational point $-T + \mathbf{T} \in \mathbf{U}_K(K_p)$.*

Proof. — 1. Obvious.

2. If f_1 and f_2 are in Ξ_K, we have $f_2 = u f_1$ for some $u \in \mathrm{Ker}\big(\mathbf{K}^\times((\mathcal{O}_K)_p) \to \mathbf{K}^\times(k)\big) = \mathbf{K}^\times((\mathfrak{p}_K)_p)$. Such u is unique since $\mathbf{K}((\mathcal{O}_K)_p)$ is a domain. We have $\mathbf{K}^\times((\mathfrak{p}_K)_p) = \mathbf{U}_K((\mathfrak{p}_K)_p)$ since $\mathbb{Z}((\mathfrak{p}_K)_p) = 0$. Thus we have Assertion 2.

3. This follows from Assertions 1 and 2.

4. By Assertion 3, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Spec}\,K_p & \longrightarrow & \mathrm{Spec}\,k \\
\downarrow{\scriptstyle\varphi_f} & & \downarrow \\
\mathbf{K}^\times & \longrightarrow & \mathbb{Z},
\end{array}$$

where the right vertical morphism corresponds to $-1 \in \mathbb{Z}(k)$. The maps induced on the fundamental groups give the required commutative diagram. The top row of the diagram in Assertion 4 is obviously exact. The bottom row of is exact as well since the sequence $0 \to \mathbf{U}_K \to \mathbf{K}^\times \to \mathbb{Z} \to 0$ is split exact.

5. The prime element T gives a splitting $\mathbf{K}^\times \cong \mathbf{U}_K \times \mathbb{Z}$. The rational point $1 - T\mathbf{T}^{-1} \in \mathbf{K}^\times(K_p)$ corresponds to the rational point $(-T + \mathbf{T}, -1) \in \mathbf{U}_K(K_p) \times \mathbb{Z}(K_p)$. This implies Assertion 5. $\qquad\square$

2.3. Independence of the choice of the prime element. — We show that the morphisms $\mathrm{Spec}\,K_p \to \mathbf{K}^\times$ corresponding to the rational points $1 - T\mathbf{T}^{-1} \in \mathbf{K}^\times(K_p)$ for various primes T actually induce the same homomorphism $\mathrm{Gal}(K^{\mathrm{ab}}/K) \to \pi_1^k(\mathbf{K}^\times)$ on the fundamental groups. More generally, we prove the following.

PROPOSITION 2.3. — *The homomorphism $\eta_f \colon \mathrm{Gal}(K^{\mathrm{ab}}/K) \to \pi_1^k(\mathbf{K}^\times)$ for $f \in \Xi_K$ is independent of the choice of f.*

Thus we may write η instead of η_f. To prove the proposition, we need the following lemma, which will also be used in the proof of Proposition 3.4 and in Section 5.

LEMMA 2.4. — *Let* $0 \to N \to A \to A' \to 0$ *be an exact sequence of sheaves of abelian groups on* $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$ *with* N *representable by an affine scheme. Then the induced map* $A((\mathfrak{p}_K)_p) \to A'((\mathfrak{p}_K)_p)$ *is surjective.*

Proof. — Taking cohomology, we have a commutative diagram with exact rows

$$
\begin{array}{ccccccc}
0 \longrightarrow & N((\mathcal{O}_K)_p) \longrightarrow & A((\mathcal{O}_K)_p) \longrightarrow & A'((\mathcal{O}_K)_p) \longrightarrow & H^1_{\mathrm{fpqc}}(\mathcal{O}_K, N) \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
0 \longrightarrow & N(k) \longrightarrow & A(k) \longrightarrow & A'(k) \longrightarrow & H^1_{\mathrm{fpqc}}(k, N),
\end{array}
$$

where H_{fpqc} denotes the cohomology with respect to the site $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$.

We show that the last vertical homomorphism is an isomorphism. Since N is affine, we can write $N = \mathrm{proj}\lim N_\lambda$, where each N_λ is an affine algebraic group scheme over k viewed as a sheaf on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$ ([7, III, §3, 7.5, Cor.]). We have $H^1_{\mathrm{fpqc}}(\mathcal{O}_K, N) \cong \mathrm{proj}\lim H^1_{\mathrm{fpqc}}(\mathcal{O}_K, N_\lambda)$. Since any infinitesimal group becomes zero over $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$, we can take N_λ to be smooth. Then we have $H^1_{\mathrm{fpqc}}(\mathcal{O}_K, N_\lambda) \cong H^1_{\mathrm{et}}(\mathcal{O}_K, N_\lambda) \cong H^1_{\mathrm{et}}(k, N_\lambda)$ by [10, Chapter III, Theorem 3.9 and Remark 3.11]. Thus $H^1_{\mathrm{fpqc}}(\mathcal{O}_K, N) \cong H^1_{\mathrm{fpqc}}(k, N)$.

On the other hand, the horizontal homomorphisms have natural sections corresponding to the Teichmüller section $k \hookrightarrow \mathcal{O}_K$. These facts imply the required surjectivity. □

Proof of Proposition 2.3. — By Assertion 2 of Proposition 2.2, it is enough to show that $\eta_f \colon \mathrm{Gal}(K^{\mathrm{ab}}/K) \to \pi_1^k(\mathbf{K}^\times)$ is a zero map for $f \in \mathbf{U}_K((\mathfrak{p}_K)_p) = \mathbf{K}^\times((\mathfrak{p}_K)_p)$. Let $0 \to N \to A \to \mathbf{K}^\times \to 0$ be an exact sequence of sheaves of abelian groups on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$ with N finite constant. Then the morphism $\varphi_f \colon \mathrm{Spec}\, K_p \to \mathbf{K}^\times$ lifts to a morphism $\mathrm{Spec}\, K_p \to A$ by Lemma 2.4. This implies that $\eta_f = 0$. □

2.4. Reduction to the case of algebraically closed residue fields. — We assume that Theorem A is true for algebraically closed k and show that Theorem A is true for general perfect k. By Assertion 4 of Proposition 2.2, the problem is reduced to that for the restricted part $\eta \colon I(K^{\mathrm{ab}}/K) \to \pi_1^k(\mathbf{U}_K)$. For this, it is enough to see that the $\mathrm{Gal}(\overline{k}/k)$-coinvariants of $\mathrm{Gal}((K^{\mathrm{ur}})^{\mathrm{ab}}/K^{\mathrm{ur}})$ (for the conjugation action) is $I(K^{\mathrm{ab}}/K)$ and the $\mathrm{Gal}(\overline{k}/k)$-coinvariants of $\pi_1^{\overline{k}}(\mathbf{U}_K)$ is $\pi_1^k(\mathbf{U}_K)$. The first assertion follows from the fact that the natural surjection $\mathrm{Gal}(K^{\mathrm{sep}}/K) \twoheadrightarrow \mathrm{Gal}(\overline{k}/k)$ admits a section ([15, Chapter II, §4.3, Exercises]). For the second assertion, we use the natural spectral sequence

$H^i(k, \operatorname{Ext}_{\overline{k}}^j(\mathbf{U}_K, \mathbb{Q}/\mathbb{Z})) \Rightarrow \operatorname{Ext}_k^{i+j}(\mathbf{U}_K, \mathbb{Q}/\mathbb{Z})$. We have $\operatorname{Hom}_{\overline{k}}(\mathbf{U}_K, \mathbb{Q}/\mathbb{Z}) = 0$ by the connectedness of $\mathbf{U}_K$. Thus $H^0(k, \operatorname{Ext}_{\overline{k}}^1(\mathbf{U}_K, \mathbb{Q}/\mathbb{Z})) = \operatorname{Ext}_k^1(\mathbf{U}_K, \mathbb{Q}/\mathbb{Z})$. Hence $\operatorname{Gal}(\overline{k}/k)$-coinvariants of $\pi_1^{\overline{k}}(\mathbf{U}_K)$ is $\pi_1^k(\mathbf{U}_K)$. Thus we get the result.

3. First proof of Theorem A

3.1. Proof of the part "η is an isomorphism". — In this subsection, we assume that the residue field k is algebraically closed and fix a prime element T of $\mathcal{O}_K$ to prove that $\eta \colon I(K^{\mathrm{ab}}/K) = \operatorname{Gal}(K^{\mathrm{ab}}/K) \to \pi_1^k(\mathbf{K}^\times) = \pi_1^k(\mathbf{U}_K)$ is an isomorphism. By Assertion 5 of Proposition 2.2, this homomorphism is induced by the morphism $\varphi^0 \colon \operatorname{Spec} K_p \to \mathbf{U}_K$ given by the rational point $-T + \mathbf{T} \in \mathbf{U}_K(K_p)$. Both groups $\operatorname{Gal}(K^{\mathrm{ab}}/K)$ and $\pi_1^k(\mathbf{U}_K)$ are profinite abelian groups. The Pontryagin dual of $\operatorname{Gal}(K^{\mathrm{ab}}/K)$ is $H^1(K, \mathbb{Q}/\mathbb{Z})$. The Pontryagin dual of $\pi_1^k(\mathbf{U}_K)$ is $\operatorname{Ext}_k^1(\mathbf{U}_K, \mathbb{Q}/\mathbb{Z})$. Therefore the problem is equivalent to showing that the dual map $\eta^\vee \colon \operatorname{Ext}_k^1(\mathbf{U}_K, \mathbb{Q}/\mathbb{Z}) \to H^1(K, \mathbb{Q}/\mathbb{Z})$ is an isomorphism. Since $\mathbf{U}_K \cong \mathbf{G}_m \times \mathbf{U}_K^1$, we have

$$\operatorname{Ext}_k^1(\mathbf{U}_K, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Ext}_k^1(\mathbf{G}_m, \mathbb{Q}/\mathbb{Z}) \oplus \operatorname{Ext}_k^1(\mathbf{U}_K^1, \mathbb{Q}/\mathbb{Z}).$$

3.1.1. The prime-to-p part. — We show that $\eta^\vee \colon \operatorname{Ext}_k^1(\mathbf{U}_K, \mathbb{Q}/\mathbb{Z}) \to H^1(K, \mathbb{Q}/\mathbb{Z})$ induces an isomorphism on the prime-to-p parts. It is enough to show that $\eta^\vee \colon \operatorname{Ext}_k^1(\mathbf{U}_K, n^{-1}\mathbb{Z}/\mathbb{Z}) \to H^1(K, n^{-1}\mathbb{Z}/\mathbb{Z})$ is an isomorphism for each integer n prime to p. Since k is algebraically closed, $n^{-1}\mathbb{Z}/\mathbb{Z}$ is isomorphic as a Galois module over k to the group μ_n of the n-th roots of unity. The group $\operatorname{Ext}_k^1(\mathbf{U}_K^1, \mu_n)$ is zero since the n-th power map is an automorphism on $\mathbf{U}_K^1$ while μ_n is killed by n. The group $\operatorname{Ext}_k^1(\mathbf{G}_m, \mu_n)$ is a cyclic group of order n generated by the extension class $0 \to \mu_n \to \mathbf{G}_m \xrightarrow{n} \mathbf{G}_m \to 0$. The group $H^1(K, \mu_n)$ is a cyclic group of order n generated by the Kummer character $\sigma \mapsto \sigma((-T)^{1/n})/(-T)^{1/n}$. The morphism $\varphi^0 \colon \operatorname{Spec} K_p \to \mathbf{U}_K$ followed by the projection $\mathbf{U}_K \twoheadrightarrow \mathbf{G}_m$ corresponds to the rational point $-T$. These generators correspond each other via the homomorphism $\eta^\vee \colon \operatorname{Ext}_k^1(\mathbf{U}_K, \mu_n) \to H^1(K, \mu_n)$ since we have a cartesian diagram

$$
\begin{CD}
\operatorname{Spec} K_p((-T)^{1/n}) @>>> \mathbf{G}_m \\
@VVV @VV{n}V \\
\operatorname{Spec} K_p @>>> \mathbf{G}_m.
\end{CD}
$$

Hence $\eta^\vee \colon \operatorname{Ext}_k^1(\mathbf{U}_K, \mu_n) \to H^1(K, \mu_n)$ is an isomorphism for any integer $n \geq 1$ prime to p.

3.1.2. *The p-primary part.* — We show that the homomorphism on the p-primary parts $\eta^\vee \colon \mathrm{Ext}^1_k(\mathbf{U}_K, \mathbb{Q}_p/\mathbb{Z}_p) \to H^1(K, \mathbb{Q}_p/\mathbb{Z}_p)$ is an isomorphism as well. We need the following lemmas.

LEMMA 3.1. — *The Artin-Hasse exponential map* $\prod_{p\nmid n \geq 1} W \to \mathbf{U}^1_K$ *sending*

$$a = (a_n)_{p\nmid n \geq 1} \in \prod_{p\nmid n \geq 1} W \quad with \quad a_n = (a_{n0}, a_{n1}, \dots) \in W$$

to

$$\prod_{p\nmid n \geq 1, \, m \geq 0} F(a_{nm} \mathbf{T}^{np^m}) \in \mathbf{U}^1_K$$

is an isomorphism of proalgebraic groups. Here we denote by W *the additive group of Witt vectors and set* $F(t) = \exp(-\sum_{e \geq 0} t^{p^e}/p^e) \in \mathbb{Z}_p[[t]]$.

Proof. — See [14, Chapter V, §3, 16]. □

LEMMA 3.2. — *Let* $f \colon A \to B$ *be a homomorphism between abelian groups* A *and* B. *If both* A *and* B *are p-divisible and p-power torsion, and f induces an isomorphism on the p-torsion parts, then f itself is an isomorphism.*

The group $\mathrm{Ext}^1_k(\mathbf{G}_m, \mathbb{Q}_p/\mathbb{Z}_p)$ is zero since the p-th power map is an automorphism on $\mathbf{G}_m$ as a perfect group scheme while $\mathbb{Q}_p/\mathbb{Z}_p$ is a union of p-power torsion groups. The group $\mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Q}_p/\mathbb{Z}_p) = \pi^k_1(\mathbf{U}^1_K)^\vee$ is p-divisible since $\mathbf{U}^1_K \cong \prod_{p\nmid n \geq 1} W$ by Lemma 3.1 and $\pi^k_1(W)$ is p-torsion-free by [12, §8.5, Prop. 5]. The group $H^1(K, \mathbb{Q}_p/\mathbb{Z}_p)$ is p-divisible since the largest pro-p quotient of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ is pro-p free ([15, Chapter II, §2.2, Corollary 1]). Thus, using Lemma 3.2, we are reduced to showing that $\eta^\vee \colon \mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Z}/p\mathbb{Z}) \to H^1(K, \mathbb{Z}/p\mathbb{Z})$ is an isomorphism. We have

$$\mathrm{Ext}^1_k(\mathbf{U}^1_K, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{p\nmid n \geq 1} \mathrm{Ext}^1_k(W, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{p\nmid n \geq 1} \mathrm{Ext}^1_k(\mathbf{G}_a, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{p\nmid n \geq 1} k.$$

Here the last isomorphism comes from the isomorphism $k \xrightarrow{\sim} \mathrm{Ext}^1_k(\mathbf{G}_a, \mathbb{Z}/p\mathbb{Z})$ sending an element $a \in k^\times$ to the extension class given by

$$(1) \qquad\qquad 0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbf{G}_a \xrightarrow{a^{-1}\wp} \mathbf{G}_a \longrightarrow 0,$$

where $\wp(x) = x^p - x$ ([12, §8.3, Prop. 3]). On the other hand, the map defined by

$$\bigoplus_{p\nmid n \geq 1} kT^{-n} \to H^1(K, \mathbb{Z}/p\mathbb{Z}),$$

$$aT^{-n} \mapsto \left(\sigma \mapsto \sigma(\wp^{-1}(aT^{-n})) - \wp^{-1}(aT^{-n})\right)$$

is an isomorphism by Artin-Schreier theory. To show that

$$\eta^\vee\colon \operatorname{Ext}^1_k(\mathbf{U}_K, \mathbb{Z}/p\mathbb{Z}) \cong \bigoplus_{p\nmid n\geq 1} k \to \bigoplus_{p\nmid n\geq 1} kT^{-n} \cong H^1(K, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism, we need to calculate the following morphism:

$$\operatorname{Spec} K_p \to \mathbf{U}^1_K/(\mathbf{U}^1_K)^p \cong \prod_{p\nmid n\geq 1} W/pW \cong \prod_{p\nmid n\geq 1} \mathbf{G}_a.$$

The morphism $\operatorname{Spec} K_p \to \mathbf{U}^1_K/(\mathbf{U}^1_K)^p$ corresponds to the K_p-rational point $1 - T^{-1}\mathbf{T}$ of $\mathbf{U}^1_K/(\mathbf{U}^1_K)^p$. The isomorphism $\prod_{p\nmid n\geq 1} \mathbf{G}_a \xrightarrow{\sim} \mathbf{U}^1_K/(\mathbf{U}^1_K)^p$ sends each element $(a_n)_{p\nmid n\geq 1}$ of the left-hand side to $\prod_{p\nmid n\geq 1} F(a_n\mathbf{T}^n)$ of the right-hand side.

PROPOSITION 3.3. — 1. *The inverse of the isomorphism $\prod_{p\nmid n\geq 1} \mathbf{G}_a \xrightarrow{\sim} \mathbf{U}^1_K/(\mathbf{U}^1_K)^p$ is given by the map*

$$\mathbf{U}^1_K/(\mathbf{U}^1_K)^p \xrightarrow{\operatorname{dlog}} \prod_{n\geq 1} \mathbf{G}_a\mathbf{T}^n \operatorname{dlog}\mathbf{T} \xrightarrow{\alpha} \prod_{p\nmid n\geq 1} \mathbf{G}_a,$$

where $\alpha(\sum_{n\geq 1} b_n\mathbf{T}^n \operatorname{dlog}\mathbf{T}) = (-b_n/n)_{p\nmid n\geq 1}$.
2. *The rational point $1 - T^{-1}\mathbf{T}$ corresponds to $(1/(nT^n))_{p\nmid n\geq 1}$ via the isomorphism $\mathbf{U}^1_K/(\mathbf{U}^1_K)^p(K_p) \cong \prod_{p\nmid n\geq 1} \mathbf{G}_a(K_p)$.*
3. *The map $\operatorname{Spec} K_p \to \prod_{p\nmid n\geq 1} \mathbf{G}_a$ gives the K_p-rational point $(1/(nT^n))_{p\nmid n\geq 1}$ of $\prod_{p\nmid n\geq 1} \mathbf{G}_a$.*

Proof. — 1. Using the identity $\operatorname{dlog} F(t) = -\sum_{e\geq 0} t^{p^e} \operatorname{dlog} t$, we have

$$\operatorname{dlog}\left(\prod_{p\nmid n\geq 1} F(a_n\mathbf{T}^n)\right) = -\sum_{\substack{e\geq 0 \\ p\nmid n\geq 1}} (a_n\mathbf{T}^n)^{p^e} \operatorname{dlog}(a_n\mathbf{T}^n)$$

$$= \sum_{\substack{e\geq 0 \\ p\nmid n\geq 1}} (-n)(a_n\mathbf{T}^n)^{p^e} \operatorname{dlog}\mathbf{T}.$$

Thus the map $\alpha \circ \operatorname{dlog}$ sends $\prod_{p\nmid n\geq 1} F(a_n\mathbf{T}^n)$ to $(a_n)_{p\nmid n\geq 1}$, as desired.

2. A simple calculation shows that $(\alpha \circ \operatorname{dlog})(1 - T^{-1}\mathbf{T}) = (1/(nT^n))_{p\nmid n\geq 1}$.

3. This follows from Assertion 2. □

Now we calculate $\eta^\vee$. Let $n \geq 1$ be an integer prime to p and $a \neq 0$ be an element of k regarded as an element of $\bigoplus_{p\nmid n\geq 1} k$ by the inclusion $k \hookrightarrow \bigoplus_{p\nmid n\geq 1} k$

into the n-th summand. The corresponding extension of $\mathbf{G}_a$ is given by (1). We have a cartesian diagram

$$
\begin{array}{ccc}
\operatorname{Spec} K_p(\wp^{-1}(a/nT^n)) & \longrightarrow & \mathbf{G}_a \\
\downarrow & & \downarrow{\scriptstyle a^{-1}\wp} \\
\operatorname{Spec} K_p & \longrightarrow & \mathbf{G}_a.
\end{array}
$$

Thus $\eta^\vee \colon \bigoplus_{p\nmid n\geq 1} k \to \bigoplus_{p\nmid n\geq 1} kT^{-n}$ preserves the direct factors and the map on the n-th factor is given by multiplication by $1/n$. This shows that $\eta^\vee \colon \operatorname{Ext}^1_k(\mathbf{U}_K, \mathbb{Z}/p\mathbb{Z}) \to H^1(K, \mathbb{Z}/p\mathbb{Z})$ is an isomorphism.

This completes the proof of Assertion 1 of Theorem A.

3.2. Proof of the part "$\eta^{-1} = \theta$". — In this section, we assume that the residue field k is a general perfect field of characteristic $p > 0$ (see Remark 3.5 below for the point this generality helps for). To prove Assertion 2 of Theorem A, we first quickly recall the construction of the reciprocity map $\theta \colon \pi^k_1(\mathbf{U}_K) \xrightarrow{\sim} I(K^{\mathrm{ab}}/K)$ of the local class field theory of Serre and Hazewinkel. Let L be a finite totally ramified Galois extension of K. We can apply the constructions in Section 2.1 to L to get sheaves $\mathbf{U}_L$ and $\mathbf{L}^\times$ on $(\operatorname{Perf}/k)_{\mathrm{fpqc}}$. Let $\mathbf{V}_{L/K}$ be the subgroup of $\mathbf{U}_L$ generated by elements of the form $\sigma(u)/u$ for $\sigma \in \operatorname{Gal}(L/K)$ and $u \in \mathbf{U}_L$. By [7, Appendix, 4.2], we have exact sequences

$$
(2) \qquad 0 \to \operatorname{Gal}(L/K)^{\mathrm{ab}} \to \mathbf{U}_L/\mathbf{V}_{L/K} \xrightarrow{N_{L/K}} \mathbf{U}^\times_K \to 0,
$$

$$
(3) \qquad 0 \to \operatorname{Gal}(L/K)^{\mathrm{ab}} \to \mathbf{L}^\times/\mathbf{V}_{L/K} \xrightarrow{N_{L/K}} \mathbf{K}^\times \to 0,
$$

where the second map $N_{L/K}$ is the norm map and the first map sends σ to $\sigma(\boldsymbol{\pi}_L)/\boldsymbol{\pi}_L$ for any choice of a prime element π_L of $\mathcal{O}_L$, with $\boldsymbol{\pi}_L$ defined as $1 \otimes \pi_L \in k \hat{\otimes}_k \mathcal{O}_L = \mathbf{O}_L(k)$ and σ acting on $\mathbf{L}^\times$ as a morphism of sheaves. To write the isomorphism $\theta \colon \pi^k_1(\mathbf{U}_K) \xrightarrow{\sim} I(K^{\mathrm{ab}}/K)$, note that the Pontryagin dual groups of $\pi^k_1(\mathbf{U}_K)$, $I(K^{\mathrm{ab}}/K)$ are $\operatorname{Ext}^1_k(\mathbf{U}_K, \mathbb{Q}/\mathbb{Z})$, $H^1(I(K^{\mathrm{ab}}/K), \mathbb{Q}/\mathbb{Z})$, respectively. Now the dual map $\theta^\vee \colon H^1(I(K^{\mathrm{ab}}/K), G) \xrightarrow{\sim} \operatorname{Ext}^1_k(\mathbf{U}_K, G)$ for a finite abelian group G is given by sending L totally ramified abelian with $\operatorname{Gal}(L/K) = G$ to the extension class (2) ([7, Appendix, 7.1]).

On the other hand, the dual $\eta^\vee \colon \operatorname{Ext}^1_k(\mathbf{K}^\times, G) \xrightarrow{\sim} H^1(K, G)$ of our isomorphism $\eta \colon \operatorname{Gal}(K^{\mathrm{ab}}/K) \xrightarrow{\sim} \pi^k_1(\mathbf{K}^\times)$ sends the extension class (3) to the G-torsor on $\operatorname{Spec} K_p$ given by pulling back (3) by the morphism $\varphi_f \colon \operatorname{Spec} K_p \to \mathbf{K}^\times$, where $f \in \Xi_K$. Therefore our claim $\eta^{-1} = \theta$ is interpreted as the claim that the G-torsor on $\operatorname{Spec} K_p$ thus obtained is $\operatorname{Spec} L_p$. Hence Assertion 3 of the following proposition proves Assertion 2 of Theorem A:

PROPOSITION 3.4. — *Assume that the residue field k is perfect of characteristic $p > 0$. Let L/K be a finite totally ramified Galois extension.*

1. *For any $f \in \Xi_K$ and any $g \in \Xi_K$, the diagram*

$$
\begin{array}{ccc}
\operatorname{Spec} L_p & \xrightarrow{\varphi_g} & \mathbf{L}^\times \\
\downarrow & & \downarrow{\scriptstyle N_{L/K}} \\
\operatorname{Spec} K_p & \xrightarrow{\varphi_f} & \mathbf{K}^\times
\end{array}
$$

is commutative "up to a $(\mathfrak{p}_L)_p$-rational point", namely the ratio of the two elements of $\mathbf{K}^\times(L_p)$ corresponding to the two morphisms $\operatorname{Spec} L_p \rightrightarrows \mathbf{K}^\times$ coming from the above possibly non-commutative diagram is in $\mathbf{K}^\times((\mathfrak{p}_L)_p) = \operatorname{Ker}(\mathbf{K}^\times((\mathcal{O}_L)_p) \twoheadrightarrow \mathbf{K}^\times(k))$.

2. *For any $f \in \Xi_K$, there exists $g \in \Xi_L$ such that the above diagram is indeed commutative.*

3. *Assume that L/K is abelian. For any $f \in \Xi_K$ and any $g \in \Xi_L$ that makes the diagram commutative, the induced diagram*

$$
\begin{array}{ccc}
\operatorname{Spec} L_p & \xrightarrow{\varphi_g} & \mathbf{L}^\times/\mathbf{V}_{L/K} \\
\downarrow & & \downarrow{\scriptstyle N_{L/K}} \\
\operatorname{Spec} K_p & \xrightarrow{\varphi_f} & \mathbf{K}^\times
\end{array}
$$

is cartesian, and is equivariant under the action of $\operatorname{Gal}(L/K)$ on the right vertical arrow coming from the exact sequence (3) *and the natural action of $\operatorname{Gal}(L/K)$ on the left vertical arrow.*

Proof. — 1. The two elements of $\mathbf{K}^\times(L_p)$ mentioned in Assertion 1 are given by f and $N_{L/K}g$. It is enough to show that $f/N_{L/K}g \in \mathbf{K}^\times((\mathcal{O}_L)_p)$ since then we immediately have $f/N_{L/K}g \in \mathbf{K}^\times((\mathfrak{p}_L)_p)$ by Condition 2 of Definition 2.1. The claim $f/N_{L/K}g \in \mathbf{K}^\times((\mathcal{O}_L)_p)$ is equivalent to the equality $(N_{L/K}g) = (f)$ of ideals of the ring $\mathbf{K}((\mathcal{O}_L)_p)$. Let $\Delta_K \subset \operatorname{Spec} \mathbf{K}((\mathcal{O}_K)_p)$ be the diagonal divisor defined by the kernel of the multiplication map

$$
\mathbf{K}((\mathcal{O}_K)_p) = ((\mathcal{O}_K)_p \hat{\otimes}_k \mathcal{O}_K) \otimes_{\mathcal{O}_K} K \to K_p.
$$

Then we have $\Delta_K = (f)$ as divisors on $\operatorname{Spec} \mathbf{K}((\mathcal{O}_K)_p)$ by Condition 1 of Definition 2.1. Similarly, for $\sigma \in \operatorname{Gal}(L/K)$, let $\sigma(\Delta_L) \subset \operatorname{Spec} \mathbf{L}((\mathcal{O}_L)_p)$ be the σ-twist of the diagonal divisor defined by the kernel of the map

$$
\mathbf{L}((\mathcal{O}_L)_p) = ((\mathcal{O}_L)_p \hat{\otimes}_k \mathcal{O}_L) \otimes_{\mathcal{O}_L} L \xrightarrow{(1 \otimes \sigma) \otimes \sigma} L_p.
$$

Then we have $p_* \Delta_L = (N_{L/K}g)$ as divisors on $\operatorname{Spec} \mathbf{K}((\mathcal{O}_L)_p)$, where $p \colon \operatorname{Spec} \mathbf{L}((\mathcal{O}_L)_p) \to \operatorname{Spec} \mathbf{K}((\mathcal{O}_L)_p)$ is the natural map. Therefore what we have to show is the equality $p_* \Delta_L = \Delta_K$ of divisors on $\operatorname{Spec} \mathbf{K}((\mathcal{O}_L)_p)$.

This equality follows from the equalities

$$p_* \Delta_L = \sum_{\sigma \in \mathrm{Gal}(L/K)} \sigma(\Delta_L) = \Delta_K$$

as divisors on $\mathrm{Spec}\, \mathbf{L}((\mathcal{O}_L)_p)$. This proves Assertion 1.

2. First take $g \in \Xi_L$ to be arbitrary. Then by Assertion 1, we have $f/N_{L/K}g \in \mathbf{K}^\times((\mathfrak{p}_L)_p)$. The homomorphism $N_{L/K} \colon \mathbf{L}^\times \to \mathbf{K}^\times$ and its restriction $\mathbf{U}_L \to \mathbf{U}_K$ are surjective and have the same kernel; see (2) and (3). Since $\mathbf{U}_L$ and $\mathbf{U}_K$ are affine, so is $\mathrm{Ker}(N_{L/K})$. Thus we can choose an element $u \in \mathbf{L}^\times((\mathfrak{p}_L)_p)$ so that $N_{L/K}u = f/N_{L/K}g$ by Lemma 2.4. Replacing g by ug, we have $N_{L/K}g = f$. This choice of g makes the diagram commutative.

3. Before the proof, note that the field $\mathbf{L}(L_p)$ has two different actions of $\mathrm{Gal}(L/K)$. One comes from the action on the sheaf $\mathbf{L}$, which we have already denoted by $h \mapsto \sigma(h)$, $\sigma \in \mathrm{Gal}(L/K)$. The other comes from the action on the coefficient field L_p, which we denote by $h \mapsto [\sigma](h)$, $\sigma \in \mathrm{Gal}(L/K)$.

Now we show the equivariance first. This is the same, in terms of the rational point g, as the equality $[\sigma](g) = (\sigma(\pi_L)/\pi_L) \cdot g$ in $(\mathbf{L}^\times/\mathbf{V}_{L/K})(L_p)$. Below we use the exponential notation, so the equality to be shown can be written as $g^{[\sigma]-1} = \pi_L^{\sigma-1}$. We have

$$g^{[\sigma]-1} = (g^{-[\sigma]})^{\sigma-1} g^{\sigma[\sigma]-1}.$$

By Assertion 3 of Proposition 2.2, the element $g^{-[\sigma]}$ is a prime element of the complete discrete valuation ring $\mathbf{O}_L(L_p)$. Hence $(g^{-[\sigma]})^{\sigma-1} = \pi_L^{\sigma-1}$ in $(\mathbf{L}^\times/\mathbf{V}_{L/K})(L_p)$. Therefore it is enough to show that $g^{\sigma[\sigma]-1} \in \mathbf{V}_{L/K}(L_p)$.

By Condition 1 of Definition 2.1, the element $g^{\sigma[\sigma]}$ generates as an ideal of $\mathbf{L}((\mathcal{O}_L)_p)$ the kernel of the map

$$\mathbf{L}((\mathcal{O}_L)_p) = ((\mathcal{O}_L)_p \hat{\otimes}_k \mathcal{O}_L) \otimes_{\mathcal{O}_L} L \xrightarrow{(\sigma \otimes \sigma) \otimes \sigma} L_p,$$

which is the same as the kernel of the usual multiplication map $\mathbf{L}((\mathcal{O}_L)_p) = ((\mathcal{O}_L)_p \hat{\otimes}_k \mathcal{O}_L) \otimes_{\mathcal{O}_L} L \to L_p$. Therefore we have $g^{\sigma[\sigma]-1} \in \mathbf{L}^\times((\mathcal{O}_L)_p)$. Since $g \in \mathbf{L}^\times(L_p)$, we have $g^{\sigma[\sigma]-1} \in \mathbf{U}_L((\mathcal{O}_L)_p)$. Moreover, we have $g^{\sigma[\sigma]-1} \in \mathbf{U}_L((\mathfrak{p}_L)_p)$ by Condition 2 of Definition 2.1. Furthermore, we have $g^{\sigma[\sigma]-1} \in (\mathrm{Ker}\, N_{L/K})((\mathfrak{p}_L)_p)$ since f is K-rational and therefore $N_{L/K}(g^{\sigma[\sigma]-1}) = f^{[\sigma]-1} = 1$. The exact sequence (2) leads an exact sequence $0 \to \mathbf{V}_{L/K} \to \mathrm{Ker}\, N_{L/K} \to \mathrm{Gal}(L/K)^{\mathrm{ab}} \to 0$, hence $(\mathrm{Ker}\, N_{L/K})((\mathfrak{p}_L)_p) = \mathbf{V}_{L/K}((\mathfrak{p}_L)_p)$. Thus we have $g^{\sigma[\sigma]-1} \in \mathbf{V}_{L/K}((\mathfrak{p}_L)_p)$. This proves the equivariance of the diagram.

Finally we show that the diagram is cartesian. The pullback of the exact sequence (3) by $\varphi_f \colon \mathrm{Spec}\, K_p \to \mathbf{K}^\times$ gives a $\mathrm{Gal}(L/K)$-torsor Z on $\mathrm{Spec}\, K_p$. The diagram factors through a morphism from $\mathrm{Spec}\, L_p$ to Z by universality. The equivariance just proved says that this is a morphism of $\mathrm{Gal}(L/K)$-torsors

on Spec K_p. Such a morphism is automatically an isomorphism. Therefore $Z =$ Spec L_p and the diagram is cartesian. $\qquad\square$

REMARK 3.5. — In [16, §4], the reciprocity isomorphism $\theta\colon I(K^{\mathrm{ab}}/K) \xrightarrow{\sim} \pi_1^k(\mathbf{U}_K)$ of Serre and Hazewinkel was extended to an isomorphism $\mathrm{Gal}(K^{\mathrm{ab}}/K) \xrightarrow{\sim} \pi_1^k(\mathbf{K}^\times)$. A full discussion about this result was taken place in [17]. The dual map $\theta^\vee\colon H^1(K,G) \xrightarrow{\sim} \mathrm{Ext}_k^1(\mathbf{K}^\times, G)$ sends a finite totally ramified abelian extension L of Galois group G to the extension class (3). The above proposition also proves the equality $\eta^{-1} = \theta$ as isomorphisms between full $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ and $\pi_1^k(\mathbf{K}^\times)$.

4. Second proof of Theorem A: Lubin-Tate theory

In this section, we assume that the residue field k is finite and give another proof of Theorem A for finite k using Lubin-Tate theory. Assume k is the finite field with q elements. We fix a prime element T of $\mathcal{O}_K$. This defines a splitting $\mathbf{K}^\times \cong \mathbf{U}_K \times \mathbb{Z}$. By Assertion 5 of Proposition 2.2, the homomorphism $\mathrm{Gal}(K^{\mathrm{ab}}/K) \xrightarrow{\eta} \mathbf{K}^\times \twoheadrightarrow \mathbf{U}_K$ comes from the morphism $\varphi^0\colon \mathrm{Spec}\, K_p \to \mathbf{U}_K$ corresponding the rational point $-T + \mathbf{T} \in \mathbf{U}_K(K_p)$. We consider the short exact sequence $0 \to U_K \to \mathbf{U}_K \xrightarrow{F-1} \mathbf{U}_K \to 0$, where F is the q-th power Frobenius morphism. The isogeny $F - 1$, called the Lang isogeny, is universal among isogenies onto $\mathbf{U}_K$ with (pro-) finite constant kernels ([14, Chap. VI, §1, Prop. 6]), hence it identifies U_K as $\pi_1^k(\mathbf{U}_K)$.

PROPOSITION 4.1. — *There is a cartesian diagram*

$$
\begin{array}{ccc}
\mathrm{Spec}(K_T^{\mathrm{ram}})_p & \longrightarrow & \mathbf{U}_K \\
\downarrow & & \downarrow{\scriptstyle F-1} \\
\mathrm{Spec}\, K_p & \xrightarrow{\varphi^0} & \mathbf{U}_K.
\end{array}
$$

Here K_T^{ram} is the field K adjoining all the T^m-torsion points (where m runs through the integers ≥ 1) of the Lubin-Tate formal group F_f ([9]) whose equation of formal multiplication by T is equal to $f(X) = TX + X^q$. The morphism $\mathrm{Spec}(K_T^{\mathrm{ram}})_p \to \mathbf{U}_K$ corresponds to the rational point $\sum_{m=0}^\infty \alpha_{m+1}\mathbf{T}^m$, where α_m is a generator of the module of T^m-torsion points of F_f such that $f(\alpha_{m+1}) = \alpha_m$. The induced isomorphism $\mathrm{Gal}(K_T^{\mathrm{ram}}/K) \cong U_K$ coincides with the one given by Lubin-Tate theory.

Proof. — We calculate the geometric fiber of $F - 1$ over $-T + \mathbf{T}$. Let $g = \sum a_m \mathbf{T}^m$ be an element of $\mathbf{U}_K(\overline{K})$. The equation $F(g)/g = -T + \mathbf{T}$ is equivalent to the system of equations $f(a_0) = 0$, $f(a_{m+1}) = a_m$, $m \geq 0$. Thus, for each $m \geq 0$, a_m is a generator of the module of T^{m+1}-torsion points of F_f. This

proves the existence of the above cartesian diagram. Next we calculate the action of $\mathrm{Gal}(K_T^{\mathrm{ram}}/K)$ on the fiber of $F-1$ over $-T+\mathbf{T}$. The Lubin-Tate group F_f for $f(X) = TX + X^q$ is the formal completion $\widehat{\mathbf{G}}_a$ of the additive group with formal multiplication of each element $\sum b_m T^m$ of $\mathcal{O}_K$ being given by the power series $\sum b_m f^{\circ m}(X) \in \mathrm{End}\,\widehat{\mathbf{G}}_a$, where $f^{\circ m}$ is the m-th iteration of f. Hence, if σ corresponds to $u(T) = \sum b_m T^m$ via the isomorphism $\mathrm{Gal}(K_T^{\mathrm{ram}}/K) \cong U_K$ of Lubin-Tate theory, we have

$$\sigma\left(\sum_{m\geq 0} \alpha_{m+1}\mathbf{T}^m\right) = \sum_{m\geq 0}\sigma(\alpha_{m+1})\mathbf{T}^m = \sum_{0\leq k\leq m<\infty} b_k\alpha_{m+1-k}\mathbf{T}^m$$

$$= u(\mathbf{T})\sum_{m\geq 0}\alpha_{m+1}\mathbf{T}^m.$$

Thus the action of σ on the fiber of $F-1$ over $-T+\mathbf{T}$ is given by multiplication by $u(\mathbf{T})$, as required. □

Therefore the map $\eta\colon \mathrm{Gal}(K^{\mathrm{ab}}/K) \to \pi_1^k(\mathbf{K}^\times)$ factors through the restriction map $\mathrm{Gal}(K^{\mathrm{ab}}/K) \twoheadrightarrow \mathrm{Gal}(K_T^{\mathrm{ram}}K^{\mathrm{ur}}/K)$, the isomorphism of Lubin-Tate theory $\mathrm{Gal}(K_T^{\mathrm{ram}}K^{\mathrm{ur}}/K) \xrightarrow{\sim} (K^\times)^\wedge$ (the profinite completion of $K^\times$) and the natural isomorphism $(K^\times)^\wedge \cong \pi_1^k(\mathbf{K}^\times)$. Thus the assertion that η is an isomorphism is equivalent to the local Kronecker-Weber theorem for Lubin-Tate extensions: $K^{\mathrm{ab}} = K_T^{\mathrm{ram}}K^{\mathrm{ur}}$. Since the isomorphism θ of Serre and Hazewinkel for finite k coincides with the one given by Lubin-Tate theory, the equality $\eta^{-1} = \theta$ for such k follows.

REMARK 4.2. — This proof also shows that the proof for η being an isomorphism given in Section 3.1 gives another proof of the local Kronecker-Weber theorem for Lubin-Tate extensions in the equal characteristic case.

5. Third proof of Theorem A: geometry

In this section, we give a geometric proof of Theorem A. This method was suggested by the referee to the author. As in Section 3, we assume the residue field k is algebraically closed and fix a prime element T of $\mathcal{O}_K$. Both the morphism $\varphi\colon \mathrm{Spec}\,K_p \to \mathbf{K}^\times$, which corresponds to the rational point $1 - T\mathbf{T}^{-1}$, and the morphism $\varphi^0\colon \mathrm{Spec}\,K_p \to \mathbf{U}_K$, which corresponds to the rational point $-T+\mathbf{T}$, give the same homomorphism $\eta\colon \mathrm{Gal}(K^{\mathrm{ab}}/K) \to \pi_1^k(\mathbf{K}^\times) = \pi_1^k(\mathbf{U}_K)$. We use the same letter to express morphisms and corresponding rational points. It is enough to study the part $\eta^\vee\colon \mathrm{Ext}_k^1(\mathbf{U}_K, N) \to H^1(K, N)$ for each finite constant group N.

For the proof of Assertion 1 of Theorem A, we need the following result due to Contou-Carrere ([6]). Let A be a commutative quasi-algebraic group over k. Then the homomorphism

$$\mathrm{Hom}(\mathbf{U}_K, A) \to A(K_p)/A((\mathcal{O}_K)_p)$$

induced by the morphism $\varphi^0\colon \mathrm{Spec}\, K_p \to \mathbf{U}_K$ is an isomorphism ([6, Théorèm (1.5)]). If A is a commutative proalgebraic group over k, the homomorphism $\mathrm{Hom}(\mathbf{U}_K, A) \to A(K_p)/A((\mathcal{O}_K)_p)$ induced by the morphism $\varphi^0\colon \mathrm{Spec}\, K_p \to \mathbf{U}_K$ is still an isomorphism.

First we show that $\eta^\vee\colon \mathrm{Ext}^1_k(\mathbf{U}_K, N) \to H^1(K, N)$ is surjective, that is, every finite abelian extension L of K with Galois group N is obtained by the pullback of an extension $0 \to N \to A \to \mathbf{U}_K \to 0$ by the morphism $\varphi^0\colon \mathrm{Spec}\, K \to \mathbf{U}_K$. Since k is algebraically closed, the abelian extension L/K is a composite of a Kummer extension of degree prime to p and Artin-Schreier-Witt extensions. A Kummer extension of K of degree n, $p \nmid n$, is the pullback of $\mathbf{G}_m \xrightarrow{n} \mathbf{G}_m$ by some morphism $\mathrm{Spec}\, K_p \to \mathbf{G}_m$. An Artin-Schreier-Witt extension of K of degree p^n is the pullback of $W_n \xrightarrow{\wp} W_n$ by some morphism $\mathrm{Spec}\, K_p \to W_n$, where W_n is the additive group of Witt vectors of length n, $\wp(x) = Fx - x$ and F is the Frobenius. Combining them, we can find an exact sequence $0 \to N \to A' \xrightarrow{\alpha} A'' \to 0$ of commutative quasi-algebraic groups and a cartesian diagram

$$
\begin{array}{ccc}
\mathrm{Spec}\, L_p & \xrightarrow{\ \beta\ } & A' \\
\downarrow & & \alpha\downarrow \\
\mathrm{Spec}\, K_p & \xrightarrow{\ \gamma\ } & A''.
\end{array}
$$

Since $\varphi^0\colon \mathrm{Spec}\, K_p \to \mathbf{U}_K$ induces an isomorphism $\mathrm{Hom}(\mathbf{U}_K, A'') \xrightarrow{\sim} A''(K_p)/A''((\mathcal{O}_K)_p)$ by the result of Contou-Carrere quoted above, we have $\gamma + \varepsilon = \varphi^0 \circ \delta$ for some $\delta\colon \mathbf{U}_K \to A''$ and $\varepsilon \in A''((\mathcal{O}_K)_p)$. Since $\alpha\colon A' \to A''$ is surjective and k is algebraically closed, Lemma 2.4 shows that $\alpha\colon A'((\mathcal{O}_K)_p) \to A''((\mathcal{O}_K)_p)$ is also surjective. Hence $\varepsilon = \alpha(\zeta)$ for some $\zeta \in A'((\mathcal{O}_K)_p)$. The above diagram is still commutative and cartesian if we add ζ to β and ε to γ. The resulting cartesian diagram splits into the following commutative diagram with cartesian squares:

$$
\begin{array}{ccccc}
\mathrm{Spec}\, L_p & \longrightarrow & A & \longrightarrow & A' \\
\downarrow & & \downarrow & & \alpha\downarrow \\
\mathrm{Spec}\, K_p & \xrightarrow{\ \varphi^0\ } & \mathbf{U}_K & \xrightarrow{\ \gamma\ } & A''
\end{array}
$$

Thus we get the surjectivity of $\eta^\vee\colon \mathrm{Ext}^1_k(\mathbf{U}_K, N) \to H^1(K, N)$.

Next we show that $\eta^\vee$ is injective. Let $0 \to N \to A \xrightarrow{\alpha} \mathbf{U}_K \to 0$ be an extension whose image in $H^1(K, N)$ by $\eta^\vee$ is trivial, i.e. $\alpha \circ \beta = \varphi^0$ for some $\beta \in A(K_p)$. The diagram

$$
\begin{array}{ccc}
\mathrm{Hom}(\mathbf{U}_K, A) & =\!=\!= & A(K_p)/A((\mathcal{O}_K)_p) \\
\downarrow{\scriptstyle \alpha} & & {\scriptstyle \alpha}\downarrow \\
\mathrm{Hom}(\mathbf{U}_K, \mathbf{U}_K) & =\!=\!= & \mathbf{U}_K(K_p)/\mathbf{U}_K((\mathcal{O}_K)_p)
\end{array}
$$

is commutative. Let $\gamma \in \mathrm{Hom}(\mathbf{U}_K, A)$ correspond $\beta \in A(K_p)$ via the top horizontal map. Then we have $\alpha \circ \gamma = \mathrm{id}$ by the equality $\alpha \circ \beta = \varphi^0$ and this diagram. Hence the extension $0 \to N \to A \xrightarrow{\alpha} \mathbf{U}_K \to 0$ is trivial. Thus we get the injectivity of $\eta^\vee \colon \mathrm{Ext}^1_k(\mathbf{U}_K, N) \to H^1(K, N)$.

Finally we show that $\eta^{-1} = \theta$. For this, we need generalized Jacobians ([14]). We denote by $\mathbb{P}^1$ the projective line over k with coordinate T. For each effective divisor $\mathfrak{m}$ on $\mathbb{P}^1$ with support $\mathrm{Supp}(\mathfrak{m}) \subset \mathbb{P}^1$, let $J_{\mathbb{P}^1, \mathfrak{m}}$ be the moduli of line bundles on X with level $\mathfrak{m}$ structure, which is isomorphic to the direct product of $\mathbb{Z}$ and the generalized Jacobian of $\mathbb{P}^1$ with modulus $\mathfrak{m}$. Let $\Phi_{\mathfrak{m}} \colon \mathbb{P}^1 - \mathrm{Supp}(\mathfrak{m}) \to J_{\mathbb{P}^1, \mathfrak{m}}$ be the Abel-Jacobi map times -1. Since $\mathbb{P}^1$ has trivial Jacobian, $J_{\mathbb{P}^1, \mathfrak{m}}$ is an affine algebraic group times $\mathbb{Z}$. The groups $J_{\mathbb{P}^1, \mathfrak{m}}$, the schemes $\mathbb{P}^1 - \mathrm{Supp}(\mathfrak{m})$ and the morphisms $\Phi_{\mathfrak{m}} \colon \mathbb{P}^1 - \mathrm{Supp}(\mathfrak{m}) \to J_{\mathbb{P}^1, \mathfrak{m}}$ form projective systems with respect to $\mathfrak{m}$. Let $J_{\mathbb{P}^1}$ (resp. $\Phi \colon \mathrm{Spec}\, k(T) \to J_{\mathbb{P}^1}$) be the projective limit of the $J_{\mathbb{P}^1, \mathfrak{m}}$ (resp. the $\Phi_{\mathfrak{m}}$), the limit being taken over all effective divisors $\mathfrak{m}$ on $\mathbb{P}^1$. They are naturally regarded as sheaves on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$ and morphisms of sheaves on $(\mathrm{Perf}/k)_{\mathrm{fpqc}}$. The inclusion $k(T) \subset k((T)) = K$ identifies K as the fraction field of the completion of the local ring of $\mathbb{P}^1$ at $T = 0$. This identification gives a natural morphism $\mathrm{Spec}\, K_p \to \mathrm{Spec}\, k(T)_p$ and a natural homomorphism $\mathbf{K}^\times \to J_{\mathbb{P}^1}$. Consider the following (non-commutative) diagram:

$$
\begin{array}{ccc}
\mathrm{Spec}\, K_p & \xrightarrow{\ \varphi\ } & \mathbf{K}^\times \\
\downarrow & & \downarrow \\
\mathrm{Spec}\, k(T)_p & \xrightarrow{\ \Phi\ } & J_{\mathbb{P}^1}.
\end{array}
$$

This diagram gives two morphisms $\mathrm{Spec}\, K_p \to J_{\mathbb{P}^1}$, or equivalently, two K_p-rational points of $J_{\mathbb{P}^1}$. Under the natural identification $J_{\mathbb{P}^1} = (\prod_{a \in \mathbb{P}^1(k) - \{0\}} \mathbf{U}_a \times \mathbf{K}^\times)/\mathbf{G}_m$ with $\mathbf{U}_a$ defined by $\mathbf{U}_a(R) = R[[\mathbf{T}^{-1} - a^{-1}]]^\times$, the difference of these two rational points is given by $((1 - T\mathbf{T}^{-1})_{a \in \mathbb{P}^1(k) - \{0\}}, 1)$, which is in $J_{\mathbb{P}^1}((\mathfrak{p}_K)_p)$. Hence this diagram induces, on fundamental groups, a

commutative diagram

$$
\begin{array}{ccc}
\mathrm{Gal}(K^{\mathrm{ab}}/K) & \xrightarrow{\ \eta\ } & \pi_1^k(\mathbf{K}^\times) \\
\downarrow & & \downarrow \\
\mathrm{Gal}(k(T)^{\mathrm{ab}}/k(T)) & \longrightarrow & \pi_1^k(J(\mathbb{P}^1))
\end{array}
$$

by the same argument as in the proof of Proposition 2.3. The top horizontal map η is an isomorphism by Assertion 1 of Theorem A. The bottom horizontal map is an isomorphism as well by [14]. Regarding this diagram, Serre proved the global version of the equality $\eta^{-1} = \theta$ and the local-global compatibility ([13, §5.2, Prop. 2 and §5.3, Lem. 2]; actually Serre's Proposition 2, which says $\psi^{-1} = \theta$ in his notation, should be corrected as $-\psi^{-1} = \theta$). This implies the local equality $\eta^{-1} = \theta$.

6. A two-dimensional analog

In this section, we discuss an analog of the above theory for a field of the form $K = k((S))((T))$. We denote by K_2 the functor of the second algebraic K-group ([1]). For each perfect k-algebra R, we have the abelian group $K_2(R[[\mathbf{S}, \mathbf{T}]])$. This gives a group functor that we denote by $K_2[[\mathbf{S}, \mathbf{T}]]$. The K_p-rational point

$$
\{-S + \mathbf{S}, -T + \mathbf{T}\} \in K_2[[\mathbf{S}, \mathbf{T}]](K_p) = K_2\big(k((S))((T))_p[[\mathbf{S}, \mathbf{T}]]\big)
$$

gives a morphism $\varphi^0 \colon \mathrm{Spec}\, K_p \to K_2[[\mathbf{S}, \mathbf{T}]]$, where $\{\cdot, \cdot\}$ denotes the symbol map. This is an analog of the map $\mathrm{Spec}\, k((T))_p \to \mathbf{U}_{k((T))}$ we have defined and studied before this section.

Now assume k is the finite field $\mathbb{F}_q$ with q elements. Define a functor X by the following cartesian diagram:

$$
\begin{array}{ccc}
X & \longrightarrow & K_2[[\mathbf{S}, \mathbf{T}]] \\
\downarrow & & \downarrow{\scriptstyle F-1} \\
\mathrm{Spec}\, K_p & \xrightarrow{\ \varphi^0\ } & K_2[[\mathbf{S}, \mathbf{T}]],
\end{array}
$$

where F is the q-th power Frobenius morphism over k. Then we expect that $K_2[[\mathbf{S}, \mathbf{T}]]$ can be viewed as a sort of an "algebraic group over k" and the equation $(F - 1)x = \{-S + \mathbf{S}, -T + \mathbf{T}\}$ gives a two-dimensional analog of Lubin-Tate theory so that X is the Spec of the perfect closure of a large totally ramified abelian extension of K (compare with Proposition 4.1).

Although this hope is hard to be made precise, we can formulate and prove a rigorous partial result (Proposition 6.1 below). For each perfect k-algebra R,

we have the space of 1-forms $\Omega^1_{R[[\mathbf{S},\mathbf{T}]]/R}$ and the space of 2-forms $\Omega^2_{R[[\mathbf{S},\mathbf{T}]]/R}$. We denote these functors by $\Omega^1_{[[\mathbf{S},\mathbf{T}]]}$, $\Omega^2_{[[\mathbf{S},\mathbf{T}]]}$, respectively. We have

$$\Omega^1_{[[\mathbf{S},\mathbf{T}]]} = \prod_{i,j\geq 0} \mathbf{G}_a\mathbf{S}^i\mathbf{T}^j d\mathbf{S} \times \mathbf{G}_a\mathbf{S}^i\mathbf{T}^j d\mathbf{T},$$

$$\Omega^2_{[[\mathbf{S},\mathbf{T}]]} = \prod_{i,j\geq 1} \mathbf{G}_a\mathbf{S}^i\mathbf{T}^j \, \mathrm{dlog}\,\mathbf{S} \wedge \mathrm{dlog}\,\mathbf{T},$$

where $\mathrm{dlog}\,\mathbf{S} = d\mathbf{S}/\mathbf{S}$, $\mathrm{dlog}\,\mathbf{T} = d\mathbf{T}/\mathbf{T}$. Consider the following sequence:

$$0 \longrightarrow K_2[[\mathbf{S},\mathbf{T}]]/pK_2[[\mathbf{S},\mathbf{T}]] \xrightarrow{\mathrm{dlog}} \Omega^2_{[[\mathbf{S},\mathbf{T}]]} \xrightarrow{C^{-1}-1} \Omega^2_{[[\mathbf{S},\mathbf{T}]]}/d\Omega^1_{[[\mathbf{S},\mathbf{T}]]}.$$

Here dlog is the dlog map of algebraic K-theory and C^{-1} is the inverse Cartier operator. We do not claim the exactness of this sequence; however, it should be compared with Bloch-Kato-Gabber's theorem ([8, Chapter 2, §2.4, Theorem 5]). We study $\mathrm{Ker}(C^{-1}-1)$ instead of $K_2[[\mathbf{S},\mathbf{T}]]/pK_2[[\mathbf{S},\mathbf{T}]]$. We have a commutative diagram

$$
\begin{array}{ccccc}
X & \longrightarrow & K_2[[\mathbf{S},\mathbf{T}]] & \xrightarrow{\mathrm{dlog}} & \mathrm{Ker}(C^{-1}-1) \\
\downarrow & & \downarrow{\scriptstyle F-1} & & \downarrow{\scriptstyle F-1} \\
\mathrm{Spec}\,K_p & \xrightarrow{\varphi^0} & K_2[[\mathbf{S},\mathbf{T}]] & \xrightarrow{\mathrm{dlog}} & \mathrm{Ker}(C^{-1}-1).
\end{array}
$$

We put $\varphi^1 = \mathrm{dlog} \circ \varphi^0$.

PROPOSITION 6.1. —   1. *There is a cartesian diagram*

$$
\begin{array}{ccc}
\mathrm{Spec}\,L_p & \longrightarrow & \mathrm{Ker}(C^{-1}-1) \\
\downarrow & & \downarrow{\scriptstyle F-1} \\
\mathrm{Spec}\,K_p & \xrightarrow{\varphi^1} & \mathrm{Ker}(C^{-1}-1).
\end{array}
$$

Here we put $L := K[x_{ij} \,|\, i,j \geq 1,\, p \nmid \gcd(i,j)]/(x_{ij}^q - x_{ij} - S^{-i}T^{-j})$.
2. *The ring L is actually a field. The extension L/K is an abelian (infinite) extension with Galois group isomorphic to the group of k-rational points of $\mathrm{Ker}(C^{-1}-1)$ (see (4) below for the structure of this group).*

Proof. — 1. First we show that the projection gives an isomorphism

$$(4) \qquad \mathrm{Ker}(C^{-1}-1) \xrightarrow{\sim} \prod_{i,j\geq 1,\, p\nmid\gcd(i,j)} \mathbf{G}_a\mathbf{S}^i\mathbf{T}^j \, \mathrm{dlog}\,\mathbf{S} \wedge \mathrm{dlog}\,\mathbf{T}.$$

We have $\Omega^2_{[[\mathbf{S},\mathbf{T}]]}/d\Omega^1_{[[\mathbf{S},\mathbf{T}]]} = \prod_{i,j\geq 1}\mathbf{G}_a\mathbf{S}^{pi}\mathbf{T}^{pj}\,\mathrm{dlog}\,\mathbf{S}\wedge\mathrm{dlog}\,\mathbf{T}$. The map $C^{-1}-1$ sends an element $\sum_{i,j\geq 1}a_{ij}\mathbf{S}^i\mathbf{T}^j\,\mathrm{dlog}\,\mathbf{S}\wedge\mathrm{dlog}\,\mathbf{T}$ of $\Omega^2_{[[\mathbf{S},\mathbf{T}]]}$ to

$$\sum_{i,j\geq 1}(a_{ij}^p\mathbf{S}^{pi}\mathbf{T}^{pj}-a_{ij}\mathbf{S}^i\mathbf{T}^j)\,\mathrm{dlog}\,\mathbf{S}\wedge\mathrm{dlog}\,\mathbf{T}$$

$$=\sum_{i,j\geq 1}(a_{ij}^p-a_{pi\,pj})\mathbf{S}^{pi}\mathbf{T}^{pj}\,\mathrm{dlog}\,\mathbf{S}\wedge\mathrm{dlog}\,\mathbf{T}$$

$$\in\Omega^2_{[[\mathbf{S},\mathbf{T}]]}/d\Omega^1_{[[\mathbf{S},\mathbf{T}]]}=\prod_{i,j\geq 1}\mathbf{G}_a\mathbf{S}^{pi}\mathbf{T}^{pj}\,\mathrm{dlog}\,\mathbf{S}\wedge\mathrm{dlog}\,\mathbf{T}.$$

Hence $\sum_{i,j\geq 1}a_{ij}\mathbf{S}^i\mathbf{T}^j\,\mathrm{dlog}\,\mathbf{S}\wedge\mathrm{dlog}\,\mathbf{T}$ is in $\mathrm{Ker}(C^{-1}-1)$ if and only if $a_{ij}^p=a_{pi\,pj}$ for any $i,j\geq 1$. This gives the required description of $\mathrm{Ker}(C^{-1}-1)$.

The map $\varphi^1\colon \mathrm{Spec}\,K_p\to\mathrm{Ker}(C^{-1}-1)$ corresponds to the rational point

$$\mathrm{dlog}\{-S+\mathbf{S},-T+\mathbf{T}\}=\frac{d(-S+\mathbf{S})}{-S+\mathbf{S}}\wedge\frac{d(-T+\mathbf{T})}{-T+\mathbf{T}}$$

$$=\sum_{i,j\geq 1}S^{-i}T^{-j}\mathbf{S}^i\mathbf{T}^j\,\mathrm{dlog}\,\mathbf{S}\wedge\mathrm{dlog}\,\mathbf{T},$$

which corresponds to $\sum_{i,j\geq 1,\,p\nmid\gcd(i,j)}S^{-i}T^{-j}\mathbf{S}^i\mathbf{T}^j\,\mathrm{dlog}\,\mathbf{S}\wedge\mathrm{dlog}\,\mathbf{T}$ via the above isomorphism. If $\sum_{i,j\geq 1,\,p\nmid\gcd(i,j)}x_{ij}\mathbf{S}^i\mathbf{T}^j\,\mathrm{dlog}\,\mathbf{S}\wedge\mathrm{dlog}\,\mathbf{T}$ lies in the geometric fiber of $F-1$ over this rational point, it must satisfy

$$(F-1)\sum_{i,j\geq 1,\,p\nmid\gcd(i,j)}x_{ij}\mathbf{S}^i\mathbf{T}^j\,\mathrm{dlog}\,\mathbf{S}\wedge\mathrm{dlog}\,\mathbf{T}$$

$$=\sum_{i,j\geq 1,\,p\nmid\gcd(i,j)}(x_{ij}^q-x_{ij})\mathbf{S}^i\mathbf{T}^j\,\mathrm{dlog}\,\mathbf{S}\wedge\mathrm{dlog}\,\mathbf{T}$$

$$=\sum_{i,j\geq 1,\,p\nmid\gcd(i,j)}S^{-i}T^{-j}\mathbf{S}^i\mathbf{T}^j\,\mathrm{dlog}\,\mathbf{S}\wedge\mathrm{dlog}\,\mathbf{T}$$

This proves Assertion 1.

2. By Artin-Schreier theory, we have an isomorphism $K/\wp_q(K)\xrightarrow{\sim}$ $\mathrm{Hom}(\mathrm{Gal}(K^{\mathrm{sep}}/K),\mathbb{F}_q)$ which sends an element $f\in K/\wp_q K$ to the character $\sigma\to(\sigma-1)\wp_q^{-1}(f)$, where $\wp_q(x)=x^q-x$. Consider the (non-commutative) $\mathbb{F}_q$-algebra $\mathbb{F}_q\langle\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)\rangle$ with $\mathbb{F}_q$-basis $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and relations $\sigma a=\sigma(a)\sigma$ for each $a\in\mathbb{F}_q$ and $\sigma\in\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$. It naturally acts on $\mathbb{F}_q$ and hence on $\mathrm{Hom}(\mathrm{Gal}(K^{\mathrm{sep}}/K),\mathbb{F}_q)$. The action of the p-th power Frobenius $\in\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ corresponds to the p-th power map on $K/\wp_q(K)$ via the isomorphism $K/\wp_q(K)\xrightarrow{\sim}\mathrm{Hom}(\mathrm{Gal}(K^{\mathrm{sep}}/K),\mathbb{F}_q)$. By the following lemma, the proof for L being a field is reduced to showing that the elements $S^{-i}T^{-j}$ for $i,j\geq 1$, $p\nmid\gcd(i,j)$ generate a free $\mathbb{F}_q\langle\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)\rangle$-submodule of $K/\wp_q(K)$.

LEMMA 6.2. — *Let $\chi_1, \ldots, \chi_n$ be n elements of $\mathrm{Hom}(\mathrm{Gal}(K^{\mathrm{sep}}/K), \mathbb{F}_q)$ and let K_i correspond to $\mathrm{Ker}(\chi_i)$ via Galois theory, $1 \leq i \leq n$. Assume $\chi_1, \ldots, \chi_n$ generate a free $\mathbb{F}_q \langle \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) \rangle$-submodule of $\mathrm{Hom}(\mathrm{Gal}(K^{\mathrm{sep}}/K), \mathbb{F}_q)$. Then the fields $K_1, \ldots, K_n$ have degree q and are linearly disjoint.*

Proof of the lemma. — The natural map $\mathbb{F}_q \langle \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) \rangle \to \mathrm{End}_{\mathbb{F}_p\text{-linear}}(\mathbb{F}_q)$ is an $\mathbb{F}_q$-algebra isomorphism since it is injective by the linear independence of field automorphisms and they have the same dimension over $\mathbb{F}_q$. Let $\psi = (\psi_1, \ldots, \psi_n)$ be an $\mathbb{F}_p$-linear map $\mathbb{F}_q^{\oplus n} \to \mathbb{F}_q$ which vanishes on the image of $\bigoplus_i \chi_i \colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathbb{F}_q^{\oplus n}$. Then $\sum_i \psi_i \circ \chi_i = 0$. By this and the assumption on χ_i, we have $\psi_i = 0$ for any i. Hence $\psi = 0$. This implies the surjectivity of $\bigoplus_i \chi_i \colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathbb{F}_q^{\oplus n}$. This proves the lemma. $\qquad \square$

The additive group K can be written as the direct sum of $V := \bigoplus_{i,j \geq 1} \mathbb{F}_q S^{-i} T^{-j}$ and $\{ \sum_{\max\{i,j\} \geq 0} a_{ij} S^i T^j \in K \}$, both being stable under the action of $\wp_q$. The space $V/\wp_q(V)$ injects into $K/\wp_q(K)$ and has $S^{-i} T^{-j}$ for $i, j \geq 1$, $q \nmid \gcd(i,j)$ as an $\mathbb{F}_q$-basis. Hence $V/\wp_q(V)$ is a free $\mathbb{F}_q \langle \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) \rangle$-module with basis $S^{-i} T^{-j}$ for $i, j \geq 1$, $p \nmid \gcd(i,j)$. This implies that L is a field.

The latter half of Assertion 2 follows from the cartesian diagram of Assertion 1. $\qquad \square$

REMARK 6.3. — By a similar argument to the proof of Assertion 2, we have

$$\mathrm{Hom}(\mathrm{Gal}(K^{\mathrm{sep}}/K), \mathbb{F}_q) \cong K/\wp_q K$$

$$\cong \mathbb{F}_q \oplus \bigoplus_{q \nmid i \geq 1} \mathbb{F}_q S^{-i} \oplus \bigoplus_{q \nmid j \geq 1} \mathbb{F}_q T^{-j} \oplus$$

$$\bigoplus_{i \geq 1} \prod_{q \nmid j \geq 1} \mathbb{F}_q S^i T^{-j} \oplus \bigoplus_{q \nmid i \geq 1} \prod_{j \geq 1} \mathbb{F}_q S^i T^{-j} \oplus \bigoplus_{i,j \geq 1, \, q \nmid \gcd(i,j)} \mathbb{F}_q S^{-i} T^{-j}.$$

The last summand $\bigoplus \mathbb{F}_q S^{-i} T^{-j}$ corresponds to the abelian extension L in the proposition.

7. $\mathcal{D}$-modules on two-dimensional K

In this section, we give an analog of the result of [2, §2.6] on $\mathcal{D}$-modules, for a field of the form $K = k((S))((T))$ with k characteristic zero. This is also regarded as an analog of what we have done in the previous section.

We formulate our result (Proposition 7.1 below). For a k-algebra R, we define $\Omega^2_{[[\mathbf{S}, \mathbf{T}]]}(R)$ to be the projective limit of the space of 2-forms of $R[\mathbf{S}, \mathbf{T}]/(\mathbf{S}, \mathbf{T})^n$ over R, the limit being taken for $n \geq 0$. Let $\varphi^1 \colon \mathrm{Spec}\, K \to \Omega^2_{[[\mathbf{S}, \mathbf{T}]]}$ be the k-morphism corresponding to the rational point $\mathrm{dlog}\{-S + \mathbf{S}, -T + \mathbf{T}\} =$

$\mathrm{dlog}(-S + \mathbf{S}) \wedge \mathrm{dlog}(-T + \mathbf{T})$. We say that a $\mathcal{D}$-module M of $\mathcal{O}$-rank one on $\Omega^2_{[[\mathbf{S},\mathbf{T}]]}$ is compatible with the group structure if $\mu^* M \cong \mathrm{pr}_1^* M \otimes \mathrm{pr}_2^* M$, where $\mu\colon \Omega^2_{[[\mathbf{S},\mathbf{T}]]} \times \Omega^2_{[[\mathbf{S},\mathbf{T}]]} \to \Omega^2_{[[\mathbf{S},\mathbf{T}]]}$ is the addition map and pr_i is the i-th projection for each $i = 1, 2$. Let $\mathcal{C}$ be the category of $\mathcal{D}$-modules of $\mathcal{O}$-rank one on $\Omega^2_{[[\mathbf{S},\mathbf{T}]]}$ that are compatible with the group structure and let $\mathcal{C}'$ be the category of $\mathcal{D}$-modules of $\mathcal{O}$-rank one on $\mathrm{Spec}\, K$. The categories $\mathcal{C}$ and $\mathcal{C}'$ are k-linear Picard categories under the tensor operation. The endomorphism ring of any object of $\mathcal{C}$ or $\mathcal{C}'$ is equal to k. Let $\mathrm{Isom}(\mathcal{C})$ (resp. $\mathrm{Isom}(\mathcal{C}')$) be the abelian group of isomorphism classes of objects of $\mathcal{C}$ (resp. $\mathcal{C}'$). Since $\Omega^2_{[[\mathbf{S},\mathbf{T}]]}$ and $\mathrm{Spec}\, K$ are the Spec's of unique factorization domains, line bundles on them can be trivialized. We can associate connection forms to objects of C and C' whose underlying line bundles are trivial. Thus $\mathrm{Isom}(\mathcal{C})$ (resp. $\mathrm{Isom}(\mathcal{C}')$) can identified with a subquotient (as an abelian group) of the space of 1-forms on the k-scheme $\Omega^2_{[[\mathbf{S},\mathbf{T}]]}$ (resp. $\mathrm{Spec}\, K$).

PROPOSITION 7.1. —   1. *The pullback by $\varphi^1\colon \mathrm{Spec}\, K \to \Omega^2_{[[\mathbf{S},\mathbf{T}]]}$ gives a fully faithful embedding $\mathcal{C} \hookrightarrow \mathcal{C}'$ of k-linear Picard categories.*
   2. *Under the above identification of $\mathcal{C}'$, we have*

$$\mathrm{Isom}(\mathcal{C}') \cong (k((T))/\mathbb{Z})\,\mathrm{dlog}\, S \oplus (k((S))/\mathbb{Z})\,\mathrm{dlog}\, T$$

$$\oplus\, d\Big(S^{-1} k[S^{-1}] \oplus T^{-1} k[T^{-1}]$$

$$\oplus\, T^{-1} S k[[S]][T^{-1}] \oplus T^{-1} S^{-1} k[S^{-1}, T^{-1}] \Big).$$

   3. *The image of $\mathrm{Isom}(\mathcal{C})$ in $\mathrm{Isom}(\mathcal{C}')$ by the pullback functor by φ^1 corresponds to the last summand $d(T^{-1} S^{-1} k[S^{-1}, T^{-1}])$ via the isomorphism of Assertion 2.*

Proof. — 2. The group $\mathrm{Isom}(\mathcal{C}')$ is identified with $\Omega^{1,d=0}_{K/k}/\,\mathrm{dlog}\, K^\times$. A straightforward calculation shows that $\Omega^{1,d=0}_{K/k}/\,\mathrm{dlog}\, K^\times$ is equal to the right-hand side of Assertion 2.

   3. The group $\Omega^2_{[[\mathbf{S},\mathbf{T}]]}$ is the product of copies of $\mathbf{G}_a$ labeled by $n, m \geq 1$ with coordinates $z_{nm} := \mathbf{S}^n \mathbf{T}^m\, \mathrm{dlog}\, \mathbf{S} \wedge \mathrm{dlog}\, \mathbf{T}$. Hence a connection form of any object M of $\mathcal{C}$ is of the form $\sum_{n,m\geq 1} a_{nm} dz_{nm}$ for some $a_{nm} \in k$ that are uniquely determined by the isomorphism class of M. Therefore $\mathrm{Isom}(\mathcal{C}) \cong \bigoplus_{n,m\geq 1} k\, dz_{nm}$. Since $\mathrm{dlog}\{-S + \mathbf{S}, -T + \mathbf{T}\} = \sum_{n,m\geq 1} S^{-n} T^{-m} z_{nm}$, the pullback of a connection form $\sum_{n,m\geq 1} a_{nm} dz_{nm}$ with $a_{nm} \in k$ is $\sum_{n,m\geq 1} a_{nm} d(S^{-n} T^{-m}) = d(\sum_{n,m\geq 1} a_{nm} S^{-n} T^{-m})$. Thus we get the result.

1. The above proof of Assertion 3 also shows that the homomorphism $\text{Isom}(\mathscr{C}) \to \text{Isom}(\mathscr{C}')$ induced by the functor $\mathscr{C} \to \mathscr{C}'$ is injective. The result follows from this. $\square$

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