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NATURAL ENDOMORPHISMS OF QUASI-SHUFFLE HOPF ALGEBRAS

by Jean-Christophe Novelli, Frédéric Patras & Jean-Yves Thibon

Abstract. — The Hopf algebra of word-quasi-symmetric functions (**WQSym**), a noncommutative generalization of the Hopf algebra of quasi-symmetric functions, can be endowed with an internal product that has several compatibility properties with the other operations on **WQSym**. This extends constructions familiar and central in the theory of free Lie algebras, noncommutative symmetric functions and their various applications fields, and allows to interpret **WQSym** as a convolution algebra of linear endomorphisms of quasi-shuffle algebras. We then use this interpretation to study the fine structure of quasi-shuffle algebras (MZVs, free Rota-Baxter algebras...). In particular, we compute their Adams operations and prove the existence of generalized Eulerian idempotents, that is, of a canonical left-inverse to the natural surjection map to their indecomposables, allowing for the combinatorial construction of free polynomial generators for these algebras.

Résumé (*Sur les endomorphismes naturels des algèbres de quasi-shuffle*)
L'algèbre de Hopf des fonctions quasi-symétriques sur les mots (**WQSym**), une généralisation non commutative de l'algèbre de Hopf des fonctions quasi-symétriques,

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peut être munie d'un produit interne qui a des propriétés remarquables de compatibilité aux autres opérations sur **WQSym**. Cette construction étend des constructions familières et centrales de la théorie des algèbres de Lie libres, des fonctions non commutatives symétriques et de leurs nombreux domaines d'application. Elle permet aussi d'interpréter **WQSym** comme algèbre de convolution des endomorphismes linéaires des algèbres quasi-shuffle. Nous utilisons cette interprétation pour étudier la structure fine des algèbres quasi-shuffle (MZVs, algèbres de Rota-Baxter libres...). En particulier, nous étudions leurs opérations d'Adams et prouvons l'existence d'un inverse à gauche canonique à la surjection naturelle vers les indécomposables ; elle donne lieu à une construction combinatoire de leurs générateurs polynomiaux.

1. Introduction

Quasi-shuffles appeared as early as 1972 in the seminal approach by P. Cartier to Baxter algebras (now most often called Rota-Baxter algebras) [3]. Their study was revived and intensified during the last 10 years, for a variety of reasons. The first one was the study of MZVs (multiple zeta values), for example in the works of Hoffman, Minh, Racinet or Zagier, since quasi-shuffles encode one representation of their products. Another line of study, largely motivated by the recent works of Connes and Kreimer on the structures of quantum field theories, was the revival of the theory of Rota-Baxter algebras initiated by M. Aguiar, K. Ebrahimi-Fard, L. Guo, and others. We refer to [17, 29, 5, 18, 1, 13, 9, 10], also for further bibliographical and historical references on these subjects. Quasi-shuffle algebras are also the free commutative tridendriform algebras [20]. A systematic presentation of Rota-Baxter algebras and of the above relations can be found in the forthcoming book [12].

The present work arose from the project to understand the combinatorial structure of "natural" operations acting on the algebra of MZVs and, more generally on quasi-shuffle algebras. It soon became clear to us that the Hopf algebra of word quasi-symmetric functions (**WQSym**), was the right setting to perform this analysis and that many properties of the classical Lie calculus (incorporated in the theory of free Lie algebras and connected topics) could be translated into this framework.

This article is a first step in that overall direction. It shows that word quasi-symmetric functions act naturally on quasi-shuffle algebras and that some key ingredients of the classical Lie calculus such as Solomon's Eulerian idempotents can be lifted to remarquable elements in **WQSym**. In the process, we show that **WQSym** is the proper analogue of the Hopf algebra **FQSym** of free quasi-symmetric functions (also known as the Malvenuto-Reutenauer Hopf algebra) in the setting of quasi-shuffle algebras. Namely, we prove a Schur-Weyl duality theorem for quasi-shuffle algebras extending naturally the classical one (which states that the linear span of permutations is the commutant of endomorphims of the tensor algebra over a vector space V induced by linear endomorphims of V).

The main ingredient of this theory is that the natural extension of the internal product on the symmetric group algebras to a product on the linear span of surjections between finite sets, which induces a new product on **WQSym**, is a lift in **WQSym** of the composition of linear endomorphisms of quasi-shuffle algebras. This simple observation yields ultimately the correct answer to the problem of studying the formal algebraic structure of quasi-shuffle algebras from the Lie calculus point of view.

2. Word quasi-symmetric functions

In this section, we briefly survey the recent theory of noncommutative quasi-symmetric functions and introduce its fundamental properties and structures. The reader is referred to [15, 6, 16] for details and further information. Let us mention that the theory of word quasi-symmetric functions is very closely related to the ones of Solomon-Tits algebras and twisted descents, the development of which was motivated by the geometry of Coxeter groups, the study of Markov chains on hyperplane arrangements and Joyal's theory of tensor species. We will not consider these application fields here and refer to [35, 2, 31, 28].

Let us first recall that the Hopf algebra of noncommutative symmetric functions [11] over an arbitrary field $\mathbb{K}$ of characteristic zero, denoted here by **Sym**, is defined as the free associative algebra over an infinite sequence $(S_n)_{n\geq 1}$, graded by $\deg S_n = n$, and endowed with the coproduct

$$\Delta S_n = \sum_{k=0}^{n} S_k \otimes S_{n-k} \quad (\text{where } S_0{=}1)\,. \tag{1}$$

It is naturally endowed with an internal product $*$ such that each homogeneous component $\mathbf{Sym}_n$ gets identified with the (opposite) Solomon descent algebra of $\mathfrak{S}_n$, the symmetric group of order n. Some bigger Hopf algebras containing **Sym** in a natural way are also endowed with internal products, whose restriction to **Sym** coincides with $*$. An almost tautological example is **FQSym**, which, being based on permutations, with the group law of $\mathfrak{S}_n$ as internal product, induces naturally the product of the descent algebra [7, 21].

A less trivial example [22] is $\mathbf{WQSym}^*$, the graded dual of **WQSym** (Word Quasi-symmetric functions, the invariants of the quasi-symmetrizing action on words [6]). It can be shown that each homogenous component $\mathbf{WQSym}^*_n$ can be endowed with an internal product (with a very simple combinatorial definition), for which it is anti-isomorphic with the Solomon-Tits algebra, so that it contains $\mathbf{Sym}_n$ as a $*$-subalgebra in a non-trivial way. The internal product of $\mathbf{WQSym}^*$

is itself induced by the one of **PQSym** (parking functions), whose restriction to the Catalan subalgebra **CQSym** again contains **Sym** in a nontrivial way [23].

Let us recall the relevant definitions. We denote by $A = \{a_1 < a_2 < \dots\}$ an infinite linearly ordered alphabet and A^* the corresponding set of words. The *packed word* $u = \mathrm{pack}(w)$ associated with a word $w \in A^*$ is obtained by the following process. If $b_1 < b_2 < \cdots < b_r$ are the letters occuring in w, u is the image of w by the homomorphism $b_i \mapsto a_i$. A word u is said to be *packed* if $\mathrm{pack}(u) = u$. We denote by PW the set of packed words. With such a word, we associate the noncommutative polynomial

$$\mathbf{M}_u(A) := \sum_{\mathrm{pack}(w)=u} w\,. \tag{2}$$

For example, restricting A to the first five integers,

$$\begin{aligned}\mathbf{M}_{13132}(A) = {} & 13132 + 14142 + 14143 + 24243 \\ & + 15152 + 15153 + 25253 + 15154 + 25254 + 35354.\end{aligned} \tag{3}$$

As for classical symmetric functions, the nature of the ordered alphabet A chosen to define word quasi-symmetric functions $\mathbf{M}_u(A)$ is largely irrelevant provided it has enough elements. We will therefore often omit the A-dependency and write simply $\mathbf{M}_u$ for $\mathbf{M}_u(A)$, except when we want to emphasize this dependency (and similarly for the other types of generalized symmetric functions we will have to deal with).

Under the abelianization $\chi:\ \mathbb{K}\langle A\rangle \to \mathbb{K}[A]$, the $\mathbf{M}_u$ are mapped to the monomial quasi-symmetric functions M_I, where $I = (|u|_a)_{a\in A}$ is the composition (that is, the sequence of integers) associated with the so-called evaluation vector $\mathrm{ev}(u)$ of u ($\mathrm{ev}(u)_i := |u|_{a_i} := |\{j, u_j = a_i\}|$). Recall, for the sake of completeness, that the M_I are defined, for $I = (i_1, \dots, i_k)$, by:

$$M_I := \sum_{j_1 < \dots < j_k} a_{j_1}^{i_1} \dots a_{j_k}^{i_k}. \tag{4}$$

The polynomials $\mathbf{M}_u$ span a subalgebra **WQSym** of $\mathbb{K}\langle A\rangle$ [14]. This algebra can be understood alternatively as the algebra of invariants for the noncommutative version [6] of Hivert's quasi-symmetrizing action, which is defined in such a way that two words are in the same $\mathfrak{S}(A)$-orbit (where $\mathfrak{S}(A)$ is the group of set automorphisms of A) iff they have the same packed word. We refer to [15] for details on the quasi-symmetrizing action.

As for **Sym**, **WQSym** carries naturally a Hopf algebra structure. Its simplest definition is through the use of two ordered countable alphabets, say $A = \{a_1 < \dots < a_n < \dots\}$ and $B := \{b_1 < \dots < b_n < \dots\}$. Let us write $A + B$ for the ordinal sum of A and B (so that for arbitrary i, j, we have $a_i < b_j$). The unique associative algebra map μ from $\mathbb{K}\langle A + B\rangle$ to $\mathbb{K}\langle A\rangle \otimes \mathbb{K}\langle B\rangle$ acting as the identity map on A and B induces a map from $\mathbf{WQSym}(A+B)$ to $\mathbf{WQSym}(A)\otimes\mathbf{WQSym}(B)\cong\mathbf{WQSym}(A)\otimes\mathbf{WQSym}(A)$,

$$\Delta(\mathbf{M}_u(A+B)) := \sum_{\mathrm{pack}(w)=u} w_{|A}\otimes w_{|B}, \tag{5}$$

which can be shown to define a Hopf algebra structure on **WQSym**. Here, for an arbitrary subset S of $A+B$, $u_{|S}$ stands for the word obtained from u by erasing all the letters that do not belong to S.

The explicit formula for the coproduct Δ generalizes the usual one for the algebra of Free Quasi-Symmetric functions **FQSym**, (a polynomial realization of the Malvenuto-Reutenauer algebra contained in **WQSym**), and reads, for a packed word u on the interval $[1,n]$:

$$\Delta(\mathbf{M}_u) := \sum_{i=0}^{n} \mathbf{M}_{u_{|[1,i]}}\otimes\mathbf{M}_{\mathrm{pack}(u_{|[i+1,n]})}. \tag{6}$$

Noncommutative symmetric functions (the elements of **Sym**), although they can be defined abstractly in terms of a family of algebraically free generators S_n (see the begining of this section), also do admit a standard realization in terms of an ordered alphabet A by

$$S_n(A) = \sum_{w\in A^n,\ \mathrm{Des}\,(w)=\varnothing} w, \tag{7}$$

where $\mathrm{Des}\,(w)=\{i|w_i>w_{i+1}\}$ denotes the descent set of w. Thus, there is a natural embedding of **Sym** into **WQSym**:

$$S_n = \sum_{\mathrm{Des}\,(u)=\varnothing} \mathbf{M}_u, \tag{8}$$

where the summation is implicitely restricted to packed words u of the suitable length, that is, here, of length n (and similarly in the forthcoming formulas). This embedding extends multiplicatively:

$$S_{n_1}...S_{n_k} = \sum_{\mathrm{Des}\,(u)\subset\{n_1,...,n_1+...+n_{k-1}\}} \mathbf{M}_u. \tag{9}$$

In terms of the realization of noncommutative symmetric functions and word quasi-symmetric functions over ordered alphabets A, these equalities are indeed equalities: both sides are formal sum of words with certain shapes.

For later use, let us also mention that the last formula implies (by a standard Möbius inversion argument that we omit, see [11] for details on the ribbon basis) that the elements

$$\sum_{\mathrm{Des}\,(u)=\{n_1,...,n_1+...+n_{k-1}\}} \mathbf{M}_u \tag{10}$$

belong to **Sym**. They form a basis, the ribbon basis R_I of **Sym**: for $I = (i_1, ..., i_k)$,

$$R_I := \sum_{\mathrm{Des}\,(u)=\{i_1,...,i_1+...+i_{k-1}\}} \mathbf{M}_u. \tag{11}$$

At last, the embedding of **Sym** in **WQSym** we just considered is a Hopf algebra embedding: one can either check directly that the image $\mathbf{S}$ of the series $\sum_{n=0}^{\infty} S_n$ in **WQSym** is grouplike ($\Delta(\mathbf{S}) = \mathbf{S} \otimes \mathbf{S}$), or think in terms of ordered alphabets and notice that the the coproduct of **WQSym** given by the ordinal sum $A+B$ restricts to a coproduct on **Sym** that agrees with the one introduced at the begining of the section.

3. Extra structures on WQSym

A natural question arises from our previous account of the theory: does there exist an internal product on **WQSym** extending the one of **Sym**? The question will appear later to be closely connected to the problem of using **WQSym** in order to investigate Hopf algebraic properties, very much as **Sym** (and the dual notion of descent algebras) is classically used to investigate the properties of tensor spaces and connected graded commutative or cocommutative Hopf algebras.

It turns out that if we want, for example, an interpretation of **WQSym** analogous to that of **FQSym** as a convolution algebra of endomorphisms of tensor spaces [30, 21, 7], we have to relax the requirement that the internal product extend the one of **Sym**. The two internal products will coincide only on a certain remarquable subalgebra of infinite series (see Section 7). Moreover, the construction does not work with the standard embedding (8), and therefore we will also have to relax the condition that the embedding of **Sym** in **WQSym** is compatible with realizations in terms of ordered alphabets. This results into a new picture of the relations between **Sym** and **WQSym**, where one has to map the complete symmetric functions as follows

$$S_n \mapsto \mathbf{M}_{12...n}\,. \tag{12}$$

Requiring the map to be multiplicative defines a new morphism of algebras -for an element $T \in$ **Sym**, we will write $\hat{T}$ its image in **WQSym**.

Proposition 3.1. — *This map is still a Hopf algebra embedding. Its action on the monomial basis generated by the S_n and on the ribbon basis is given respectively by: For $I = (i_1, ..., i_k)$ with $i_1 + ... + i_k = n$,*

$$\hat{S}^I := \hat{S}_{i_1} ... \hat{S}_{i_k} = \sum_{\mathrm{Des}\,(u)\supseteq[n]-\{i_k, i_k+i_{k-1},...,i_k+...+i_1\}} \mathbf{M}_{\breve{u}}\,, \tag{13}$$

$$(14)\qquad \hat{R}_I = \sum_{\mathrm{Des}\,(u)=[n]-\{i_k,i_k+i_{k-1},\dots,i_k+\dots+i_1\}} \mathbf{M}_{\breve{u}}\,,$$

where $w = a_{j_1}\cdots a_{j_n} \mapsto \breve{w} = a_{j_n}\cdots a_{j_1}$ *is the anti-automorphism of the free associative algebra reversing the words.*

The first assertion follows from the observation that two series $\sum_n S_n$ and $\sum_n \mathbf{M}_{12\dots n}$ are grouplike and generate two free associative algebras (respectively **Sym** and a subalgebra of **WQSym** isomorphic to **Sym**).

Let us compute the action on the monomial basis (the action on the ribbon basis follows by Möbius inversion since $\hat{S}^I = \sum_{J\subset I} \hat{R}_J$). From $\hat{S}_n = \sum_{i_1<\dots<i_n} a_{i_1}\dots a_{i_n}$, we get:

$$(15)\qquad \hat{S}_n = \sum_{\mathrm{Des}\,(u)=[n-1]} \mathbf{M}_{\breve{u}}.$$

The same principle applies in general (if a word of length n is strictly increasing in position i, then the reverse word has a descent in position $n-i$) and we get:

$$(16)\qquad \hat{S}^I = \sum_{j_1^1<\dots<j_{i_1}^1,\dots,j_1^k<\dots<j_{i_k}^k} a_{j_1^1}\dots a_{j_{i_k}^k}$$

$$= \sum_{\mathrm{Des}\,(u)\supseteq[n]-\{i_k,i_k+i_{k-1},\dots,i_k+\dots+i_1\}} \mathbf{M}_{\breve{u}},$$

(17)

from which the Proposition follows.

Moving beyond the relation to **Sym**, recall that, in general,

$$(18)\qquad \mathbf{M}_u\mathbf{M}_v = \sum_{w=u'v',\mathrm{pack}(u')=u,\ \mathrm{pack}(v')=v} \mathbf{M}_w\,.$$

An interesting feature of **WQSym** is the quasi-shuffle nature of this product law. This feature explains many of its universal properties with respect to quasi-shuffle algebras.

To understand it, first notice that packed words u over the integers (recall that the alphabet can be chosen arbitrarily provided it is "big enough") can be interpreted as surjective maps

$$(19)\qquad u:\ [n]\longrightarrow[k]\,,\quad (k=\max(u))\quad u(i):=u_i$$

or, equivalently, as ordered partitions (set compositions) of $[n]$: let us write $\overline{u}$ for $(u^{-1}(1),\dots,u^{-1}(k))$. We will use freely these two interpretations of packed words from now on to handle computations in **WQSym** as computations involving surjective morphisms or ordered partitions.

Now, let us define recursively a combinatorial operation which is actually a special case of the general notion of quasi-shuffle product introduced in Section 5. For two finite sequences of sets $U = (U_1, ..., U_k)$ and $V = (V_1, ..., V_l)$ we define $U \uplus V$ (which is a formal sum of sequences of sets) by:

$$U \uplus V := (U_1, U' \uplus V) + (V_1, U \uplus V') + (U_1 \cup V_1, U' \uplus V'), \tag{20}$$

where $U' := (U_2, ..., U_k)$ and $V' := (V_2, ..., V_l)$. Then, if u (resp. v) encodes the set composition U (resp. V), in **WQSym**:

$$\mathbf{M}_u \mathbf{M}_v = \mathbf{M}_t, \tag{21}$$

where the set of packed words t encodes $T = U \uplus V[n]$, n is the length of u, and for an arbitrary sequence S of subsets of the integers, $S[p] := (S_1 + p, ..., S_k + p)$. We will write abusively $\uplus$ for the operation on the linear span of packed words induced by the "shifted shuffle product" $\overline{u} \uplus \overline{v}[n]$ so that, with our previous conventions, $t = u \uplus v$ and (with a self-explanatory notation for $\mathbf{M}_{u \uplus v}$) $\mathbf{M}_u \mathbf{M}_v = \mathbf{M}_{u \uplus v}$.

To conclude this section, let us point out that a candidate for the internal product we were looking for is easily described using the interpretation of packed words as surjections:

Definition 3.2. — *The internal product of* **WQSym** *is defined in the* **M**-*basis by*

$$\mathbf{M}_u * \mathbf{M}_v = \mathbf{M}_{v \circ u} \quad \text{whenever } l(v) = \max(u) \text{ and } 0 \text{ otherwise.} \tag{22}$$

The following sections show that this product has the expected properties with respect to the other structures of **WQSym** and with respect to arbitrary quasi-shuffle algebras.

4. A relation between internal and external products

In **Sym**, there is a fundamental compatibility relation between the internal product, the usual product and the coproduct. It is called the splitting formula [11], and is essentially a Hopf-algebraic interpretation of the noncommutative Mackey formula discovered by Solomon [33]. It can be extended to **FQSym**, with certain restrictions [6]. The key ingredient for doing this is an expression of the product of **FQSym** in terms of shifted concatenation and internal product with an element of **Sym**. This can again be done here.

The natural notion of shifted concatenation in **WQSym** is not the same as in **FQSym**: indeed, if u and v are packed words, one would like that $u \bullet v$ be a packed word. The correct way to do this is to shift the letters of v by the maximum of u:

$$u \bullet v = u \cdot v[k]\,, \quad \text{where } k = \max(u)\,. \tag{23}$$

For example, $11 \bullet 21 = 1132$. We consistently set

$$\mathbf{M}_u \bullet \mathbf{M}_v = \mathbf{M}_{u\bullet v}. \tag{24}$$

Lemma 4.1. — *We have the distributivity property:*

$$(\mathbf{M}_u \bullet \mathbf{M}_t) * (\mathbf{M}_v \bullet \mathbf{M}_w) = (\mathbf{M}_u * \mathbf{M}_v) \bullet (\mathbf{M}_t * \mathbf{M}_w), \tag{25}$$

whenever $l(v) = \max(u), l(w) = \max(t)$.

We also have the following crucial lemma (compare with [6, Eq. (2)]):

Lemma 4.2. — *Let $u_1, \dots, u_r$ be packed words, and define a composition $I = (i_1, \dots, i_r)$ by $i_k = \max(u_k)$. Then, if* **Sym** *is embedded in* **WQSym** *by means of (12), and the internal product is defined by (22),*

$$\mathbf{M}_{u_1}\mathbf{M}_{u_2}\cdots\mathbf{M}_{u_r} = (\mathbf{M}_{u_1} \bullet \mathbf{M}_{u_2} \bullet \cdots \bullet \mathbf{M}_{u_r}) * S^I\,. \tag{26}$$

For example,

$$\mathbf{M}_{11}\mathbf{M}_{21} = \mathbf{M}_{1132} + \mathbf{M}_{1121} + \mathbf{M}_{2231} + \mathbf{M}_{2221} + \mathbf{M}_{3321} \tag{27}$$

is obtained from

$$\mathbf{M}_{11} \bullet \mathbf{M}_{21} = \mathbf{M}_{1132} \tag{28}$$

by internal product on the right by

$$S^{12} = S_1S_2 = \mathbf{M}_1\mathbf{M}_{12} = \mathbf{M}_{123} + \mathbf{M}_{112} + \mathbf{M}_{213} + \mathbf{M}_{212} + \mathbf{M}_{312}\,. \tag{29}$$

Proof. — The Lemma is most easily proven by switching to the langage of surjections. Let us notice first that, by construction of the shifted quasi-shuffle product of set compositions, $u_1 \uplus u_2$ is the formal sum of all surjections from $[l(u_1) + l(u_2)]$ to $[i_1 + i_2 - p]$, where p runs from 0 to $\inf(i_1, i_2)$, that can be obtained by composition of $u \bullet v$ with a surjective map ϕ from $[i_1 + i_2]$ to $[i_1 + i_2 - p]$ such that ϕ is (strictly) increasing on $[i_1]$ and $\{i_1 + 1, ..., i_1 + i_2\}$.

Let us write γ_{i_1,i_2} for the formal sum of these surjections with domain $[i_1+i_2]$ and codomain $[i_1 + i_2 - p]$, $p = 0, \dots, \inf(i_1, i_2)$. We get, as a particular case: $1_{i_1} \uplus 1_{i_2} = \gamma_{i_1,i_2}$, where 1_n stands for the identity in $\mathfrak{S}_n$, the symmetric group of rank n. In general, we have therefore:

$$\mathbf{M}_{u_1}\mathbf{M}_{u_2} = (\mathbf{M}_{u_1} \bullet \mathbf{M}_{u_2}) * (\mathbf{M}_{1\dots i_1}\mathbf{M}_{1\dots i_2}) = (\mathbf{M}_{u_1} \bullet \mathbf{M}_{u_2}) * S^{i_1,i_2}, \tag{30}$$

with the notation

$$S^{i_1\dots i_k} := S_{i_1} \dots S_{i_k} = \mathbf{M}_{1\dots i_1} \dots \mathbf{M}_{1\dots i_k}. \tag{31}$$

The same reasoning applies to an arbitrary number of factors. □

5. Operations on quasi-shuffle algebras

Let us recall first the definition of the quasi-shuffle algebra $QS(A)$ on a commutative algebra A (without a unit and over $\mathbb{K}$). The underlying vector space is the tensor algebra over A: $QS(A) = \bigoplus_n A^{\otimes n}$, where $A^{\otimes 0} := \mathbb{K}$. The product is defined recursively by:

(32)
$$\begin{aligned}(a_1 \otimes ... \otimes a_n) \uplus (b_1 \otimes ... \otimes b_m) =& a_1 \otimes ((a_2 \otimes ... \otimes a_n) \uplus (b_1 \otimes ... \otimes b_m)) \\ &+ b_1 \otimes ((a_1 \otimes ... \otimes a_n) \uplus (b_2 \otimes ... \otimes b_m)) \\ &+ a_1 b_1 \otimes ((a_2 \otimes ... \otimes a_n) \uplus (b_2 \otimes ... \otimes b_m)).\end{aligned}$$

LEMMA 5.1. — *The quasi-shuffle algebra is a right module over* **WQSym** *equipped with the internal product. The action is defined by:*

$$(a_1 \otimes ... \otimes a_n)\mathbf{M}_u := \delta_m^n b_1 \otimes ... \otimes b_k, \quad b_i := \prod_{u(j)=i} a_j, \tag{33}$$

where we used the surjection interpretation of packed words, u is a surjective map from $[m]$ to $[k]$, and δ_m^n is the Kronecker symbol ($\delta_m^n = 1$ if $m = n$ and $= 0$ else).

This right-module structure allows to rewrite the definition of the quasi-shuffle product as (see, *e.g.*, [3], also for a proof that $QS(A)$ is actually a commutative algebra):

$$(a_1 \otimes ... \otimes a_n) \uplus (b_1 \otimes ... \otimes b_m) = (a_1 \otimes ... \otimes a_n \otimes b_1 \otimes ... \otimes b_m) S^{n,m}. \tag{34}$$

Interpreting $S^{n,m}$ as an element of **FQSym** instead of **WQSym** and using the standard right action of permutations on tensors, we would get the ordinary shuffle product.

LEMMA 5.2. — *The quasi-shuffle algebra is endowed with a Hopf algebra structure by the deconcatenation coproduct*

$$\Delta(a_1 \otimes ... \otimes a_n) := \sum_{i=0}^{n} (a_1 \otimes ... \otimes a_i) \otimes (a_{i+1} \otimes ... \otimes a_n) : \tag{35}$$

$$\Delta((a_1 \otimes ... \otimes a_n) \uplus (b_1 \otimes ... \otimes b_m)) = \Delta(a_1 \otimes ... \otimes a_n)(\uplus \otimes \uplus)\Delta(b_1 \otimes ... \otimes b_m). \tag{36}$$

This Lemma, due to Hoffman [17], amounts to checking that both sides of this last identity are equal to:

$$\sum_{i \leq n, j \leq m} ((a_1 \otimes ... \otimes a_i) \uplus (b_1 \otimes ... \otimes b_j)) \otimes ((a_{i+1} \otimes ... \otimes a_n) \uplus (b_{j+1} \otimes ... \otimes b_m)), \tag{37}$$

which follows immediately from the definition of Δ and $\uplus$.

PROPOSITION 5.3. — *The right module structure of $QS(A)$ over* **WQSym** *is compatible with the outer product (*i.e.*, the usual graded product of* **WQSym**, *induced by concatenation of words), in the sense that this product coincides with the convolution product $\star$ in* $\mathrm{End}(QS(A))$ *induced by the Hopf algebra structure of $QS(A)$.*

Indeed, by definition of the convolution product of Hopf algebras linear endomorphisms, we have, for u and v surjections from $[n]$ (resp. $[m]$) to $[p]$ (resp. $[q]$):

$$\begin{aligned}(a_1 \otimes ... \otimes a_{n+m})(\mathbf{M}_u \star \mathbf{M}_v) & \\ &= ((a_1 \otimes ... \otimes a_n)\mathbf{M}_u) \uplus (a_{n+1} \otimes ... \otimes a_{n+m})\mathbf{M}_v \\ &= (((a_1 \otimes ... \otimes a_n)\mathbf{M}_u) \otimes ((a_{n+1} \otimes ... \otimes a_{n+m})\mathbf{M}_v))S^{p,q} \\ &= (a_1 \otimes ... \otimes a_{n+m})(\mathbf{M}_u \bullet \mathbf{M}_v)S^{p,q} \\ &= (a_1 \otimes ... \otimes a_{n+m})\mathbf{M}_u\mathbf{M}_v\ .\end{aligned} \tag{38}$$

by Lemma 4.2, or, since the identity does not depend on $a_1, ..., a_{n+m}$:

$$\mathbf{M}_u \star \mathbf{M}_v = \mathbf{M}_u\mathbf{M}_v. \tag{39}$$

Let us formalize, for further use, our last observation on the dependency on $a_1, ..., a_{n+m}$ into a general recognition principle that will prove useful to deduce properties in **WQSym** from its action on quasi-shuffle algebras.

LEMMA 5.4. — *Let f and g be two elements in* **WQSym** *and let us assume that, for an arbitrary commutative algebra A and arbitrary $a_1, ..., a_n, ...$ in A, $(a_1 \otimes ... \otimes a_n)f = (a_1 \otimes \cdots \otimes a_n)g$ for all n. Then, $f = g$.*

The Lemma follows, *e.g.*, by letting $a_1, ..., a_n, ...$ run over an infinite ordered alphabet and letting A be the free commutative algebra over this alphabet.

6. Nonlinear Schur-Weyl duality

The same kind of argument can actually be used to characterize **WQSym** as a universal endomorphism algebra in the same way as **FQSym** is a universal endomorphism algebra according to the classical Schur-Weyl duality. Recall the latter: for an arbitrary vector space V, let us write $T(V) := \bigoplus_{n\in\mathbb{N}} T_n(V) := \bigoplus_{n\in\mathbb{N}} V^{\otimes n}$. Linear morphisms between vector spaces $f : V \longmapsto W$ induce maps $T(f) : T(V) \longmapsto T(W)$ compatible with the graduation ($T_n(f)$ maps $T_n(V)$ to $T_n(W)$: $T_n(f)(v_1 \otimes ... \otimes v_n) := f(v_1) \otimes ... \otimes f(v_n)$). In categorical langage, Schur-Weyl duality characterizes natural transformations of the functor T (or, equivalently, of the subfunctors T_n) from vector spaces to graded vector spaces and reads: the only family of maps $\mu_V : T_n(V) \longmapsto T_n(V)$ (where V runs over vector spaces over $\mathbb{K}$) such that, for any map f as above,

$$T_n(f) \circ \mu_V = \mu_W \circ T_n(f), \tag{40}$$

are linear combination of permutations: $\mu_V \in \mathbb{K}[S_n]$ (the converse statement is obvious: permutations and linear combinations of them acting on tensors always satisfy this equation).

We consider here the corresponding nonlinear problem and characterize natural transformations of the functor

$$T(A) := \bigoplus_{n\in\mathbb{N}} T_n(A) := \bigoplus_{n\in\mathbb{N}} A^{\otimes n} \tag{41}$$

viewed now as a functor from commutative algebras without a unit to vector spaces. Concretely, we look for families of linear maps μ_A from $T_n(A)$ to $T_m(A)$, where A runs over commutative algebras without a unit and m and n are arbitrary integers such that, for any map f of algebras from A to B,

$$T_m(f) \circ \mu_A = \mu_B \circ T_n(f). \tag{42}$$

Let us say that such a family μ_A satisfies nonlinear Schur-Weyl duality (with parameters n, m). The purpose of the section is to prove:

Proposition 6.1. — *Let* **Nat** *be the vector space spanned by families of linear maps that satisfy the nonlinear Schur-Weyl duality. Then* **Nat** *is canonically isomorphic to* **WQSym**.

Equivalently, the vector space $\mathbf{Nat}_{n,m}$ *of families of linear maps that satisfy non linear Schur-Weyl duality with parameters* n, m *is canonically isomorphic to the linear span of surjections from* $[n]$ *to* $[m]$.

The results in the previous section imply that **WQSym** is canonically embedded in **Nat**. Let us show now that the converse property holds. We write $\mathbf{Q}[x_1,...,x_n]^+$ for the vector space of polynomials in the variables $x_1,...,x_n$ without constant term and notice that, for an arbitrary family $a_1,...,a_n$ of elements of a commutative algebra A, the map $f(x_i) := a_i$ extends uniquely to an algebra map from $\mathbf{Q}[x_1,...,x_n]^+$ to A. In particular, if μ_A is a family of linear maps that satisfy the nonlinear Schur-Weyl duality, we have:

$$\mu_A(a_1\otimes ...\otimes a_n) = \mu_A \circ T(f)(x_1\otimes ...\otimes x_n) = T(f)(\mu_{\mathbf{Q}[x_1,...,x_n]^+}(x_1\otimes ...\otimes x_n)), \tag{43}$$

so that the knowledge of $\mu_{\mathbf{Q}[x_1,...,x_n]^+}(x_1 \otimes ... \otimes x_n)$ determines entirely the other maps μ_A.

Let $\mu_A \in \mathbf{Nat}_{n,m}$. Then, $\mu_{\mathbf{Q}[x_1,...,x_n]^+}(x_1 \otimes ... \otimes x_n) \in (\mathbf{Q}[x_1,...,x_n]^+)^{\otimes m}$. The latter vector space has a basis $\mathscr{B}$ whose elements are the tensors $\mathbf{p} = p_1 \otimes ... \otimes p_m$, where the p_is run over all the nontrivial monomials in the x_is (for example $x_1^2x_3 \otimes x_2^2 \otimes x_1x_5$ is a basis element for $n = 5$ and $m = 3$):

$\mu_{\mathbf{Q}[x_1,...,x_n]^+}(x_1 \otimes ... \otimes x_n)$ can therefore be written uniquely as a linear combination of these basis elements.

Now, the commutation property (42) implies that

$$Y := \mu_{\mathbf{Q}[x_1,...,x_n]^+}(x_1 \otimes ... \otimes x_n) \in (\mathbf{Q}[x_1, ..., x_n]^+)^{\otimes m} \tag{44}$$

must be linear in each variable x_i (take f such that $x_i \mapsto ax_i$ and $x_j \mapsto x_j$ for $j \neq i$).

This implies in particular that $\mathbf{Nat}_{n,m} = 0$ when $m > n$. For $m = n$, Y must be a linear combination of permutations $x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$, and for $m < n$,

$$Y = \sum_{\mathbf{p} \in \mathscr{B}, deg(\mathbf{p})=n} \lambda_{\mathbf{p}} \mathbf{p} \tag{45}$$

with $p_1 \ldots p_n = x_1 \ldots x_n$, which implies that

$$\mu_{\mathbf{Q}[x_1,...,x_n]^+}(x_1 \otimes \cdots \otimes x_n) = \sum_{f \in \mathrm{Surj}(n,m)} \lambda_f (\prod_{f(i)=1} x_i \otimes \cdots \otimes \prod_{f(i)=m} x_i). \tag{46}$$

Thus, μ_A can necessarily be written as a linear combination of (maps induced by) surjections from n to m.

7. The Characteristic subalgebra of WQSym

The existence of two algebra maps from **WQSym** (equipped with the product of word quasi-symmetric functions and the internal product) to End($QS(A)$) (equipped with the convolution product and the composition product) extends a classical result. There are indeed two analogous maps from **FQSym** to the endomorphism algebra of the tensor algebra over an alphabet X [21]: this corresponds roughly to the case where one considers $QS(A)$ with A the linear span of X equipped with the null product and can be understood as a particular case of the constructions we are interested in here.

In the "classical" situation, it is however well-known that, from the Hopf algebraic point of view, most relevant informations are contained in a very small convolution subalgebra of **FQSym**, namely the one generated by the identity of the algebra [25, 26]. In the present section, we investigate the structure of the corresponding subalgebra of **WQSym** and deduce from this study that many essential objects in Lie theory (Solomon's idempotents...) have a quasi-shuffle analogue in **WQSym**. Most results in this section are direct applications of [25, Chap. 1] (published in [26]), to which we refer for proofs and details.

We make implicit use of the recognition principle (Lemma 5.4) to deduce these results from the existence of an action of **WQSym** of $QS(A)$. To deal with formal power series in **WQSym**, we consider the usual topology (the one associated to the graduation induced by word length, that is, the one for which words with large lengths are close to 0), and write $\widehat{\mathbf{WQSym}}$ for the corresponding completion.

LEMMA 7.1. — *The k-th characteristic endormorphism (or Adams operation) of $QS(A)$ is the following element of $\widehat{\mathbf{WQSym}}$, identified with the k-th convolution power of the identity map:*

$$\Psi^k := I^k, \ I := \hat{\sigma}_1 := \sum_{n\geq 0} \mathbf{M}_{1,\dots,n}. \tag{47}$$

The characteristic endomorphisms satisfy:

- *Ψ^k is an algebra endomorphism of $QS(A)$,*
- *$\Psi^k \Psi^l = \Psi^{k+l}$,*
- *$\Psi^k * \Psi^l = \Psi^{kl}$.*

See [26, Prop. 1.4, Prop. 1.3]. To deduce the identity $\Psi^k\Psi^l = \Psi^{k+l}$, we use the fact that the product in **WQSym** maps to the convolution product in an arbitrary $QS(A)$.

Many important structure results that hold for graded Hopf algebras [27] do not hold for quasi-shuffle bialgebras -that are not graded but only filtered: the product in $QS(A)$ maps $A^{\otimes n} \otimes A^{\otimes m}$ to $\bigoplus_{p\leq n+m} A^{\otimes p}$, whereas the coproduct respects the graduation and maps $A^{\otimes p}$ to $\bigoplus_{n+m=p} A^{\otimes n} \otimes A^{\otimes m}$. However, some properties of graded Hopf algebras hold for the quasi-shuffle algebras:

- Let f, g two linear endomorphisms of $QS(A)$ that vanish on $\bigoplus_{p\leq n} A^{\otimes p}$, resp. $\bigoplus_{p\leq m} A^{\otimes p}$, then, since the coproduct preserves the graduation, $f \star g$ vanishes on $\bigoplus_{p\leq n+m+1} A^{\otimes p}$ (we say that $f, g, f\star g$ are respectively $n, m, n+m+1$-connected).
- The element I is invertible in $\widehat{\mathbf{WQSym}}$ -this follows from

$$I^{-1} = (M_0 + \sum_{n>0} M_{1\dots n})^{-1} = \sum_{k\geq 0} (-1)^k (\sum_{n>0} M_{1\dots n})^k, \tag{48}$$

since $\sum_{n>0} M_{1\dots n}$ is 0-connected.

DEFINITION 7.2. — *We call characteristic subalgebra of $\widehat{\mathbf{WQSym}}$ and write* **Car** *for the convolution subalgebra of $\widehat{\mathbf{WQSym}}$ generated by I.*

LEMMA 7.3. — *The representation of* **Car** *on $QS^{(n)}(A) := \bigoplus_{p\leq n} A^{\otimes p}$ is unipotent of rank $n+1$. That is, for any 0-connected element f in* **Car**, *f^{n+1} acts on $QS^{(n)}(A)$ as the null operation.*

This follows from the previously established properties.

PROPOSITION 7.4. — *The action of Ψ^k on $QS^{(n)}(A)$ is polynomial in k:*

$$\Psi^k = \sum_{i=0}^{n} k^i e_i, \tag{49}$$

where the "quasi-Eulerian idempotents" are given by

$$e_i = \frac{\log(I)^i}{i!}\,. \tag{50}$$

This follows from the convolution identity $\Psi^k = I^k = \exp(\log(I^k)) = \exp(k\log(I))$ since $\log(I) = \sum\limits_{n\geq 1}(-1)^{n+1}\frac{(I-M_0)^n}{n}$ is 0-connected. One can make explicit the formula for the e_is using the Stirling coefficients of the first kind (Fla I.4.5 in [25]). The e_i are orthogonal idempotents (this follows as in the usual case from $\Psi^k\Psi^l = \Psi^{k+l}$: the proof of [25, Prop.I,4,8] [26, Prop.3.4] applies). When A is the linear span of an alphabet equipped with the null product, we recover Solomon's Eulerian idempotents.

Using the fact that

$$\Psi^k = \hat{\sigma}_1^k \tag{51}$$

and computing in **Sym**, we obtain from (14) the following expression.

PROPOSITION 7.5. — *We have:*

$$e_1 = \sum_{n\geq 1}\frac{1}{n}\sum_{I\vDash n}\frac{(-1)^{l(I)-1}}{\binom{n-1}{l(I)-1}}\sum_{\mathrm{Des}\,(u)=[n]-\{i_{l(I)},\dots,i_{l(I)}+\dots+i_1\}}\mathbf{M}_{\breve{u}}\,, \tag{52}$$

where $I\vDash n$ means that $I=(i_1,\dots,i_{l(I)})$ is a composition of n ($i_1+\dots+i_{l(I)}=n$).

For example, up to degree 3,

$$\begin{aligned} e_1 =& \mathbf{M}_1\\ &+\frac{1}{2}(\mathbf{M}_{12}-\mathbf{M}_{11}-\mathbf{M}_{21})\\ &+\frac{1}{6}(2\mathbf{M}_{123}-\mathbf{M}_{122}-\mathbf{M}_{112}+2\mathbf{M}_{111}-\mathbf{M}_{231}-\mathbf{M}_{132}\\ &+2\mathbf{M}_{221}-\mathbf{M}_{121}-\mathbf{M}_{213}-\mathbf{M}_{212}+2\mathbf{M}_{211}+2\mathbf{M}_{321}-\mathbf{M}_{132})+\cdots \end{aligned} \tag{53}$$

$$\begin{aligned} e_2 =& \frac{1}{2}(\mathbf{M}_{12}+\mathbf{M}_{11}+\mathbf{M}_{21})\\ &+\frac{1}{2}(\mathbf{M}_{123}-\mathbf{M}_{111}-\mathbf{M}_{221}-\mathbf{M}_{211}-\mathbf{M}_{321})+\cdots \end{aligned} \tag{54}$$

(55)
$$e_3 = \frac{1}{6}\sum_{|u|=3} \mathbf{M}_u + \cdots$$

8. The case of quasi-symmetric functions

The fundamental example of a quasi-shuffle algebra is $QSym$, the Hopf algebra of quasi-symmetric functions (it is the quasi-shuffle algebra over the algebra of polynomials in one variable, or in additive notation, over the nonnegative integers [34]). Thus, it is of some interest to have a closer look at the right action of **WQSym** on $QSym$ as defined in the foregoing section. Because of the duality between $QSym$ and **Sym**, this will also result into a refined understanding of the links between word quasi-symmetric functions and noncommutative symmetric functions from a Lie theoretic point of view.

The basis which realizes $QSym$ as a quasi-shuffle algebra is the quasi-monomial basis M_I whose definition was recalled in Section 2:

(56)
$$M_I M_J = M_{I \uplus J} := \sum_K (K|I \uplus J) M_K \, ,$$

where, for two compositions K and L, $(K|L) := \delta_K^L$. Let us denote by a $*$ the right action of **WQSym**: for $I = (i_1, ..., i_k)$ and u a packed word of length k,

(57)
$$M_I * \mathbf{M}_u = M_J \, , \quad j_r = \sum_{u(s)=r} i_s .$$

For example,

(58)
$$M_{21322} * \mathbf{M}_{12121} = M_{2+3+2,1+2} = M_{73} \, .$$

Hence, we have:

Lemma 8.1. — *The compatibility formula (Prop. 5.3) can be rewritten as*

(59)
$$M_I * (\mathbf{M}_u \mathbf{M}_v) = \mu[\Delta M_I *_2 (\mathbf{M}_u \otimes \mathbf{M}_v)]$$

where μ is the multiplication map. This mirrors the splitting formula for the internal product of **FQSym**

(60)
$$(\mathbf{F}_\sigma \mathbf{F}_\tau) * S^I = \mu[(\mathbf{F}_\sigma \otimes \mathbf{F}_\tau) *_2 \Delta S^I]$$

which can be extended to any number of factors on the left, and S^I be replaced by an arbitrary noncommutative symmetric function. Similarly, LEMMA 8.2. — *For any $F \in QSym$ and $\mathbf{G}_1, \ldots, \mathbf{G}_r \in \mathbf{WQSym}$,*

$$F * (\mathbf{G}_1\mathbf{G}_2\cdots\mathbf{G}_r) = \mu_r[\Delta^r F *_r (\mathbf{G}_1 \otimes \mathbf{G}_2 \otimes \cdots \otimes \mathbf{G}_r)], \tag{61}$$

where μ_r, $*_r$ and Δ^r stand for r-fold iterations of the corresponding product and coproduct maps.

One may ask whether the formula remains valid for bigger quotients of **WQSym** (recall that $QSym$ is its commutative image). It appears that one must have $\mathbf{M}_u \equiv \mathbf{M}_v$ for $\mathrm{ev}(u) = \mathrm{ev}(v)$ except when u and v are formed of an equal number of 1s and 2s, in which case other choices are allowed. Thus, $QSym$ is essentially the only interesting quotient.

The commutative image map can be expressed by means of the action $*$. Recall that $M_{1^n} = e_n$ is the n-th elementary symmetric function. For a packed word u of length n,

$$M_{1^n} * \mathbf{M}_u = M_{\mathrm{ev}(u)} \tag{62}$$

so that for any $\mathbf{G} \in \mathbf{WQSym}$, its commutative image G is

$$G = \lambda_1 * \mathbf{G} \quad \left(\lambda_1 := \sum_{n\geq 0} M_{1^n}\right). \tag{63}$$

The Adams operations Ψ^k of $QSym$ defined above are

$$\Psi^k(F) = \mu_k \circ \Delta^k(F) =: F(kX) \tag{64}$$

where the λ-ring theoretical notation $F(kX)$ is motivated by the observation that the coproduct of $QSym$ can be defined by means of ordinal sums of alphabets ($\Delta F(X) = F(X+Y)$).

By the previous theory, we have

PROPOSITION 8.3. — *The Adams operations of $QSym$ can be expressed as*

$$F(kX) = F * \hat{\sigma}_1^k = F * \left(\sum_{n\geq 0} \mathbf{M}_{12\ldots n}\right)^k. \tag{65}$$

For example, with $k = 2$, we can easily compute the first terms by hand

$$\begin{aligned} \hat{\sigma}_1^2 =& 1 + 2\mathbf{M}_1 + \mathbf{M}_1\mathbf{M}_1 + 2\mathbf{M}_{12} \\ & + \mathbf{M}_1\mathbf{M}_{12} + \mathbf{M}_{12}\mathbf{M}_1 + 2\mathbf{M}_{123} + \ldots \\ =& 1 + 2\mathbf{M}_1 \\ & + 3\mathbf{M}_{12} + \mathbf{M}_{21} + \mathbf{M}_{11} \\ & + 4\mathbf{M}_{123} + M_{112} + \mathbf{M}_{213} + \mathbf{M}_{312} + \mathbf{M}_{121} \\ & + \mathbf{M}_{122} + \mathbf{M}_{132} + \mathbf{M}_{231} + \ldots \end{aligned} \tag{66}$$

so that $\Psi^2(M_n) = 2M_n$, and $\Psi^2(M_{ij}) = 3M_{ij} + M_{ji} + M_{i+j}$, *etc.*, which agrees indeed with the direct computation $\Psi^k(M_I) = M_I(kX)$, as $M_{ij}(X+Y) = M_{ij}(X) + M_i(X)M_j(Y) + M_{ij}(Y)$ which for $Y = X$ does yield $3M_{ij} + M_{ji} + M_{i+j}$.

Note that we can have sligthly more general operators by introducing extra parameters, *e.g.*,

$$M_I * \hat{\sigma}_t = t^{\ell(I)} M_I. \tag{67}$$

Then, we would have deformations of the Adams operations, like

$$M_{ij} * (\hat{\sigma}_x \hat{\sigma}_y) = (x^2 + xy + y^2) M_{ij} + xy(M_{ji} + M_{i+j})\,. \tag{68}$$

Let us take now advantage of the duality between **Sym** and $QSym$ [11]. Take care that the following duality results are specific to $QSym$ and **Sym** and would not hold for arbitrary quasi-shuffle algebras -in particular, there is no such direct link in general between the quasi-Eulerian idempotents acting on $QS(A)$ and the usual Eulerian idempotents as the one described below.

We write ψ^k for the adjoint of Ψ^k, acting on **Sym**. Since the product and coproduct on **Sym** are dual to the ones on $QSym$, we have again, on **Sym**, $\psi^k := \mu_k \circ \Delta^k$, where now μ_k is the iterated product of order k on **Sym** and Δ_k its iterated coproduct. These are again the classical Adams operations on **Sym**, but they are not algebra morphisms, due to the noncommutativity of **Sym**: with Sweedler's notation for the coproduct ($\Delta(F) = F_{(1)} \otimes F_{(2)}$),

$$\begin{aligned} \psi^2(FG) =& \mu \circ \Delta(FG) = \sum_{(F),(G)} F_{(1)} G_{(1)} F_{(2)} G_{(2)} \\ &\neq \sum_{(F),(G)} F_{(1)} F_{(2)} G_{(1)} G_{(2)} = \psi^2(F)\psi^2(G). \end{aligned} \tag{69}$$

Similarly to what happens on $QSym$:

Lemma 8.4. — *The ψ^k are given by left internal product with the reproducing kernel $\sigma_1(kA)$:*

$$\psi^k(F(A)) = \sigma_1(kA) * F(A)\,. \tag{70}$$

Proof. — By the splitting formula,

$$\sigma_1(kA) * F(A) = \sigma_1^k * F = \mu_k[\sigma_1 \otimes \cdots \otimes \sigma_1 *_k \Delta^k F] = \mu_k \circ \Delta^k F = \psi^k(F). \tag{71}$$

□

One must pay attention to the fact that there is another family of such operations, corresponding to the right internal product with $\sigma_1(kA)$:

$$\psi^k(F(A)) = \sigma_1(kA) * F(A) \neq F(A) * \sigma_1(kA) = F(kA). \tag{72}$$

The right internal product with $\sigma_1(kA)$ is an algebra morphism (this follows from the splitting formula (60)); in terms of alphabets, this operation corresponds to the transformation $F(A) \mapsto F(kA)$ -we refer to [19] for a detailed study of transformations of alphabets in the framework of noncommutative symmetric functions. Since they are associated with the same kernels $\sigma_1(kA)$, the spectral projectors of both families are encoded by the same noncommutative symmetric functions, the only difference being that one has to take internal product on different sides.

Thus,

PROPOSITION 8.5. — *The adjoint of the quasi-Eulerian idempotent e_1 acting on $QSym$ is $F \mapsto E_1 * F$ where $E_1 = \Phi(1) = \log \sigma_1$ is the usual Eulerian idempotent.*

Recall from [11] that its action on a product of *primitive* elements $F_1 \cdots F_r$ is given by

$$E_1 * (F_1 \cdots F_r) = (F_1 F_2 \cdots) \cdot E_1 \,, \tag{73}$$

where, on the right hand-side, E_1 is the Eulerian idempotent viewed as an element of the group algebra of the symmetric group of order r acting by *permutation of the indices, e.g.*, $E_1 * (F_1F_2) = (1/2)(F_1F_2 - F_2F_1)$.

Let us now choose a basis Q_L of the primitive Lie algebra $\mathcal{L}$ of **Sym**. For the sake of definiteness, we may choose the Lyndon basis on the sequence of generators Φ_n (see [11]) and we may assume that L runs over Lyndon compositions. We can then extend it to a Poincaré-Birkhoff-Witt basis of its universal enveloping algebra $U(\mathcal{L}) =$ **Sym**, so that the Eulerian idempotent will act by $E_1 * Q_I = 0$ if I is not Lyndon, and $= Q_I$ otherwise [32]. Let now P_I be the dual basis of Q_I in $QSym$. Then, e_1 acts by $e_1(P_I) = 0$ if I is not Lyndon, and $= P_I$ otherwise. Then, $QSym$ is free as a polynomial algebra over the P_L by Radford's theorem [30].

Hence, we have proved:

PROPOSITION 8.6. — *The quasi-Eulerian idempotent e_1 maps any basis of $QSym$ to a generating set. Moreover, with our particular choice of the basis, S^I is triangular on the Q_I so that P_I is triangular on the M_I, thus $e_1(M_L)$ for L Lyndon form a free generating set.*

We shall see in the next section that it is true in general that e_1 projects the quasi-shuffle algebra onto a generating subspace, although, as we already mentioned, one can not use any more in the general situation duality together with the properties of the Eulerian idempotents acting on envelopping algebras. However, the case of $QSym$ is essentially generic for a wide class of quasi-shuffle algebras.

A possible line of argumentation (to be developed in a subsequent paper) would be as follows. It has been observed in [4] that noncommutative symmetric functions provided a good framework for understanding Ecalle's formalism of moulds in a special case. To deal with the general case, one can introduce the following straightforward generalization of noncommutative symmetric functions, which will also provide us with a better understanding of quasi-shuffle algebras in general.

Let Ω be an additive monoid, such that any element ω has only a finite number of decompositions[(1)] $\omega = \alpha + \beta$. Let $\mathbf{Sym}^{\Omega}$ be the free associative algebra over indeterminates S_ω ($\omega \in \Omega$, $S_0 = 1$), graded by $\deg(S_\omega = \omega)$, endowed with the coproduct

$$\Delta S_\omega = \sum_{\alpha+\beta=\omega} S_\alpha \otimes S_\beta \tag{74}$$

and define $QSym^{\Omega}$ as its graded dual. Let $M_{\boldsymbol{\omega}}$ be the dual basis of $S^{\boldsymbol{\omega}}$ Its multiplication rule is obviously given by the quasi-shuffle (over the algebra $\mathbb{K}[\Omega]$). Let $\Phi = \log \sum_\omega S_\omega = \sum_\omega \Phi_\omega$. From this, we can build a basis $\Phi^{\boldsymbol{\omega}}$, a basis of products of primitive elements, which multiplies by the ordinary shuffle product over the alphabet Ω.

This provides in particular a simple proof that the quasi-shuffle algebra is isomophic to the shuffle algebra [17] (here it is one and the same algebra, seen in two different bases. The isomorphism of [17] corresponds to a particular choice of generators of the primitive Lie algebra).

Now, the previous argumentation could be copied verbatim here, replacing compositions by words over $\Omega - \{0\}$. One may expect that $\mathbf{Sym}^{\Omega}$ admits an internal product, allowing to reproduce Ecalle's mould composition. This is the case for example when $\Omega = \mathbb{N}^r$ [24], the algebra $\mathbf{Sym}^{\Omega}$ being in this case the natural noncommutative version of McMahon's multisymmetric functions.

9. The generalized Eulerian idempotent as a canonical projection

The purpose of this last section is to show that the results in the previous section can be, to a large extent, generalized to arbitrary quasi-shuffle algebras.

It is well-known that quasi-shuffle algebras $QS(A)$ are free commutative algebras over a vector space $L(A)$ (for example the linear span of Lyndon words). This follows from the observation that the highest degree component of the quasi-shuffle product map $QS(A)_n \otimes QS(A)_m \longrightarrow QS(A)_{n+m} \subset QS(A)$ is simply the usual shuffle product, so that the freeness of $QS(A)$ follows from

(1) This restriction is not strictly necessary, but relaxing it would require a generalization of the notion of Hopf algebra.

the freeness of the tensor algebra over A equipped with the shuffle product by a standard triangularity argument.

For classical graded commutative connected Hopf algebras H (such as the tensor algebra over A equipped with the shuffle product), the Leray theorem asserts that H is a free commutative algebra and one can compute by purely combinatorial means a family of generators of H as a polynomial algebra [25, 26]. The same result actually holds for quasi-shuffle algebras. The section is devoted to its proof.

Recall that the characteristic subalgebra **Car** of **WQSym** is the free associative subalgebra generated by I (the identity map, when elements of **WQSym** are viewed as operations on quasi-shuffle algebras). This element I is grouplike in **WQSym** and the generalized Eulerian idempotent $e_1 = \log(I)$ is therefore a primitive element. The following proposition shows that the Hopf algebra structure inherited by **Car** from **WQSym** is actually compatible with its action on quasi-shuffle algebras. We use the Sweedler notation: $\Delta(\sigma) = \sigma^{(1)} \otimes \sigma^{(2)}$.

PROPOSITION 9.1. — *Let $\sigma \in$ **Car**, then, we have, for an arbitrary commutative algebra A:*

$$((a_1\otimes\cdots\otimes a_n)\uplus(b_1\otimes\cdots\otimes b_m))\cdot\sigma = (a_1\otimes\cdots\otimes a_n)\cdot\sigma^{(1)}\uplus(b_1\otimes\cdots\otimes b_m))\cdot\sigma^{(2)}. \tag{75}$$

The Proposition is obviously true when $\sigma = I$. From [8, Lemma 3.1], it is also true in the convolution algebra generated by I, from which the Proposition follows.

Notice that this property is not true for **WQSym** -the coproduct in **WQSym** is not compatible with the action on quasi-shuffle algebras. This property is actually already true for **FQSym**: the coproduct of **FQSym** is not compatible, in general, with the Hopf algebra structure of shuffle or tensors algebras (this corresponds, in terms of quasi-shuffle algebras, to the particular case of commutative algebras with a null product).

THEOREM 9.2. — *The generalized Eulerian idempotent e_1 is a projection onto a vector space generating $QS(A)$ as a free commutative algebra (that is, equivalently, the image of e_1 is naturally isomorphic to the indecomposables of $QS(A)$: the quotient of the augmentation ideal by its square, which is spanned by non trivial products).*

We already know that e_1 is a spectral idempotent (it maps $QS(A)$ to the eigenspace of the Adams operations associated with their lowest nontrivial eigenvalue). From the previous proposition, we have, since e_1 is primitive, for a, b elements of $QS(A)_n, QS(A)_m,\ n, m \neq 0$:

$$(a \uplus b) \cdot e_1 = (a \cdot e_1) \uplus (b \cdot \epsilon) + (a \cdot \epsilon) \uplus (b \cdot e_1), \tag{76}$$

where ϵ is the augmentation of $QS(A)$, so that $b \cdot \epsilon = a \cdot \epsilon = 0$, and finally:

$$(a \uplus b) \cdot e_1 = 0 \tag{77}$$

from which it follows that the kernel of e_1 contains the square of the augmentation ideal $QS(A)^+$. In particular, e_1 induces a well-defined map on the indecomposables $\mathrm{Ind} := QS(A)^+/(QS(A)^+)^2$.

Let us show finally that this map is the identity, from which the Theorem will follow. Let us compute first $\Psi^2(a)$ for an arbitrary $a \in QS(A)^+_n$. We have:

$$\Psi^2(a) = 2e_1(a) + ... + 2^n e_n(a), \tag{78}$$

where the e_i are higher convolution powers of $e_1 = \log(I)$ and map therefore into $(QS(A)^+)^2$, so that on the indecomposables, $\Psi^2 = 2e_1$. On the other hand (using the Sweedler notation),

$$\Psi^2(a) = I^{*2}(a) = 2a + a^{(1)}a^{(2)}, \tag{79}$$

so that, on Ind, $\Psi^2 = 2 \cdot I$ and the Theorem follows.

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