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ELENA STROESCU

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A DILATION THEOREM FOR OPERATORS ON BANACH SPACES

by

Elena STROESCU

Introduction. -

Let R^+ be the set of all non-negative real numbers and $\mathcal{B}(\mathfrak{X})$ the Banach algebra of all linear bounded operators on a Banach space $\mathfrak{X}$. In this paper, we present a dilation theorem by which an object $\{\mathfrak{X}, \Gamma, U\}$ dilates into $\{\tilde{\mathfrak{X}}, \varphi, P, \tilde{\Gamma}, V\}$; where $\mathfrak{X}$ and $\tilde{\mathfrak{X}}$ are Banach spaces, φ is a bicontinuous isomorphism of $\mathfrak{X}$ into $\tilde{\mathfrak{X}}$, P a continuous projection of $\tilde{\mathfrak{X}}$ onto $\varphi(\mathfrak{X})$, $\Gamma = \{T_t\}_{t\in R^+} \subset \mathcal{B}(\mathfrak{X})$ and $\tilde{\Gamma} = \{\tilde{T}_t\}_{t\in R^+} \subset \mathcal{B}(\tilde{\mathfrak{X}})$ are operator semi-groups, U is a $\mathcal{B}(\mathfrak{X})$-valued linear map on an arbitrary algebra $\mathfrak{A}$ estimated by a submultiplicative functional and V a $\mathcal{B}(\tilde{\mathfrak{X}})$-valued representation on $\mathfrak{A}$ such that $V_a \tilde{T}_t = \tilde{T}_t V_a$, for every $a \in \mathfrak{A}$ and $t \in R^+$. This theorem is an extension of some previous results (see [8], [9]) ; it has arisen from the concern to characterize restrictions of spectral operators on invariant subspaces (or operators which dilate in spectral operators) by a map replacing the spectral representation.

Notations. -

Throughout the following C denotes the complex plane ; $N = \{0,1,2,\ldots\}$; $\mathfrak{A}$ an arbitrary algebra over C with unit element denoted by 1 ; K a submultiplicative functional of $\mathfrak{A}$ into R^+ (i.e. $K_{ab} \leq K_a K_b$ for any $a,b \in \mathfrak{A}$) such that $K_1 = 1$; $\mathfrak{X}$ a Banach space over C ; $\mathcal{B}(\mathfrak{X})$ the Banach algebra of all linear bounded operators on $\mathfrak{X}$ over C ; I the identity operator. Let T_1, $T_2 \in \mathcal{B}(\mathfrak{X})$ two commuting operators ; then one says that T_1 is quasi-nilpotent equivalent with T_2 and denotes $T_1 \sim T_2$, if $\lim_{n\to\infty} \|(T_1 - T_2)^n\|^{1/n} = 0$. A family of operators $\{T_t\}_{t\in R^+} \subset \mathcal{B}(\mathfrak{X})$ is called semi-group if $T_0 = I$ and $T_{t+s} = T_t T_s$ for any t and $s \in R^+$.

THEOREM. - <u>Let</u> $\{T_t\}_{t\in R^+} \subset \mathcal{B}(\mathfrak{X})$ <u>be a semi-group of operators and</u> $U : \mathfrak{A} \to \mathcal{B}(\mathfrak{X})$ <u>a linear map such that</u> $U_1 = I$, $\|U_a\| \leq K_a$, <u>for any</u> $a \in \mathfrak{A}$.

Then, there exists a Banach space $\tilde{\mathfrak{X}}$, an isometric isomorphism φ of $\mathfrak{X}$ into $\tilde{\mathfrak{X}}$, a continuous projection P of $\tilde{\mathfrak{X}}$ onto $\varphi(\mathfrak{X})$, a semi-group $\tilde{\Gamma} = \{\tilde{T}_t\}_{t\in R^+} \subset \mathcal{B}(\tilde{\mathfrak{X}})$ and a representation $V : \mathfrak{A} \to \mathcal{B}(\tilde{\mathfrak{X}})$ such that :

(o) $\|P\| = 1$; $\|\tilde{T}_t\| = \|T_t\|$, for any $t \in R^+$; $V_1 = \tilde{I}$ and $\|V_\alpha\| \leq K_\alpha$, for any $\alpha \in \mathcal{A}$.

(i) $V_\alpha \tilde{T}_\tau = \tilde{T}_\tau V_\alpha$, for any $\alpha \in \mathcal{A}$, $\tau \in R^+$.

(ii) $P\tilde{T}_\tau V_\alpha \varphi(x) \doteq \varphi(T_\tau U_\alpha x)$, for any $\alpha \in \mathcal{A}$, $\tau \in R^+$, $x \in \mathfrak{X}$.

(iii) $\tilde{\mathfrak{X}}$ is the closed vector space spanned by $\{\tilde{T}_t V_\alpha \varphi(x); \alpha \in \mathcal{A}, t \in R^+, x \in \mathfrak{X}\}$.

(iv) Let $s \in R^+$; then we have the following equivalences :

1° $\tilde{T}_s \varphi(x) = \varphi(T_s x)$, for any $x \in \mathfrak{X}$;

2° $P\tilde{T}_s V_\alpha \varphi(x) = \tilde{T}_s P V_\alpha \varphi(x)$, for any $\alpha \in \mathcal{A}$, $x \in \mathfrak{X}$;

3° $U_a T_s = T_s U_a$, for any $a \in \mathcal{A}$.

(v) Let $b \in \mathcal{A}$; then $V_b \varphi(x) = \varphi(U_b x)$, for any $x \in \mathfrak{X}$ is equivalent with $U_{ab} = U_a U_b$, for any $a \in \mathcal{A}$.

(vi) Let $\sigma \in R^+$ and $\beta \in \mathcal{A}$ commuting with all the elements of $\mathcal{A}$ such that $U_{a\beta} = U_a U_\beta$, $T_\sigma U_a = U_a T_\sigma$, for any $a \in \mathcal{A}$; then $\|(\tilde{T}_\sigma - V_\beta)^n\| = \|(T_\sigma - U_\beta)^n\|$, for every $n \in N$.

<u>Proof</u> : A) Let us consider the Cartesian product $\mathfrak{X}^{R^+ \times \mathcal{A}} = \prod_{(t,a) \in R^+ \times \mathcal{A}} \mathfrak{X}^{(t,a)}$ and the direct sum $\mathfrak{X}^{(R^+ \times \mathcal{A})} = \bigoplus_{(t,a) \in R^+ \times \mathcal{A}} \mathfrak{X}^{(t,a)}$, where $\mathfrak{X}^{(t,a)} = \mathfrak{X}$, for every $t \in R^+$, $a \in \mathcal{A}$. An element $y \in \mathfrak{X}^{R^+ \times \mathcal{A}}$ is a family $(y_{t,a})_{(t,a) \in R^+ \times \mathcal{A}}$ (many times we write $y = (y_{t,a})_{t,a}$) of components $(y)_{(t,a)} = y_{t,a} \in \mathfrak{X}$, for every $t \in R^+$, $a \in \mathcal{A}$. If $y \in \mathfrak{X}^{(R^+ \times \mathcal{A})} \subset \mathfrak{X}^{R^+ \times \mathcal{A}}$, then $(y)_{t,a} = y_{t,a} \neq 0$ for only a finite number of elements $(t,a) \in R^+ \times \mathcal{A}$.

Let us consider a map :

$$\Theta = (\Theta^{t,a})_{(t,a) \in R^+ \times \mathcal{A}} \quad \text{of} \quad \mathfrak{X}^{(R^+ \times \mathcal{A})} \quad \text{into} \quad \mathfrak{X}^{R^+ \times \mathcal{A}}$$

defined by

$$\Theta\, y = (T_t \sum_{s,b} T_s U_{ab} y_{s,b})_{t,a} \quad , \quad \text{for every} \quad y \in \mathfrak{X}^{(R^+ \times \mathcal{A})} .$$

It is easy to see that Θ is a well defined linear map. Then, we denote by $\hat{\mathfrak{X}}$ the range of Θ and by $\hat{y}$ an arbitrary element of $\hat{\mathfrak{X}}$.

For every $\hat{y}\in\hat{\mathfrak{X}}$, we have :

$$\Theta^{-1}(\{\hat{y}\}) = \{y\in\mathfrak{X}^{(R^+\times \mathcal{a})} \ ; \ \Theta y = \hat{y}\} .$$

We define a function $\omega : \hat{\mathfrak{X}} \to R^+$ by $\omega(\hat{y}) = \inf_{y\in\Theta^{-1}(\{\hat{y}\})} \sum_{s,b} \| T_s \| K_b \| y_{s,b} \|$, for every $\hat{y}\in\hat{\mathfrak{X}}$; let us prove that ω is a norm on $\hat{\mathfrak{X}}$. Let $\mu\in C$ be non-zero, $\hat{y}\in\hat{\mathfrak{X}}$ and $\Delta(\mu\hat{y}) = \{\mu y \ ; \ y\in\Theta^{-1}(\{\hat{y}\})$; then we show that $\Theta^{-1}(\{\mu\hat{y}\}) = \Delta(\mu\hat{y})$. Indeed, let $\mu y\in\Delta(\mu\hat{y})$, i.e. $y\in\Theta^{-1}(\{\hat{y}\})$, then $\mu\hat{y} = (\mu T_t \sum_{s,b} T_s U_{ab}\, y_{s,b})_{t,a} = \Theta\mu y$, hence $\mu y\in\Theta^{-1}(\{\mu\hat{y}\})$. Let now $z\in\Theta^{-1}(\{\mu\hat{y}\})$, i.e. $\Theta z = \mu\hat{y}$ or $\Theta\frac{z}{\mu} = \hat{y}$, hence $y' = \frac{z}{\mu}\in\Theta(\{\hat{y}\})$ and $z = \mu y'\in\Delta(\mu\hat{y})$. Then $\omega(\mu\hat{y}) = \inf_{z\in\Theta^{-1}(\{\mu\hat{y}\})} \sum \| T_s \| K_b \| z_{s,b} \|$

$$= \inf_{z\in\Delta(\mu\hat{y})} \sum_{s,b} \|T_s \| K_b \| z_{s,b}\| = \inf_{y\in\Theta^{-1}(\{\hat{y}\})} \sum_{s,b} \|T_s \| K_b \| \mu y_{s,b} \| =$$

$$= |\mu| \inf_{y\in\Theta^{-1}(\{\hat{y}\})} \sum_{s,b} \| T_s \| K_b \| y_{s,b} \| = |\mu| \ \omega(\hat{y}),$$ i.e. $\omega(\mu\hat{y}) = |\mu|\omega(\hat{y})$;

whence one deduces also that $\omega(\hat{0}) = 0$. Then, for $\mu = 0$ we have $\omega(0\hat{y}) = 0$ and $0\omega(\hat{y}) = 0$, for any $\hat{y}\in\hat{\mathfrak{X}}$. Hence $\omega(\mu\hat{y}) = |\mu|\omega(\hat{y})$, for any $\hat{y}\in\hat{\mathfrak{X}}$, $\mu\in C$.

Let $\hat{y}^1$, $\hat{y}^2\in\hat{\mathfrak{X}}$ and

$$\Delta(\hat{y}^1 + \hat{y}^2) = \{y^1+y^2 \ ; \ y^1\in\Theta^{-1}(\{\hat{y}^1\}),\ y^2\in\Theta^{-1}(\{\hat{y}^2\})\},$$

then obviously we have $\Delta(\hat{y}^1+ \hat{y}^2)\subset\Theta^{-1}(\{\hat{y}^1+ \hat{y}^2\})$ and

$$\omega(\hat{y}^1+ \hat{y}^2) = \inf_{z\in\Theta^{-1}(\hat{y}^1+ \hat{y}^2)} \sum_{s,b} \| T_s \| K_b \| z_{s,b} \| \leq$$

$$\leq \inf_{z\in\Delta(\hat{y}^1+ \hat{y}^2)} \sum_{s,b} \| T_s \| K_b \| z_{s,b} \|$$

$$= \inf_{y^1\in\Theta^{-1}(\{\hat{y}^1\}),\ y^2\in\Theta^{-1}(\{\hat{y}^2\})} \sum_{s,b} \| T_s \| K_b \| y^1_{s,b} + y^2_{s,b} \| \leq$$

$$\leq \inf_{y^1\in\Theta^{-1}(\{\hat{y}^1\})} \sum_{s,b} \|T_s \| K_b \| y^1_{s,b} \| + \inf_{y^2\in\Theta^{-1}(\{\hat{y}^2\})} \sum_{s,b} \| T_s \| K_b \| y^2_{s,b} \|$$

i.e. $(\hat{y}^1+ \hat{y}^2) \leq \omega(\hat{y}^1)+ \omega(\hat{y}^2)$, for all $\hat{y}^1$, $\hat{y}^2\in\hat{\mathfrak{X}}$.

Then, from the definition of ω , for every $\hat{y}\in\hat{\mathfrak{X}}$, we have :

1) $\omega(\hat{y}) \leq \sum_{s,b} \| T_s \| K_b \| y_{s,b}$, for any $y\in\Theta^{-1}(\{\hat{y}\})$ and

2) $\| \hat{y}_{t,a} \| \leq \| T_t \| K_a \omega(\hat{y})$, for $t\in R^+$, $a\in\mathcal{a}$.

Hence ω is a norm on $\hat{\mathfrak{X}}$; we denote by $\tilde{\mathfrak{X}}$ the ω-completion of $\mathfrak{X}$ and the norm on $\tilde{\mathfrak{X}}$ also by ω .

B) We define an isomorphism φ of $\mathfrak{X}$ into $\mathfrak{X}^{R^+\times \mathfrak{a}}$ by $\varphi(x)=(T_t U_a x)_{t,a}=$ $= (T_t \sum_{s,b} T_s U_{ab} \delta_{os} \delta_{1b} x)_{t,a} \in \hat{\mathfrak{X}}$, for every $x\in \mathfrak{X}$.

Applying 1) and 2) we get

3) $\|x\| \leqslant \omega(\varphi(x)) \leqslant \|x\|$, for any $x\in\mathfrak{X}$.

Therefore φ is an isometric isomorphism of $\mathfrak{X}$ into $\tilde{\mathfrak{X}}$.

We define a projection P of $\hat{\mathfrak{X}}$ onto $\varphi(\mathfrak{X})$, by $P\hat{y} = \varphi(\hat{y}_{o,1})$, for every $\hat{y}\in\hat{\mathfrak{X}}$. Applying 3) and 2), we get $\omega(P\hat{y}) = \omega(\varphi(\hat{y}_{o,1})) \leqslant \|\hat{y}_{o,1}\| \leqslant \omega(\hat{y})$, i.e.

4) $\omega(P\hat{y}) \leqslant \omega(\hat{y})$, for any $\hat{y}\in\hat{\mathfrak{X}}$. Hence, P can be extended by continuity to a continuous projection of $\tilde{\mathfrak{X}}$ onto $\varphi(\mathfrak{X})$, that will be denoted by the same symbol.

Let now $\tau\in R^+$; then for every $\hat{y}\in\hat{\mathfrak{X}}$ we put

$$\tilde{T}_\tau \hat{y} = (T_t \sum_{s,b} T_{s+\tau} U_{ab} y_{s,b})_{t,a} = (T_t \sum_{\sigma,b} T_\sigma U_{ab} y_{\sigma-\tau,b})_{t,a} =$$
$$= (T_t \sum_{\sigma,b} T_\sigma U_{ab} \mathfrak{z}_{\sigma,b})_{t,a} = \Theta_z = \hat{\mathfrak{z}} \in \hat{\mathfrak{X}} ,$$

where we denote $s + \tau = \sigma$; $z_{\sigma,b} = y_{\sigma-\tau,b}$ for $\sigma\geqslant\tau$ and $\mathfrak{z}_{\sigma,b}= 0$, for $0\leqslant\sigma<\tau$, with $b\in\mathfrak{a}$.

We see easily that $\tilde{T}_\tau$ is a well defined linear map of $\hat{\mathfrak{X}}$ into $\hat{\mathfrak{X}}$. Let us prove that also it is continuous.

For every $\hat{y}\in\hat{\mathfrak{X}}$, denoting $\Delta(\tau, \hat{y}) = \{\mathfrak{z}\in \mathfrak{X}^{(R^+\times \mathfrak{a})} ; \mathfrak{z}_{\sigma,b} = y_{\sigma-\tau,b}$ for $\sigma\geqslant\tau$ and $\mathfrak{z}_{\sigma,b} = 0$ for $0\leqslant\sigma<\tau$, $b\in\mathfrak{a}$, $y\in\Theta^{-1}(\{\hat{y}\})\}$, we see that $\Delta(\tau, \hat{y})\subset \Theta^{-1}(\{\tilde{T}_\tau \hat{y}\})$. Then, we have $\omega(\tilde{T}_\tau \hat{y}) = \inf_{\mathfrak{z}\in\Theta^{-1}(\{\tilde{T}_\tau\hat{y}\})} \sum_{\sigma,b} \|T_\sigma\| K_b \|\mathfrak{z}_{\sigma,b}\| \leqslant$

$$\inf_{\mathfrak{z}\in\Delta(\tau,\hat{y})} \sum_{\sigma,b} \|T_\sigma\| K_b \|\mathfrak{z}_{\sigma,b}\| = \inf_{\mathfrak{z}\in\Theta^{-1}(\{\tilde{T}_\tau\hat{y}\})} \sum_{\sigma,b} \|T_\sigma\| K_b \| y_{\sigma-\tau,b}\|$$
$$= \inf_{y\in\Theta^{-1}(\{\hat{y}\})} \sum_{s,b} \|T_{s+\tau}\| K_b \|y_{s,b}\| \leqslant \|T_\tau\| \omega(\hat{y}), \text{ i.e.}$$

5) $\omega(\tilde{T}_\tau \hat{y}) \leqslant \|T_\tau\| \omega(\hat{y})$, for any $\hat{y}\in\hat{\mathfrak{X}}$.

Thus, for every $\tau\in R^+$, $\tilde{T}_\tau$ can be extended by continuity to an element of $\mathfrak{B}(\mathfrak{X})$, that will be denoted by the same symbol. Then, we see easily that $P\tilde{T}_\tau \varphi(x) = \varphi(T_\tau x)$, for any $x\in\mathfrak{X}$.

Hence $\|T_\tau\, x\| = \omega(\varphi(T_\tau\, x)) = \omega(P\tilde{T}_\tau\, \varphi(x)) \leqslant \omega(\tilde{T}_\tau\, \varphi(x)) \leqslant \|\tilde{T}_\tau\|\, \omega(\varphi(x)) = \|\tilde{T}_\tau\|\, \|x\|$, i.e.

6) $\|T_\tau\, x\| \leqslant \|\tilde{T}_\tau\|\, \|x\|$, for any $x \in \mathfrak{X}$. At last, we see easily that $\{\tilde{T}_\tau\}_{\tau \in R^+}$ is a semi-group of operators, that we denote by $\tilde{\Gamma}$.

C) Let us define a representation V. Let $\alpha \in \mathcal{A}$; then for every $\hat{y} \in \hat{\mathfrak{X}}$, we put

$$V_\alpha\, \hat{y} = (T_t \sum_{s,b} T_s\, U_{a\alpha b}\, y_{s,b})_{t,a} = (T_t \sum_{s,c} T_s\, U_{ac} \sum_{b \in \mathcal{A}_c} y_{s,b})_{t,a} =$$

$$= (T_t \sum_{s,c} T_s\, U_{ac}\, u_{s,c})_{t,a} = \Theta u = \hat{u} \in \hat{\mathfrak{X}} \text{ , where}$$

$$\mathcal{A}_c = \{b \in \mathcal{A} \; ; \; \alpha b = c\} \text{ and } u_{s,c} = \sum_{b \in \mathcal{A}_c} y_{s,b} \text{ , for } s \in R^+ \text{ , } c \in \mathcal{A} \text{ .}$$

The map $V_\alpha : \hat{\mathfrak{X}} \to \hat{\mathfrak{X}}$ is well defined. Indeed, let $\hat{y}^1 = \hat{y}^2 \in \hat{\mathfrak{X}}$; then there exists y^1, $y^2 \in \mathfrak{X}^{(R^+ \times \mathcal{A})}$ such that $\hat{y}^1 = \Theta y^1$ and $\hat{y}^2 = \Theta y^2$, hence

$$T_t \sum_{s,b} T_s\, U_{ab}\, y^1_{s,b} = T_t \sum_{s,b} T_s\, U_{ab}\, y^2_{s,b} \text{ , for any } t \in R^+ \text{ , } a \in \mathcal{A} \text{ .}$$

Then, $T_t \sum_{s,b} T_s\, U_{a'b}\, y^1_{s,b} = T_t \sum_{s,b} T_s\, U_{a'b}\, y^2_{s,b}$, for $t \in R^+$ and $a' = a\alpha \in \mathcal{A}$ with $a \in \mathcal{A}$. We see easily that for every $\alpha \in \mathcal{A}$, $V_\alpha : \hat{\mathfrak{X}} \to \hat{\mathfrak{X}}$ is a linear map and $V_1\, \hat{y} = \hat{y}$, for any $\hat{y} \in \hat{\mathfrak{X}}$. Moreover, $V : \mathcal{A} \to \mathcal{L}(\hat{\mathfrak{X}})$ is a representation (see [4]; for a vector space X, $\mathcal{L}(X)$ denotes the algebra of all linear maps of X into X). Now, we prove that, $V_\alpha : \hat{\mathfrak{X}} \to \hat{\mathfrak{X}}$ is continuous, for every $\alpha \in \mathcal{A}$. Let $\alpha \in \mathcal{A}$, $\hat{y} \in \hat{\mathfrak{X}}$ and $\Delta(\alpha, \hat{y}) = \{u \in \mathfrak{X}^{(R^+ \times \mathcal{A})} \; ; \; u_{s,c} = \sum_{b \in \mathcal{A}_c} y_{s,b} \text{ , } y \in \Theta^{-1}(\{\hat{y}\})\}$, then we see $\Delta(\alpha, \hat{y}) \subset \Theta^{-1}(\{V_\alpha\, \hat{y}\})$. Therefore, we have :

$$\omega(V_\alpha\, \hat{y}) = \inf_{u \in \Theta^{-1}(\{V_\alpha \hat{y}\})} \sum_{s,c} \|T_s\|\, K_b \|\, u_{s,c}\| \leqslant$$

$$\leqslant \inf_{u \in \Delta(\alpha,\hat{y})} \sum_{s,c} \|T_s\|\, K_c\, \|u_{s,c}\| = \inf_{y \in \Theta^{-1}(\{\hat{y}\})} \sum_{s,c} \|T_s\|\, K_c\, \|\sum_{b \in \mathcal{A}_c} y_{s,b}\| \leqslant$$

$$\leqslant \inf_{y \in \Theta^{-1}(\{\hat{y}\})} \sum_{s,b} \|T_s\|\, K_{\alpha b}\, \|y_{s,b}\| \leqslant K_\alpha \inf_{y \in \Theta^{-1}(\{\hat{y}\})} \sum_{s,b} \|T_s\|\, K_b\, \|y_{s,b}\| = K_\alpha \omega(\hat{y});$$

i.e. for every $\alpha \in \mathcal{A}$ we get

7) $\omega(V_\alpha\, \hat{y}) \leqslant K_\alpha\, \omega(\hat{y})$, for any $\hat{y} \in \hat{\mathfrak{X}}$. Hence, V_α can be extended by continuity to an element of $\mathcal{B}(\tilde{\mathfrak{X}})$ that will be denoted by V_α, for every $\alpha \in \mathcal{A}$.

Thus, (0) is completely proved. The property (i) is immediate, since for every $\alpha\in\mathcal{A}$ and $\tau\in R^+$, we have $\tilde{T}_\tau V_\alpha \hat{y} = (T_t \sum_{s,b} T_{s+\tau} U_{a\alpha b}\, y_{s,b})_{t,a} = V_\alpha \tilde{T}_\tau \hat{y}$, for any $\hat{y}\in\hat{\mathfrak{X}}$. Using the definitions of φ, P, V_α and $\tilde{T}_\tau$, for $\alpha\in\mathcal{A}$, $\tau\in R^+$, we obtain immediately (ii), (iii) and (v).

D) Let us prove (iv). From $\tilde{T}_s\varphi(x) = (T_t T_s U_a x)_{t,a}$ and $\varphi(T_s x) = (T_t U_a T_s x)_{t,a}$, we see that 1° and 3° are equivalent.

Now chosing $\alpha = 1$ in 2°, and using $P\tilde{T}_\tau\varphi(x) = \varphi(T_\tau x)$ for $\tau\in R^+$, $x\in\mathfrak{X}$ (see (ii)), we get 1°.

Conversely, taking into account of (ii) and writting 1° with $U_\alpha x$ instead of x, for $\alpha\in\mathcal{A}$, we get 2°.

At last, we show (vi). Let $\sigma\in R^+$, and $\beta\in\mathcal{A}$, as in the assumption, also let $n\in N$ and $\hat{y}\in\hat{\mathfrak{X}}$; then, we write :

$$(\tilde{T}_\sigma - V_\beta)^n\hat{y} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \tilde{T}_\sigma^k V_\beta^{n-k}\hat{y} =$$

$$= \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} (T_t \sum_{s,b} T_s U_{ab} T_\sigma^k U_\beta^{n-k} y_{s,b})_{t,a} = \Theta v = \hat{v}\in\hat{\mathfrak{X}} ,$$

where v is defined by

$$v_{s,b} = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} T_\sigma^k U_\beta^{n-k} y_{s,b} \quad , \text{ for } y\in\Theta^{-1}(\{\hat{y}\}) , \ s\in R^+ , \text{ and}$$

$b\in\mathcal{A}$.

Denoting by $\Delta(\sigma, \beta, n, \hat{y})$ = the set of all element v so defined, we see that :

$$\Delta(\sigma, \beta, n, \hat{y}) \subset \Theta^{-1}(\{(\tilde{T}_\sigma - V_\beta)^n\hat{y}\}).$$

Then, we have :

$$\omega((\tilde{T}_\sigma - V_\beta)^n\hat{y}) = \inf_{v\in\Theta^{-1}(\{(\tilde{T}_\sigma - V_\beta)^n.\hat{y}\})} \sum_{s,b} \|T_s\|\, K_b \|v_{s,b}\| \leqslant$$

$$\leqslant \inf_{v\in\Delta(\sigma,\beta,n,\hat{y})} \sum_{s,b} \|T_s\|\, K_b \|v_{s,b}\| =$$

$$= \inf_{y\in\Theta^{-1}(\{\hat{y}\})} \sum_{s,b} \|T_s\|\, K_b \|\sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} T^k U_\beta^{n-k} y_{s,b}\| \leqslant$$

$$\leqslant \|\sum_{k=0} (-1)^{n-k}\binom{n}{k} T_\sigma^k U_\beta^{n-k}\| \inf_{y\in\Theta^{-1}(\{\hat{y}\})} \sum_{s,b} \|T_s\|\, K_b \|y_{s,b}\| =$$

$$= \left\| \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} T_\sigma^k U_\beta^{n-k} \right\| \omega(\hat{y})$$. Therefore, for every $n \in N$,

we have $\omega((\tilde{T}_\sigma - V_\beta)^n \hat{y}) \leqslant \|(\tilde{T}_\sigma - U_\beta)^n\| \ \omega(\hat{y})$, for any $\hat{y} \in \hat{\mathfrak{X}}$; hence $\|(\tilde{T}_\sigma - V_\beta)^n\| \leqslant \|(T_\sigma - U_\beta)^n\|'$. Conversely, since $(\tilde{T}_\sigma - V_\beta)^n \ \varphi(x) = \varphi((T_\sigma - U_\beta)^n x)$, for any $x \in \mathfrak{X}$, we get easily $\|(\tilde{T}_\sigma - V_\beta)^n\| \leqslant \|(T_\sigma - U_\beta)^n\|$.

DEFINITION. - Let $\{\mathfrak{X}, \Gamma, U\}$ be an object, where $\mathfrak{X}$ is a Banach space, $\Gamma = \{T_t\}_{t \in R^+} \subset \mathcal{B}(\mathfrak{X})$ a semi-group of operators and $U : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ a linear map as in the above theorem. Then, an object $\{\tilde{\mathfrak{X}}, \varphi, P, \tilde{\Gamma}, V\}$ where $\tilde{\mathfrak{X}}$ is a Banach space, φ a bicontinuous isomorphism of $\mathfrak{X}$ into $\tilde{\mathfrak{X}}$, P a continuous projection of $\tilde{\mathfrak{X}}$ onto $\varphi(\mathfrak{X})$, $\tilde{\Gamma} = \{\tilde{T}_t\}_{t \in R^+} \subset \mathcal{B}(\tilde{\mathfrak{X}})$ a semi-group of operators and $V : \mathcal{A} \to \mathcal{B}(\tilde{\mathfrak{X}})$ a representation such that $V_1 = \tilde{I}$, $V_\alpha \tilde{T}_\tau = \tilde{T}_\tau V_\alpha$, for any $\alpha \in \mathcal{A}$, $\tau \in R^+$, is called an $\mathcal{A}$-spectral dilation of $\{\mathfrak{X}, \Gamma, U\}$ if the property (ii) is satisfyed. An $\mathcal{A}$-spectral dilation is called minimal if also we have (iii).

Remark 1. - When $\mathcal{A}$ is a Michael algebra and $U : \mathcal{A} \to \mathcal{B}(\mathfrak{X})$ a linear continuous map, then K is the seminorm which estimates U .

Remark 2. - Let $T \in \mathcal{B}(\mathfrak{X})$; then the above theorem is obviously true with $\{T^n\}_{n \in N}$ instead of $\{T_t\}_{t \in R^+}$.

Application. - Let $\mathcal{U}$ be an admissible algebra in the sense of [1]. Then, an operator $T \in \mathcal{B}(\mathfrak{X})$ is called $\mathcal{U}$-subspectral (see [9]) if there is a Banach space containing $\mathfrak{X}$ as a closed subspace, a continuous projection P of $\tilde{\mathfrak{X}}$ onto $\mathfrak{X}$, a $\mathcal{U}$-spectral operator $\tilde{T} \in \mathcal{B}(\mathfrak{X})$ having a $\mathcal{U}$-spectral representation $V : \mathcal{A} \to \mathcal{B}(\tilde{\mathfrak{X}})$ with the properties $V_z \mathfrak{X} \subset \mathfrak{X}$ and $P\tilde{T}V_f x = \tilde{T}PV_f x$, for any $f \in \mathcal{U}$, $x \in \mathfrak{X}$, such that $\tilde{T}|_{\mathfrak{X}} = T$.

We have the following characterization for $\mathcal{U}$-subspectral operators : an operator $T \in \mathcal{B}(\mathfrak{X})$ is $\mathcal{U}$-subspectral if and only if there is a linear map $U : \mathcal{U} \to \mathcal{B}(\mathfrak{X})$ with the properties :

(1) $U_1 = I$,

(2) $U_{fz} = U_f U_z$,

(3) $\|U_f\| \leqslant M L_f$ for any $f \in \mathcal{U}$,

(where M is a positive constant and $L : \mathcal{U} \to \mathcal{B}(\mathcal{Y})$, a linear map satisfying

(j) $\|L_{fg}\| \leqslant \|L_f\| \; \|L_g\|$, for any $f, g \in \mathcal{U}$ and the function

(jj) $\xi \to L_{f\xi}$ is analytic in $\complement$ supp f , for every $f \in \mathcal{U}$;

$\mathcal{Y}$ is a Banach space), such that $TU_f = U_f T$, for any $f \in \mathcal{U}$ and $U_z \sim T$, (see [8] and [9]).

If $\mathcal{U}$ is an admissible topologic algebra with the topology of Michael algebra, then the property (3) of U is replaced by its continuity.

For instance, let $\gamma = \{z \in C \; ; \; |z| = 1\}$; one denotes by $L^p(\gamma)(p < \infty)$ the Banach space of the all complex-valued functions f on γ such that $|f|^p$ is integrable with respect to the Lebesgue measure. (Thus a function $f \in L^p(\gamma)$ if and only if the function $\tilde{f}$ defined by $\tilde{f}(\theta) = f(e^{i\theta})$ for $\theta \in [-\pi, +\pi]$ belongs to $L^p(\frac{1}{2\pi} \, d\theta))$.

In the same way one considers the Banach algebra $L^\infty(\gamma)$ of all complex-valued essential bounded functions with respect to the Lebesgue measure on γ , (i.e. a function $f \in L^\infty(\gamma)$ if and only if the function $\tilde{f}$ defined by $\tilde{f}(\theta) = f(e^{i\theta})$ belongs to $L^\infty(\frac{1}{2\pi} \, d\theta))$.

Let $p \geqslant 1$, as usual, the space H^p is the set of analytic functions in $D = \{z \; ; \; |z| < 1\}$ such that f_r defined by $f_r(\theta) = f(re^{i\theta})$, for $\theta \in [-\pi, +\pi]$, belongs to $L^p(\frac{1}{2\pi} \, d\theta)$ for every $0 \leqslant r \leqslant 1$, or with the other words, H^p is a closed subspace of functions f of $L^p(\gamma)$ such that $\int_{-\pi}^{+\pi} e^{in\theta} f(e^{i\theta}) = d\theta = 0$, $n = 1, 2, 3, \ldots$

Taking $\mathfrak{X} = L^p(\gamma)$ and $\mathcal{U} = L^\infty(\gamma)$, we define a representation $V : \mathcal{U} \to \mathcal{B}(\mathfrak{X})$ by :

$$V_\varphi f = \varphi f \text{ , for every } \varphi \in L^\infty(\gamma) \text{ , } f \in L^p(\gamma) \text{ .}$$

From the theorem of M. Riesz ([3], cap. IX) we have $L^p(\gamma) = H^p \oplus \overline{H}^p_0$, $1 < p < \infty$, where $\overline{H}^p_0$ is the space of complex-conjugate functions of H^p becoming zero at $z = 0$. Let P be the continuous projection of $L^p(\gamma)$ onto H^p . We define the continuous linear map $U : L^\infty(\gamma) \to \mathcal{B}(H^p)$ by :

$$U_\varphi f = P \, V_\varphi \, f \text{ , for every } \varphi \in L^\infty(\gamma) \text{ , } f \in H^p \text{ .}$$

Obviously, U is a continuous linear map with the above properties (1) and (2). Then an operator $T \in \mathcal{B}(H^p)$ such that $U_\varphi T = T U_\varphi$, for $\varphi \in L^\infty(\gamma)$ and $T \sim U_{e^{i\theta}}$ is a $L^\infty(\gamma)$-subspectral operator. For $p = 2$, $V_{e^{i\theta}}$ is the bilateral shift and $U_{e^{i\theta}}$ is the unilateral shift (see [2]) .

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Academia R.S. Romania
Institutul de Matematicà
Calea Grivitei 21
BUCURESTI 12 (Roumanie)