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# APPROXIMATION THEOREMS AND NASH CONJECTURE

by Alberto TOGNOLI

Summary :

The purpose of this lecture is to illustrate some applications of Weierstrass' and Whitney's approximation theorems in their relative form.

In particular it will be mentioned how from these descends a theorem which affirms that the classification of the analytic fiber bundle on a coherent real analytic space coincides with the topological one.

Then, using Weierstrass' relative approximation theorem, an outline of the proof of the following fact will be given : every compact differentiable variety admits a structure of regular algebraic variety.

## § 1 . THE RELATIVE APPROXIMATION THEOREMS

a) Some definitions.

In this article we shall study only entities defined on the real field. Let $U$ be an open set of $\mathbb{R}^n$ , $\mathcal{O}_U$ denotes the sheaf of germs of the real analytic functions on $U$ and $\Gamma(\mathcal{O}_U)$ the ring of (global) sections of $\mathcal{O}_U$ .

A function $f \in \Gamma(\mathcal{O}_U)$ is said algebraic if for any point $x \in U$ there exists a neighbourhood $U_{x_o}$ and some polynomials $\alpha_i : \mathbb{R}^n \to \mathbb{R}$ such that

$$\sum_{i=0} (f(x))^i \alpha_i(x) = 0 \ , \forall x \in U_{x_o}$$

Let $\mathfrak{A}_U$ denote the sheaf of germs of algebraic functions.

Let $V$ be a closed subset of $U$ , $V$ is said an analytic subset of $U$ if the following condition is satisfied : for every $a \in V$ there exists an open neighbourhood $U_a$ such that :

$$V \cap U_a = \{x \in U_a | f_1(x) = \dots = f_q(x) = 0 \ , f_i \in \Gamma(\mathcal{O}_{U_a})\} \ .$$

Let $V$ be an analytic subset of $U$ and $\mathfrak{J}_V$ denote the ideal subsheaf of $\mathcal{O}_U$ of germs of the analytic functions that are identically zero on $V$ .

Finally we denote $O_V = O_{U/\mathcal{J}_V}$ , the sheaf $O_V$ is said the sheaf of germs of analytic functions on V .

In such a way, to any analytic set V of U , is associated a local ringed space.

Then a local ringed space $(X, O_X)$ is said a real analytic space if :

I) X is paracompact.

II) $(X, O_X)$ is locally isomorphic to a ringed space associated to an analytic subset of an open set of $\mathbb{R}^n$ .

In a similar way we define algebraic set of U any closed set that, locally, is the set of zeros of algebraic functions, and we associate to any algebraic set V the sheaf $\mathcal{A}_V = \mathcal{A}_{U/\mathcal{J}_V^a}$ of germs of algebraic functions restricted to V .

Finally a local ringed space $(X, O_X)$ is said an algebraic space if it is paracompact and locally isomorphic to a ringed space associated to an algebraic set.

A closed set V of $\mathbb{R}^n$ is said an affine variety if there exist some polynomials $f_i : \mathbb{R}^n \to \mathbb{R}$ $i = 1,\dots,q$ such that $V = \{x \in \mathbb{R}^n | f_1(x) = \dots = f_q(x) = 0\}$ .

Let V be an affine variety, we shall denote $R_V$ the sheaf of germs of regular functions on V . Using affine varieties $(V, R_V)$ as local models one defines algebraic varieties (see [1]).

If X , Y are real analytic spaces or algebraic spaces or algebraic varieties we shall use the term morphism (and isomorphism) of X into Y instead of morphism (and isomorphism) of ringed spaces. If X , Y are analytic spaces a morphism is usually said an analytic map.

Let U be an open set of $\mathbb{R}^n$ , V an analytic set, $x_o \in V$ and $V_{x_o}$ the germ of V at $x_o$ .

We shall say that V is regular in the point $x_o$ if it is possible to find $q = n - \dim V_{x_o}$ analytic functions $f_1,\dots,f_q$ , defined on a neighbourhood $U_{x_o}$ of $x_o$ , such that :

I) $V \cap U_{x_o} = \{x \in U_{x_o} | f_1(x) = \dots = f_q(x) = 0\}$

II) $(df_1)(x_o),\dots,(df_q)(x_o)$ are linearly independent.

Let $(X, O_X)$ be a real analytic space, we shall say that $x_o \in X$ is a regular point if there exists a neighbourhood $B_{x_o}$ of $x_o$ that is isomorphic to an analytic set containing only regular points. A point that is not regular is called singular. A similar definition of regular point is given for algebraic spaces and algebraic varieties.

Let $(X , O_X)$ be a real analytic space (real algebraic variety) containing only regular points then $X$ is called an analytic (algebraic) real manifold. An algebraic space that contains only regular points is called a regular algebraic space.

Let $U$ be an open set of $\mathbb{R}^n$ and $V$ an analytic (algebraic) subset of $U$ ; it is a well known fact, (see [2],[3]), that in general the sheaf $\mathfrak{J}_V$ $(\mathfrak{J}^a_V)$ is not coherent considered as $O_U$ - module $(\mathfrak{a}_U$ - module).

We shall say that an analytic (algebraic) subset of $U$ is coherent if the sheaf $\mathfrak{J}_V$ $(\mathfrak{J}^a_V)$ is a coherent $O_U$ - module $(\mathfrak{a}_U$ - module).

An analytic (algebraic) space is called coherent if any point $x_o \in X$ has a neighbourhood isomorphic to an analytic (algebraic) coherent subset of some open set of $\mathbb{R}^n$ .

It is known that an algebraic space is coherent if and only if the associated real analytic space is coherent (see [3]). Finally we remember that any real analytic manifold and any regular algebraic space is coherent.

Let $V$ be an affine variety of $\mathbb{R}^n$ , $x_o \in V$ and $\mathfrak{J}(V_{x_o})$ , $(I(V_{x_o}))$ the rings of germs of analytic functions (and of polynomials) that are zero on the germ $V_{x_o}$ of $V$ at $x_o$ (on $V$) .

Let $O_{x_o}$ be the ring of germs of analytic functions defined in some neighbourhoods of $x_o$ in $\mathbb{R}^n$ .

We shall say that the point $x_o$ is an almost regular point of $V$ if $\mathfrak{J}(V_{x_o})$ is generated, as $O_{x_o}$ - module, by $I(V_{x_o})$ .

An affine variety $V$ is said almost regular if $V$ is almost regular in any point.

It is easy to prove that $x_o$ is an almost regular point of $V$ if, and only if, the intersection of all the germs of complex analytic sets of $\mathbb{C}^n$ that contains $V_{x_o}$ is the germ of a complex affine variety that contains $V$ (see [4]). As a consequence we have that any regular point of $V$ (considered as affine variety) is almost regular.

b) The approximation theorems.

In the suite we will give some applications of the following theorems :

THEOREM 1. - Let $U$ be open in $\mathbb{R}^n$ , $V$ a coherent analytic subset of $U$ and $g \in \Gamma(O_V)$ an analytic function on $V$ . Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of compact sets in U such that :

$$K_n \subset \overset{\circ}{K}_{n+1} \quad , \quad \bigcup_{n\in\mathbb{N}} K_n = U \ .$$

Let $\{n_t\}_{t\in\mathbb{N}}$ be a sequence of positive integers.

Finally let $\{\varepsilon_t\}_{t\in\mathbb{N}}$ be a sequence of positive numbers.

Then for any function $f : U \to \mathbb{R}$ of class $C^\infty$ such that $f|_V = g|_V$ there exists an analytic function $h : U \to \mathbb{R}$ with the following properties :

I) $\left| \dfrac{\partial^\alpha (f-h)(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right| < \varepsilon_p$ for any $x \in K_{p+1} - K_p$ and $0 \leqslant \alpha < n_p$

II) $f|_V = h|_V$

THEOREM 2. - Let $U$ be an open set of $\mathbb{R}^n$ , $V$ a compact affine almost regular variety contained in $U$ . Suppose that $V$ , considered as analytic set, is coherent and denote by $p : \mathbb{R}^n \to \mathbb{R}$ a polynomial function.

Let $f : U \to \mathbb{R}$ be a function of class $C^\infty$ such that $f|_V = p|_V$ , $H$ a compact set of $U$ and $\varepsilon$ a positive number.

Then, for every positive integrer $q$ , there exists a polynomial $h : \mathbb{R}^n \to \mathbb{R}$ such that :

I) $\left| \dfrac{\partial^\alpha (f-h)(x)}{\partial x_1^{\alpha_1} \cdots \cdots \partial x_n^{\alpha_n}} \right| < \varepsilon$ , for any $x \in H$ , $0 \leqslant \alpha < q$

II) $f|_V = h|_V$

THEOREM 3. - Let $U$ be an open set of $\mathbb{R}^n$ , $V$ a compact, coherent affine almost regular variety contained in $U$ and $p : U \to \mathbb{R}$ an algebraic function.

Let $f : U \to \mathbb{R}$ be a function of class $C^\infty$ such that $f|_V = p|_V$ , $H$ a compact set of $U$ and $\varepsilon$ a positive number.

Then, for every positive integer $q$ , there exists an algebraic function $h : U \to \mathbb{R}$ such that conditions I) and II) of theorem 2 are satisfied.

We shall give a sketch of the proof of theorem 2 .

Let $\mathbb{R}\{X_1,\ldots,X_n\}$ , $\mathbb{R}[[X_1,\ldots,X_n]]$ be the ring of convergent power series and formal power series.

In the following on local rings we shall consider the $M$-adic topology and we shall denote by $\hat{A}$ the completion of $A$ .

A ring A is said analytic (or formal) if $A = \mathbb{R}\{X_1,\dots,X_n\}/\mathfrak{I}$ ($A = \mathbb{R}[[X_1,\dots,X_n]]/\mathfrak{I}$) where $\mathfrak{I}$ is an ideal.

It is known that analytic and formal rings are local noetherian rings and Hausdorff spaces (with respect to M-adic topology).
From the last assertion the following equality is clear : for any ideal $\mathfrak{I}$ of an analytic or formal ring A we have

$$\hat{\mathfrak{I}} = \hat{A} \,.\, \mathfrak{I} \overset{\text{def}}{=} \{x \in A \mid x = \sum_{i=1}^{q} \alpha_i\, g_i \;,\; \alpha_i \in \hat{A} \;,\; g_i \in \mathfrak{I}\} \,.$$

($\hat{A}\cdot\mathfrak{I}$ is dense in $\hat{\mathfrak{I}}$, but $\hat{A}\cdot\mathfrak{I}$ is an ideal, then closed, and we have $\hat{\mathfrak{I}} = \hat{A}\cdot\mathfrak{I}$).

Let U be an open set of $\mathbb{R}^n$, 0 the origin and suppose $0 \in U$. Let E be a set contained in U and g a function of class $C^\infty$ defined in a neighbourhood of 0 ; we shall say that g has on E , in 0 , a <u>zero</u> of <u>infinite order</u> if for any $p \in \mathbb{N}$ there exists a positive number $C_p$ and a neighbourhood $B_p$ of 0 such that on $B_p \cap E$ we have : $|g(x)| < C_p \,.\, \|x\|^p$ where

$$x = (x_1,\dots,x_n) \;,\; \|x\| = \sum_{i=i}^{n} x_i^2 \,.$$

We remark that if g has a zero of infinite order on E in 0 then any function h having the same formal development has the same property.

Finally we shall denote by $\mathfrak{J}(E_o)$ the subset of $\mathbb{R}[[X_1,\dots,X_n]]$ of the elements associated to a germ of a $C^\infty$ - function having a zero of infinite order on E in 0 .

If $E_o$ is a germ of analytic set (algebraic variety) we shall denote by $\mathfrak{I}(E_o)$ ($P(E_o)$) the ring of germs of analytic functions (polynomials) that are zero on $E_o$ .

It is clear that in the above definitions the choice of the origin as fixed point is inessential.

Using the above notation we have the following

LEMMA 1. - <u>Let</u> V <u>be an affine variety of</u> $\mathbb{R}^n$ <u>and</u> $x \in V$ <u>be an almost regular point, then we have</u>

$$\widehat{P(V_x)} = \widehat{\mathfrak{I}(V_x)} = \mathfrak{J}(V_x)$$

<u>Proof</u> : The first equality is a consequence of the definition of almost regular point, the second is proved in [6].

LEMMA 2. - Let $V$ be an affine variety of $\mathbb{R}^n$ , $x \in V$ be an almost regular point and suppose that $V$ , considered as an analytic space, is coherent in $x$ .
Let $f : U(x) \to \mathbb{R}$ be a function of $C^\infty$ class defined on a neighbourhood $U(x)$ of $x$ in $\mathbb{R}^n$ .
If $f|_{U(x) \cap V} = 0$ there exist some polynomials $g_1,\dots,g_q$ and some $C^\infty$ functions $\alpha_1,\dots,\alpha_q$ , defined on a neighbourhood $U'(x)$ of $x$ , such that :

$$f(y) = \sum_{i=i}^{q} \alpha_i(y)\, g_i(y) \quad , \quad \forall\, y \in U'(x) \quad \text{and} \quad g_i|_V \equiv 0 .$$

Proof : By hypothesis there exists a neighbourhood $D(x)$ of $x$ in $V$ and some polynomials $g_1,\dots,g_q$ such that : $g_i|_V = 0$ , $i = 1,\dots,q$ , for any $y \in D(x)$ the ring $\mathcal{J}(V_y)$ is generated by $g_1,\dots,g_q$ .
For any $y \in D(x)$ the germ $f_y$ of $f$ is, in virtue of lemma 1, of the form

$$(1) \qquad f_y = \sum_{i=i}^{q} (\alpha_i)_y\, (g_i)_y \qquad \text{where} \quad (\alpha_i)_y \in \mathbb{R}[[X_1,\dots,X_n]] .$$

By a result of Malgrange (see [8]) from (1) we deduce that $f_x$ is a linear combination of $(g_i)_x$ with $C^\infty$ coefficients and the lemma is proved.

LEMMA 3. - Let $V$ be an affine, compact, almost regular subvariety of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a function of class $C^\infty$ defined on a neighbourhood $U$ of $V$ .
Let $K$ be a compact set of $U$ , and suppose $f|_V \equiv 0$ , then there exist some polynomials $g_1,\dots,g_q$ and some functions $\alpha_1,\dots,\alpha_q$ of class $C^\infty$ defined on a neighbourhood $U_K$ of $K$ such that :

$$f(x) = \sum_{i=i}^{q} \alpha_i(x)\, g_i(x) \quad , \quad \forall\, x \in U_K \quad \text{and} \quad g_i|_V \equiv 0 \quad , \quad i = 1,\dots,q .$$

Proof : $V$ is almost regular and compact then there exist some polynomials $g_1,\dots,g_q$ such that : $g_i|_V = 0$ , $\{g_i\}_{i=1,\dots,q}$ generate $\mathcal{J}(V_x)$ for any $x \in V$ and if $x \notin V$ then there exists $i$ such that $g_i(x) \neq 0$ .
For any $x \in U$ there exists a neighbourhood $U_x$ and some functions of class $C^\infty$ : $\{\alpha_i^x\}_{i=1,\dots,q}$ such that

$$(1) \qquad f(y) = \sum_{j=1}^{q} \alpha_j^x(y)\, g_j(y) \quad , \quad \forall\, y \in U_x .$$

In fact, if $x \in V$ , (1) is a consequence of lemma 2 , if $x \notin V$ then there exists $g_i$ such that $g_i(x) \neq 0$ and we can write $f(y) = f(y)/g_i(y) \,.\, g_i(y)$.

So we have proved that there exists a finite open, (in $\mathbb{R}^n$) , covering $\{U_i\}_{i=1,\dots,s}$ of K and functions $\{\alpha_j^i\}_{\substack{j=1,\dots,q \\ i=1,\dots,s}}$ of class $C^\infty$ , such that we have : $f(y) = \sum_{j=1}^{q} \alpha_j^i(y)\, g_j(y)$ , $\forall y \in U_i$ .

Let $\{\rho_i\}_{i=1,\dots,s}$ be a partition of unity of class $C^\infty$ relative to the covering $\{U_i\}_{i=1,\dots,s}$ .

The we have :

$$f(x) = f(x) \,.\, \sum_{i=1}^{s} \rho_i(x) = \sum_{i=1}^{s} \rho_i(x) \,.\, \sum_{j=1}^{q} \alpha_j^i(x)\, g_j(x) =$$

$$= \sum_{i,j} \rho_i(x)\, \alpha_j^i(x) g_j(x) = \sum_{j=1}^{q} g_j(x) \,.\, \sum_{i=1}^{s} \alpha_j^i(x)\, \rho_i(x) =$$

$$= \sum_{j=1}^{q} \alpha_j(x)\, g_j(x)$$

where $\alpha_j = \sum_{i=1}^{s} \alpha_j^i\, \rho_i$ .

The functions $\alpha_j$ are of class $C^\infty$ and the lemma is proved.

<u>Proof of theorem 2</u>. : We have $f - p|_V \equiv 0$ then it is enough to prove the theorem for the function $g = f - p$ such that $g|_V = 0$ .

Lemma 3 affirms that there exist some polynomials $g_1,\dots,g_q$ and $C^\infty$ functions $\alpha_1,\dots,\alpha_q$ defined on a neighbourhood $U_K$ of K such that :

$$g(x) = \sum_{j=1}^{q} \alpha_j(x)\, g_j(x) \quad , \quad x \in U_K \quad \text{and} \quad g_j|_V \equiv 0 \ , \ j = 1,\dots,q \ .$$

It is now possible, by the classical Weierstrass approximation theorem, to choose polynomials $\hat{\alpha}_j$ such that the polynome $\sum_{j=1}^{q} \hat{\alpha}_j\, g_j + p$ satisfies the conditions of theorem 2.

<u>Remark</u> : The proof of theorem 3 is quite similar.

The proof of theorem 1 is of the same type but more difficult because in general we need infinitely many elements of $\Gamma_V(\mathcal{J})$ to generate $\mathcal{J}(V_x)$ , $x \in V$ .

After we use Whitney's approximation theorem instead of Weierstrass theorem. Theorem 1 is contained in [20].

## § 2 . APPROXIMATION THEOREMS IN THE CASE OF MANIFOLDS

It is a natural problem to see if it is possible to deduce from theorems 1,2,3 some results of the following type :

1') let $X$ , $Y$ be two real analytic spaces and $f : X \to Y$ a continuous map, then $f$ can be approached by analytic maps $f_i : X \to Y$ such that any $f_i$ is in the same homotopy class of $f$ .

2') let $X$ , $Y$ be two affine, compact varieties and $f : X \to Y$ a continuous map, then $f$ can be approached by a sequence of morphisms.

3') let $X$ , $Y$ be two compact algebraic spaces and $f : X \to Y$ a continuous map, then $f$ can be approached by a sequence of morphisms $f_n : X \to Y$ such that any $f_n$ is in the same homotopy class of $f$ .

It is also possible to see for "relative problem" of type 1'), 2'), 3').

In the next proposition we shall give a partial solution to problem 1').

PROPOSITION 1. - Let $X$ be a coherent real analytic space and suppose that for any connected component $X_i$ of $X$ we have $\dim X_i < +\infty$ .

Let $Y$ be a real analytic manifold, $d : Y \times Y \to \mathbb{R}$ a continuous metric and $f : X \to Y$ a continuous map.

Then, for any $\varepsilon > 0$ , there exists an analytic map $h : X \to Y$ such that : $d(f(x) , h(x)) < \varepsilon$ , $\forall x \in X$ and $h$ is homotopic to $f$ .

Proof : We may suppose $X$ connected.

There exists an analytic proper injective map $j : X \to \mathbb{R}^n$ , $n = 2 \dim X + 1$ , such that $j : X \to j(X)$ is homeomorphism and $j(X)$ is a coherent real analytic space (see [9]).

It is then clear that it is enough to solve the problem for the analytic subspace $j(X)$ of $\mathbb{R}^n$ and the function $f' = f \circ j^{-1}$ , so in the following we shall suppose $X$ subspace of $\mathbb{R}^n$ .

It is known that $Y$ may be considered as a submanifold of $\mathbb{R}^m$ , $m \geqslant 2 \dim Y+1$ and there exists a tubular neighbourhood $U$ of $Y$ in $\mathbb{R}^m$ .

By definition of tubular neighbourhood there exists an analytic map $p : U \to Y$ such that : $p(x) = x$ , if $x \in Y$ , and $p$ is homotopic to the identity map $i : U \to U$ .

Any continuous map $f : X \to Y$ may be approached by $C^\infty$ maps $f'_i : X \to U \subset \mathbb{R}^m$ (see [10]) ; theorem 1 asserts that we can approach $f'_i$ by analytic maps $f''_i : X \to U \subset \mathbb{R}^m$ .

If $f'_i$ is close enough to $f$ and $f''_i$ to $f'_i$ the analytic map $f_i = p \circ f''_i : X \to Y$ approaches $f$ in the required sense.

Finally it is easy to verify that if $f''_i$ approaches enough $f$ then $f_i$ is homotopic to $f$ .

The proposition is now proved.

The demonstration of proposition 1 points out that we obtain results of type 1'), 2'), 3') if the following conditions are satisfied :

a) $X$ and $Y$ are imbedded in some euclidian space ;

b) $Y$ has a tubular neighbourhood.

So we can affirm that (at last following this way) we cannot solve the problem 1') if $Y$ is singular (it is known that, if $Y$ has at least a singular point, it is impossible to find a tubular neighbourhood).

Analogously we cannot solve problem 2') and we can solve problem 3') only if $X$ and $Y$ are isomorphic to algebraic subspaces of some euclidian space(*) and $Y$ is regular at any point (the existence of tubular neighbourhoods for algebraic regular subspaces of $\mathbb{R}^n$ is proved in [3]).

It is not difficult to convince ourself that result 1'), if $Y$ is singular, result 2'), result 3') if $X$ or $Y$ are not imbedded are false (at least in general)

For example let :

$$X = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 - 9 = 0\}$$

$$Y_1 = \{(x,y) \in \mathbb{R}^2 | x^2 + (y-1)^2 - 1 = 0\}$$

$$Y_2 = \{(x,y) \in \mathbb{R}^2 | x^2 + (y+1)^2 - 1 = 0\}$$

$Y = Y_1 \cup Y_2$ and $f : X \to Y$ the projection of $X$ into $Y$ from the origin $0$ of $\mathbb{R}^2$ .

It is easy to verify that :

$f$ is continuous but for any analytic map $f' : X \to Y$ we have $f'(X) \subset Y_1$ or $f'(X) \subset Y_2$ .

(*) In general a regular compact algebraic space is not isomorphic to a subspace of an euclidian space (see [3]).

So we conclude that f cannot be approximated by analytic map and any analytic map $f' : X \to Y$ is not homotopic to f .

About the problem 2') we remark the following : if it should be possible to obtain results of type 2') then we shall also have that two compact regular affine varieties are isomorphic if and only if they are $C^\infty$ - isomorphic and this is false(*) (in fact for proving this last result we need a stronger version of 2') involving approximation of derivatives).

About the problem 3') we remark the circle $S^1$ may be considered as a real algebraic subspace of $\mathbb{R}^2$ , and also with the algebraic structure induced by $\mathbb{R}$ identifying $S^1$ with $\mathbb{R}/\mathbb{Z}$ . It is easy to verify that $S^1$ , endowed with the last structure, has no global algebraic function ; we shall denote $\hat{S}^1$ the circle with this last structure.

It is now clear that the identity map $i : \hat{S}^1 \to S^1$ cannot be approximated by morphisms of algebraic structures and any morphism is not homotopic to i .

Using theorem 1 in the relative form we can strenghten proposition 1 and we obtain :

THEOREM 4. - Let X be a real coherent analytic space, X' a coherent analytic subspace of X such that $\dim X' < +\infty$ .

Let Y be a real analytic manifold, $d : Y \times Y \to \mathbb{R}$ a continuous metric and $f : X \to Y$ a continuous map such that $f|_{X'}$ is analytic.

Then for any $\varepsilon > 0$ there exists an analytic map $h : X \to Y$ such that :

$f|_{X'} = h|_{X'}$ , $d(f(x) , h(x)) < \varepsilon$ , $\forall x \in X$ and f is homotopic to h .

The idea for proving theorem 4 is the following : let $X = \bigcup_{n \in \mathbb{N}} X_n$ the decomposition of X into irreducible components ; then one, using proposition 1 , approximate $f|_{X_1}$ by $f^1 : X_1 \to Y$ , after, without changing $f^1|_{X_1 \cap X_2}$ , one approximate $f|_{X_1 \cup X_2}$ ....

The family $\{X_n\}_{n \in \mathbb{N}}$ is locally finite so we can construct an analytic approximation of f .

Theorem 4 is proved in [11].

A problem tied to problem 1') is the following

1") Let X be a coherent real analytic space and $(B \xrightarrow{\pi} X , G , F)$ an analytic fiber bundle with structural Lie group G and fiber F . Suppose F is an analy-

(*) In fact one proves that if two affine varieties X , X' are isomorphic then their complexifications are birationally equivalent.

tic manifold and $\gamma : X \to B$ be a continuous cross section.
We ask if it is possible to approach $\gamma$ by analytic cross sections. A partial affirmative answer is given by

PROPOSITION 2. - Let $X$ be an analytic manifold and $B \xrightarrow{\pi} X$ an analytic fiber bundle the fiber of which is a manifold. Let $d : B \times B \to \mathbb{R}$ be a continuous metric on $B$, $X'$ a coherent analytic subspace of $X$ and $\gamma : X \to B$ a continuous cross section such that $\gamma|_{X'}$ is analytic.
Then for any $\varepsilon > 0$ there exists an analytic cross section $\gamma_a : X \to B$ such that :
$\gamma_{a|X'} = \gamma_{|X'}$ , $d(\gamma(x), \gamma_a(x)) < \varepsilon$ , $\forall x \in X$ and $\gamma_a$ is homotopic to $\gamma$ .

Proof : $B$ is an analytic manifold then, by proposition 1 , the map $\gamma : X \to B$ may be approached by analytic maps $\gamma_i : X \to B$ such that $\gamma_{|X'} = \gamma_{i|X'}$ .

In general the maps $\alpha_i = \pi \circ \gamma_i : X \to X$ are not the identity but, if $\gamma_i$ is close enough to $\gamma$ (*) , we know that $\alpha_i$ is an isomorphism of analytic manifolds.

It is now clear that $\hat{\gamma}_i = \gamma_i \circ \alpha_i^{-1} : X \to B$ is an analytic cross section of $B$ and, if $\gamma_i$ is close enough to $\gamma$ , then $\hat{\gamma}_i$ satisfies the condition $d(\hat{\gamma}_i(x), \gamma(x)) < \varepsilon$ , $\forall x \in X$ .
If $x \in X'$ we have $\alpha_i(x) = x$ then $\hat{\gamma}_i(x) = \gamma_i(x) = \gamma(x)$ . The proposition 1 asserts that, if $\gamma_i$ is close enough to $\gamma$ , there exists a homotopy $\gamma_i^t$ tieing $\gamma_i$ to $\gamma$ ; it is clear that $\gamma_i^t$ ties $\hat{\gamma}_i$ to $\gamma$ .

The proof is acquired.

As a consequence of the theorem 4 and the proposition 2 we can prove the following

PROPOSITION 3. - Let $X$ , $\dim X < +\infty$ , be a real coherent analytic space and $B_t \xrightarrow{\pi} X$ a topological principal fibre bundle of structural group $G$ . If $G$ is a connected (or a compact) Lie group then there exists an analytic fiber bundle $B_a \xrightarrow{\pi_a} X$ that is topologically equivalent to $B_t$ .

Let $X$ be a real analytic manifold and $G$ a Lie group.

Let $B_i \xrightarrow{\pi_i} X$ , $i = 1,2$ , be two analytic principal fiber bundles with structural group $G$ , then $B_1$ is analytically isomorphic to $B_2$ if and only if $B_1$ is topologically isomorphic to $B_2$ .

(*) Here we need that $\gamma_i$ and their "first derivative" approach $\gamma$ and its first derivative and this is possible by theorem 1 .

Proof : It is known (see [10]), that if the Lie group G is connected then, in the bundle $B_t \to X$, the structural group may be reduced to a compact subgroup $G'$.

For any $n \in \mathbb{N}$ there exists a universal bundle $U(G',n) \to D(G',n)$ relative to the group $G'$ ; it is known (see [10]) that the universal bundle $U(G',n) \to D(G',n)$ may be endowed of real analytic structure.

To prove the first part of the proposition it is enough to show that any continuous map $\varphi : X \to D(G',n)$, $n = \dim X$, is homotopic to an analytic map $\varphi_a : X \to D(G',n)$ and this is proved in proposition 1.

To prove the second part of proposition we recall that, given the fiber bundles $B_1$, $B_2$, there exists another fiber bundle : $B_{1,2} \to X$ such that $B_1$ is topologically (analytically) isomorphic to $B_2$ if and only if $B_{1,2}$ has at least one continuous (analytic) section (for the construction of $B_{1,2}$ see [13]).

It is now clear that the proposition 2 proves the second part of this proposition.

Proposition 2 is a particular case of the following

THEOREM 5. - Let $X$ be a real coherent analytic space, $\dim X < \infty$ and $X'$ a coherent analytic subspace.

Let $B \xrightarrow{\pi} X$ be a real analytic fiber bundle of structural Lie group $G$ and fiber the analytic manifold $F$

Let $d : B \times B \to \mathbb{R}$ a continuous metric, $\gamma : X \to B$ a continuous cross section such that $\gamma|_{X'}$ is analytic.

Then, if $G$ is connected, for any $\varepsilon > 0$ there exists an analytic section $\gamma_a : X \to B$ such that :

$$\gamma|_{X'} = \gamma_a|_{X'} \ , \quad d(\gamma(x) \ , \ \gamma_a(x)) < \varepsilon \ , \quad \forall x \in X \text{ and } \gamma \text{ is homotopic to } \gamma_a .$$

Remark : It is possible to prove a version of proposition 1 and 2 for compact regular algebraic sets of $\mathbb{R}^n$ (the proofs are formally the same).

Also a weak form of proposition 3 may be proved for the compact algebraic subsets of $\mathbb{R}^n$.

## § 3. AN APPLICATION OF THEOREM 2

Let $V$ be a compact differentiable submanifold of $\mathbb{R}^n$ ; J. Nash in [14], has put the following problems :

I) does it exist an affine regular variety $V_a$ isomorphic (as differentiable manifold) to $V$ ?

II) if there exists $V_a$, is it possible to realize $V_a$ as a submanifold of $\mathbb{R}^n$ close to $V$ ?

Nash has proved that there exists an affine variety $V'_a$ such that $V'_a$ has an analytic component $V_a$ that solves problems I) and II). In the terminology we have introduced we can say that Nash has solved problems I) and II) with a regular compact algebraic set $V_a$. Using theorem 2 we can prove that the problem I) has an affirmative resolution and problem II) can be solved if $n > 2 \dim V$ (*). We now shall give some definitions to explain problem II). Let $L$, $L'$ be two linear r-dimensional subspaces of $\mathbb{R}^n$ and $x_1,\ldots,x_r$, $y_1,\ldots,y_{n-r}$ a system of orthogonal coordinates of $\mathbb{R}^n$ such that : $L = \{(x_1,\ldots,x_r , y_1,\ldots,y_{n-r}) | y_1=\ldots=y_{n-r}=0\}$. We shall say that $L'$ is an $\varepsilon$-approximation of $L$ if $L'$ has equations of the form

$$y_i = \sum_{j=1}^{r} a_{ij} x_j + c_i , \quad i = 1,\ldots,n-r$$

with the condition $\sum_{i,j} |a_{ij}|^2 + \sum c_i^2 < \varepsilon$

Let $V$ be a compact differentiable manifold of dimension $r$ differentiably embedded in $\mathbb{R}^n$. At each point $x \in V$ take the disc $D_x$ of radius $\delta$ contained in the $n-r$ dimensional linear space orthogonal to $V$.

If $\delta$ is small enough it is known that the union of all these discs has the structure of a fibre bundle over $V$.

This bundle is called the <u>normal bundle of radius</u> $\delta$ (**) and it is denoted by $B(\delta)$.

The set $B(\delta)$ is an open neighbourbood of $V$ in $\mathbb{R}^n$ and the projection $p : B(\delta) \to V$ defined by : $p(y) = x$ if $y \in D_x$ is a differentiable map.

Let $V'$ be a differentiable manifold of $\mathbb{R}^n$, we shall say that $V'$ <u>is an $\varepsilon$-approximation of</u> $V$ if :

1°/ $V'$ is contained in the tubular neighbourbood $B(\varepsilon)$ of $V$

2°/ $p : V' \to V$ is an isomorphism of the differentiable structures

3°/ for any $x \in V'$ the tangent linear variety to $V'$ at $x$ is an $\varepsilon$-approximation of the tangent linear variety to $V$ at $p(x)$.

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(*) The author conjectures that problem II) can be solved without any restriction on the codimension of $V$.

(**) $B(\delta)$ is also called the tubular neighbourbood of radius $\delta$.

Let $V$ be a differentiable submanifold of $\mathbb{R}^n$ we shall say that $V$ has, (in $\mathbb{R}^n$) an algebraic $\varepsilon$-approximation if there exists an affine regular subvariety $V'$ of $\mathbb{R}^n$ that is an $\varepsilon$-approximation of $V$.

We shall say that $V$ admits algebraic approximation if, for any $\varepsilon > 0$, $V$ has an algebraic $\varepsilon$-approximation.

A formulation of problem II is the following :

Any compact differentiable submanifold of $\mathbb{R}^n$ admits algebraic approximation ?

It is possible to prove the following

THEOREM 6. - Let $V$ be a compact differentiable submanifold of $\mathbb{R}^n$, $n > 2 \dim V$, then $V$ admits algebraic approximation.

COROLLARY. - Any compact differentiable manifold is isomorphic to a regular affine variety.

Theorem 6 is proved in [4] we shall give here an idea of the proof. We need the following

LEMMA. - Any compact differentiable manifold is in the same cobordism class of a compact, regular affine variety.

Proof : Let $P_n(\mathbb{R})$ be the n-projective space on the real numbers. We denote by $z_o,\dots,z_n$, $w_o,\dots,w_m$, $m \leqslant n$ two systems of coordinates of $P_n(\mathbb{R})$ and $P_m(\mathbb{R})$.

We put :

$$H_{n,m}(\mathbb{R}) = \{\{z_j\}\times\{w_j\} \in P_n(\mathbb{R}) \times P_m(\mathbb{R}) | w_o z_o + w_1 z_1 + \dots + w_m z_m = 0\}$$

It is known (see [16]) that the manifolds $P_n(\mathbb{R})$, $H_{nm}(\mathbb{R})$ are generators of cobordism ring.

Then to prove the lemma it is enough to show that $P_n(\mathbb{R})$ has a structure of regular affine variety.

Let us consider the map $\chi_{ik} : P_n(\mathbb{R}) \to \mathbb{R}$ defined by

$$\chi_{ik}(x_j) = x_i x_k / \sum_{j=o}^{n} x_1^2$$

It is easy to verify that the map $\chi : P_n(\mathbb{R}) \to \mathbb{R}^{(n+1)^2}$ defined by $\chi(x) = \{\chi_{ik}(x)\}_{i,k=o,\dots,n}$ is injective, of maximum rank at any point and the set $\chi(P_n(\mathbb{R}))$ is the regular affine subvariety $W$ of $\mathbb{R}^{(n+1)}$ defined by the equations :

$$\sum_{i=o}^{n} \chi_{ii} = 1$$

$$\chi_{ik}\, \chi_{lr} = \chi_{il}\, \chi_{kr}$$

$$\chi_{ik} = \chi_{ki} \quad i,k,l,r = o,\dots,n .$$

So we have proved that $P_n(\mathbb{R})$ is isomorphic to $W$ ; it is now easy to verify that $W$ is regular affine subvarity of $\mathbb{R}^{(n+1)^2}$ .

Let $V_1$ , $V_2$ be two differentiable manifolds and suppose that $V_1$ is in the same cobordism class of $V_2$ .

By Whitney's embedding theorems, (see [17]) , we may suppose that there exists a differential submanifold, with boundary $W$ of $\mathbb{R}^{n+1}$ such that, if $x_1,\dots,x_{n+1}$ are coordinates in $\mathbb{R}^{n+1}$ , we have :

1°/ $W \subset \{\{x_i\} | x_{n+1} > 0\}$ , the boundary $\partial W = V_1 \cup V_2$ of $W$ is equal to $W \cap \{\{x_i\} | x_{n+1} = 0\}$ .

2°/ the set $\hat{W} = W \cup \{(x_1,\dots,x_{n+1}) | (x_1,\dots,-x_{n+1}) \in W\}$ is a differentiable submanifold of $\mathbb{R}^{n+1}$ .

3°/ the hyperplane $x_{n+1} = 0$ cuts transversally $\hat{W}$ .

Furthermore if $V_1$ is an affine regular variety we may suppose that $W$ is the disjoint union of a regular affine subvariety $V_1'$ of $\mathbb{R}^{n+1}$ , isomorphic to $V_1$ , and of a differentiable submanifold $V_2'$ isomorphic to $V_2$ .

The manifold $W$ shall be said the torus constructed on $V_1$ and $V_2$ .

The idea of the proof of theorem 6 is the following : let $V_2$ be a compact differentiable manifold and $V_1$ a regular compact affine variety in the same cobordism class. Let $\hat{W}$ be the torus constructed on $V_1$ and $V_2$ . Then we approach $\hat{W}$ by an affine regular variety $W'$ in such a way that the intersection of $W'$ with the hyperplane $x_{n+1} = 0$ is composed by two analytic compact manifolds $V_1'$ , $V_2'$ that are $\varepsilon$ -approximation of $V_1$ , $V_2$ for some $\varepsilon$ .

But if in the approximation process we use theorem 2 instead of the classical Weierstrass theorem we can obtain $V_1 = V_1'$ . So we have that $V_1' \cup V_2'$ is a regular affine subvariety of $\mathbb{R}^n$ , $V_1' = V_1$ is an affine regular subvariety and we can conclude that $V_2'$ is affine and an $\varepsilon$ -approximation of $V_2$ .