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QUADRATIC FORMS AND SESQUILINEAR FORMS IN INFINITE DIMENSIONAL SPACES

WITT TYPE THEOREMS IN SPACES OF DENUMERABLY INFINITE DIMENSION

par

H. GROSS

0. - Introduction

All forms considered here are forms over divisionrings.

If $\phi : E \times E \to k$ is a sesquilinear form on the vector-space E over the divisionring k (with antiautomorphism $\alpha \to \alpha^*$) then we shall always tacitly assume ϕ to be orthosymmetric, i.e. $\phi(x,y) = 0$ if and only if $\phi(y,x) = 0$. Provided that $\dim E/E^{\perp}$ is at least 2 the automorphism $\alpha \to \alpha^{**}$ ist then inner and there exist nonzero $\gamma \in k$ such that $\phi\gamma$ is hermitean or antihermitean (with respect to an involution of k). An ε-hermitean form ϕ (i.e. $\phi(y,x) = \varepsilon\phi(x,y)^*$ for some $\varepsilon \in$ center(k) and all x,y) is said to be tracevalued iff for every $x \in E$ there is $\alpha \in k$ such that $\phi(x,x) = \alpha + \varepsilon\alpha^*$. An arbitrary form ψ on a space F with $\dim F/F^{\perp} \geqslant 2$ is said to be tracevalued iff some (and hence every) ε-hermitean multiple $\psi\gamma$, $\gamma \neq 0$, is tracevalued.

If Φ is a form on the space E then $\|\phi\|$ or $\|E\|$ is the set $\{\phi(x,x) | x \in E\}$; an alternate form has $\|\phi\| = \{0\}$.

Here I shall mainly be concerned with Witt-type theorems. The celebrated theorem of Witt states that an isometry $T_o : F \to \overline{F}$ between finite dimensional subspaces of a non degenerate tracevalued sesquilinear space E always extends to a metric automorphism on all of E ([3], p. 71). The classical theory of forms and its associated groups pivots on this theorem ; it is therefore not necessary to discuss the importance of our matter.

It is easy to discover that the theorem as stated above is false when $\dim F$ is infinite ([9], chap. 3). When trying to describe the state of affairs in this case it is first of all necessary to distinguish between two problems of a rather different nature :

Problem 1. Given isometric subspaces $F,\overline{F}$ of a sesquilinear space E when does there exist a metric automorphism T of E with $TF = \overline{F}$? In other words, when will there be at least some isometry $T_o : F \to \overline{F}$ which extends to all of E ?

Problem 2. Describe conditions which are sufficient for a given isometry T_o : $F \to \overline{F}$ to admit an extension to all of E.

Theorems 1, 2, 10 below concern Problem 1, theorems 6, 7, 11 and Remark 4 concern Problem 2.

We shall consider two extreme situations here. On one hand we shall discuss forms which admit "many" isotropic vectors ; on the other hand we shall discuss definite forms over ordered fields. The differences as regards the answers to Problem 1 and Problem 2 are astonishingly different for the two classes of sesquilineare spaces (see e.g. Remark 6 below).

I. - Witt type theorems in the case of many isotropic vectors

I. 1. The Main Theorem

Let E be a non degenerate sesquilinear space of dimension $\aleph_o$ and L(E) the lattice of all subspaces of E. Consider sublattices $\mathcal{V}, \overline{\mathcal{V}}$ of E that are stable under the operation $\perp$ (taking the orthogonal). We are interested in situations where lattice isomorphisms $\tau : \mathcal{V} \to \overline{\mathcal{V}}$ must be induced by metric automorphisme of the sesquilinear space E. For this to be the case there are many obvious conditions ; we mention two of them

(0) $\quad (X^{\perp})^{\tau} = (X^{\tau})^{\perp} \quad , \quad X \in \mathcal{V}$

(1) $\quad \dim X/\Sigma \{Y \in \mathcal{V} | Y \subsetneq X\} = \dim X^{\tau}/\Sigma \{Y^{\tau} | Y \subsetneq X\} \quad , \quad X \in \mathcal{V}$

Notice that $\mathcal{V}$ is not assumed to be complete, so $\Sigma \{Y \in \mathcal{V} | Y \subsetneq X\}$ is a subspace of E which need not be an element of $\mathcal{V}$.

In the proof of the main theorem the elements $X \in \mathcal{V}$ with

(2) $$X \neq \Sigma \{Y \in \mathcal{V} | Y \subsetneq X\}$$

are of primary importance (they might be called "Σ-inaccessible" elements and must not be confused with the join-inaccessible elements of [2]). We shall impose the following condition on the lattices $\mathcal{V}$.

(3) For all $X \in \mathcal{V}$ satisfying (2) the principal filter generated by X in $\mathcal{V}$ is prime.

Examples where (3) always holds are provided by the distributive lattices : for, every $X \in \mathcal{V}$ with (2) is a join-irreducible element of $\mathcal{V}$ and in a distributive lattice a principal filter is prime if and only if the generator is join-irreducible.

A condition on τ which is quite obvious is that τ preserve indices, i.e. dimensions of quotients of neighbouring elements in $\mathcal{V}$.

The easiest spaces to work with when discussing Witt-type theorems are the alternate spaces (since E is non degenerate the involution must be the identity and thus k commutative). For non alternate spaces the following imposes a restriction (cf. [8], p. 159) :

(4) The set $\|E\| = \{\phi(x,x) | x \in E\}$ is an additive subgroup of k. If W is any degenerate infinite dimensional subspace of D, $D \in \mathcal{V}$, then for every $\alpha \in \|E\|$ there is a $w \in W$ with $\Phi(w,w) = \alpha$.

For tracevalued forms Φ condition (4) may be formulated more convenienly by postulating that every W contain a totally isotropic subspace of infinite dimension. However, it turns out that in the tracevalued case this condition would be unnecessarily severe ; it will be sufficient to require

(4_{tr}) If $\dim D/D^{\perp} \cap D = \aleph_0$, $D \not\subset D^{\perp}$, then there exists an infinite dimensional totally isotropic subspace $Y \subset D$ with $(D^{\perp} \cap D) \cap Y = (0)$ or else $D = D^{\perp\perp}$ and the principal ideal generated by D in the lattice $\mathcal{V}$ is $\{D, D^{\perp}, (0)\}$.

For the sake of easier formulation of our results we put down one more condition (cf. remark 1).

(5) If the form is not alternate then $\dim D/D^{\perp} \cap D \in \{0, \aleph_0\}$ for all $D \in \mathcal{V}$.

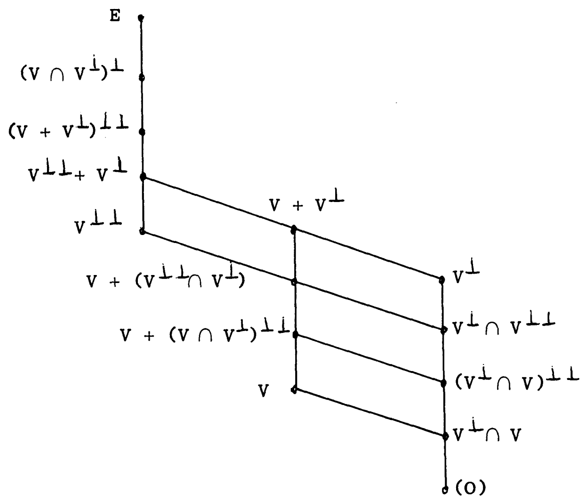
Theorem 1 ("Main theorem"). Let E be a non degenerate sesquilinear space of dimension $\aleph_0$. Let $\mathcal{V}$ and $\overline{\mathcal{V}}$ be lattices of subspaces of E which contain the spaces (0), E and with every element X the space $X^{\perp}$ as well. Assume (4) or -if the form is tracevalued- (4_{tr}). Let $\tau : \mathcal{V} \to \overline{\mathcal{V}}$ be a lattice isomorphism which respects indices and satisfies (0) and (1). In order that τ be induced by a metric automorphism of E it is sufficient that $\mathcal{V}$ satisfies (3), (5) and the descending chain condition.

Remark 1. In applications of the main theorem condition (5) can often be bypassed by chopping off finite dimensional orthogonal summands. A good example is theorem 2 below.

Remark 2. There are many involutorial divisionrings k such that every form on an $\aleph_0$-dimensional k-space will automatically satisfy condition (4). We shall list a few examples here. For the sake of illustration we shall stick to symmetric forms (see [6] for further examples) ; the (commutative) fields which we shall mention all share the following condition (mentioned in Theorem 3 of [10]).

(6) There exists $m \in \mathbb{N}$ depending on k solely such that every symmetric form in $n > m$ variables possesses a non banal zero.

Example 1. All fields k with "char $k \neq 2$ & k non formally real & k^*/k^{*2} (the multiplicative group of k modulo square factors) finite". Here m is the order of the group k^*/k^{*2} (necessarily a power of 2). Fields of power series provide examples for $|k^*/k^{*2}|$ any power of 2. Further examples are the Hilbertfields of [5].



The lattice $\mathcal{V}(V)$ of Chap. I.2. generated by a subspace V of the non degenerate sesquilinear space E.

Example 2. The functionfields k in r variables over a finite constant field. Here $m = 2^{r+1}$ (for $r = 1$ this is a classical result of Hasse theory, for arbitrary r it is a result of [12]).

I. 2. Witt's Theorem

As an application to Theorem 1 we let $\mathcal{V} = \mathcal{V}(V)$ be the lattice generated by one single subspace V of the sesquilinear space E under the operations $+$, $\cap$, $\perp$. $\mathcal{V}(V)$ contains 14 elements [10] ; in fact, it is the union of two chains,

$$\begin{aligned}(7)\qquad \mathcal{V}(V) = \{(0) &\subset V \cap V^{\perp} \subset (V \cap V^{\perp})^{\perp\perp} \subset (V^{\perp} \cap V^{\perp\perp}) \subset V^{\perp} \subset V + V^{\perp} \subset V^{\perp\perp} + \\ &+ V^{\perp} \subset (V + V^{\perp})^{\perp\perp} \subset (V \cap V^{\perp})^{\perp} \subset E\} \cup \{V \subset V + (V \cap V^{\perp})^{\perp\perp} \subset V + \\ &+ (V^{\perp} \cap V^{\perp\perp}) \subset V^{\perp\perp}\}\end{aligned}$$

so that $\mathcal{V}(F)$ must be distributive. From the main theorem one can deduce the following result concerning Problem 1 of the introduction (an earlier more direct although less perspicuous proof of Theorem 2 is contained in [8]) :

Theorem 2. Let E be a non degenerate sesquilinear space of dimension $\aleph_0$, V and $\overline{V}$ isometric subspaces of E satisfying

0) $V^{\perp} \cong \overline{V}^{\perp}$ (isometrically)
1) $\dim(V \cap V^{\perp})^{\perp\perp}/V \cap V^{\perp} = \dim(\overline{V} \cap \overline{V}^{\perp})^{\perp\perp}/\overline{V} \cap \overline{V}^{\perp}$
2) $\dim(V^{\perp} \cap V^{\perp\perp})/(V \cap V^{\perp})^{\perp\perp} = \dim(\overline{V}^{\perp} \cap \overline{V}^{\perp\perp})/(\overline{V} \cap \overline{V}^{\perp})^{\perp\perp}$
3) $\dim(V^{\perp} + V^{\perp\perp})/(V^{\perp} + V) = \dim(\overline{V}^{\perp} + \overline{V}^{\perp\perp})/(\overline{V}^{\perp} + \overline{V})$
4) $\dim(V + V^{\perp})^{\perp\perp}/(V^{\perp} + V^{\perp\perp}) = \dim(\overline{V} + \overline{V}^{\perp})^{\perp\perp}/(\overline{V}^{\perp} + \overline{V}^{\perp\perp})$

In order that there exist a metric automorphism T of E with $TV = \overline{V}$ the following conditions are sufficient

5) if $\dim V/V \cap V^{\perp} = \aleph_0$ then condition (4) is satisfied with $D = V$ or -if the form is tracevalued- (4_{tr}) holds for $D = V$ or $V = \Sigma\{Z \in \mathcal{V}(V) \mid Z \subsetneq V\}$
6) if $\dim V^{\perp}/V^{\perp} \cap V^{\perp\perp} = \aleph_0$ then condition (4) is satisfied with $D = V$ or -if the form is tracevalued- (4_{tr}) holds for $D = V$. or $V = \Sigma\{Z \in \mathcal{V}(V) \mid Z \subsetneq V\}$

Remark 3. Conditions 0) though 4) are obviously necessary for an automorphism of the required sort to exist. They are not, in general, sufficient. See § 3.7 in [8].

Corollary 1. Let V be a subspace of the nondegenerate alternate space E, $\dim E = \aleph_0$. The finitely many cardinal numbers defined by the lattice $\mathcal{V}(V)$ (dimensions of quotients of neighbouring elements) are a complete set of orthogonal invariants for the subspace V.

This proves (cf. 8 , p. 162) an old conjecture of Kaplansky ([10], p. 11). The corresponding statement is false when $\dim V > \aleph_0$, counter examples may be found in ([7], p. 132) (cf. question 3 in [10]).

Corollary 2 [10]. Let E be a non degenerate tracevalued sesquilinear space of dimension $\aleph_0$ and R a totally isotropic subspace. If $R = R^{\perp\perp}$ then there exists a totally isotropic subspace $R' \subset E$ such that $R + R'$ is an orthogonal summand of E ("Witt decomposition"); $R + R'$ is then a sum of hyperbolic planes with R, R' spanned by the two halves of a symplectic basis.

The two corollaries are typical for a host of applications which can be made of Theorem 2. We shall proceed with further applications of the Main Theorem.

I. 3. Orthogonal and Symplectic Separation

Notation. If in a direct sum $F_1 \oplus F_2$ of sesquilinear spaces both summands F_1, F_2 are totally isotropic we shall write "$F_1 \overset{0}{\oplus} F_2$".

Definition. Let F_1, F_2 be subspaces of the nondegenerate sesquilinear space E. The pair F_1, F_2 is said to be orthogonally [resp. symplectically] separated in E if and only if there exists a decomposition $E = E_1 \overset{\perp}{\oplus} E_2$ [resp. $E = E_1 \overset{0}{\oplus} E_2$] with $F_i \subset E_i$ $(i = 1,2)$.

Notice that F_1, F_2 are separated if and only if $F_1^{\perp\perp}, F_2^{\perp\perp}$ are separated (in either sense) ; we shall therefore assume without loss of generality that

$$F_1 = F_1^{\perp\perp} \quad , \quad F_2 = F_2^{\perp\perp} \quad .$$

In order that F_1, F_2 be separated in either sense it is evidently necessary that $F_1 \cap F_2 = (0)$ ("disjoint pair") and

(8) $\qquad (F_1 + F_2)^{\perp\perp} = F_1 + F_2$

(9) $\qquad F_1^{\perp} + F_2^{\perp} = E (= (F_1 \cap F_2)^{\perp})$

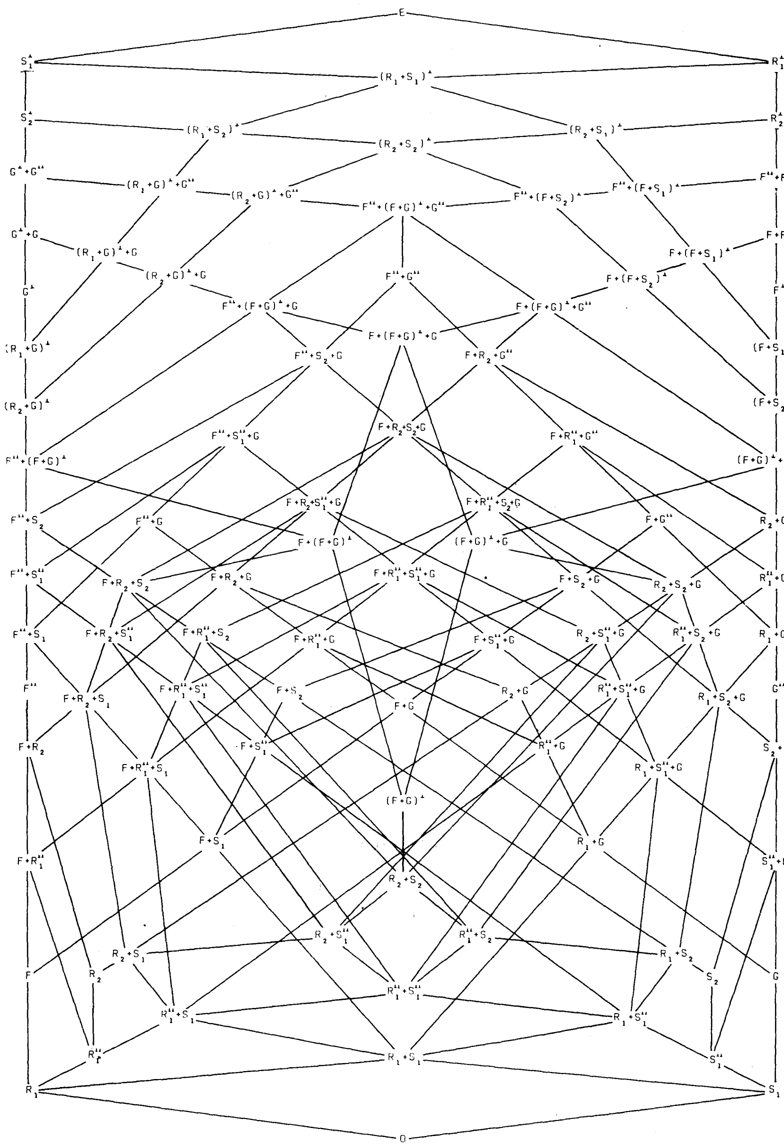
(8) and (9) may conveniently be interpreted in the lattice $L_{\perp\perp}(E)$ of all $\perp$-closed subspace X of $E(X^{\perp\perp} = X)$. We first remark that $L_{\perp\perp}(E)$ happens to be a sublattice of the lattice $L(E)$ of all subspaces of E if and only if trivially so by the following.

Theorem 3. Let E be any nondegenerate sesquilinear space. $L_{\perp\perp}(E)$ is modular if and only if $\dim E$ is finite.

For char $k \neq 2$ and hermitean forms this was proved in [11] ; using the same technique a proof can also be given in the general case. This theorem considerably generalizes a fact well known in the case of Hilbertspace (cf. Thm (32.17) in [13]). Now (8) and (9) say that F_1, F_2 is a disjoint modular and dual modular pair in the lattice $L_{\perp\perp}(E)$ (cf. Thm (33.4) in [13]).

Our result is that under certain general conditions such pairs F_1, F_2 must always be separated. In order to obtain this result via Theorem 1 we need

Theorem 4 [4]. Let E be a nondegenerate sesquilinear space of dimension $\aleph_0$ and F_1, F_2 an orthogonal [resp. totally isotropic] modular and dual modular pair



The lattice $\mathcal{V}(F,G)$ of Theorem 4. $\mathcal{V}$ is generated by an orthogonal pair $F;G$ of subspaces of the sesquilinear space E (E non degenerate and of countably infinite dimension) with $(F+G)^{\perp\perp} = F^{\perp\perp} + G^{\perp\perp}$ and $F^{\perp} + G^{\perp} = E$.

(Abbrev. $R_1 = F^{\perp} \cap F$, $R_2 = F^{\perp\perp} \cap F^{\perp}$, $S_1 = G^{\perp} \cap G$, $S_2 = G^{\perp\perp} \cap G^{\perp}$)

in $L_{\perp\perp}(E)$. Let $\mathcal{V}(F_1,F_2)$ be the smallest sublattice of $L(E)$ (the lattice of all subspaces of E) which is stable under the operation $\perp$ and which contains (0), E, F_1,F_2. $\mathcal{V}(F_1,F_2)$ is finite and distributive, in fact, it has 100 elements (in general) and is generated by two chains.

For F_1 and F_2 arbitrary subspaces of E the lattice $\mathcal{V}(F_1,F_2)$ will not, in general, be finite [6].

Theorem 5. Let E be a nondegenerate alternate space of dimension $\aleph_0$ and F_1,F_2 $\perp$-closed subspaces with $F_1 \perp F_2$ [resp. $F_1 \perp F_1$, $F_2 \perp F_2$]. If (8) and (9) are satisfied then F_1,F_2 is orthogonally [resp. symplectically] separated.

From Theorem 5 we obtain an answer to question 4 in [10], namely

Corollary 1. Let E be as in the theorem. If F_1,F_2 is a disjoint modular and dual modular pair and $F_1 \perp F_2$ [resp. $F_1 \perp F_1$, $F_2 \perp F_2$] then the finitely many cardinal numbers defined by the lattice $\mathcal{V}(F_1,F_2)$ are a complete set of orthogonal invariants for the pair F_1,F_2.

One may also formulate theorem 5 for non alternate spaces ; one then has to put down some conditions in the vein of (4) or (4_{tr}). Direct proofs for these situations as well as for theorem 5 are given in [4].

Remark 4. Theorem 5 can be used to solve Problem 2 of the introduction for algebraic isometries $T_o : F \to F \subset E$ whose polynomials split into different linear factors. See chap. II in [4].

I. 4. Extending Isometries

We give here some results concerning Problem 2 of the Introduction. We shall make use of the weak linear topology $\sigma(E)$ of a sesquilinear space E ; $\sigma(E)$ has $\{X^{\perp} | X$ linear subspace of E & $\dim X < \infty\}$ as a 0-neighbourhoodbasis. A linear subspace Y of E is $\sigma(E)$-closed if and only if it is $\perp$-closed ($Y^{\perp\perp} = Y$).

Theorem 6 ([1], p. 8). An isometry $T_o : F \to \overline{F}$ between $\perp$-closed subspaces $F,\overline{F}$ of the nondegenerate $\aleph_o$-dimensional alternate space E can be extended to all of E if and only if the following two conditions hold.

(10) T_o is homeomorphic with respect to $\sigma(E)$

(11) $\dim F^{\perp}/F^{\perp} \cap F = \dim \overline{F}^{\perp}/\overline{F}^{\perp} \cap \overline{F}$

Theorem 7 ([1], p. 16). Let E be as in Thm 6 and $T_o : F \to \overline{F}$ an isometry between $\perp$-dense subspaces (i.e. $F^{\perp} = \overline{F}^{\perp} = (0)$). T_o can be extended to an isometry of all of E if and only if

(12) $U^{\perp}$ and $(T_oU)^{\perp}$ are isometric for all $U \subset F$ with $\dim F/U \leqslant 2$.

Remark 5. Theorem 6 can be generalized to non alternate forms Φ. The following conditions, together with (10) and (11), prove sufficient for an extension of T_o to exist

(13) If $\dim F^{\perp}/F^{\perp}\cap F < \infty$ then $F^{\perp}$ and $\overline{F}^{\perp}$ are isometric and Φ is tracevalued.

(14) If $\dim F^{\perp}/F^{\perp}\cap F = \infty$ then $||E||$ is an additive subgroup of k. If $\alpha \in ||E||$, H a finite dimensional subspace of E and W a subspace of $F^{\perp}$ with $\dim F^{\perp}/W + (F^{\perp}\cap F) < \infty$ there exists $x \in W \cap H^{\perp}$ with $\Phi(x,x) = \alpha$ (if Φ is tracevalued this condition is equivalent to the existence of an infinite dimensional totally isotropic subspace $G \subset F^{\perp}$ with $G \cap (F^{\perp}\cap F) = (0)$).

II. - Witt type theorems in the case of definite forms

II.1. Definite Forms

Let $(k,*)$ be an involutorial divisionring and $(k_o,<)$ an ordered subdivisionring. If (E,Φ) is a hermitean space over $(k,*)$ such that $||E|| \subset k_o$ then we say that Φ is definite on the line $k(x_o) \subset E$ $(0 \neq x_o \in E)$ if and only if $\Phi(x_o,x_o).\Phi(\lambda x_o,\lambda x_o) > 0$ for all $0 \neq \lambda \in k$. We say that ϕ is positive definite if $\phi(x,x) > 0$ for all nonzero $x \in E$.

There exist non commutative involutorial divisionrings $(k,*)$, $* \neq 1$, which are ordered and which have the property "$\Sigma \lambda_i\lambda_i^* = 0 \Rightarrow \lambda_i = 0$" ([6]). Clearly, such fields admit anisotropic hermitean forms, however, they do not admit definite forms by the following

Theorem 8 ([6]). Let $(k,*)$ and k_o be as above and (E,ϕ) a non degenerate hermitean space over k with $||E|| \subset k_o$. If $* \neq 1$ then the following are equivalent. (i) ϕ is definite on all lines of E. (ii) ϕ is anisotropic and definite on at least one line of E. (iii) ϕ is definite and either $(k,*)$ is a quaternion algebra $(\frac{\alpha,\beta}{k_o})$ with $\alpha,\beta < 0$ and $*$ the usual "conjugation" or else $(k,*)$ is commutative and a quadratic extension of k_o, $k = k_o(\sqrt{\gamma})$ for some $\gamma < 0$ and $(\alpha + \beta\sqrt{\gamma})^* = \alpha - \beta\sqrt{\gamma}$ for all $\alpha,\beta \in k_o$. If on the other hand $* = 1$ then k is commutative and $k = (k_o,<)$ is ordered.

In [6] we have treated Problem 1 of the introduction for arbitrary subspaces of definite spaces as defined here. As we can give but an illustration in this short survey we shall make the following simplifications here :

1. k_o is archimedean ordered, hence $k_o \subseteq \mathbb{R}$ without loss of generality ;
2. $k_o = k$ and the form is symmetric.

Finally we put down the following condition (cf. Theorem 4 in [10]).

(15) There exists $m \in \mathbb{N}$ depending on k solely such that every positive symmetric form in $n \geqslant m$ variables represents 1 or -1 (or both).

Examples. Algebraic numberfields ($m = 4$).
A consequence of (14) is that every $\aleph_0$-dimensional positive definite space admits an orthonormal basis.

II. 2. Invariants for $\perp$-dense subspaces

The general case is treated in [6] ; here we shall merely consider subspaces $V,\overline{V}$ of a positive definite space (E,Φ) which satisfy $V^{\perp} = \overline{V}^{\perp} = (0)$. By a standardbasis for the embedding $V \subset E$ we mean a basis $\mathcal{B} = (v_i)_{i\in\mathbb{N}} \cup (f_\iota)_{\iota\in J\subset\mathbb{N}}$ such that $(v_i)_{\mathbb{N}}$ is an orthonormal basis for V and $(f_\iota)_J$ an orthonormal basis for some supplement of V in E. With respect to a fixed basis $\mathcal{B}$ we set

(16) $\quad \alpha_{\iota i} = \Phi(f_\iota, v_i) \in k \qquad (\iota \in J \; ; \; i \in \mathbb{N})$

(17) $\quad A_{\iota\chi n} = \sum_{i=1}^{n} \alpha_{\iota i}\alpha_{\chi i} \in k \qquad (\iota, \chi \in J \; ; \; n \in \mathbb{N})$

(18) $\quad A_{\iota\chi} = \lim_{n\to\infty} A_{\iota\chi n} \in \mathbb{R} \qquad (\iota, \chi \in J)$

One proves that the real matrix $A = (A_{\iota\chi})_{J\times J}$ is positive definite and $A - \mathbb{1}$ is negative semidefinite. We call A the matrix associated with $\mathcal{B}$. A may be interpreted as a point with coordinates $A_{\iota\chi}$ in a real space of dimension $\frac{1}{2}n(n+1)$ where $n = \text{card } J = \dim E/V \leqslant \aleph_0$. Thus to every standardbasis there corresponds a point of the convexe region $\mathcal{R}$ which is the intersection of the two cones

(19)
$$C_1 \; : \; (A_{\iota\chi}) \quad \text{positive definite}$$
$$C_2 \; : \; (A_{\iota\chi}) - \mathbb{1} \quad \text{negative semidefinite.}$$

Conversely one proves the following

Theorem 9 ([14], [6]). Let A be any positive $J\times J$ matrix over $\mathbb{R}$, card $J \leqslant \aleph_0$ such that $A - \mathbb{1}$ is negative semidefinite. There exists a positive definite symmetric space (E,ϕ) over k which contains a standardbasis $\mathcal{B} = (v_i)_{i\in\mathbb{N}} \cup (f_\iota)_{\iota\in J\subset\mathbb{N}}$ with $\perp$-dense span of the v_i and with A the associated matrix.

Thus, conversely, to every point of the convex region $\mathcal{R} = C_1 \cap C_2$ there corresponds a dense embedding $V \subset E$ (E spanned by an orthonormal basis). The orthogonal invariants which we set up for the $\perp$-dense $V \subset E$ will enable us to replace the study of orbits in the set of $\perp$-dense $V \subset E$ under the orthogonal group of E by the study of orbits of points in the region $\mathcal{R}$ under some more accessible group.

"Quantities" appropriate for the description of a $\perp$-dense embedding $V \subset E$ are not the matrices A associated with standard bases but rather the matrices $A - \mathbb{1}$. Here is how they transform :

Let $\mathcal{B} = (v_i)_{i\in\mathbb{N}} \cup (f_\iota)_{\iota\in J}$, $\overline{\mathcal{B}} = (\overline{v}_i)_{i\in\mathbb{N}} \cup (\overline{f}_\iota)_{\iota\in J}$ be two standard bases for the embedding $V \subset E$, $V^\perp = (0)$ (over the field $k \subseteq \mathbb{R}$). We have

$$(20) \qquad \overline{f}_\iota = \sum_{\chi=1}^{\chi(\iota)} \gamma_{\iota\chi} f_\chi + \sum_{i=1}^{i(\iota)} \xi_{\iota i} v_i$$

for certain row finite matrices $(\gamma_{\iota\chi}), (\xi_{\iota i})$ over k ; $(\gamma_{\iota\chi})$ is invertible. Let furthermore $(A_{\iota\chi}), (\overline{A}_{\iota\chi})$ be the matrices over $\mathbb{R}$ associated with $\mathcal{B}$ and $\overline{\mathcal{B}}$ respectively. Then

$$(21) \qquad \overline{A}_{\iota\chi} - \delta_{\iota\chi} = \sum_{\upsilon,\mu=1}^{\upsilon(\iota),\mu(\chi)} \gamma_{\iota\upsilon}(A_{\upsilon\mu} - \delta_{\upsilon\mu})\gamma_{\chi\mu} \qquad (\gamma_{\iota\upsilon} \in k)$$

where $(\delta_{\iota\chi})$ is the unit matrix.

Our principal result in the present case is

Theorem 10 ([14], [6]). Let $V, \overline{V} \subset E$ be $\perp$-dense subspaces. There is a metric automorphisme T of E with $TV = \overline{V}$ if and only if there are standardbases for $V \subset E$, $\overline{V} \subset E$ such that (21) holds, i.e. if and only if the real matrices $\overline{A} - \mathbb{1}$, $A - \mathbb{1}$ are equivalent over the subfield k.

Corollary 1. Let E be the usual inner product space over $\mathbb{R}$ of dimension $\aleph_0$. For every $n \leqslant \aleph_0$ there are precisely $n+1$ orbits (under the orthogonal group of E) of $\perp$-dense subspaces $V \subset E$ with $\dim E/V = n$. The nullity of the semidefinite $n\times n$ matrix $A - \mathbb{1}$ and n are the only invariants.

Corollary 2. Let E be as in Corollary 1. Every $\perp$-dense embedding $V \subset E$ splits, i.e. E is an orthogonal sum of $\dim E/V$ copies of E with each copy containing a $\perp$-dense hyperplane V_i such that $V = \Sigma V_i$.

Corollary 3. If $k \subsetneq \mathbb{R}$ then the embedding $V \subset E$ splits if and only if the real matrix A associated with any standardbasis for V in E can be diagonalized over the subfield k.

Corollary 4. If $k \subsetneq \mathbb{R}$ (e.g. k the real closure of $\mathbb{Q}$) then there are $2^{\aleph_0}$ orbits of $\perp$-dense subspaces $V \subset E$ with $\dim E/V = n$; among them there are $2^{\aleph_0}$ orbits whose representatives do not split.

$\mathbb{R} \otimes_k E$ is a normed vector space under the norm $\sqrt{\phi(x,x)}$ for $x \in \mathbb{R} \otimes_k E$. We endow E with the induced topology and let $\hat{V}$ be the closure of the subspaces $V \subset E$. Dim $\hat{V}/V$ is an obvious orthogonal invariant of the subspace V. If $V^\perp = (0)$ and $m \leqslant \aleph_0$ is the nullity of a matrix A associated with the embedding $V \subset E$ then one proves that $m = \dim \hat{V}/V$. In particular we have

Corollary 5. Let E be the usual inner product space over $\mathbb{R}$ of dimension $\aleph_0$. If V is a $\perp$-dense subspace of E then the pair $\{\dim E/\hat{V}, \dim \hat{V}/V\}$ is a complete set of orthogonal invariants for V (here $\hat{V}$ is the closure of V in the "natural" topology of E).

Remark 6. If E is the symmetric space spanned by an orthonormal basis over $k = \mathbb{C}$, then in contrast to corollary 1 there is only 1 orbit of $\perp$-dense subspaces $V \subset E$ for each $n = \dim E/V \leqslant \aleph_0$. (This is an immediate consequence of Theorem 2.)

II. 3. Extending Isometries

Let the field $k \subseteq \mathbb{R}$ be as in II. 1. and, as usual, $\dim E = \aleph_0$.

Theorem 11. Let $V, \overline{V} \subset E$ be dense subspaces with respect to the natural topology in E. A given isometry $T_o : V \to \overline{V}$ admits an (isometric) extension to all of E if and only if T_o is homeomorphic with respect to the weak linear topologies $\sigma(E)|_V$, $\sigma(E)|_{\overline{V}}$.

Even when $k = \mathbb{R}$ there does not seem to exist an "obvious" proof for Theorem 11 (suggested, say, by Hilbert space arguments). A proof for Theorem 11 is contained in [14], [6] ; it proceeds by a recursive construction of the required extension. This also explains why a "topological" theorem of this sort carries along with it arithmetical assumptions such as (15).

III. - Bibliography

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