

MÉMOIRES DE LA S. M. F.

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Mémoires de la S. M. F., tome 59 (1979), p. 125-129

http://www.numdam.org/item?id=MSMF_1979__59__125_0

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SOME FUNCTORS RELATED TO POLYNOMIAL THEORY, II

by

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1. Introduction. We consider the following natural transformation :

$$T^m : \mathcal{P}_R^m(X, Y) \rightarrow \text{Map}(X, Y), \quad T^m(f) = f_R$$

where R denotes a commutative ring with 1, X, Y - R -modules, and $\mathcal{P}_R^m(X, Y)$ is the R -module of all forms of degree m on the pair (X, Y) (in the sense of N.Roby [2]).

An element of $\mathcal{P}_R^m(X, Y)$ is a system $f = (f_A)$ indexed by all commutative R -algebras A , where $f_A : X \otimes A \rightarrow Y \otimes A$ are mappings satisfying the following conditions :

(i) $(1 \otimes u) \circ f_A = f_B \circ (1 \otimes u)$ for any R -algebra homomorphism $u : A \rightarrow B$,

(ii) $f_A(\underline{x}a) = f_A(\underline{x})a^m$ for any R -algebra A , any $\underline{x} \in X \otimes A$ and $a \in A$.

It is proved in [1] that in the case $X = R^n, Y = R$ we obtain :

$$T^m : R[T_1, \dots, T_n]_m \rightarrow \text{Map}(R^n, R), \quad T^m(F)(x_1, \dots, x_n) = F(x_1, \dots, x_n).$$

It is well-known that the above homomorphism is not always injective ; this is the starting point and the motivation of the following considerations.

It is known from [2] that the functor $\mathcal{P}_R^m(X, -)$ is represented by the m -th divided power $\Gamma_R^m(X)$ of the module X . Similarly, it is proved in [1] that $\mathcal{P}_R^m(X, -) = \text{Ker } T^m$ is represented by $\widehat{\Gamma}_R^m(X)$ where :

$$\widehat{\Gamma}_R^m(X) = \Gamma_R^m(X) / R\{x^{(m)}; x \in X\}.$$

The above module is generated by the classes of elements :

$$\gamma_{m_1, \dots, m_k}^{(m)}(x_1, \dots, x_k) = x_1^{(m_1)} \dots x_k^{(m_k)}, \quad m_i \geq 0, \quad m_1 + \dots + m_k = m, \quad x_1, \dots, x_k \in X,$$

which are denoted by $\widetilde{\gamma}_{m_1, \dots, m_k}^{(m)}(x_1, \dots, x_k)$.

It is easy to see that $\widehat{\Gamma}_R^m$ is an endo-functor of the category $R\text{-Mod}$.

We recall the following results contained in [1] :

Lemma 1.1. $\widehat{\Gamma}_R^m$ commutes with direct limits.

Lemma 1.2. $\widehat{\Gamma}_R^m(X)$ is finitely generated if so is X .

Theorem 1.3. There exist the natural isomorphisms :

- (1) $\widehat{\Gamma}_{R_S}^m(X_S) \approx \widehat{\Gamma}_R^m(X)_S$ for any multiplicative set S in R
 (2) $\widehat{\Gamma}_{R/I}^m(X/IX) \approx \widehat{\Gamma}_R^m(X)/I \widehat{\Gamma}_R^m(X)$ for any ideal I in R
 (3) $\widehat{\Gamma}_{R \times R'}^m(X \times X') \approx \widehat{\Gamma}_R^m(X) \times \widehat{\Gamma}_{R'}^m(X')$.

Theorem 1.4. For a finitely generated R-module X, the following conditions are equivalent :

- (i) $\widehat{\Gamma}_R^m(X) = 0$
 (ii) $\widehat{\Gamma}_{R/P}^m(X/PX) = 0$ for any $P \in \text{Max}(R)$
 (iii) For any $P \in \text{Max}(R)$: either $\dim_{R/P}(X/PX) \leq 1$ or $m \leq |R/P|$.
 In particular, $\widehat{\Gamma}_R^m = 0$ iff $m \leq d(R) : \inf \{|R/P| ; P \in \text{Max}(R)\}$.

2. The structure of $\widehat{\Gamma}_R^m(X)$. We shall give some structural informations on $\widehat{\Gamma}_R^m(X)$ which generalize results contained in [1]. The first step is the following

Lemma 2.1. If $P \in \text{Spec}(R) - \text{Max}(R)$ then $\widehat{\Gamma}_R^m(X)_P = 0$ for any R-module X. Moreover, is X is finitely generated then $\text{Ann}(\widehat{\Gamma}_R^m(X)) \not\subseteq P$.

Proof : Observe that R/P is an infinite domain (it is not a field!) and hence $d(R_P) = \infty$. It follows from Theorem 1.3 and 1.4 that $\widehat{\Gamma}_R^m(X)_P = \widehat{\Gamma}_{R_P}^m(X_P) = 0$. Then the second part of the lemma follows from Lemma 1.2.

Corollary 2.2. If $\dim(R) > 0$ then :

- (1) $\widehat{\Gamma}_R^m(X)$ are torsion modules.
 (2) $\widehat{\Gamma}_R^m(X)$ is free iff it is zero.

If $\dim(R_P) > 0$ for any $P \in \text{Max}(R)$ then :

- (3) $\widehat{\Gamma}_R^m(X)$ is projective iff it is zero.

Now we explain the structure of $\widehat{\Gamma}_R^m(X)$ over Noetherian rings.

Theorem 2.3. Let R be a Noetherian ring and let X be a finitely generated R-module. Then there exists a natural R-isomorphism :

$$\widehat{\Gamma}_R^m(X) \approx \bigoplus_{P \in \text{Max}(R)} \widehat{\Gamma}_{R/P^{k_P}}^m(X/P^{k_P}X)$$

induced by $X \rightarrow X/P^{k_P}X$, for all sufficiently large k_P .

Proof : We can assume that $\text{Ann}(\widehat{\Gamma}_R^m(X)) \neq R$. Let $\text{Ann}(\widehat{\Gamma}_R^m(X)) = Q_1 \cap \dots \cap Q_s$ be a primary decomposition, and let $P_i = \text{rad}(Q_i)$. Observe that $P_i^{k_i} \subset Q_i$ for all sufficiently large k_i . Denote $I = P_1^{k_1} \dots P_s^{k_s} \subset \text{Ann}(\widehat{\Gamma}_R^m(X))$. Since $P_1, \dots, P_s \in \text{Max}(R)$ by Lemma 2.1, it follows that $R/I \approx \prod_{i=1}^s R/P_i^{k_i}$ and hence :

$$\widehat{\Gamma}_R^m(X) = \widehat{\Gamma}_R^m(X) / I \widehat{\Gamma}_R^m(X) \approx \widehat{\Gamma}_{R/I}^m(X/IX) \approx \bigoplus_{i=1}^s \widehat{\Gamma}_{R/P_i}^m(X/P_i^k X).$$

If $P \in \text{Max}(R) - \{P_1, \dots, P_s\}$ then $I + P^k = R$ for each natural k , and hence :

$$\widehat{\Gamma}_{R/P^k}^m(X/P^k X) = \widehat{\Gamma}_R^m(X)/P^k \widehat{\Gamma}_R^m(X) = 0.$$

This completes the proof.

Corollary 2.4. If R is a Noetherian ring then there exists a natural R -isomorphism

$$\widehat{\Gamma}_R^m(X) \approx \bigoplus_{P \in \text{Max}(R)} \widehat{\Gamma}_P^m(X_P)$$

induced by $X \rightarrow X_P$.

Proof: Compare the decompositions from Theorem 2.3 for X and X_P in the case if X is finitely generated. Next apply Lemma 1.1.

The same argument prove the following

Corollary 2.5. If R is a local Noetherian ring then there exists a natural R -isomorphism :

$$\widehat{\Gamma}_R^m(X) \approx \widehat{\Gamma}_{\hat{R}}^m(X \otimes \hat{R})$$

induced by $X \rightarrow X \otimes \hat{R}$.

Observe that the above two corollaries reduce the computation of $\widehat{\Gamma}_R^m(X)$ for Noetherian R to the case when R is local and complete. Theorem 2.3 reduces this problem (for finitely generated X) to the case when R is local Artinian. This case will be studied in the next section.

3. The Artinian case. Let (R, P) be an Artinian local ring. Then $P^k = 0$ for some natural k . Observe that $r^2 = 0$ for any $r \in P^{k-1}$ (if $k > 1$). This is the motivation of the following.

Proposition 3.1. If $r^2=0$ in R and $m \leq 5$ then $r \widehat{\Gamma}_R^m(X) = 0$ for any R -module X .

Proof : To start with, we give some general formulas. It follows from [1] that :

$$\sum_{m_i > 0} \widetilde{\gamma}_{m_1, \dots, m_n}^{m_i}(x_1, \dots, x_n) = 0 \text{ for any } x_1, \dots, x_n \in X.$$

Denote $\gamma_{m_1, \dots, m_n} = \widetilde{\gamma}_{m_1, \dots, m_n}(x_1, \dots, x_n)$ for $m_i > 0, \sum m_i = m$. We must prove that r annihilates all this generators. We have :

$$(1) \quad \gamma_{m_1, \dots, m_n} = 0.$$

Replacing x_1 by rx_1 and $(1+r)x_1$ we get :

$$(2) \quad r\gamma_{1, m_2, \dots, m_n} = 0$$

$$(2') \quad \gamma_{(1+rm_1), m_1, \dots, m_n} = 0$$

since $r^2 = 0$ and $(1+r)^k = 1+kr$. In view of (1) and (2) we get from (2') :

$$(3) \quad r \sum_{k=3}^{m-n+1} (k-2) \Sigma/k, m_2, \dots, m_n / = 0.$$

In particular, it follows that :

- (a) $/1, \dots, 1/ = 0$ by (1) ($n=m$)
- (b) $r/2, 1, \dots, 1/ = 0$ by (1) and (2) ($n=m-1$)
- (c) $r/3, 1, \dots, 1/ = 0$ by (3) ($n=m-2$)
- (d) $r/1, m-1/ = 0$ by (2) ($n=2$).

For $m \leq 2$ there is nothing to prove. For $m=3$ we utilize (a), (b). For $m=4$ we get $r/3, 1/ = r/1, 3/ = r/2, 1, 1/ = r/1, 2, 1/ = r/1, 1, 2/ = r/1, 1, 1, 1/ = 0$. Hence also $r/2, 2/ = 0$ by (1). For $m=5$ we have $r/1, 4/ = r/3, 1, 1/ = r/2, 1, 1, 1/ = r/1, 1, 1, 1, 1/ = 0$ and analogously for any permutation. Then (2) and (3) get us $r/1, 2, 2/ = r/3, 2/ = 0$. This completes the proof.

Remark 3.2. Using the same formulas (when we also replace x_2 by $-x_2$) we can prove the above proposition for $m \leq 7$ with the assumption that 2 is invertible in R .

Corollary 3.3. Let R be a Noetherian ring and $m \leq 5$ (or $m \leq 7$ and 2 is invertible in R). Then there exists a natural R -isomorphism :

$$\widehat{\Gamma}_R^m(X) \approx \bigoplus_{P \in \text{Max}(R)} \widehat{\Gamma}_{R/P}^m(X/PX)$$

induced by $X \rightarrow X/PX$.

Proof : It can be assumed that X is finitely generated. In view of Theorem 2.3, it suffices to prove that $\widehat{\Gamma}_R^m(X) \approx \widehat{\Gamma}_{R/P}^m(X/PX)$ for any Artinian local (R, P) . If $P^k = 0$, $P^{k-1} \neq 0$ and $k > 1$ (i.e. R is not a field) then :

$$\widehat{\Gamma}_R^m(X) = \widehat{\Gamma}_R^m(X) / P^{k-1} \widehat{\Gamma}_R^m(X) \approx \widehat{\Gamma}_{R/P^{k-1}}^m(X/P^{k-1}X)$$

by Proposition 3.1 and Remark 3.2. Induction on k completes the proof.

Remark 3.4. The assumptions of the above corollary are necessary. In fact, it can be computed that :

$$\widehat{\Gamma}_{Z_4}^6(Z_4^2) = Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_4, \quad \widehat{\Gamma}_{Z_9}^6(Z_9^2) = Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_3 \oplus Z_9.$$

Remark 3.5. Since the dimensions of $\widehat{\Gamma}^m(X)$ over fields are known (see [1]), Corollary 3.3 solves the problem of computation of $\widehat{\Gamma}^m(X)$ over Noetherian rings for small m . For example, it can be proved that :

$$\begin{aligned} \widehat{\Gamma}_Z^3(Z^n) &= \binom{n}{2} Z_2 \\ \widehat{\Gamma}_Z^4(Z^n) &= 2 \binom{n+1}{3} Z_2 \oplus \binom{n}{2} Z_3 \end{aligned}$$

$$\mathbb{F}_Z^5(Z^n) = (3 \binom{n}{2} + 5 \binom{n}{3} + 3 \binom{n}{4})Z_2 \oplus 2 \binom{n+1}{3}Z_3$$

where $\binom{n}{k} = 0$ for $n < k$. However, the problem is open for large m .

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