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# *ON SOME STRUCTURAL DESIGN PROBLEMS*

*C. JOURON*

Analyse Numérique et Fonctionnelle
C.N.R.S. et Université de Paris-Sud Département de Mathématiques
Bâtiment 425, 91405 - Orsay

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## 1. INTRODUCTION

Some structural design problems may be stated as

$$(\mathcal{P}) \qquad \inf_{\substack{u \in C \\ \lambda(u) = \lambda_1}} J(u)$$

where $u$ is the thickness of the structure, $J(u)$ its weight, $\lambda(u)$ its fundamental frequency of vibration, $\lambda_1$ the fundamental frequency of the structure with uniform thickness and $C$ is the convex of constraints on $u$.

Here we consider the problems $(\mathcal{P})$ when $J$ and $\lambda$ have the following properties

$$J \text{ is convex on } C$$
$$\lambda \text{ is pseudoconcave on } C .$$

We shall state necessary and sufficient conditions of optimality for an abstract problem $(\mathcal{P})$. And we shall apply these results in structural design. We refer to [4] [5] for the proof of these results and for more details.

## 2. PROBLEM $(\mathcal{P})$ : ABSTRACT CASE.

Let $E$ a locally convex Hausdorff space, $C$ a convex of $E$ , $J$ and $\lambda$

two real valued functions defined on $C$. We consider the two following problems

$$(\mathcal{P}) \qquad \operatorname*{Inf}_{\substack{u \in C \\ \lambda(u) = \lambda_1}} J(u)$$

$$(\mathcal{Q}) \qquad \operatorname*{Inf}_{\substack{u \in C \\ \lambda(u) \geqslant \lambda_1}} J(u)$$

where $\lambda_1 = \lambda(u_1)$ $\quad u_1 \in C$.

Proposition 2.1. Assume

(2.1) $\quad J$ pseudoconvex on $C$

(2.2) $\quad \lambda$ continuous on $C$

(2.3) $\quad \exists\, u_\alpha \in C$ such that $\lambda(u_\alpha) < \lambda_1$ and

$$J(u_\alpha) < J(u) \qquad \forall\, u \in C\ ,\ u \neq u_\alpha$$

then the problems $(\mathcal{P})$ and $(\mathcal{Q})$ are equivalent (i.e. if $u$ is a solution of $(\mathcal{P})$ $u$ is a solution of $(\mathcal{Q})$ and reciprocally).

Let's recall the definition of a pseudoconvex function. $J$ is a pseudoconvex on $C$ if $J$ is Gateaux-differentiable on $C$ and if

$$\forall\ (u,v) \in C \times C \quad , \quad J'(u).(v-u) \geqslant 0 \implies J(v) \geqslant J(u) .$$

The reader can find the properties of pseudoconvex and quasiconvex functions in MANGASARIAN [6].

We make use of these properties for the proof of Proposition 2.1.

Remark. $(\mathcal{Q})$ is a convex problem.

Proposition 2.2. Assume (2.1),

(2.4) $\quad \lambda$ pseudoconcave on $C$ $\quad$ and

(2.5) $\exists \; u_\gamma \in C$ <u>such that</u> $\lambda(u_\gamma) > \lambda_1$ .

<u>Then if</u> $\bar{u} \in C$ <u>and if</u> $\lambda(\bar{u}) \geqslant \lambda_1$ , $\bar{u}$ <u>is a solution of</u> $(Q)$ <u>if and only if</u>

$\exists \; \bar{\eta} \leqslant 0$ <u>such that</u>

(2.6) $$J'(\bar{u}).(v-\bar{u}) + \bar{\eta}\, \lambda'(\bar{u}).(v-\bar{u}) \geqslant 0 \qquad \forall v \in C$$

(2.7) $$\bar{\eta}(\lambda(\bar{u})-\lambda_1) = 0 \quad .$$

The proof of the necessary condition is based on the results of HALKIN [2] and (2.4)-(2.5) assumptions.

The proof of the sufficiency is easy.

<u>Corollary 2.1</u>. <u>Assume</u> (2.1)-(2.3)-(2.4)-(2.5). <u>Let</u> $\bar{u} \in C$ , $\lambda(\bar{u}) = \lambda_1$ <u>then</u> $\bar{u}$ <u>is a solution of</u> $(\mathcal{P})$ <u>if and only if</u>

$$\exists \; \bar{\eta} < 0 \text{ such that } J'(\bar{u}).(v-\bar{u}) + \bar{\eta}\, \lambda'(\bar{u}).(v-\bar{u}) \geqslant 0 \qquad \forall \, v \in C \; .$$

## 3. <u>THE PROBLEM</u> $(\mathcal{P})$ <u>IN STRUCTURAL DESIGN</u>.

Let $\Omega$ be an open bounded connected set in $\mathbb{R}^n$ let $\mathbf{a}$ and $\mathbf{b}$ two functions

$$\mathbf{a} : \; ]0,+\infty[ \; \longrightarrow \; ]0,+\infty[ \qquad \mathcal{C}^2 \text{ , concave}$$

$$\mathbf{b} : \; ]0,+\infty[ \; \longrightarrow \; ]0,+\infty[ \qquad \mathcal{C}^2 \text{ , convex} .$$

Denote by $(.,.)$ the scalar product of $L^2(\Omega)$

$<.,.>$ the bilinear canonical pairing over $H_0^1(\Omega) \times H^{-1}(\Omega)$ .

$U = \{u \in L^\infty(\Omega) \; ; \exists \alpha(u) > 0$ such that $u(x) \geq \alpha(u)$ a.e. in $\Omega\}$

(3.1) $C = \{u \in L^\infty(\Omega) \; ; \; 0 < \alpha \leq u(x) \leqslant \beta$ a.e. in $\Omega\}$ $\alpha$, $\beta$ given with $0 < \alpha < 1 < \beta < +\infty$

$a$ and $b$ the maps from $U$ to $U$ defined by

$$a(u)(x) = \mathbf{a}(u(x)) \; , \; b(u)(x) = \mathbf{b}(u(x)).$$

$A_u$ the isomorphism from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ defined by

$$< A_u\, w, \phi > = \int_\Omega a(u)(x)\, \nabla w(x) \,.\, \nabla\phi\,(x)\, dx \;, \quad (w,\phi) \in H_0^1(\Omega)^2$$

$B_u$ the operator of $L^2(\Omega)$ defined by

$$B_u\, w = b(u)\, w \,.$$

3.1. <u>The spectral problem.</u>

It is known that the first eigenvalue $\lambda(u)$ of

$$A_u\, w = \lambda\, B_u\, w$$

is simple and that the associated eigenvector $w(u)$ satisfies

$$w(u;x) \geqslant 0 \qquad \text{a.e. in } \Omega$$

$$\lambda(u) = \inf_{\substack{w \in H_0^1(\Omega) \\ w \neq 0}} \frac{< A_u\, w,w >}{(B_u\, w,w)} = \frac{< A_u\, w(u),w(u) >}{(B_u\, w(u),w(u))}$$

We suppose that $w(u)$ is normalized in $L^2(\Omega)$ .

Using the Implicit Function Theorem we can prove that the function

$$u \in U \longrightarrow \lambda(u)$$

is differentiable on $U$ and that

$$\text{(3.2)} \quad \lambda'(u).v = \frac{\int_\Omega \{(a'(u).v)(x)\, |\nabla w(u;x)|^2 - \lambda(u)(b'(u).v)(x)\, w^2(u;x)\}\, dx}{\int_\Omega b(u)\, w^2(u;x)\, dx} \,.$$

From the concavity of $a$ and $(-b)$ , it follows that :

$$\text{(3.3)} \quad \lambda(v) \leqslant \lambda(u) + \lambda'(u).(v-u)\, \frac{\int_\Omega b(u)(x)\, w^2(u;x)\, dx}{\int_\Omega b(v)(x)\, w^2(u;x)\, dx} \qquad \forall\, (u,v) \in U \times U \,.$$

Using this inequality we get

<u>Theorem 3.1.</u> $\lambda$ <u>is pseudo concave on</u> $U$ . $\lambda$ <u>is upper semicontinuous on</u> $C$ <u>for the weak* topology of</u> $L^\infty(\Omega)$ .

3.2. Statement of the problem and results.

$\Omega$ is the form of the structure, u is the thickness and therefore the weight is

$$J(u) = \int_\Omega u(x)\,dx .$$

Let $u_1$ be the function in C defined by

$$u_1(x) = 1 \quad \forall\, x \in \Omega ,$$

and

$$\lambda_1 = \lambda(u_1) .$$

We shall consider the following optimisation problem :

minimize $J(u)$ subject to $\lambda(u) = \lambda_1$ and $u \in C$ i.e.

$(\mathcal{P})$
$$\inf_{\substack{\lambda(u)=\lambda_1 \\ u\in C}} J(u) .$$

Let $u_\alpha$ be the function defined on C by

(3.4)
$$u_\alpha(x) = \alpha \quad \forall\, x \in \Omega .$$

This function satisfies

$$J(u_\alpha) < J(u) \qquad \forall\, u \in C ,\ u \neq u_\alpha .$$

Introduce the problem

$(\mathcal{Q})$
$$\inf_{\substack{u\in C \\ \lambda(u)\leq\lambda_1}} J(u) .$$

Then using the previous results it is easy to prove

Theorem 3.2. Assume

(3.5) $\lambda(u_\alpha) < \lambda_1$

(3.6) $\exists\, u_\gamma \in C$ such that $\lambda(u_\gamma) > \lambda_1$

then i) $(\mathcal{P})$ and $(\mathcal{Q})$ are equivalent ;

ii) $(\mathcal{S})$ <u>has at least one solution</u> ;

iii) $\bar{u} \in C$ <u>is a solution of</u> $(\mathcal{S})$ <u>if</u> $\lambda(\bar{u}) = \lambda_1$ <u>and</u>

(3.7) $\exists\ \bar{\eta} < 0$ <u>such that</u>

$$J(v) - J(\bar{u}) + \bar{\eta}\ \lambda'(u).(v-\bar{u}) \geqslant 0 \quad \forall\ v \in C\ ;$$

iv) <u>the set</u> $S(\mathcal{S})$ <u>of solutions of</u> $(\mathcal{S})$ <u>is convex and</u> $\{w(u);u \in S(\mathcal{S})\}$ <u>is reduced to one function.</u>

<u>Remark</u>. The condition (3.7) is equivalent to (3.8) $\exists\ e > 0$ such that

$$g(\bar{u};x) \leqslant e \quad \text{a.e. in } \Omega_\beta = \{x \in \Omega;\bar{u}(x) < \beta\}$$

and

$$g(\bar{u},x) \geqslant e \quad \text{a.e. in } \Omega_\alpha = \{x \in \Omega;\bar{u}(x) > \alpha\}$$

where

$$g(\bar{u};x) = \mathbb{a}'(u(x))\ |\nabla w(u;x)|^2 - \lambda_1\ \mathbb{b}'(u(x))\ w^2(u;x)\ .$$

3.3. <u>Examples</u>.

The previous results can be applied to numerous examples and in particular to

| | | |
|---|---|---|
| Example 1 | $a(u) = u$ | $b(u) = 1$ |
| Example 2 | $a(u) = u$ | $b(u) = u + \delta$ where $\delta$ is a constant positive |
| Example 3 | $a(u) = 1$ | $b(u) = \frac{1}{u^2}$ . |

We refer to [ 1 ] and [ 3 ] for the motivation of these problems.

In these examples the function $u_\alpha$ defined by (3.4) satisfies (3.5) and the function $u_\gamma$ defined by

$$u_\gamma(x) \equiv \gamma \quad \text{where} \quad 1 < \gamma \leqslant \beta$$

satisfies (3.6) and we can apply Theorem3.2. In particular in Example 3, (3.8) gives $\exists\ e > 0$ such that

$$\frac{w^2(\bar{u};x)}{\bar{u}^3(x)} \leqslant e \quad \text{a.e. in } \Omega_\beta$$

$$\frac{w^2(\bar{u};x)}{\bar{u}^3(x)} \geqslant e \quad \text{a.e. in } \Omega_\alpha$$

and this implies that, in this example $(\mathcal{P})$ has only one solution $\bar{u}$ . Moreover if $\Omega = ]0,1[$ we can construct this solution.

Remark. We have used the same method to study problems of the following form

$$\sup_{\substack{J(u)=J_1 \\ u\in C}} \lambda(u) .$$

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