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THE PERIODS OF ABELIAN VARIETIES WITH COMPLEX MULTIPLICATION AND THE SPECIAL VALUES OF CERTAIN ZETA FUNCTIONS

by

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Let K be a CM-field of degree 2n and I_K the free $\mathbb{Z}$-module generated by all embeddings of K into $\mathbb{C}$. Given a CM-type $\varphi = \sum_{i=1}^{n} \tau_i$ of K, take a $\overline{\mathbb{Q}}$-rational abelian variety of type (K,φ) and a $\overline{\mathbb{Q}}$-rational holomorphic 1-form ω_i on A such that $\omega_i \circ a = a^{\tau_i}\omega_i$ for all $a \in K$. As shown in [2, p.383], there is a non-zero complex number $p_K(\tau_i,\varphi)$ depending only on K,φ, and τ_i such that

$$[\pi . p_K(\tau_i,\varphi)]^{-1} \int_c \omega_i \in \overline{\mathbb{Q}}$$

for every $c \in H_1(A,\mathbb{Z})$. The quantity $p_K(\tau_i,\varphi)$ can actually be chosen to be a positive real number; it is also given as the value of a certain $\overline{\mathbb{Q}}$-rational (meromorphic) Hilbert modular form at a CM-point (see [2]). Now denote by ρ the complex conjugation, and put $p_K(\tau_i\rho,\varphi) = p_K(\tau_i,\varphi)^{-1}$. Then we have

<u>Theorem 1</u> : <u>If</u> $\varphi_1, \ldots, \varphi_m$ <u>are CM-types of</u> K <u>and</u> τ <u>is an embedding of</u> K <u>into</u> $\mathbb{C}$, <u>the product</u> $\prod_{i=1}^{m} p_K(\tau, \varphi_i)^{s_i}$ <u>with</u> $s_i \in \mathbb{Z}$, <u>up to algebraic factors, depends only on</u> τ <u>and</u> $\sum_{i=1}^{m} s_i \varphi_i$. <u>Moreover, if</u> L <u>is a CM-field containing</u> K <u>and</u> ψ <u>is a CM-type of</u> L <u>whose restriction to</u> K <u>is</u> $\sum_{i=1}^{m} s_i \varphi_i$, <u>then the above product equals, up to algebraic factors, to</u> $\prod_{\sigma} p_L(\sigma, \psi)$, <u>where</u> σ <u>runs over all embeddings of</u> L <u>into</u> $\mathbb{C}$, <u>which coincide with</u> τ <u>on</u> K.

The proof is given in [3] . To express this theorem in a different way, we consider two linear maps

$$\mathrm{Res}_{L/K} : I_L \longrightarrow I_K \quad , \quad \mathrm{Inf}_{L/K} : I_K \longrightarrow I_L.$$

Here $\mathrm{Res}_{L/K}(\sigma)$ is the sum of all restrictions of σ to K; $\mathrm{Inf}_{L/K}(\tau)$ is the sum of all extensions of τ to L.

<u>Theorem 2</u> : <u>The above</u> p_K <u>can be extended to a bilinear map of</u> $I_K \times I_K$ <u>into</u> $\mathbb{C}^\times/\overline{\mathbb{Q}}^\times$ <u>with the following properties</u> :

1) $p_K(\alpha\rho, \beta) = p_K(\alpha, \beta\rho) = p_K(\alpha, \beta)^{-1}$ <u>for</u> $\alpha, \beta \in I_K$;

2) $p_K(\alpha, \mathrm{Res}_{L/K}\beta) = p_L(\mathrm{Inf}_{L/K}\alpha, \beta)$, $p_K(\mathrm{Res}_{L/K}\beta, \alpha) = p_L(\beta, \mathrm{Inf}_{L/K}\alpha)$ <u>for</u> $\alpha \in I_K$, $\beta \in I_L$, <u>and</u> $K \subset L$;

3) $p_M(\gamma\alpha, \gamma\beta) = p_K(\alpha, \beta)$ <u>if</u> γ <u>is an isomorphism of</u> M <u>onto</u> K.

<u>Theorem 3</u> : <u>If</u> (L, ψ) <u>is the reflex of</u> (K, φ), <u>we have</u> $p_K(\sigma, \varphi) = p_L(\psi\sigma, \mathrm{id}_L)$ <u>for every embedding</u> σ <u>of</u> K <u>into</u> $\mathbb{C}$.

These theorems imply various algebraic relations among the periods. For example, we have :

Theorem 4 : For $\alpha \in I_K$, let $t(\alpha)$ denote the rank of the module $\sum_{\gamma \in G} \mathbb{Z}\alpha\gamma$, where G is the Galois group over $\mathbb{Q}$ of the Galois closure of K. If $\sum_{i=1}^{n} \tau_i$ is a CM-type of K, then for every $\beta \in I_K$, the module

$$\{(e_1,\ldots,e_n) \in \mathbb{Z}^n \mid \prod_{i=1}^{n} p_K(\tau_i,\beta)^{e_i} = 1\}$$

has rank at least $n-t(\beta-\beta\rho)$.

If β is a CM-type, we have $t(\beta-\beta\rho) = t(\beta) - 1$. Theorems 2, 3 and 4 will be proved in [4].

The quantities p_K occur as the values of an L-function of a CM-field with an algebraic valued Hecke character of infinite order (see [1, Theorem 2]). As a new example of a zeta function whose values are given by p_K, we consider

$$D(s) = \sum_{0 \neq x \equiv a(\Lambda)} \mu\,(\mathrm{Tr}_{K/\mathbb{Q}}(yxx^{\rho}))\,x^{\Phi}(x^{\tau})^{-k}|x^{\tau}|^{-2s} \qquad (s \in \mathbb{C}).$$

Here Λ is a lattice in K and $a \in K$; $0 < k \in \mathbb{Z}$; τ is an embedding of K into $\mathbb{C}$; μ denotes the Fourier coefficients of an elliptic modular form $g(z) = \Sigma\mu(b)e^{2\pi ibz}$; y is a real element of K such that y^{τ} is its only positive conjugate; Φ is an element of I_K with non-negative coefficients.

Theorem 5 : The series D is convergent for sufficiently large $\mathrm{Re}(s)$ and can be continued to a meromorphic function on the whole plane.

Theorem 6 : Suppose that g is a cusp form of weight ℓ, $\mu(b)$ are all algebraic, and τ and $\tau\rho$ occur in Φ with the same multiplicity, say q. Let m be an integer such that

$$(2n - 1 - k + \ell + \deg(\Phi))/2 < m \leq q .$$

Then $D(m)$ is $\pi^k p_K(k\tau - \Phi, 2\tau)$ times an algebraic number.

A more general result holds for a series of a similar type with a Hilbert modular form (which is not necessarily a cusp form) in place of g. The details will be given in [4].

References

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