SPECTRAL PROPERTIES OF SELF-SIMILAR LATTICES AND ITERATION OF RATIONAL MAPS

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Abstract. — In this text we consider discrete Laplace operators defined on lattices based on finitely-ramified self-similar sets, and their continuous analogous defined on the self-similar sets themselves. We are interested in the spectral properties of these operators. The basic example is the lattice based on the Sierpinski gasket. We introduce a new renormalization map which appears to be a rational map defined on a smooth projective variety (more precisely, this variety is isomorphic to a product of three types of Grassmannians: complex Grassmannians, Lagrangian Grassmannian, orthogonal Grassmannians). We relate some characteristics of the dynamics of its iterates with some characteristics of the spectrum of our operator. More specifically, we give an explicit formula for the density of states in terms of the Green current of the map, and we relate the indeterminacy points of the map with the so-called Neumann-Dirichlet eigenvalues which lead to eigenfunctions with compact support on the unbounded lattice. Depending on the asymptotic degree of the map we can prove drastically different spectral properties of the operators. Our formalism is valid for the general class of finitely ramified self-similar sets (*i.e.* for the class of p.c.f. self-similar sets of Kigami). Hence, this work aims at a generalization and a better understanding of the initial work of the physicists Rammal and Toulouse on the Sierpinski gasket.

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Résumé (**Propriétés spectrales des réseaux auto-similaires et itération d'applications rationelles**)

Dans ce texte, nous considérons le laplacien discret, défini sur un réseau construit à partir d'un ensemble auto-similaire finiment ramifié, et son analogue continu défini sur l'ensemble auto-similaire lui-même. Nous nous intéressons aux propriétés spectrales de ces opérateurs. L'exemple le plus classique est celui du triangle de Sierpinski (Sierpinski gasket) et du réseau discret associé. Nous introduisons une nouvelle application de renormalisation qui se trouve être une application rationnelle définie sur une variété projective lisse (plus précisément, cette variété est un produit de grassmanniennes de trois types : grassmanniennes classiques, grassmanniennes lagrangiennes, grassmanniennes orthogonales). Nous relions certaines propriétés spectrales de ces opérateurs avec la dynamique des itérés de cette application. En particulier, nous donnons une formule explicite de la densité d'états en termes du courant de Green de l'application, et nous caractérisons le spectre de Neumann-Dirichlet (qui correspond aux fonctions propres à support compact sur l'ensemble infini) à l'aide des points d'indétermination de l'application. Suivant le degré asymptotique de l'application nous pouvons prouver que les propriétés spectrales de l'opérateur sont très différentes. Notre formalisme s'applique à la classe des ensembles auto-similaires finiment ramifiés (ou autrement dit à la classe des « p.c.f. self-similar sets » de Kigami). Ainsi, ce travail généralise et donne une compréhension plus profonde des résulats obtenus initialement par Rammal et Toulouse dans le cas du triangle de Sierpinski.

CONTENTS

INTRODUCTION

In this text we investigate the spectral properties of Laplace operators defined on hierarchical lattices based on finitely ramified self-similar sets, and their continuous analogs. The basic example is the lattice based on the Sierpinski gasket. These operators have much to do with the operators considered in the context of Schrödinger operators with random or quasi-periodic potential. Here, the disorder is not in the potential but in the lattice itself. It is well-known that in the context of Schrödinger operators on the line the spectral properties are intimately related to the dynamics of the propagator of the underlying differential equation (*cf.* for example [**8**], [**33**]). In comparison, in our models we will show that the characteristics of the spectrum of our operator are related to the dynamics of the iterates of a certain renormalization map that we explicitly define and that appears to be a rational self-map of a compact complex manifold.

The interest in such lattices and in their spectral properties comes from physicists (*cf.* [**35**], [**34**], [**1**] and [**4**]) because they present interesting computable models, with peculiar properties. In [**35**], [**34**], on the particular lattice based on the Sierpinski gasket, Rammal and Toulouse discovered interesting relations between the spectrum of the discrete Laplace operator and the dynamics of the iteration of some rational map on $\mathbb{C}$. More precisely, they exhibited a polynomial map on $\mathbb{C}$ that relates the spectrum of the operator on successive scales: they remarked that if λ is an eigenvalue at level $n + 1$ then $\lambda(5 - \lambda)$ is an eigenvalue at level n. Traditionally, this law was called the spectral decimation of the Sierpinski gasket, *i.e.* this terminology reflects the existence of a 1-dimensional map that relates the spectrum of the operator on successive scales. Starting from this, Rammal ([**34**]) gave a fairly complete description of the spectrum of the discrete operator on this lattice. In particular, he computed explicitly the eigenvalues and showed the existence of the so-called molecular states (that we call Neumann-Dirichlet eigenfunctions in this text) which are eigenfunctions with compact support. This was made rigorous and generalized to the continuous

operator defined on the Sierpinski gasket itself by Fukushima and Shima (*cf.* [**19**]). The spectral type of the operator on the Sierpinski lattice, has been analyzed by Teplyaev, *cf.* [**49**].

In general, the spectral decimation that works for the Sierpinski gasket is not valid, and the question of generalizing the initial work of Rammal remained unsolved. In [**20**] a class of lattices for which the spectral decimation works is exhibited. In [**38**], for the particular example of a Sturm-Liouville operator defined on $\mathbb{R}$, the author made explicit some relations between the spectral properties of the operator and the properties of the dynamics of the iterates of a rational map; this map is no longer 1-dimensional but is defined on the 2-dimensional projective space.

This text aims at a generalization of these previous works. Besides the interest of the generalization, this brings new understanding of the models. In particular, the renormalization map involved is now multidimensional and certain notions which are specific to the dynamics in higher dimension and which were hidden in the case of the Sierpinski gasket (where the renormalization map involved was 1-dimensional), such as the notion of indeterminacy points (which corresponds to the singularities of the map), the degree of the iterates, enter the discussion and play an important role. In comparison with our previous work, [**38**], the main progress that allows us to handle the general case is the construction of a new renormalization map. This renormalization map is a rational map defined on some compact Kähler manifold. It is of the type of the maps considered in [**13**], [**12**], and our techniques rely heavily on recent works of Fornaess Sibony, Diller Favre, Guedj (*cf.* [**45**], [**16**], [**13**], [**12**], [**14**]) on the dynamics of rational maps in higher dimensions. It is interesting to note that many of the key notions in this field (such as the degrees of the iterates, the indeterminacy points, the Green current) find a significance related to the spectral properties of our operators. In particular, we are able to give an explicit expression for the density of states in terms of the Green current of the map and we prove that the molecular states of Rammal (called Neumann-Dirichlet eigenvalues in the text) correspond exactly to the indeterminacy points of the map.

Since the text is long, we first describe the model and our results on the particular example of the lattice associated with the Sierpinski gasket. Let $F \subset \mathbb{C}$, $F = \{0, 1, \frac{1}{2} + i\frac{\sqrt{3}}{2}\}$, be the vertices of a unit triangle, and Ψ_1, Ψ_2, Ψ_3 be the three homotheties with ratio $\frac{1}{2}$ and centers the points $0, 1, \frac{1}{2} + i\frac{\sqrt{3}}{2}$, respectively. It is well-known that there exists a unique proper subset X of $\mathbb{C}$ self-similar with respect to Ψ_1, Ψ_2, Ψ_3, *i.e.* such that $X = \cup_{i=1}^3 \Psi_i(X)$, and that it is the celebrated Sierpinski gasket, represented on Figure 1.

Fix now a sequence $\omega \in \{1, 2, 3\}^{\mathbb{N}}$, called the blow-up, and define $X_{\langle 0 \rangle} = X$ and

$$X_{\langle n \rangle} = \Psi_{w_1}^{-1} \circ \cdots \circ \Psi_{w_n}^{-1}(X).$$

It is clear that $X_{\langle n \rangle}$ is an increasing sequence of sets and that $X_{\langle n+1 \rangle}$ is a scaled copy of X that contains $X_{\langle n \rangle}$ as one of the three subcells; more precisely, we have
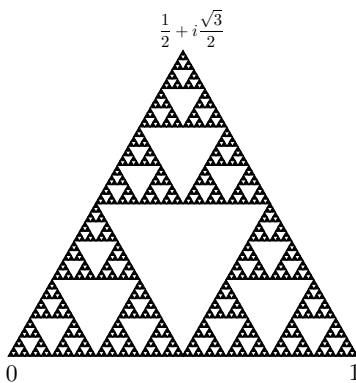
$$\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}$$

0   1

$X_{\langle n\rangle} = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_{n+1}}^{-1}(\Psi_{\omega_{n+1}}(X))$, which is clearly a subset of $X_{\langle n+1\rangle}$. Remark that the position of the cell $X_{\langle p\rangle}$ in $X_{\langle n\rangle}$ for $n > p$ depends on the blow-up ω. We then set

$$X_{\langle \infty\rangle} = \cup_{n=0}^{\infty} X_{\langle n\rangle}.$$

We define the boundary of $X_{\langle n\rangle}$ by $\partial X_{\langle 0\rangle} = F$ and

$$\partial X_{\langle n\rangle} = \Psi_{w_1}^{-1} \circ \cdots \circ \Psi_{w_n}^{-1}(F).$$

There is a natural discrete sequence of lattices associated with this structure. The lattice at level 0 is $F_{\langle 0\rangle} = F$, the vertices of the unit triangle in $X_{\langle 0\rangle}$. The lattice at level n, is the set of vertices of the unit triangles in $X_{\langle n\rangle}$. More precisely,

$$F_{\langle n\rangle} = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(\cup_{j_1,\ldots,j_n} \Psi_{j_1} \circ \cdots \circ \Psi_{j_n}(F)).$$

The position of $F_{\langle 0\rangle}$ in the lattice at level n depends on ω, and we represent on figure 2 the lattice at level 4, F_4. The bolded small triangle is the set $F_{\langle 0\rangle}$ for the blow-up starting from $(\omega_1, \ldots, \omega_4) = (1, 1, 1, 1)$ on the left and $(1, 3, 1, 2)$ on the right. The sequence $F_{\langle n\rangle}$ is increasing and we set

$$F_{\langle \infty\rangle} = \cup_{n=0}^{\infty} F_{\langle n\rangle},$$

and $\partial F_{\langle n\rangle} = \partial X_{\langle n\rangle}$.

It is important to realize that the infinite lattices $F_{\langle \infty\rangle}$ obtained from different blow-ups ω and ω' are a priori not isomorphic (except when ω and ω' are equal after a certain level). To understand this, one can compare the constant blow-up $(1, \ldots, 1, \ldots)$ with a non-stationary blow-up: the first one contains a point with only 2 neighbors (which is the point 0, center of the homothety Ψ_1), on the second one all points have 4 neighbors (indeed, the boundary points $\partial F_{\langle n\rangle}$ are sent to infinity when n goes to infinity).
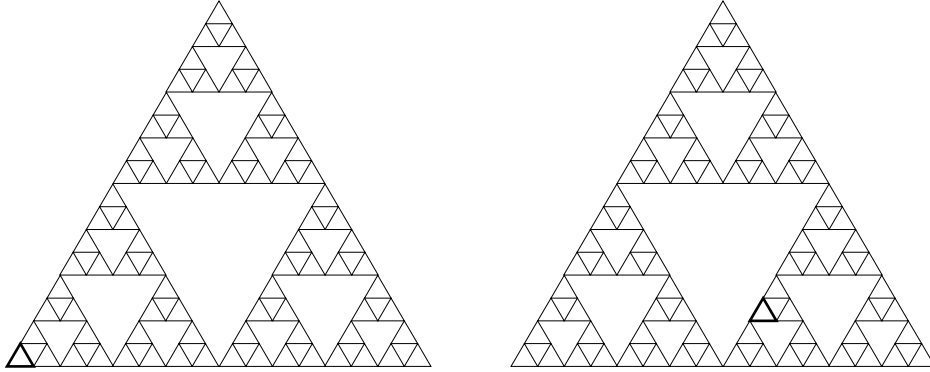
The aim of this text is to investigate the spectral properties of some natural Laplace operator defined either on the infinite lattice $F_{\langle\infty\rangle}$ or on the unbounded set $X_{\langle\infty\rangle}$. The class of lattices or self-similar sets we consider is issued from the class of finitely-ramified self-similar sets (also called p.c.f. self-similar sets in [**25**]) described in section 1.1, and is much larger than the Sierpinski gasket. Although the classical examples have a natural geometrical embedding, these sets are defined abstractly from a very simple finite structure: one starts from a finite set F and one constructs $F_{\langle 1\rangle}$ as the union of N copies of F, glued together according to a prescribed rule (represented by an equivalence relation $\mathcal{R}$ on $\{1,\dots,N\}\times F$), then $F_{\langle 2\rangle}$ is defined as the union of N copies of $F_{\langle 1\rangle}$ glued together according to the same rule, and so on. From this discrete structure, one can construct an increasing sequence of sets $F_{\langle n\rangle}$, and also a self-similar set X (*cf.* section 1.2 for precise definitions).

To take into account the eventual symmetries of the picture, we fix a group of symmetries acting on each $F_{\langle n\rangle}$ (but in general not on $F_{\langle\infty\rangle}$). For the Sierpinski gasket we can see that the group $G \sim S_3$ (S_3 denotes the group of permutation of F) of isometries of the regular triangle $\partial F_{\langle n\rangle}$ leaves globally invariant the lattice $F_{\langle n\rangle}$. We fix this group G as the group of symmetries of the structure (*i.e.* this means that we will only consider G-invariant objects).

Note that for consistency with the notations of the main text, we denote by N the number of subcells of $F_{\langle 1\rangle}$. Here, we have $N = 3$.

We now define the type of operators we will consider in this text. We restrict to the discrete setting in this introduction and we present the definitions only in the case of the Sierpinski gasket. On $F_{\langle n\rangle}$ we define the difference operator $A_{\langle n\rangle}$ as the operator on $\mathbb{R}^{F_{\langle n\rangle}}$ defined by

$$(1) \qquad A_{\langle n\rangle}f(x) = -\sum_{y\sim x}(f(y) - f(x)), \quad \forall f \in \mathbb{R}^{F_{\langle n\rangle}},$$

where $y \sim x$ means that y is in the same triangle of unit size as x. The measure $b_{\langle n \rangle}$ is defined as the measure which gives mass 1 to the points of $\partial F_{\langle n \rangle}$ and mass 2 to the points in $F_{\langle n \rangle} \smallsetminus \partial F_{\langle n \rangle}$. The choice of this particular operator and of this measure comes from the fact that they are self-similar in the sense that $A_{\langle n \rangle}$ (resp. $b_{\langle n \rangle}$) is the sum of N^n copies of the operator $A_{\langle 0 \rangle}$ (resp. the measure $b_{\langle 0 \rangle}$) on each of the small triangles of $F_{\langle n \rangle}$). The operators we will be interested in are the operators $H_{\langle n \rangle}^+$ defined on $\mathbb{R}^{F_{\langle n \rangle}}$ by

$$(2) \qquad H_{\langle n \rangle}^+ f(x) = -\frac{1}{b_{\langle n \rangle}(\{x\})} A_{\langle n \rangle} f(x), \quad \forall\, f \in \mathbb{R}^{F_{\langle n \rangle}}, \ \ \forall\, x \in F_{\langle n \rangle}.$$

It is clear that $H_{\langle n \rangle}^+$ is self-adjoint on $L^2(F_{\langle n \rangle}, b_{\langle n \rangle})$ and semi-negative. The operator with Dirichlet boundary condition on $\partial F_{\langle n \rangle}$, $H_{\langle n \rangle}^-$, is defined as the restriction to $\{f \in \mathbb{R}^{F_{\langle n \rangle}}, \ f_{|\partial F_{\langle n \rangle}} = 0\}$ of $H_{\langle n \rangle}^+$, and is self-adjoint on $L^2(F_{\langle n \rangle} \smallsetminus \partial F_{\langle n \rangle}, b_{\langle n \rangle})$. The measure $b_{\langle n \rangle}$ and the operators $H_{\langle n \rangle}^\pm$ naturally extend to a measure $b_{\langle \infty \rangle}$ on $F_{\langle \infty \rangle}$ and to semi-negative self-adjoint linear operators $H_{\langle \infty \rangle}^\pm$ on $L^2(F_{\langle \infty \rangle}, b_{\langle \infty \rangle})$.

There are two measures which play a crucial role in this text. The first one is the classical density of states: for each n, denote by $\nu_{\langle n \rangle}^\pm$ the counting measures of the eigenvalues of $H_{\langle n \rangle}^\pm$. The density of states is defined as the limit

$$\mu = \lim_{n \to \infty} \frac{1}{N^n} \nu_{\langle n \rangle}^\pm.$$

(In general, this measure exists and does not depend on the boundary condition.) The second measure is the measure that counts the asymptotic number of the so-called Neumann-Dirichlet eigenvalues: we say that a function is a Neumann-Dirichlet eigenfunction (or N-D eigenfunction for short) of $H_{\langle n \rangle}$ with eigenvalue λ if it is both an eigenfunction of $H_{\langle n \rangle}^+$ and $H_{\langle n \rangle}^-$ (with eigenvalue λ), *i.e.* if it is an eigenfunction of $H_{\langle n \rangle}^+$ null on $\partial F_{\langle n \rangle}$. These particular eigenfunctions play an important role since, when extended to $F_{\langle \infty \rangle}$ by 0, they are eigenfunctions with compact support of the operators $H_{\langle \infty \rangle}^\pm$ on the infinite lattice. One can define the counting measure of N-D eigenvalues $\nu_{\langle n \rangle}^{ND}$ and show that $\nu_{\langle n+1 \rangle}^{ND} \geqslant N \nu_{\langle n \rangle}^{ND}$. This implies that the limit

$$\mu^{ND} = \lim_{n \to \infty} \frac{1}{N^n} \nu_{\langle n \rangle}^{ND}$$

exists and is pure point. We call it the density of N-D eigenvalues.

As we said before, the lattice $F_{\langle \infty \rangle}$ depends on the particular choice of the blow-up ω, hence the topological spectrum $\Sigma^\pm$ of $H_{\langle \infty \rangle}^\pm$ and the Lebesgue decomposition, $\Sigma_{\mathrm{ac}}^\pm$, $\Sigma_{\mathrm{sc}}^\pm$, $\Sigma_{\mathrm{pp}}^\pm$, of the spectrum of the operator on the infinite lattice $H_{\langle \infty \rangle}^\pm$ a priori depend on ω. By contrast, the measures μ and μ^{ND} do not depend on ω (since the measures $\nu_{\langle n \rangle}^\pm$ and $\nu_{\langle n \rangle}^{ND}$ obviously do not depend on ω). In [**40**], we proved basic results regarding the spectral properties of the operators $H_{\langle \infty \rangle}^\pm$ and, in particular, on their relations with the measures μ and μ^{ND}. First we showed that the topological spectrum $\Sigma^\pm(\omega)$

and the Lebesgue decomposition $\Sigma^{\pm}_{\mathrm{ac}}(\omega)$, $\Sigma^{\pm}_{\mathrm{sc}}(\omega)$, $\Sigma^{\pm}_{\mathrm{pp}}(\omega)$ of the operators $H^{\pm}_{\langle\infty\rangle}(\omega)$ (we write $\Sigma(\omega), \ldots$ to show the a priori dependence in the blow-up ω) are constant almost surely in the blow-up ω, for the measure on $\{1, \ldots, N\}^{\mathbb{N}}$ equal to the product of the uniform measure on $\{1, \ldots, N\}$. Hence, we can talk about the almost sure spectrum and the almost sure Lebesgue decomposition of the spectrum of the operator $H^{\pm}_{\langle\infty\rangle}$. We also proved that the almost sure spectrum is equal to the support of the measure μ, *i.e.* that we have $\Sigma^{\pm}(\omega) = \mathrm{supp}(\mu)$ for almost all blow-up ω (actually, our result is more precise than this). Finally, we showed that when the density of states is completely created by the N-D eigenvalues, *i.e.* when $\mu^{ND} = \mu$, then the spectrum of $H^{\pm}_{\langle\infty\rangle}$ is pure point with compactly supported eigenfunctions, almost surely in ω.

So, the measures μ and μ^{ND} give important information on the spectral properties of the operators $H^{\pm}_{\langle\infty\rangle}$. Hence, these two measures deserve to be understood and our aim in this text is to describe the relations that exist between these measures and the dynamics of a certain renormalization map that we construct.

There are two renormalization maps, closely related, which play an important role. We do not define precisely these maps in this introduction, we just describe some of their properties. The first one, denoted by T, is defined on the space Sym^G, the space of G-invariant symmetric operators on $\mathbb{C}^F$, and is a rational map (*i.e.* TQ, for Q in Sym^G, is rational in the coefficients of Q). For example, in the case of the Sierpinski gasket $\mathrm{Sym}^G \sim \mathbb{C}^2$: indeed, if W_0 is the space of constant functions on F and W_1 its orthogonal complement (for the natural scalar product on $\mathbb{C}^F$), then any element of Sym^G can be written thanks to 2 coordinates $(u_0, u_1) \in \mathbb{C}^2$ under the form $u_0 p_{|W_0} + u_1 p_{|W_1}$ where $p_{|W_0}$ and $p_{|W_1}$ are the orthogonal projections respectively on W_0 and W_1. We do not define explicitly the map T in this introduction (*cf.* section 2 and 3.1), but this map is easy to compute; for example, in the case of the Sierpinski gasket, in coordinates (u_0, u_1) we have $T(u_0, u_1) = 3(\frac{u_0 u_1}{2u_0 + u_1}, \frac{u_1(u_0 + u_1)}{5u_1 + u_0})$. This map is the one that was considered in earlier work of the author, [**38**]; it is also very closely related to the renormalization map that was introduced initially by Rammal and Toulouse in the case of the Sierpinski gasket. The iterates of this map contain some information on the spectrum of the operators on the n-th level lattice $H^{\pm}_{\langle n\rangle}$. This explains why it was useful in the understanding of the spectral properties of these operators.

Nevertheless, this map fails to give enough information. The main progress in this text is the construction of a new renormalization map defined on a bigger space: more precisely, it is defined on a projective space that contains Sym^G as a smooth subvariety and it coincides with T on Sym^G. We consider two sets of variables $(\overline{\eta}_x)_{x \in F}$ and $(\eta_x)_{x \in F}$, and the Grassmann algebra generated by these variables (*i.e.* the algebra generated by $(\overline{\eta}_x)_{x \in F}$ and $(\eta_x)_{x \in F}$, with the relation of anticommutation between all these variables). We denote by $\mathcal{A}$ the subalgebra generated by the monomials containing the same number of variables $\overline{\eta}$ and η. We also denote by $\mathcal{P}(\mathcal{A})$ the

projective space associated with $\mathcal{A}$ and by $\pi : \mathcal{A} \to \mathcal{P}(\mathcal{A})$ the canonical projection. We set

$$\overline{\eta}Q\eta = \sum_{x,y \in F^2} Q_{x,y}\overline{\eta}_x \eta_y$$

for any $F \times F$ matrix Q. Then there is a natural embedding of Sym^G into $\mathcal{P}(\mathcal{A})$ given by

$$Q \longmapsto \pi(\exp \overline{\eta}Q\eta),$$

where exp denotes the exponential of the Grassmann algebra. We denote by $\mathbb{L}^G$ the closure in $\mathcal{P}(\mathcal{A})$ of Sym^G (*i.e.* of the points of the type $\pi(\exp \overline{\eta}Q\eta)$). We can show (*cf.* section 2.2) that $\mathbb{L}^G$ is a smooth analytic subvariety of $\mathcal{P}(\mathcal{A})$ and it defines a compactification of Sym^G. For example, in the case of the Sierpinski gasket $\mathbb{L}^G$ is equal to $\mathbb{P}^1 \times \mathbb{P}^1$ (and indeed, $\mathbb{P}^1 \times \mathbb{P}^1$ is a compactification of $\mathbb{C}^2$). The key point in our work is the following: we construct a homogeneous polynomial map $R : \mathcal{A} \to \mathcal{A}$ of degree N. This map naturally induces a rational map on the projective space $\mathcal{P}(\mathcal{A})$: a fundamental property of this map is that it leaves invariant the subvariety $\mathbb{L}^G \subset \mathcal{P}(\mathcal{A})$, and thus induces by restriction a map $g : \mathbb{L}^G \to \mathbb{L}^G$. Actually, the map g coincides with T on the subset Sym^G of $\mathbb{L}^G$. More precisely, we have the following formula

$$g(\pi(\exp(\overline{\eta}Q\eta))) = \pi(R(\exp(\overline{\eta}Q\eta))),$$

when both expressions are well-defined. Hence, the restriction of the map g to $\mathbb{L}^G$ extends T to the compactification $\mathbb{L}^G$ of Sym^G. The measures μ and μ^{ND} are related to the properties of the map R and g, and in particular, to the dynamics of the restriction of g to $\mathbb{L}^G$.

Let us now state our main results. We define the Green function of R (introduced by Fornaess and Sibony, *cf.* [**45**] or appendix B) as the function $G : \mathcal{A} \to \mathbb{R} \cup \{-\infty\}$ given by

$$G(x) = \lim_{n \to \infty} \frac{1}{N^n} \ln \|R^n(x)\|, \quad \forall x \in \mathcal{A}.$$

The limit exists for all x in $\mathcal{A}$ and is a plurisubharmonic function (this essentially means that G is subharmonic when restricted to a complex line). This function G contains important information on the dynamics of the map induced by R on $\mathcal{P}(\mathcal{A})$. We denote by $\rho_n(x)$, for $x \in \mathbb{L}^G$, the order of vanishing of the restriction of the function R^n to $\mathbb{L}^G$ (*cf.* section 3.2). Since R is homogeneous of degree N, we have $\rho_{n+1}(x) \geqslant N\rho_n(x)$ and we set

$$\rho_\infty(x) = \lim_{n \to \infty} \frac{1}{N^n}\rho_n(x).$$

We define $\phi : \mathbb{C} \to \mathcal{A}$ by $\phi(\lambda) = \exp \overline{\eta}(A_{\langle 0 \rangle} - \lambda \mathrm{Id})\eta$. Theorem (3.6) gives the following explicit expressions for μ and μ^{ND}:

$$(3) \qquad \qquad \mu = \frac{1}{2\pi} \Delta G \circ \phi,$$

$$(4) \qquad \qquad \mu^{ND} = \sum_{\lambda} \rho_{\infty}(\phi(\lambda)) \delta_{\lambda},$$

where Δ in (3) is the distributional Laplacian, and δ_{λ} in (4) is the Dirac mass at λ ($\rho_{\infty}(\phi(\lambda))$ is null except on a countable set of λ's, so that the sum (4) is well-defined).

In section 4, we investigate the structure of the Green function on $\pi^{-1}(\mathbb{L}^G)$. This is important since the Green function $G_{|\pi^{-1}(\mathbb{L}^G)}$ is related to the dynamics of the map g. On the other hand, we see from (3) and (4) that it is also related to the measures μ and μ^{ND}. The function $G_{|\pi^{-1}(\mathbb{L}^G)}$ is the potential of a unique closed, positive, (1, 1)-current on $\mathbb{L}^G$: precisely, if s is a local section of the projection π on an open subset $U \subset \mathbb{L}^G$ the current $dd^c G \circ s$ does not depend on s and defines a positive closed current on all $\mathbb{L}^G$ (*cf.* appendix, we recall that

$$dd^c G \circ s = \frac{i}{\pi} \partial \overline{\partial}(G \circ s) = \frac{i}{\pi} \sum_{i,j} \frac{\partial^2}{\partial z_i \partial \overline{z}_j}(G \circ s) dz_i \wedge d\overline{z}_j,$$

where the derivatives are taken in the sense of distributions). The current S is intimately related to the dynamics of the map g and to the structure of the measures μ and μ^{ND}. With the iterates of g we can associate an asymptotic degree d_{∞} (called the dynamical degree, *cf.* section 4.3) and we show in theorem (4.3) that the following dichotomy holds:

– If $d_{\infty} < N$, then S is a countable sum of currents of integration on hypersurfaces of $\mathbb{L}^G$. In this case $\mu^{ND} = \mu$, thus the spectrum is pure point with compactly supported eigenfunctions, almost surely in ω. (We also show that the number of Neumann only eigenvalues, *i.e.* $|\nu_{\langle n \rangle}^+ - \nu_{\langle n \rangle}^{ND}|$, grows like $n^{d_{\infty}}$).

– If $d_{\infty} = N$, then the current S does not charge hypersurfaces; it is the Green current of the map g. In particular, it is null on the Fatou set of g. Moreover, generically (in a sense made precise in theorem (4.3)), we have $\mu^{ND} = 0$, *i.e.* there does not exist N-D eigenfunctions.

Note that a similar dichotomy theorem was shown in [**20**] for the particular class of decimable fractals, for which there exists a 1-dimensional renormalization map that relate the spectrum on different scales (but the relation with the N-D spectrum was not made).

Let us now make some remarks. The renormalization map we consider here is not the same as the one considered in our previous work [**38**]. The introduction of this new map is the key point that allows us to handle the case of lattices based on general finitely ramified self-similar sets. In particular, this map is not defined on a projective space. This induces several difficulties, for example, the notion of degree

is more complicated and is related to the action of the map on some cohomology groups. But there are several facts that seem to indicate that the map we consider is the good renormalization map: the first one is that the indeterminacy points of g, which have a crucial influence on the dynamical properties of g, have a clear meaning in terms of the Neumann-Dirichlet eigenvalues of our operator. The second is that the map g behaves well in the non-degenerate case, *i.e.* when $d_\infty = N$: to be more precise, the map g is algebraically stable (*cf.* definition in appendix), and this allows us to define the Green current; this is not the case for other maps we could consider (*cf.* section 4.5 where we compare different renormalization maps) and which are birationally equivalent to g. Finally, let us mention that considering expressions like $\exp(\overline{\eta} Q \eta)$ in the Grassmann algebra is very natural in the context of supersymmetry, and that the theory of supersymmetry appeared to be very useful in the context of random Schrödinger operators (*cf.* for example [**28**], [**47**], [**51**]).

The important question of determining the type of spectrum of the operator on the infinite lattice remains largely open: with our techniques we are only able to characterize a small part of the spectrum, namely the part of the spectrum which corresponds to compactly supported eigenfunctions (*i.e.* N-D eigenfunctions). It would be very interesting to determine the almost sure type of the spectrum of the operator (for example, to determine whether it is continuous or purely punctual) in terms of characteristic of the dynamics of the map (for example, in the spirit of Kotani's theorem where the Lyapounov exponent can characterize the type of the spectrum). There are very few examples where results in this direction are known: in the case of the Sierpinski gasket Teplyaev, *cf.* [**49**], gave fairly complete results (depending on the blow-up); for a self-similar Sturm-Liouville operator on $\mathbb{R}$ we investigate the type of the spectrum for different blow-ups, *cf.* [**41**]. Another interesting question would be to consider random potential on $F_{\langle\infty\rangle}$ (or random fractal lattices as in [**23**]) and to determine whether Anderson localization occurs as for 1-dimensional Schrödinger operators.

Let us now describe the organization of the paper. In the first part, we introduce the models and recall three elementary results obtained in [**40**], concerning the spectrum and the density of states. In the second part, we give some preliminary results, which are crucial in the rest of the text. We introduce the Grassmann algebra and the Lagrangian Grassmannian. The third part is devoted to the proof of the main formulas (3) and (4). In the fourth part, we analyze the structure of the current S on $\mathbb{L}^G$. Finally, in part 5 we illustrate our results by several examples. In the appendix A, B, C, we recall some of the results from pluricomplex analysis and rational dynamics that we need in the text. In appendix E, we describe the topological structure of G-Lagrangian Grassmannians.

We treat both the lattice case and the case of operators defined on continuous self-similar sets. For a reader not familiar with the subject it is better, upon a first

reading, to skip the discussion of the continuous case which is of a more technical nature.

CHAPTER 1

DEFINITIONS AND BASIC RESULTS

1.1. Finitely ramified self-similar sets and associated lattices

We introduce here an abstract definition of the notion of finitely ramified self-similar sets and of the associated lattices. Although all classical examples have a natural geometrical representation we choose to present the abstract setting since the procedure of construction is simple and natural. Let us describe it briefly: first choose two integers N and N_0, $1 < N_0 \leqslant N$. The basic cell is $F_{\langle 0 \rangle} = F = \{1, \ldots, N_0\}$. The set at level 1 is defined as the union of N copies of F, glued together according to a prescribed law $\mathcal{R}$ (formally defined as an equivalent relation on $\{1, \ldots, N\} \times F$). In $F_{\langle 1 \rangle}$ we define the boundary set $\partial F_{\langle 1 \rangle}$ as the set of points of the type (x, x) for $x \in F$ which can be identified with F (if $\mathcal{R}$ satisfies some minor properties). Then we define the set at level 2, $F_{\langle 2 \rangle}$, as the union of N copies of $F_{\langle 1 \rangle}$ glued together by their boundary points, $\partial F_{\langle 1 \rangle}$, according to the law $\mathcal{R}$, and so on. To define the infinite lattice we blow-up the structure, according to a sequence ω in $\{1, \ldots, N\}^{\mathbb{N}}$, *i.e.* at each level, $F_{\langle n \rangle}$ is the sublattice ω_{n+1} of $F_{\langle n+1 \rangle}$. To construct the self-similar set we refine the structure instead of blowing it up. Let us now present precise definitions.

1.1.1. The lattice case. — Let N and N_0 be two positive integers such that $1 < N_0 \leqslant N$. We set $F = \{1, \ldots, N_0\}$. The set F will represent the basic cell and N the number of cells at level 1. We suppose given an equivalence relation $\mathcal{R}$ on $\{1, \ldots, N\} \times F$ (this equivalence relation will describe the connections in the set at level 1). For some reasons that will appear clearly later we assume that $\mathcal{R}$ satisfies

  – $(i, x)\mathcal{R}(i, y)$ implies $x = y$.
  – The class of (i, i) for i in F is a singleton.
  – For any (i, i') in $\{1, \ldots, N\}$ there exists a sequence $i_1 = i, \ldots i_p = i'$ of $\{1, \ldots, N\}$ such that for all $k \leqslant p - 1$ there exists j and j' such that $(i_k, j)\mathcal{R}(i_{k+1}, j')$.

We first give the formal definition of the infinite set $F_{\langle \infty \rangle}$ and of its subsets $F_{\langle n \rangle}$. The lattice structure on these sets will be apparent only in section 1.2 and will be

inherited from the discrete Laplace operator we construct on these sets. Let us fix an element $\omega = (\omega_1, \ldots, \omega_k, \ldots)$ in $\{1, \ldots, N\}^{\mathbb{N}}$ called the blow-up. As explained in the introduction, the blow-up ω describes how the sets $F_{\langle n \rangle}$ fit into one each other. Denote by $\widetilde{F}_{\langle \infty \rangle}$ the set of backward sequences $(\ldots, j_{-k}, \ldots, j_{-1}, x)$ in $\{1, \ldots, N\}^{\mathbb{N}} \times F$ such that $j_{-k} = \omega_k$ after a certain level. On $\widetilde{F}_{\langle \infty \rangle}$ we define the relation $\mathcal{R}_{\langle \infty \rangle}$ given by $(\ldots, j_{-k}, \ldots, j_{-1}, x)\mathcal{R}_{\langle \infty \rangle}(\ldots, j'_{-k}, \ldots, j'_{-1}, x')$ if and only if there exists k_0 such that

(5) $$j_{-k} = j'_{-k}, \quad \text{for } k \geqslant k_0 + 1,$$

(6) $$j_{-k} = x, \; j'_{-k} = x', \quad \text{for } k \geqslant 1 \text{ and } k \leqslant k_0 - 1,$$

(7) $$\text{and } (j_{-k_0}, x) \; \mathcal{R} \; (j'_{-k_0}, x').$$

Using the second property of $\mathcal{R}$, we easily check that $\mathcal{R}_{\langle \infty \rangle}$ is an equivalence relation. Then we define $F_{\langle \infty \rangle}$ as the set $\widetilde{F}_{\langle \infty \rangle}$ quotiented by $\mathcal{R}_{\langle \infty \rangle}$. The increasing sequence of subsets $F_{\langle 0 \rangle} \subset \cdots \subset F_{\langle n \rangle} \subset \cdots \subset F_{\langle \infty \rangle}$ is defined by

$$F_{\langle n \rangle} = \{(\ldots, j_{-k}, \ldots, j_{-1}, x) \in F_{\langle \infty \rangle} \mid j_{-k} = \omega_k \text{ for all } k \geqslant n\}.$$

(N.B.: in the last expression and in the following, we simply write $(\ldots, j_{-k}, \ldots, j_{-1}, x)$ for the class of $(\ldots, j_{-k}, \ldots, j_{-1}, x)$ in $F_{\langle \infty \rangle}$). It is clear that

$$F_{\langle \infty \rangle} = \cup_{n=0}^{\infty} F_{\langle n \rangle}.$$

Denote by $\mathcal{R}_{\langle n \rangle}$ the equivalence relation on $\{1, \ldots, N\}^n \times F$ exactly as $\mathcal{R}$, *i.e.* by $(j_{-n}, \ldots, j_{-1}, x)\mathcal{R}_{\langle n \rangle}(j'_{-n}, \ldots, j'_{-1}, x')$ iff there exists $k_0 \leqslant n$ for which (5), (6), (7) are satisfied. In the definition of $F_{\langle n \rangle}$ we see that only the terms $(j_{-n}, \ldots, j_{-1}, x)$ count — the others are fixed to ω_k — and it is easy to see that $F_{\langle n \rangle}$ can be identified with $\{1, \ldots, N\}^n \times F/\mathcal{R}_{\langle n \rangle}$. (More precisely, if $\widetilde{F}_{\langle n \rangle}$ is defined as the set of points $(\ldots, j_{-k}, \ldots, x)$ such that $j_{-k} = \omega_k$ for all $k \geqslant n$, then it is clear, thanks to the first property of $\mathcal{R}$, that the natural bijection $\Theta : \widetilde{F}_{\langle n \rangle} \to \{1, \ldots, N\}^n \times F$, commutes with $\mathcal{R}_{\langle \infty \rangle}$ and $\mathcal{R}_{\langle n \rangle}$, *i.e.* that $X\mathcal{R}_{\langle \infty \rangle}Y$ iff $\Theta(X)\mathcal{R}_{\langle n \rangle}\Theta(Y)$. Hence, $F_{\langle n \rangle}$ can be identified with $\{1, \ldots, N\}^n \times F/\mathcal{R}_{\langle n \rangle}$.) For example, the set $F_{\langle 0 \rangle}$ is equal to F and the set $F_{\langle 1 \rangle}$ is equal to $\{1, \ldots, N\} \times F/\mathcal{R}$. The boundary of the set $F_{\langle n \rangle}$ is defined as

$$\partial F_{\langle n \rangle} = \{(x, \ldots, x) \in F_{\langle n \rangle}, \text{ for some } x \text{ in } F = \{1, \ldots, N_0\}\}.$$

Thanks to the second property of $\mathcal{R}$, we see that $\partial F_{\langle n \rangle}$ can be identified with F (*i.e.* the map $x \in F \mapsto (x, \ldots, x) \in \partial F_{\langle n \rangle}$ is bijective). The boundary set $\partial F_{\langle \infty \rangle}$ is defined as

$$\partial F_{\langle \infty \rangle} = \cap_{n=0}^{\infty} \cup_{m \geqslant n} \partial F_{\langle m \rangle}$$
$$= \{(\ldots, j_{-k}, \ldots, j_{-1}, x) \in F_{\langle \infty \rangle} \mid j_{-k} = x \text{ for all } k \geqslant 0\}.$$

We set $\mathring{F}_{\langle n \rangle} = F_{\langle n \rangle} \smallsetminus \partial F_{\langle n \rangle}$ and $\mathring{F}_{\langle \infty \rangle} = F_{\langle \infty \rangle} \smallsetminus \partial F_{\langle \infty \rangle}$.

REMARK 1.1. — By definition $\partial F_{\langle \infty \rangle} \neq \varnothing$ if and only if ω_k is stationary to a certain x in F: in this case $\partial F_{\langle \infty \rangle}$ contains the unique point $(\ldots, x, \ldots, x)$.

For all n, p and $\{i_1, \ldots, i_p\}$ in $\{1, \ldots, N\}^p$ we set

$$F_{\langle n+p \rangle, i_1, \ldots, i_p} = \{(j_{-(n+p)}, \ldots, j_{-1}, x) \in F_{\langle n+p \rangle} \mid j_{-(n+p)} = i_1, \ldots, j_{-(n+1)} = i_p\}.$$

It is clear that

$$F_{\langle n \rangle} = F_{\langle n+p \rangle, \omega_{n+p}, \ldots, \omega_{n+1}}$$

and that

$$F_{\langle n+p \rangle} = \cup_{i_1, \ldots, i_p = 1}^{N} F_{\langle n+p \rangle, i_1, \ldots, i_p}.$$

In this sense, $F_{\langle n+p \rangle}$ is the (non-disjoint) union of N^p copies of $F_{\langle n \rangle}$. The subsets $F_{\langle n+p \rangle, i_1, \ldots, i_p}$ are called the $\langle n \rangle$-cells of $F_{\langle n+p \rangle}$. It is clear that the $\langle n \rangle$-cells can only intersect by their boundary sets $\partial F_{\langle n+p \rangle, i_1, \ldots, i_p}$ (with the obvious definition $\partial F_{\langle n+p \rangle, i_1, \ldots, i_p} = \{(i_1, \ldots, i_p, x, \ldots, x) \in F_{\langle n+p \rangle}, \text{ for } x \in F\}$).

REMARK 1.2. — At this point we did not construct any lattice structure: the lattice structure will be induced by the discrete Laplace operator we shall construct on $F_{\langle n \rangle}$ and $F_{\langle \infty \rangle}$ (in fact, as we shall see, two points will be neighbors for this operator if they belong to the same $\langle 0 \rangle$-cell).

To take into account the eventual symmetries of the structure we suppose given a finite group G (eventually trivial) acting on $\{1, \ldots, N\}$ and leaving invariant the subset $F = \{1, \ldots, N_0\}$. We suppose that the relation $\mathcal{R}$ is G-invariant, for the action of G on the product $\{1, \ldots, N\} \times F$. The relation $\mathcal{R}_{\langle n \rangle}$ is then clearly G-invariant for the action of G on $\{1, \ldots, N\}^n \times F$. Thus, the group G acts on the quotient $F_{\langle n \rangle}$, leaving globally invariant its boundary set $\partial F_{\langle n \rangle}$ (remark that if we consider the action of G on $F_{\langle n+1 \rangle}$ then it does not leave the subset $F_{\langle n \rangle}$ invariant in general. For this reason there is no natural action of G on the lattice $F_{\langle \infty \rangle}$). This symmetry group will play the following role: all the objects we will consider will be G-invariant, in particular the discrete Laplace operator we will construct on $F_{\langle n \rangle}$.

1.1.2. The continuous (or fractal) case. — It is easy to construct a self-similar set from the previous discrete structure. The definition we introduce here is a bit less general than the classical definition of p.c.f. self-similar sets introduced by Kigami (*cf.* [**25**]), but a bit more constructive. Formally, we define X as the set $\{1, \ldots, N\}^{\mathbb{N}}$ quotiented by the equivalence relation $\sim$ given by: $(j_1, \ldots, j_k, \ldots) \sim (j'_1, \ldots, j'_k, \ldots)$ if and only if there exists an integer k_0 and two elements x and x' in $F = \{1, \ldots, N_0\}$ such that

$$(j_1, \ldots, j_{k_0-1}) = (j'_1, \ldots, j'_{k_0-1}),$$
$$j_k = x \text{ and } j'_k = x' \text{ for all } k \geqslant k_0 + 1,$$
$$\text{and } (j_{k_0}, x) \, \mathcal{R} \, (j'_{k_0}, x').$$

If we equip $\{1, \ldots, N\}^{\mathbb{N}}$ with the usual metric d given by $d((j_k), (j'_k)) = 1/2^{\inf\{k, j_k \neq j'_k\}}$, then X is compact for the quotient topology (and metrizable). It is also easy to check

that the third property of $\mathcal{R}$ implies that X is connected. The boundary set of X is define as the set

$$\partial X = \{(x, \ldots, x, \ldots), \ \text{for a } x \text{ in } F\}.$$

It is clear that ∂X can be identified with F thanks to the second hypothesis on $\mathcal{R}$ (so we usually write $\partial X = F$). We denote by $\Psi_i \ : \ X \to X$ the application

$$\Psi_i((i_1, \ldots, i_n, \ldots)) = (i, i_1, \ldots, i_n, \ldots).$$

Thanks to the first hypothesis on $\mathcal{R}$, each Ψ_i is injective. We see that

$$X = \cup_{i=1}^{N} \Psi_i(X),$$

$$\Psi_i(X) \cap \Psi_j(X) = \Psi_i(F) \cap \Psi_i(F), \quad \forall i \neq j.$$

Hence the set X is self-similar with respect to the injections Ψ_i. From the second relation we see that the connections between the subsets $\Psi_i(X)$ of X are contained in the image of F by the application Ψ_i: this justifies that we consider F as the boundary of the set X. We now construct an infinite sequence of sets $X_{\langle 0 \rangle} \subset \cdots \subset X_{\langle n \rangle} \subset \cdots \subset X_{\langle \infty \rangle}$ as in the discrete case. Remind that we fixed ω in $\{1, \ldots, N\}^{\mathbb{N}}$ called the blow-up. We set:

$$\widetilde{X}_{\langle \infty \rangle} = \{(j_k) \in \{1, \ldots, N\}^{\mathbb{Z}} \mid \text{there exists } k_0 \geqslant 0 \text{ s.t. } j_{-k} = \omega_k \text{ for all } k \geqslant k_0\}.$$

Then we define $X_{\langle \infty \rangle}$ as the quotient of $\widetilde{X}_{\langle \infty \rangle}$ by the equivalence relation $\sim_{\langle \infty \rangle}$ defined exactly as $\sim$, *i.e.* by: $(j_k)_{k \in \mathbb{Z}} \sim_{\langle \infty \rangle} (j_k')_{k \in \mathbb{Z}}$ if and only if there exists $k_0 \in \mathbb{Z}$ and two elements x and x' in F

$$j_k = j_k' \text{ for all } k \leqslant k_0 - 1,$$
$$j_k = x \text{ and } j_k' = x' \text{ for all } k \geqslant k_0 + 1,$$
$$\text{and } (j_{k_0}, x) \ \mathcal{R} \ (j_{k_0}', x').$$

Then we set for all $n \geqslant 0$

$$X_{\langle n \rangle} = \{(j_k) \in X_{\langle \infty \rangle} \mid j_{-k} = \omega_k \text{ for all } k \geqslant n\}.$$

It is clear that for $X_{\langle n \rangle}$ only the terms $(j_{-n}, \ldots)$ counts, thus the set $X_{\langle n \rangle}$ can be considered as the set $\{1, \ldots, N\}^{[-n, \infty)}$ quotiented by the equivalent relation induced by $\sim_{\langle \infty \rangle}$ (and in the following we will represent the points in $X_{\langle n \rangle}$ only by the sequence $(j_{-n}, \ldots)$). It is then clear that $X_{\langle 0 \rangle} = X$ (actually all $X_{\langle n \rangle}$ can be identified with X, just by shifting the indices), and that

$$X_{\langle \infty \rangle} = \cup_{n=0}^{\infty} X_{\langle n \rangle}.$$

As in the discrete case we set

$$X_{\langle n+p \rangle, i_1, \ldots, i_p} = \{(j_k)_{k=-(n+p)}^{\infty} \in X_{\langle n+p \rangle} \mid (j_{-(n+p)}, \ldots, j_{-(n+1)}) = (i_1, \ldots, i_p)\}.$$

It is clear that

$$X_{\langle n \rangle} = X_{\langle n+p \rangle, \omega_{n+p}, \ldots, \omega_{n+1}}$$

and that the set $X_{\langle n+p \rangle}$ is the non-disjoint union

$$X_{\langle n+p \rangle} = \cup_{i_1,\ldots,i_p} X_{\langle n+p \rangle, i_1,\ldots,i_p}.$$

The boundary of $X_{\langle n \rangle}$, is defined by

$$\partial X_{\langle n \rangle} = \{(x, \ldots, x, \ldots), \text{ for a } x \text{ in } F\}.$$

We set $\overset{\circ}{X}_{\langle n \rangle} = X_{\langle n \rangle} \smallsetminus \partial X_{\langle n \rangle}$. Finally we set

$$\partial X_{\langle \infty \rangle} = \cup_{n=0}^{\infty} \cap_{m \geqslant n} \partial X_{\langle m \rangle}$$
$$= \{(j_k) \in X_{\langle \infty \rangle} \mid j_k = x \text{ for some } x \text{ in } F\},$$

and $\overset{\circ}{X}_{\langle \infty \rangle} = X_{\langle \infty \rangle} \smallsetminus \partial X_{\langle \infty \rangle}$.

REMARK 1.3. — The set $X_{\langle n \rangle}$ (resp. $X_{\langle \infty \rangle}$) contains the discrete set $F_{\langle n \rangle}$ (resp. $F_{\langle \infty \rangle}$) as the set of sequences (j_k) such that $j_k = x$ for a x in F and all $k \geqslant 0$. With this identification we clearly have $\partial X_{\langle n \rangle} = \partial F_{\langle n \rangle}$ and $\partial X_{\langle \infty \rangle} = \partial F_{\langle \infty \rangle}$.

1.1.3. Geometric embedding. — All classical examples come from self-similar sets which have a natural embedding in $\mathbb{R}^d$. We now describe how such a structure appears in geometrical examples of self-similar sets. It is essentially related to the property of finite ramification. Suppose given $\Psi_1, \ldots, \Psi_N$, N strictly contractive similitude of $\mathbb{R}^d$ with different fixed points $x_1, \ldots, x_N$. It is well-known that there exists a unique proper subset X of $\mathbb{R}^d$ such that

$$(8) \qquad\qquad X = \cup_{i=1}^{N} \Psi_i(X).$$

The set X is compact and actually equal to the set of limits $\lim_{n \to \infty} \Psi_{i_1} \circ \cdots \circ \Psi_{i_n}(y)$ for $(i_k) \in \{1, \ldots, N\}^{\mathbb{N}}$. It is easy to see that the former limit does not depend on y, so that it defines a mapping from $\{1, \ldots, N\}^{\mathbb{N}}$ onto X. Hence, X can be written $\{1, \ldots, N\}^{\mathbb{N}} / \sim$ for a certain equivalence relation $\sim$. We can easily check that $\sim$ can be constructed as previously if the set X is connected and if there exists a subset F of the set of fixed points of $\Psi_1, \ldots, \Psi_N$ such that

$$(9) \qquad\qquad \Psi_i(X) \cap \Psi_j(X) = \Psi_i(F) \cap \Psi_j(F), \quad \forall\, i \neq j.$$

(This last condition is usually called the condition of finite ramification.) Indeed we can as well suppose that $F = \{x_1, \ldots, x_{N_0}\}$ for $N_0 = |F| \leqslant N$ and identify F with $\{1, \ldots, N_0\}$. We define the relation $\mathcal{R}$ on $\{1, \ldots, N\} \times F$ by $(i,j)\mathcal{R}(i',j')$ if and only if $\Psi_i(x_j) = \Psi_{i'}(x_{j'})$. It is then easy to see that the relation $\sim$ we just defined is also the relation obtained as in section 1.1.2 from the relation $\mathcal{R}$. In this case the sequences $X_{\langle n \rangle}$ and $F_{\langle n \rangle}$ are naturally embedded in $\mathbb{R}^d$ as

$$X_{\langle n \rangle} = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(X),$$
$$F_{\langle n \rangle} = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(\cup_{j_1,\ldots,j_n} \Psi_{j_1} \circ \cdots \circ \Psi_{j_n}(F)).$$

The sets $\partial F_{\langle n \rangle}$ and $\partial X_{\langle n \rangle}$ are just scaled copies of F, given by

$$\partial F_{\langle n \rangle} = \partial X_{\langle n \rangle} = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(F).$$

1.1.4. Examples

The Sierpinski gasket. — In this case $N_0 = N = 3$, $F = \{1, 2, 3\}$ and the relation $\mathcal{R}$ is given on Figure 3. The set at level 4 is represented on Figure 2 in the introduction:
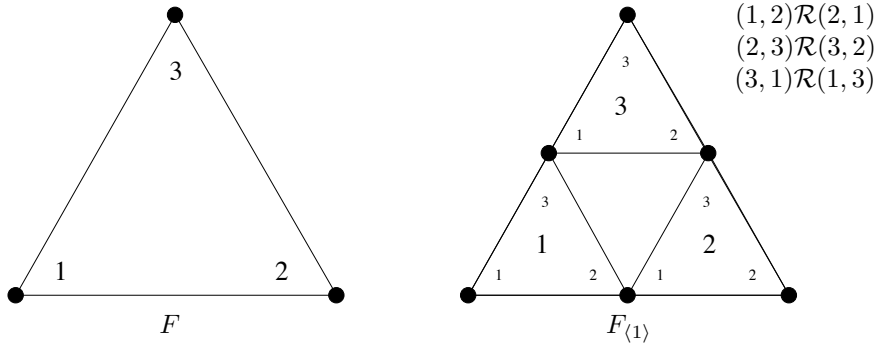
FIGURE 3

the initial cell $F_{\langle 0 \rangle}$ is represented by the bolded triangle for the blow-up starting from $(1, 1, 1, 1)$ on the left and for the blow-up starting from $(1, 3, 1, 2)$ on the right. Usually we take for G the group of permutations of $F = \{1, 2, 3\}$ (*i.e.* geometrically, G acts on $F_{\langle n \rangle}$ as the group of isometries of the boundary triangle $\partial F_{\langle n \rangle}$). But we could also consider the trivial group as the group of symmetries (this is considered in section 6). As it is well-known, the Sierpinski gasket is traditionally considered as a self-similar subset of $\mathbb{C}$. Let us now describe this and the relations with the discrete structure we just introduced. Consider the 3 homotheties

$$\Psi_1(x) = \frac{x}{2}; \quad \Psi_2(x) = \tfrac{1}{2}(x - 1) + 1;$$
$$\Psi_3(x) = \tfrac{1}{2}(x - (\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2})) + \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}.$$

There exists a unique proper subset of $\mathbb{C}$ that satisfies equation (8) and it is the celebrated Sierpinski gasket represented on picture 1 of the introduction. Remark that the set of fixed points $F = \{0, 1, \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\}$ of Ψ_1, Ψ_2, Ψ_3 satisfies (9) and defines the relation $\mathcal{R}$ as explained in section 1.1.3 by $(i, j)\mathcal{R}(i', j')$ if and only if $\Psi_i(x_j) = \Psi_{i'}(x_{j'})$ (if we denote by x_1, x_2, x_3 the fixed points of Ψ_1, Ψ_2, Ψ_3). The natural geometric representation of $X_{\langle n \rangle}$ is given by the sequence of preimages

$$X_{\langle n \rangle} = \Psi_{\omega_1}^{-1} \circ \cdots \circ \Psi_{\omega_n}^{-1}(X).$$

The set $F_{\langle n \rangle}$ is the set of vertices of the triangles of size 1 in $X_{\langle n \rangle}$.

The unit interval. — Let us first describe the continuous model, by its geometric representation. Consider $X = [0, 1]$ and a real $0 < \alpha < 1$. It is clear that X is self-similar with respect to the $N = 2$ homotheties $\Psi_1(x) = \alpha x$ and $\Psi_2(x) = 1 + (x-1)(1-\alpha)$. Remark that $F = \{0, 1\}$ satisfies equation (9): the abstract self-similar set would be constructed from the equivalence relation $\mathcal{R}$ such that $(1, 2)\mathcal{R}(2, 1)$. (Remark that when $\alpha = 1/2$ the abstract definition of the self-similar set $[0, 1]$ as a quotient of $\{1, 2\}^{\mathbb{N}}$ corresponds exactly to the expression of a point in $[0, 1]$ in base 2). We take for G the trivial group $G = \{\text{Id}\}$. We remark that for different values of α we just have different geometric representations of the same self-similar structure as defined in 1.1.2. We will consider on these sets different operators depending on α, which have a natural expression in these geometric representations. Concerning the blow-up we remark that $\partial X_{\langle \infty \rangle}$ is non empty if and only if ω is stationary. More precisely, if ω is stationary to 1 (resp. to 2) then $X_{\langle \infty \rangle}$ is a half-line bounded from the left (resp. from the right) (for example, if $\omega_n = 1$ then $X_{\langle n \rangle} = [0, \alpha^{-n}]$). If ω is not stationary then $X_{\langle \infty \rangle} = \mathbb{R}$.

The nested fractals. — The nested fractals define a class of finitely ramified self-similar sets, introduced by Lindström (*cf.* [**31**]), embedded in $\mathbb{R}^d$, which are invariant by a large group of symmetries. We refer to [**31**] for the definitions. Note that the Sierpinski gasket is the basic example of nested fractals.

1.2. Construction of a self-similar Laplacian

We fix for the rest of the text two N-tuples $(\alpha_1, \dots, \alpha_N)$ and $(\beta_1, \dots, \beta_N)$ of positive real numbers. The N-tuple $(\alpha_1, \dots, \alpha_N)$, resp. $(\beta_1, \dots, \beta_N)$ will represent the scaling in energy, resp. in measure in our structure. We suppose moreover that $(\alpha_1, \dots, \alpha_N)$, $(\beta_1, \dots, \beta_N)$ are G-invariant, *i.e.* that $(\alpha_{g \cdot 1}, \dots, \alpha_{g \cdot N}) = (\alpha_1, \dots, \alpha_N)$, $(\beta_{g \cdot 1}, \dots, \beta_{g \cdot N}) = (\beta_1, \dots, \beta_N)$. We set $\gamma_i = (\alpha_i \beta_i)^{-1}$ and we make the following assumption

(H) We suppose that $(\beta_1, \dots, \beta_N)$ is proportional to $(\alpha_1^{-1}, \dots, \alpha_N^{-1})$ so that γ_i does not depend on i. We denote by γ the common value of the γ_i.

1.2.1. Discrete difference operators. — We suppose given A, a semi-positive symmetric endomorphism of $\mathbb{R}^F$ of the form

$$(10) \qquad Af(x) = -\sum_{y \in F, y \neq x} a_{x,y}(f(y) - f(x)), \quad \forall f \in \mathbb{R}^F, \forall x \in F,$$

where $a_{x,y}$, $x \neq y$, are non negative reals such that $a_{x,y} = a_{y,x}$. We suppose moreover that A is irreducible, *i.e.* that the graph on F defined by strictly positive $a_{x,y}$ is connected and that A is G-invariant, *i.e.* that $a_{g \cdot x, g \cdot y} = a_{x,y}$ for all g in G. We suppose also given a G-invariant positive measure b on F.

REMARK 1.4. — A typical example is the discrete Laplace operator $Af(x) = -\sum_{y \neq x}(f(y) - f(x))$ and b the uniform measure on F. We denote by $A_{\langle n \rangle, i_1, \ldots, i_n}$ the symmetric operator on $\mathbb{R}^{F_{\langle n \rangle}}$ defined as the copy of A on the cell $F_{\langle n \rangle, i_1, \ldots, i_n}$, *i.e.* the operator defined for f in $\mathbb{R}^{F_{\langle n \rangle}}$ by

$$\begin{cases} (A_{\langle n \rangle, i_1, \ldots, i_n} f)_{|F_{\langle n \rangle, i_1, \ldots, i_n}} = A(f_{|F_{\langle n \rangle, i_1, \ldots, i_n}}), \\ A_{\langle n \rangle, i_1, \ldots, i_n} f(x) = 0 \ \text{ if } x \notin F_{\langle n \rangle, i_1, \ldots, i_n}. \end{cases}$$

N.B.: In the first line we considered $f_{|F_{\langle n \rangle, i_1, \ldots, i_n}}$ as a function on F since $F_{\langle n \rangle, i_1, \ldots, i_n}$ can be identified with F.

We denote by $b_{\langle n \rangle, i_1, \ldots, i_n}$ the measure on $F_{\langle n \rangle}$ defined as the copy of b on $F_{\langle n \rangle, i_1, \ldots, i_n}$, *i.e.* given by

$$\int_{F_{\langle n \rangle}} f \, db_{\langle n \rangle, i_1, \ldots, i_n} = \int f_{|F_{\langle n \rangle, i_1, \ldots, i_n}} \, db, \quad \forall f \in \mathbb{R}^{F_{\langle n \rangle}}.$$

Then we set

$$(11) \qquad A_{\langle n \rangle} = \sum_{i_1, \ldots, i_n = 1}^{N} \alpha_{\omega_n} \cdots \alpha_{\omega_1} \alpha_{i_1}^{-1} \cdots \alpha_{i_n}^{-1} A_{\langle n \rangle, i_1, \ldots, i_n},$$

$$(12) \qquad b_{\langle n \rangle} = \sum_{i_1, \ldots, i_n = 1}^{N} \beta_{\omega_n}^{-1} \cdots \beta_{\omega_1}^{-1} \beta_{i_1} \cdots \beta_{i_n} b_{\langle n \rangle, i_1, \ldots, i_n}.$$

REMARK 1.5. — We see from the definition that the value of $A_{\langle n \rangle}$ and $b_{\langle n \rangle}$ depend on the N-tuples $(\alpha_1, \ldots, \alpha_N)$ and $(\beta_1, \ldots, \beta_N)$ only up to a constant. Remark that $A_{\langle n \rangle}$ and $b_{\langle n \rangle}$ form an inductive sequence since, for $n \geqslant p$, if $\text{supp}(f) \subset \overset{\circ}{F}_{\langle p \rangle} \cup (\partial F_{\langle p \rangle} \cap \partial F_{\langle n \rangle})$ then

$$A_{\langle n \rangle} f = A_{\langle p \rangle} f \quad \text{and} \quad \int f \, db_{\langle n \rangle} = \int f \, db_{\langle p \rangle}.$$

(Indeed, this comes from the fact that $F_{\langle p \rangle} = F_{\langle n \rangle, \omega_n, \ldots, \omega_{p+1}}$). Therefore $b_{\langle n \rangle}$ can be extended to a measure $b_{\langle \infty \rangle}$ on $F_{\langle \infty \rangle}$. Similarly, the linear operators $A_{\langle n \rangle}$ can be extended to a linear operator $A_{\langle \infty \rangle}$ on $\mathbb{R}^{F_{\langle \infty \rangle}}$ (a priori $A_{\langle \infty \rangle}$ is defined on compactly supported functions of $F_{\langle \infty \rangle}$, but since there is only "local interactions", $A_{\langle \infty \rangle}$ can be extended to a linear operator on $\mathbb{R}^{F_{\langle \infty \rangle}}$ itself). Remark that thanks to the third property of $\mathcal{R}$ the operator $A_{\langle n \rangle}$ is conservative, *i.e.* $A_{\langle n \rangle} f = 0$ is equivalent to f constant.

Denote by $\langle \cdot, \cdot \rangle$ the usual scalar product on $\mathbb{R}^{F_{\langle n \rangle}}$. Let $H_{\langle n \rangle}^{+}$ be the operator on $L^2(F_{\langle n \rangle}, b_{\langle n \rangle})$ defined by:

$$(13) \qquad \langle A_{\langle n \rangle} f, g \rangle = -\int H_{\langle n \rangle}^{+} f g \, db_{\langle n \rangle} \quad \forall f, g \in \mathbb{R}^{F_{\langle n \rangle}}.$$

The operator $H_{\langle n \rangle}^{+}$ is semi-negative, self-adjoint and must be viewed as a discrete difference operator with Neumann boundary condition on $\partial F_{\langle n \rangle}$ (since no condition

is imposed on the value of the functions on the boundary points). The operator with Dirichlet boundary condition, denoted $H_{\langle n \rangle}^-$, is the self-adjoint operator on $\mathbb{R}^{\overset{\circ}{F}\langle n \rangle}$ defined as the restriction of $H_{\langle n \rangle}^+$ to $\mathbb{R}^{\overset{\circ}{F}\langle n \rangle} \simeq \{ f \in \mathbb{R}^{F_{\langle n \rangle}} \mid f_{|\partial F_{\langle n \rangle}} = 0 \}$. We sometimes write $\mathcal{D}_{\langle n \rangle}^+ = \mathbb{R}^{F_{\langle n \rangle}}$ and $\mathcal{D}_{\langle n \rangle}^- = \mathbb{R}^{\overset{\circ}{F}\langle n \rangle}$ for the domain of $H_{\langle n \rangle}^\pm$.

If $K > 0$ is such that $\langle Af, f \rangle \leqslant K \int f^2 db$ for all f in $\mathbb{R}^F$ then it is easy to see from (11) and (12) and assumption (H) that the same inequality is true for $A_{\langle n \rangle}$ and $b_{\langle n \rangle}$. Thus the sequence $H_{\langle n \rangle}^\pm$ is uniformly bounded for the operator norm on $L^2(b_{\langle n \rangle})$ and can be extended into a semi-negative, self-adjoint operator $H_{\langle \infty \rangle}^+$ on $\mathcal{D}_{\langle \infty \rangle}^+ = L^2(b_{\langle \infty \rangle})$. We define $H_{\langle \infty \rangle}^-$ as the restriction of $H_{\langle \infty \rangle}^+$ to $\mathcal{D}_{\langle \infty \rangle}^- = \{ f \in \mathcal{D}_{\langle \infty \rangle}^+ \mid f_{|\partial F_{\langle \infty \rangle}} = 0 \}$. Clearly, we have

$$\langle A_{\langle \infty \rangle} f, g \rangle = - \int H_{\langle \infty \rangle}^\pm f g \, db_{\langle \infty \rangle}, \quad \forall\, f, g \in \mathcal{D}_{\langle \infty \rangle}^\pm.$$

Finally note that if $\partial F_{\langle \infty \rangle} = \varnothing$ then the operators $H_{\langle \infty \rangle}^+$ and $H_{\langle \infty \rangle}^-$ are equal and in this case we simply write $H_{\langle \infty \rangle}$ for $H_{\langle \infty \rangle}^+ = H_{\langle \infty \rangle}^-$.

Let us now explain the consequences of condition (H). Let f be a function with support contained in $\overset{\circ}{F}_{\langle n \rangle}$. Denote by $\widetilde{f}$ the function on $F_{\langle \infty \rangle}$ with support in $F_{\langle n+p \rangle, i_1, \ldots, i_p}$ and which is a copy of f on $F_{\langle n+p \rangle, i_1, \ldots, i_p}$. Then from formula (11), (12) and (13) we see that

$$(14) \qquad (H_{\langle \infty \rangle}^\pm \widetilde{f})_{|F_{\langle n+p \rangle, i_1, \ldots, i_p}} = \gamma_{\omega_{n+1}} \cdots \gamma_{\omega_{n+p}} (\gamma_{i_1} \cdots \gamma_{i_p})^{-1} (H_{\langle \infty \rangle}^\pm f)_{|F_{\langle n \rangle}}$$

and that $H_{\langle \infty \rangle}^\pm \widetilde{f}$ is null on the complement of $F_{\langle n+p \rangle, i_1, \ldots, i_p}$. By (H) the coefficients $\gamma_{\omega_1} \cdots \gamma_{\omega_n} (\gamma_{i_1} \cdots \gamma_{i_p})^{-1}$ are equal to 1, which means that $H_{\langle \infty \rangle}^\pm$ is locally invariant by translation. This property is the counterpart of the property of statistical translation invariance traditionally assumed in the case of Schrödinger operator with random potential.

1.2.2. The continuous situation. — In this section we define a "natural" Laplace operator on the continuous sets $X_{\langle n \rangle}$. The problem of the construction of such an operator is not easy (*cf.* for example, [**31**], [**25**], [**30**], [**36**]) and it is now clear that the best framework to use is the framework of Dirichlet spaces. We essentially follow the definitions of [**36**]. We suppose here that $\sum_{i=1}^N \beta_i = 1$ and that $\alpha_i < 1$ for all i. We know that there exists a unique probability measure m on X such that

$$(15) \qquad \int_X f \, dm = \sum_{i=1}^N \beta_i \int_X f \circ \Psi_i \, dm.$$

We suppose given a G-invariant self-similar Dirichlet form $(a, \mathcal{D})$ in the sense of [**36**], *i.e.* an irreducible, local, conservative, regular Dirichlet form on (X, m) satisfying the conditions of theorem 2.6. of [**36**]. In particular, $(a, \mathcal{D})$ is self-similar with respect to

the weights $(\alpha_1, \ldots, \alpha_N)$, *i.e.* for all $f \in \mathcal{D}$, $f \circ \Psi_i$ is in $\mathcal{D}$ and

$$(16) \qquad a(f, f) = \sum_{i=1}^{N} (\alpha_i)^{-1} a(f \circ \Psi_i, f \circ \Psi_i),$$

and is G-invariant, *i.e.* $a(g \cdot f, g \cdot f) = a(f, f)$. In [**36**], a criterion is given for the existence and uniqueness of such a Dirichlet form. Remark that the weights $(\alpha_1, \ldots, \alpha_N)$ can be chosen only up to a constant, since the scaling factor is determined by equation (16). The problem of the existence and uniqueness of such a Dirichlet form is not trivial and has been investigated in [**31**], [**36**] (and references therein). This is related to the existence of a fixed point for a certain renormalization map (that corresponds to the restriction of the renormalization map T that we introduce in section 3 to a certain subset, *cf.* remark 3.11).

Remind that $X_{\langle n \rangle}$ is isomorphic to X so that we can define the Dirichlet form $(a_{\langle n \rangle}, \mathcal{D}_{\langle n \rangle})$ by $\mathcal{D}_{\langle n \rangle} = \mathcal{D}$ and $a_{\langle n \rangle} = \alpha_{\omega_1} \cdots \alpha_{\omega_n} a$ and the measure $b_{\langle n \rangle}$ on $X_{\langle n \rangle}$ by $b_{\langle n \rangle} = \beta_{\omega_1}^{-1} \cdots \beta_{\omega_n}^{-1} b$. If f in $\mathcal{D}_{\langle n+p \rangle}$ is such that $\mathrm{supp}(f) \subset \overset{\circ}{X}_{\langle n \rangle}$ then using (15) and (16) we see that $a_{\langle n+p \rangle}(f, f) = a_{\langle n \rangle}(f, f)$ and $\int f \, dm_{\langle n \rangle} = \int f \, dm_{\langle n+p \rangle}$. Hence, we see that $m_{\langle n \rangle}$ can be extended to a measure $m_{\langle \infty \rangle}$ on $X_{\langle \infty \rangle}$, and we set

$$\mathcal{D}_{\langle \infty \rangle} = \{ f \in L^2(X_{\langle \infty \rangle}, m_{\langle \infty \rangle}) \mid \sup_n a_{\langle n \rangle}(f_{|X_{\langle n \rangle}}, f_{|X_{\langle n \rangle}}) < \infty \}.$$

On $\mathcal{D}_{\langle \infty \rangle}$ we define $a_{\langle \infty \rangle}$ by $a_{\langle \infty \rangle}(f, f) = \lim_{n \to \infty} a_{\langle n \rangle}(f_{|X_{\langle n \rangle}}, f_{|X_{\langle n \rangle}})$. One can check that $a_{\langle \infty \rangle}$ is a local, regular, conservative and irreducible Dirichlet form (*cf.* [**18**]). We set $\mathcal{D}_{\langle n \rangle}^{-} = \{ f \in \mathcal{D}_{\langle n \rangle} \mid f_{|\partial X_{\langle n \rangle}} = 0 \}$ and $\mathcal{D}_{\langle n \rangle}^{+} = \mathcal{D}_{\langle n \rangle}$ (and idem for $\mathcal{D}_{\langle \infty \rangle}^{\pm}$). We define $H_{\langle n \rangle}^{\pm}$ and $H_{\langle \infty \rangle}^{\pm}$ as the infinitesimal generators of $(a_{\langle n \rangle}, \mathcal{D}_{\langle n \rangle}^{\pm})$ and $(a_{\langle \infty \rangle}, \mathcal{D}_{\langle \infty \rangle}^{\pm})$. Note that they satisfy the same property of local invariance by translation as in the discrete case, *i.e.* formula (14) is still valid.

1.2.3. Examples

The Sierpinski gasket. — If we take $G \sim S_3$ the group of isometries of the unit triangle F, then the values of (α_i) and (β_i) are determined up to a constant, so we can as well take in the lattice case $\alpha_i = 1$, $\beta_i = 1$. There is only one possible choice for A and b, up to a constant: we take for A the canonical discrete Laplace operator on F given by formula (10) with $a_{x,y} = 1$ if $x \neq y$, and for b the measure that gives a mass 1 to the points of F. The operator $A_{\langle n \rangle}$ is obviously given by formula (1) of the introduction and $b_{\langle n \rangle}$ is the measure that gives a mass 1 to the points of $\partial F_{\langle n \rangle}$ and 2 to the points of $\overset{\circ}{F}_{\langle n \rangle}$. Hence, the operator $H_{\langle n \rangle}^{\pm}$ is the operator defined by (2) in the introduction. In the continuous case the construction of the Laplace operator was initially done in [**2**] by probabilistic means. There is uniqueness of such an operator. The value of (α_i) and (β_i) is determined by equation (15) and (16) to $\alpha_i = 3/5$, $\beta_i = 1/3$ (*cf.* for example, [**30**]).

The unit interval. — In the lattice case we can take any α_i and β_i but since they matter only up to a constant we take $\alpha_1 = \alpha$, $\alpha_2 = (1 - \alpha)$ and $\beta_1 = 1 - \alpha$, $\beta_2 = \alpha$ (hence, assumption (H) is satisfied). In the lattice case the only possible choice for A is, up to a constant, the discrete Laplace operator

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The measure b is determined by two positive reals m_0, m_1 by $b = m_0 \delta_0 + m_1 \delta_1$.

In the continuous case we can explicitly construct the self-similar structure. Let m be the unique self-similar measure on X satisfying (15). Consider on $[0,1]$ the canonical Dirichlet form $a(f, f) = \int_0^1 (f')^2 dx$ defined on

$$\mathcal{D} = \{f \in L^2(m) \mid f' \in L^2([0,1], dx)\}.$$

By a simple change of variables it is clear that a satisfies (16) with $\alpha_1 = \alpha$ and $\alpha_2 = 1 - \alpha$. Hence we have explicitly constructed the self-similar Dirichlet space $(a, \mathcal{D})$, for all possible values of α. The operator $H^+ = H^+_{\langle 0 \rangle}$ is the operator $\frac{d}{dm}\frac{d}{dx}$ defined on

$$\{f \in L^2(X, m) \mid \exists\, g \in L^2(X, m),\ f(x) = ax + b + \int_0^x \int_0^y g(z) dm(z) dy,$$
$$f'(0) = f'(1) = 0\},$$

by $H^+ f = g$. Similarly H^- is the operator $\frac{d}{dm}\frac{d}{dx}$ with Dirichlet boundary condition on $\{0, 1\}$. The author considered this case in [**38**] and [**41**].

The nested fractals. — In general, for nested fractals we take all the α_i equal, and the β_i equal (in the continuous case the exact value of the α_i is given by the self-similar structure and the β_i must be equal to $1/N$). There is nothing special to say about the lattice case. In the continuous case, Lindström and Kusuoka constructed the self-similar Dirichlet space (*cf.* [**31**], [**30**]) and the author proved the uniqueness of such a self-similar Dirichlet space (*cf.* [**36**]).

1.3. The density of states

1.3.1. Definition

The lattice case. — Denote by $0 = \lambda^+_{\langle n \rangle, 1} > \lambda^+_{\langle n \rangle, 2} \geqslant \cdots \geqslant \lambda^+_{\langle n \rangle, |F_{\langle n \rangle}|}$ the eigenvalues of $H^+_{\langle n \rangle}$. Denote by $0 > \lambda^-_{\langle n \rangle, 1} \geqslant \cdots \geqslant \lambda^-_{\langle n \rangle, |\mathring{F}_{\langle n \rangle}|}$ the Dirichlet eigenvalues, *i.e.* the eigenvalues of $H^-_{\langle n \rangle}$.

Let $\nu^+_{\langle n \rangle}$ (resp. $\nu^-_{\langle n \rangle}$) be the counting measures of the Neumann (resp. Dirichlet spectrum) defined by:

$$(17) \qquad\qquad \nu^\pm_{\langle n \rangle} = \sum_k \delta_{\lambda^\pm_{\langle n \rangle, k}},$$

where δ_x stands for the Dirac mass at x. We write $\nu_{\langle n \rangle}^{\pm}(\lambda) = \int_{\lambda}^{0} \nu_{\langle n \rangle}^{\pm}(d\lambda)$, $\lambda \leqslant 0$, for its repartition function.

It is clear by construction that the counting measures do not depend on the blow-up ω (since the operators $H_{\langle n \rangle}^{\pm}$ are isomorphic for different blow-up ω).

DEFINITION 1.6. — If the limit

$$(18) \qquad \lim_{n \to \infty} \frac{1}{N^n} \nu_{\langle n \rangle}^{\pm}$$

exists and does not depend on the choice of the boundary condition then it is called the density of states and denoted by μ.

REMARK 1.7. — The existence of the limit is proved in [**18**] and in [**27**] but will also be a consequence of theorem (3.6).

REMARK 1.8. — The reader must be careful that our terminology is not coherent with the classical terminology of [**8**], [**33**] and with the terminology of our previous paper [**38**] where the measure μ is called the integrated density of states. Remark that despite the terminology, μ is a measure which may have no density.

The continuous case. — In this case the operator $H_{\langle n \rangle}^{\pm}$ has compact resolvent (*cf.* for example, [**30**]) and the eigenvalues form a non-increasing sequence going to $-\infty$. We adopt the same definition for the counting measures $\nu_{\langle n \rangle}^{\pm}$ and for the density of states as in the lattice case.

PROPOSITION 1.9. — *If the density of states exists then its repartition function* $\mu(\lambda) = \int_{\lambda}^{0} \mu(\lambda)(d\lambda)$ *satisfies:*

$$(19) \qquad \mu(\gamma\lambda) = N\mu(\lambda).$$

Proof. — This is clear from the scaling relations satisfied by $a_{\langle n \rangle}$ and $m_{\langle n \rangle}$. $\qquad \square$

1.3.2. The density of Neumann-Dirichlet eigenvalues. — We say that a function f is a Neumann-Dirichlet (N-D for short) eigenfunction of $H_{\langle n \rangle}$ with eigenvalue λ if it is both a Dirichlet and a Neumann eigenfunction (therefore we forget the superscript $\pm$ in $H_{\langle n \rangle}$ since it is at the same time an eigenfunction of $H_{\langle n \rangle}^{+}$ and $H_{\langle n \rangle}^{-}$), *i.e.* in the lattice case this means that

- f is in $\mathcal{D}_{\langle n \rangle}^{-}$, *i.e.* $f \in \mathbb{R}^{F_{\langle n \rangle}}$ and $f_{|\partial F_{\langle n \rangle}} = 0$,
- $\langle A_{\langle n \rangle} f, g \rangle = -\lambda \int fg db_{\langle n \rangle}$, for all function g in $\mathcal{D}_{\langle n \rangle}^{+} = \mathbb{R}^{F_{\langle n \rangle}}$.

and in the continuous case that

- f is in $\mathcal{D}_{\langle n \rangle}^{-}$,
- $a_{\langle n \rangle}(f, g) = -\lambda \int fg dm_{\langle n \rangle}$, for all function g in $\mathcal{D}_{\langle n \rangle}^{+}$.

We denote by $\nu_{\langle n \rangle}^{ND}$ the counting measure of the N-D eigenvalues of $H_{\langle n \rangle}$ (counted with multiplicity) and by $E_{\langle n \rangle}^{ND}$ the subspace of $\mathcal{D}_{\langle n \rangle}^{+}$ generated by the N-D eigenfunctions.

Remark that any function f of $E_{\langle n \rangle}^{ND}$, when extended by 0 to $F_{\langle n+p \rangle}$ (resp. $X_{\langle n+p \rangle}$) is a N-D eigenfunction of $H_{\langle n+p \rangle}$. When extended by 0 to $F_{\langle \infty \rangle}$ (resp. $X_{\langle \infty \rangle}$) it is an eigenfunction of $H_{\langle \infty \rangle}^{+}$ and $H_{\langle \infty \rangle}^{-}$, with compact support. Hence $E_{\langle n \rangle}^{ND}$ is an increasing sequence of subspaces of $\mathcal{D}_{\langle \infty \rangle}^{-}$ and we denote by $\mathcal{H}_{ND}$ the closure in $\mathcal{D}_{\langle \infty \rangle}^{-}$ of the space $\cup_n E_{\langle n \rangle}^{ND}$.

REMARK 1.10. — By definition, the restriction of $H_{\langle \infty \rangle}^{\pm}$ to the Hilbert subspace $\mathcal{H}_{ND}$ is purely punctual, more precisely, the set of Neumann-Dirichlet eigenfunctions form a Hilbert basis of compactly supported eigenfunctions.

It is easy to see that
$$\nu_{\langle n+1 \rangle}^{ND} \geqslant N \nu_{\langle n \rangle}^{ND}.$$
Indeed, if f is a N-D eigenfunction of $H_{\langle n \rangle}$ then we can construct N copies of f on the N $\langle n \rangle$-cells of $F_{\langle n+1 \rangle}$. Precisely, for all $i = 1, \ldots, N$ we consider the function f_i on $\mathbb{R}^{F_{\langle n+1 \rangle}}$ which is the copy of f on $F_{\langle n+1 \rangle, i}$ and equal to 0 on $F_{\langle n+1 \rangle} \smallsetminus F_{\langle n+1 \rangle, i}$. These functions form an orthogonal family of N-D eigenfunctions of $H_{\langle n+1 \rangle}$ with same eigenvalues (by the hypothesis (H) and formula (14)).

DEFINITION 1.11. — The limit
$$\frac{1}{N^n} \nu_{\langle n \rangle}^{ND}$$
exists and is called the density of N-D eigenvalues and denoted by μ^{ND}.

REMARK 1.12. — Obviously, the measure μ^{ND} is purely punctual. It is clear that supp μ^{ND} is the topological spectrum of the restriction of $H_{\langle \infty \rangle}^{\pm}$ to the Hilbert subspace $\mathcal{H}_{ND}$.

REMARK 1.13. — In the continuous case the repartition function $\mu^{ND}(\lambda) = \int_\lambda^0 d\mu^{ND}$, $\lambda \leqslant 0$, satisfies the same scaling relation as $\mu(\lambda)$: $\mu^{ND}(\gamma\lambda) = N\mu^{ND}(\lambda)$.

1.4. Some basic results

We recall from [**40**] three basic results on the spectrum of the operators $H_{\langle \infty \rangle}^{\pm}$ and their relations with the measures μ and μ^{ND}. For convenience, we suppose here the existence of the density of states. We denote by $\Sigma^{\pm}$ the spectrum of the operators $H_{\langle \infty \rangle}^{\pm}$ (and we simply write Σ when $\partial F_{\langle \infty \rangle} = \varnothing$). We recall that the essential spectrum is obtained from the spectrum by removing all isolated points corresponding to eigenvalues with finite multiplicity, we denote it by $\Sigma_{\mathrm{ess}}^{\pm}$.

PROPOSITION 1.14 ([**40**], proposition 1). — *For both the lattice and the continuous case we have the following:*

(i) *If the boundary set $\partial X_{\langle\infty\rangle} = \partial F_{\langle\infty\rangle}$ is empty then* $\operatorname{supp}\mu = \Sigma = \Sigma_{\mathrm{ess}}$.

(ii) *Otherwise we just have* $\operatorname{supp}\mu = \Sigma_{\mathrm{ess}}^+ = \Sigma_{\mathrm{ess}}^-$. *Moreover, the eigenvalues eventually lying in* $\Sigma^\pm \smallsetminus \operatorname{supp}(\mu)$ *have multiplicity 1.*

REMARK 1.15. — In [**41**], we show that in the case of the unit interval blown-up to the half-line $\mathbb{R}_+$ (by the constant blow-up $\omega_k = 1$) the spectrum of the operator can be pure point with isolated eigenvalues of multiplicity 1 lying in the complement of $\operatorname{supp}\mu$ and accumulating on $\operatorname{supp}\mu$. Therefore in this case the equality $\Sigma_{\mathrm{ess}}^\pm = \operatorname{supp}\mu$ is satisfied by not $\Sigma^\pm = \operatorname{supp}\mu$.

We endow $\Omega = \{1, \ldots, N\}^{\mathbb{N}}$ with the product of the uniform measure on $\{1, \ldots, N\}$. The next two propositions give almost sure results on the blow-up. Remind that the lattices $F_{\langle\infty\rangle}$ (and the sets $X_{\langle\infty\rangle}$) are not isomorphic for different blow-ups. Hence, to show the dependence of the operator $H_{\langle\infty\rangle}^\pm$ and of the spectrum $\Sigma^\pm$ on the blow-up we write $H_{\langle\infty\rangle}^\pm(\omega)$ and $\Sigma^\pm(\omega)$. We denote by $\Sigma_{\mathrm{ac}}^\pm(\omega)$, $\Sigma_{\mathrm{sc}}^\pm(\omega)$, and $\Sigma_{\mathrm{pp}}^\pm(\omega)$ resp. the absolutely continuous, singular continuous and pure point part of the Lebesgue decomposition of the spectrum of $H_{\langle\infty\rangle}^\pm(\omega)$ (*cf.* [**8**], [**33**] for definition, or [**40**]). The first result is the analogous of a classical result for ergodic families of Schrödinger operators (*cf.* [**8**] or [**33**]).

PROPOSITION 1.16 ([**40**], proposition 2). — *There exists deterministic* Σ, Σ_{ac}, Σ_{sc}, *and* Σ_{pp} *such that for almost all* ω *in* Ω *(for the product of the uniform measure on* $\{1, \ldots, N\}$) *we have* $\Sigma_{\cdot}^\pm(\omega) = \Sigma_{\cdot}$.

PROPOSITION 1.17 ([**40**], proposition 3). — *If the density of states is completely created by the N-D eigenvalues, i.e. if* $\mu^{ND} = \mu$ *then for almost all* ω *in* Ω *the set of N-D eigenfunctions is complete i.e.* $\mathcal{H}_{ND} = \mathcal{D}_{\langle\infty\rangle}^+(\omega) = \mathcal{D}_{\langle\infty\rangle}^-(\omega)$. *Thus, the spectrum is pure point with compactly supported eigenfunctions.*

CHAPTER 2

PRELIMINARIES

2.1. The notion of trace on a subset

Let F be a finite set and $F' \subset F$ a subset.

DEFINITION 2.1. — Let Q be a complex symmetric $F \times F$ matrix. We denote by $Q_{|F'}$ the restriction of Q to F', *i.e.* the $F' \times F'$ matrix defined by $(Q_{|F'})_{x,y} = Q_{x,y}$ for x, y in F'. We call trace on F' of the matrix Q, the $F' \times F'$ matrix $Q_{F'}$, given, when the expression is defined, by

$$Q_{F'} = \left((Q^{-1})_{|F'} \right)^{-1}.$$

N.B.: One must be careful that the close notations $(Q_{|F'})$ and $(Q_{F'})$ represent two different types of restriction.

N.B.: These definitions could of course be given for non symmetric matrices but we will only be concerned with the symmetric case.

REMARK 2.2. — $Q_{F'}$ is sometimes called the Schur complement and appears in several circumstances, *cf.* for example, [**32**], [**7**]. In [**9**] the properties of this operation are carefully investigated, this operation is called "la reponse du réseau".

PROPOSITION 2.3

(i) *If Q has the following block decomposition on F' and $F \smallsetminus F'$*

$$Q = \begin{pmatrix} Q_{|F'} & B \\ B^t & Q_{|F \smallsetminus F'} \end{pmatrix}$$

then

(20) $$Q_{F'} = Q_{|F'} - B(Q_{|F \smallsetminus F'})^{-1}B^t.$$

Therefore the map $Q \mapsto Q_{F'}$ is rational in the coefficients of Q with poles included in the set $\det(Q_{|F \smallsetminus F'}) = 0$.

(ii) *If* $\det(Q_{|F \smallsetminus F'}) \neq 0$, *then for any function* f *in* $\mathbb{C}^{F'}$, *we denote by* Hf, *the function of* $\mathbb{C}^F$ *given by*

$$
\begin{cases}
Hf = f & \text{on } F', \\
Hf = -(Q_{|F \smallsetminus F'})^{-1} B^t f & \text{on } F \smallsetminus F'.
\end{cases}
$$

We call Hf *the harmonic prolongation of* f *with respect to* Q *and we have* $Q_{F'}(f) = (Q(Hf))_{|F'}$.

(iii) *If moreover* Q *is real semi-positive then* $Q_{F'}$ *is characterized by*

$$
(21) \qquad \langle Q_{F'} f, f \rangle = \inf_{g \in \mathbb{R}^F,\, g_{|F'} = f} \langle Qg, g \rangle, \quad \forall\, f \in \mathbb{R}^{F'},
$$

where $\langle \cdot, \cdot \rangle$ *denotes the usual scalar product resp. on* $\mathbb{R}^{F'}$ *and* $\mathbb{R}^F$. *The infimum is reached at the unique point* Hf.

REMARK 2.4. — The terminology comes from the theory of Dirichlet forms: if Q is semi-positive and such that $\langle Q \cdot, \cdot \rangle$ is a Dirichlet form (*i.e.* Q is Markovian, *cf.* for example, [**36**]) then $\langle Q_{F'} \cdot, \cdot \rangle$ is a Dirichlet form called the trace of $\langle Q \cdot, \cdot \rangle$ on F' (*cf.* [**17**]). In particular, we remark that the Markov property is preserved by the operation of taking the trace.

Proof

(i) Let f be a function in $\mathbb{C}^F$ null on $F \subset F'$. Set $g = Q^{-1} f$. We easily get

$$
g_{|F \smallsetminus F'} = -(Q_{|F \smallsetminus F'})^{-1} B^t g_{|F'},
$$

and

$$
\begin{aligned}
f_{|F'} &= Q_{|F'} g_{|F'} + Q_{|F \smallsetminus F'} g_{|F \smallsetminus F'} \\
&= (Q_{|F'} - B(Q_{|F \smallsetminus F'})^{-1} B^t) g_{|F'}.
\end{aligned}
$$

By definition $((Q^{-1})_{|F'} f_{|F'}) = g_{|F'}$. This implies that $f_{|F'} = Q_{F'}(g_{|F'})$ and thus formula (20).

(ii) It is an immediate consequence of (i).

(iii) Classically, g realizes the infimum in (21) if and only if $(Qg)_{|F \smallsetminus F'} = 0$, *i.e.* if $B^t f + Q_{|F \smallsetminus F'} g_{|F \smallsetminus F'} = 0$. If $Q_{|F \smallsetminus F'}$ is invertible then g is unique and given on $F \smallsetminus F'$ by $g_{|F \smallsetminus F'} = -(Q_{|F \smallsetminus F'})^{-1} B^t f$. This implies that $g = Hf$ and thus concludes the proof of the proposition. $\qquad\square$

2.2. The Grassmann algebra

The operation of taking the trace of a symmetric matrix on a subset is central in our problem. It is a complicated operation since it is rational. However we can embed the space of symmetric matrices in a Grassmann algebra in such a way that this operation becomes linear. This will be crucial in our work in order to construct

a good renormalization map. This is the key tool we use to generalize some of our previous works (*cf.* [**38**]).

2.2.1. Definition. — As in 2.1, F is a finite set and $|F|$ denotes its cardinality, most of the time we identify F with $\{1, \ldots, |F|\}$. Consider $\overline{E}$ and E two copies of $\mathbb{C}^F$, with canonical basis $(\overline{\eta}_x)_{x \in F}$ and $(\eta_x)_{x \in F}$. We consider the Grassmann algebra $\bigwedge(\overline{E} \oplus E)$ defined by

$$\bigwedge(\overline{E} \oplus E) = \bigoplus_{k=0}^{2|F|} (\overline{E} \oplus E)^{\wedge k},$$

where $\wedge$ denotes the exterior product. We denote by $\mathcal{A}$ the subalgebra generated by the monomials containing the same number of variables $\overline{\eta}$ and η, *i.e.*

$$\mathcal{A} = \oplus_{k=0}^{|F|} \overline{E}^{\wedge k} \wedge E^{\wedge k}.$$

A canonical basis of $\mathcal{A}$ is

$$(1, \overline{\eta}_{i_1} \wedge \cdots \wedge \overline{\eta}_{i_k} \wedge \eta_{j_1} \wedge \cdots \wedge \eta_{j_k}, \ i_1 < \cdots < i_k, \ j_1 < \cdots < j_k, \ 1 \leqslant k \leqslant |F|).$$

We endow $\mathcal{A}$ with $\langle \cdot, \cdot \rangle$ the hermitian scalar product which makes this basis an orthonormal basis (with the convention that it is linear on the left and antilinear on the right). To simplify notations we will forget the sign $\wedge$ to denote the exterior product and simply write $\eta_i \eta_j$ for $\eta_i \wedge \eta_j$. Remark that the elements of $\mathcal{A}$ commute since $\mathcal{A}$ is generated by the monomials of even degrees.

If Q is a $F \times F$ matrix then we denote $\overline{\eta} Q \eta$ the element of $\mathcal{A}$:

$$\overline{\eta} Q \eta = \sum_{i,j \in F} Q_{i,j} \overline{\eta}_i \eta_j.$$

We will be particularly interested in terms of the type

$$\exp(\overline{\eta} Q \eta) = \sum_{k=0}^n \frac{1}{k!} \Big(\sum_{i,j} Q_{i,j} \overline{\eta}_i \eta_j \Big)^k$$

$$(22) \qquad = \sum_{k=0}^n \sum_{\substack{i_1 < \cdots < i_k \\ j_1 < \cdots < j_k}} \det\Big((Q)_{\substack{i_1, \ldots, i_k \\ j_1, \ldots, j_k}} \Big) \overline{\eta}_{i_1} \eta_{j_1} \cdots \overline{\eta}_{i_k} \eta_{j_k},$$

where $(Q)_{\substack{i_1, \ldots, i_k \\ j_1, \ldots, j_k}}$ is the $k \times k$ matrix obtained from Q by keeping only the lines $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$.

LEMMA 2.5. — *Let Q be a complex $|F| \times |F|$ matrix, then*

$$(23) \qquad \qquad \| \exp \overline{\eta} Q \eta \|^2 = \det(\mathrm{Id} + Q Q^*)$$

$$(24) \qquad \qquad = \prod_{i=1}^{|F|} (1 + \rho_i^2)$$

where $\rho_1 \leqslant \cdots \leqslant \rho_{|F|}$ are the characteristic roots of Q, i.e. the eigenvalues of $\sqrt{Q^* Q}$, and $\| \ \|$ is the norm induced by the canonical scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{A}$.

Proof. — It is well-known that we can find unitary matrices U, W such that

$$Q = W \begin{pmatrix} \rho_1 & & \\ & \ddots & \\ & & \rho_{|F|} \end{pmatrix} U.$$

Denote $\overline{\zeta} = \overline{\eta} W$ (*i.e.* $\overline{\zeta}_k = \sum_i W_{i,k} \overline{\eta}_i$ for all k) and $\psi = U\eta$ (*i.e.* $\psi_k = \sum_j U_{k,j} \psi_j$), we have

$$\exp(\overline{\eta} Q \eta) = \exp \overline{\zeta} \begin{pmatrix} \rho_1 & & \\ & \ddots & \\ & & \rho_{|F|} \end{pmatrix} \psi$$

$$= 1 + \sum_{i=1}^{|F|} \rho_i \, \overline{\zeta}_i \psi_i + \sum_{i_1 < i_2} \rho_{i_1} \rho_{i_2} \, \overline{\zeta}_{i_1} \psi_{i_1} \overline{\zeta}_{i_2} \psi_{i_2} + \cdots$$

$$+ \rho_1 \cdots \rho_{|F|} \, \overline{\zeta}_1 \psi_1 \cdots \overline{\zeta}_{|F|} \psi_{|F|}$$

But the family of vectors $\left(1, (\overline{\zeta}_i \psi_i), (\overline{\zeta}_{i_1} \psi_{i_1} \overline{\zeta}_{i_2} \psi_{i_2}), \ldots, \overline{\zeta}_1 \psi_1 \cdots \overline{\zeta}_{|F|} \psi_{|F|} \right)$ is orthonormal, so we proved (23). $\qquad \square$

If Y is in $\mathcal{A}$ we denote by i_Y the interior product by Y, *i.e.* the linear operator $i_Y : \mathcal{A} \to \mathcal{A}$ defined by

$$(25) \qquad \langle i_Y(X), Z \rangle = \langle X, YZ \rangle, \quad \forall X, Z \in \mathcal{A}.$$

In particular, remark that

$$i_{\prod_{x \in F} \overline{\eta}_x \eta_x} (\exp \overline{\eta} Q \eta)) = \det Q.$$

Suppose now that F' is a subset of F, and denote by $\mathcal{A}_{F'}$ the subalgebra of $\mathcal{A}$ generated by the variables $(\overline{\eta}_x)_{x \in F'}$, $(\eta_x)_{x \in F'}$. We define the linear operator

$$(26) \qquad \begin{aligned} R_{F \to F'} &: \mathcal{A} \longrightarrow \mathcal{A}_{F'} \\ X &\longmapsto i_{\prod_{x \in F \setminus F'} \overline{\eta}_x \eta_x}(X). \end{aligned}$$

REMARK 2.6. — The operator $R_{F \to F'}$ is often presented as an antisymmetric integral. More precisely, $R_{F \to F'}(X)$ coincides with the antisymmetric integral of X with respect to $\prod_{x \in F \setminus F'} d\eta_x \overline{\eta}_x$, *i.e.* $R_{F \to F'}(X) = \int X \prod_{x \in F \setminus F'} d\eta_x \overline{\eta}_x$, as defined in [**5**] (*cf.* also [**47**]). The antisymmetric integral appears in the context of supersymmetry, as an antisymmetric counterpart of the Gaussian integral. It is interesting to note that supersymmetry has been used in the context of random Schrödinger operators several times (*cf.* for instance [**28**], [**47**], [**51**] and references therein).

PROPOSITION 2.7. — *Let Q be a complex symmetric $F \times F$ matrix, we have*

$$(27) \qquad \det(Q) = \left\langle R_{F \to F'}(\exp \overline{\eta} Q \eta), \prod_{x \in F'} \overline{\eta}_x \eta_x \right\rangle,$$

$$(28) \qquad \det(Q_{|F \setminus F'}) = \left\langle R_{F \to F'}(\exp \overline{\eta} Q \eta), 1 \right\rangle,$$

and

$$(29) \qquad \exp(\overline{\eta}Q_{F'}\eta) = \frac{R_{F\to F'}(\exp\overline{\eta}Q\eta)}{\det(Q_{|F\smallsetminus F'})},$$

when $\det(Q_{|F\smallsetminus F'}) \neq 0$.

Proof. — The first two formulas are simple consequences of the definitions. For a matrix Q we denote by $\det(Q_{\substack{i_1,\dots,i_k \\ j_1,\dots,j_k}})$ the determinant of the matrix where we keep only the lines $i_1,\dots,i_k$ and columns $j_1,\dots,j_k$, and by $\det(Q^{\substack{i_1,\dots i_k \\ j_1,\dots,j_k}})$ the determinant of the matrix where we removed the lines $i_1,\dots i_k$ and the columns $j_1,\dots,j_k$. It is well-known that

$$\det(Q^{\substack{i_1,\dots,i_k \\ j_1,\dots,j_k}}) = \det((Q^{-1})_{\substack{i_1,\dots,i_k \\ j_1,\dots,j_k}}) \det Q.$$

Let $i_1 < \cdots < i_k, \widehat{i}_1 < \cdots < \widehat{i}_{|F'|-k}$ be elements of F' such that

$$\{i_1,\dots,i_k,\widehat{i}_1,\dots,\widehat{i}_{|F'|-k}\} = F',$$

and $j_1 < \cdots < j_k, \widehat{j}_1 < \cdots < \widehat{j}_{|F'|-k}$ be elements of F' such that

$$\{j_1,\dots,j_k,\widehat{j}_1,\dots,\widehat{j}_{|F'|-k}\} = F'.$$

We have

$$\begin{aligned}
\left\langle R_{F\to F'}(\exp\overline{\eta}Q\eta), \overline{\eta}_{i_1}\eta_{j_1}\cdots\overline{\eta}_{i_k}\eta_{j_k}\right\rangle &= \det(Q^{\substack{\widehat{i}_1,\dots,\widehat{i}_{|F'|-k} \\ \widehat{j}_1,\dots,\widehat{j}_{|F'|-k}}}) \\
&= \det Q \det((Q^{-1})_{\substack{\widehat{i}_1,\dots,\widehat{i}_{|F'|-k} \\ \widehat{j}_1,\dots,\widehat{j}_{|F'|-k}}}) \\
&= \det((Q_{F'})_{\substack{i_1,\dots,i_k, \\ j_1,\dots,j_k}}) \frac{\det Q}{\det(Q_{F'})} \\
&= \left\langle \frac{\det Q}{\det(Q_{F'})} \exp\overline{\eta}Q_{F'}\eta, \overline{\eta}_{i_1}\eta_{j_1}\cdots\overline{\eta}_{i_k}\eta_{j_k}\right\rangle.
\end{aligned}$$

Evaluating the equality for $k = 0$ we get

$$\langle R_{F\to F'}(\exp\overline{\eta}Q\eta), 1\rangle = \det(Q_{|F\smallsetminus F'}) = \frac{\det Q}{\det Q_{F'}},$$

and this is enough to conclude the proof of the proposition. $\square$

Let us introduce a notation: if f is a holomorphic function from a domain $D \subset \mathbb{C}^n$ to $\mathbb{C}^m$ then we denote by $\mathrm{ord}(f, x_0)$ the order of vanishing of f at the point $x^0 \in D$, *i.e.* the maximal integer p such that one can find an open set U containing x_0 and holomorphic functions $h_{i_1,\dots,i_p}$, $1 \leqslant i_1 \leqslant \cdots \leqslant i_p \leqslant n$ on U such that

$$f = \sum_{i_1 \leqslant \cdots \leqslant i_p} (x_{i_1} - x_{i_1}^0)\cdots(x_{i_p} - x_{i_p}^0) h_{i_1,\dots,i_p}(x), \quad \text{on } U .$$

If Q is a $F \times F$ symmetric matrix then we denote by $\ker^{ND}(Q)$ (for the Neumann-Dirichlet kernel) the subspace $\ker^{ND}(Q) = \{f \in \ker(Q), f_{|F'} = 0\}$.

PROPOSITION 2.8

(i) *If Q_0 is a symmetric $F \times F$ real matrix then:*

$$(30) \qquad \dim(\ker^{ND}(Q_0)) = \mathrm{ord}\,(Q \longmapsto R_{F \to F'}(\exp \overline{\eta} Q \eta), Q_0)\,,$$

where in the right hand side Q is taken from the set of complex symmetric $F \times F$ matrices.

(ii) *Moreover, if B is a real positive definite symmetric $F \times F$ matrix then*

$$(31) \qquad \dim(\ker^{ND}(Q_0)) = \mathrm{ord}\,(\lambda \longmapsto R_{F \to F'}(\exp \overline{\eta}(Q_0 - \lambda B)\eta), 0)\,.$$

REMARK 2.9. — The fact that the matrix Q_0 is real is crucial since this result is essentially related to the fact that this matrix is diagonalizable.

REMARK 2.10. — A priori the order of vanishing of the function

$$\lambda \longmapsto R_{F \to F'}(\exp \overline{\eta}(Q_0 - \lambda B)\eta)$$

at $\lambda = 0$ is greater or equal to the order of vanishing of $Q \mapsto R_{F \to F'}(\exp \overline{\eta} Q \eta)$ at Q_0. The point (ii) tells us that there is actually equality, *i.e.* that the line $\{Q_0 - \lambda B, \lambda \in \mathbb{C}\}$ intersects transversally at $\lambda = 0$ the analytic set $\{Q \mid R_{F \to F'}(\exp \overline{\eta} Q \eta) = 0\}$.

Proof

(i) The point (i) is a consequence of (ii). Indeed, the order of vanishing of $Q \mapsto R_{F \to F'}(\exp \overline{\eta} Q \eta)$ is equal to the order of vanishing in a generic direction. Otherwise stated, this means that there is a proper analytic subset $\mathcal{A} \subset \mathrm{Sym}_F$ such that for any direction B in $\mathrm{Sym}_F \smallsetminus \mathcal{A}$ the order of vanishing in (i) is equal to the order of vanishing of the function $\lambda \mapsto R_{F \to F'}(\exp \overline{\eta}(Q_0 - \lambda B)\eta)$ at the point $\lambda = 0$. Denote by $\mathrm{Sym}_{F,\mathbb{R}}$ the space of real symmetric $F \times F$ matrices, regarded as the set of real directions in the tangent vector space Sym_F. Since $\mathcal{A}$ is analytic, $\mathcal{A} \cap \mathrm{Sym}_{F,\mathbb{R}}$ is of empty interior. Considering that the subset of B in $\mathrm{Sym}_{F,\mathbb{R}}$ which are positive definite is open in $\mathrm{Sym}_{F,\mathbb{R}}$, we know that this set cannot be contained in $\mathcal{A}$, hence if we assume (ii) we know that $\dim \ker(Q_0)$ is the order of annulation in a generic direction. Hence (ii) implies (i).

(ii) We first derive an explicit expression for $T(Q_0 - \lambda B)$. Since Q_0 is real and B real positive definite we can diagonalize $(Q_0)_{|F \smallsetminus F'}$ in an orthonormal basis for $B_{|F \smallsetminus F'}$, *i.e.* we can find eigenvalues $\lambda_1^-, \ldots, \lambda_{|F \smallsetminus F'|}^-$ and a family of functions $f_1^-, \ldots, f_{|F \smallsetminus F'|}^-$ in $\mathbb{R}^{F \smallsetminus F'}$ such that

$$\left\langle f_k^-, B f_{k'}^- \right\rangle_{F \smallsetminus F'} = \delta_{k,k'},$$
$$(Q_0)_{|F \smallsetminus F'} f_k^- = \lambda_k^-(B_{|F \smallsetminus F'}) f_k^-,$$

where $\langle, \rangle_{F \smallsetminus F'}$ is the usual scalar product on $\mathbb{R}^{F \smallsetminus F'}$. For a real function f on F' we denote by $H_\lambda f$ the harmonic prolongation of f with respect to $Q_0 - \lambda B$. The function

$H_\lambda f$ can be written

$$H_\lambda f = f + \sum_{k=1}^{|F \smallsetminus F'|} c_k f_k^- ,$$

and we easily get

$$c_k = \frac{\left\langle f, ((Q_0 - \lambda B) f_k^-)_{|F'} \right\rangle_{F'}}{\lambda - \lambda_k^-}.$$

We set $\phi_{k,\lambda} = ((Q_0 - \lambda B) f_k^-)_{|F'}$ and we denote by $p_{k,\lambda} : \mathbb{R}^{F'} \to \mathbb{R}^{F'}$ the projector given by $p_{k,\lambda}(f) = \langle f, \phi_{k,\lambda} \rangle_{F'} \phi_{k,\lambda}$. Thus we have from proposition 2.3 (ii)

$$TQ = (Q_0 - \lambda B)_{|F'} + \sum_{k=1}^{|F \smallsetminus F'|} \frac{p_{k,\lambda}}{\lambda - \lambda_k^-}.$$

Denote now by n_0 the dimension of $\ker^{ND}(Q_0)$, and by n_0' the dimension of $\ker((Q_0)_{|F \smallsetminus F'})$, so that $n_0' \geqslant n_0$. This means that n_0' of the eigenvalues λ_k^- are null. We can as well suppose that $\lambda_1^- = \cdots = \lambda_{n_0'}^- = 0$ and that $f_1^-, \ldots, f_{n_0}^-$ form an orthonormal basis (w.r. to B) of $\ker^{ND}(Q_0)$. This implies that for $k \leqslant n_0$, $Q_0 f_k^- = 0$ and thus that $\phi_{k,\lambda} = \lambda (B f_k^-)_{|F'}$. For $k \leqslant n_0$ we denote by $\widetilde{p}_k : \mathbb{R}^{F'} \to \mathbb{R}^{F'}$ the projector given by

$$\widetilde{p}_k(f) = \left\langle f, (B f_k^-)_{|F'} \right\rangle (B f_k^-)_{|F'}$$

and we have $p_{k,\lambda} = \lambda^2 \widetilde{p}_k$. For $k \geqslant n_0 + 1$ we simply denote $\phi_k = \phi_{k,0}$ and $p_k = p_{k,0}$. The functions $\{\phi_k\}_{k=n_0+1,\ldots,n_0'}$ are linearly independent. Indeed, otherwise there would exit a linear combination of the f_k^-, $k = n_0 + 1, \ldots, n_0'$, belonging to $\ker^{ND}(Q_0)$. This is not possible since the dimension of $\ker^{ND}(Q_0)$ is n_0. Considering relation (29), we see that for small λ's the function $R_{F \to F'}(\exp(\overline{\eta}(Q_0 - \lambda B)\eta))$ can be written

$$(-\lambda)^{n_0'} \Big(\prod_{k=n_0'+1}^{|F|} (\lambda_k^- - \lambda) \Big) \exp(\overline{\eta}(Q_0 - \lambda B)_{|F'}\eta)$$

$$\exp\Big(\overline{\eta}\Big(\sum_{k=n_0+1}^{n_0'} \frac{1}{\lambda} p_{k,\lambda} \Big)\eta \Big) \exp\Big(\overline{\eta}\Big(\sum_{k=n_0'+1}^{|F \smallsetminus F'|} \frac{1}{\lambda - \lambda_k^-} p_{k,\lambda} \Big)\eta \Big) \exp\Big(\lambda\overline{\eta}\Big(\sum_{k=1}^{n_0} \widetilde{p}_k \Big)\eta \Big)$$

$$= C\lambda^{n_0'} \exp(\overline{\eta} Q_0 \eta) \exp\Big(\overline{\eta}\Big(\sum_{k=n_0+1}^{n_0'} \frac{1}{\lambda} p_k \Big)\eta \Big) \exp\Big(\overline{\eta}\Big(\sum_{k=n_0'+1}^{|F \smallsetminus F'|} \frac{-1}{\lambda_k^-} p_k \Big)\eta \Big)(1 + o(\lambda))$$

where C is a non null constant. Considering that the operators p_k have rank 1, the last expression equals

$$C\lambda^{n_0} \prod_{k=n_0+1}^{n_0'} (\lambda + \overline{\eta} p_k \eta) \exp\Big(\overline{\eta}\Big(\sum_{k=n_0'+1}^{|F \smallsetminus F'|} \frac{1}{\lambda_k^-} p_k \Big)\eta \Big)(1 + o(\lambda)).$$

From this we deduce that λ^{n_0} can be factorized in the last expression, hence that the order of vanishing of the function $\lambda \mapsto R_{F \to F'}(\exp \overline{\eta}(Q_0 - \lambda B)\eta)$ is at least n_0. In the last expression, the term of degree $n_0' - n_0$ in the variables $\overline{\eta}$ and in the variables η is

$$\lambda^{n_0} \prod_{k=n_0+1}^{n_0'} (\overline{\eta} p_k \eta) + (\text{terms of order } \lambda^k, \ k > n_0).$$

Since the p_k's are linearly independent $\prod_{k=n_0+1}^{n_0'}(\overline{\eta} p_k \eta)$ is not null. This proves that the order of vanishing of $\lambda \mapsto R_{F \to F'}(\exp \overline{\eta}(Q_0 - \lambda B)\eta)$ is exactly n_0. $\square$

2.2.2. The Lagrangian Grassmannian. — We denote by Sym_F the space of complex symmetric $|F| \times |F|$ matrices. We denote by $\mathcal{P}(\mathcal{A})$ the projective space associated with $\mathcal{A}$ and by $\pi : \mathcal{A} \to \mathcal{P}(\mathcal{A})$ the canonical projection. It is clear that the map $Q \mapsto \pi(\exp(\overline{\eta} Q \eta))$ is injective and hence defines an embedding of Sym_F in $\mathcal{P}(\mathcal{A})$. In this section we describe the subvariety defined as the closure of the set of points of the type $\pi(\exp(\overline{\eta} Q \eta))$ for Q in Sym_F. This subvariety defines a compactification of the set Sym_F, and we will see that it is a Lagrangian Grassmannian. This type of compactification already appeared in the context of electrical network, *cf.* [**9**], [**10**].

We first recall some classical notions. Let n be an integer and $(\cdot, \cdot)$ be the bilinear form on $\mathbb{C}^{2n}$ given by $(X, Y) = \sum X_i Y_i$ and J be the $2n \times 2n$ matrix given by:

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix.

Obviously $(\cdot, J \cdot)$ is an antisymmetric non-degenerate bilinear form (usually called the symplectic form when considered on $\mathbb{R}^{2n}$).

DEFINITION 2.11. — A linear subspace L of $\mathbb{C}^{2n}$ is called Lagrangian if for all x, y in L, $(x, Jy) = 0$.

We denote by $\mathbb{L}^n$ the set of n-dimensional Lagrangian subspace of $\mathbb{C}^{2n}$. It is a homogeneous space. It can be indeed described as the quotient of the complex symplectic group by the stabilizer of a point. Therefore $\mathbb{L}^n$ is a $n(n+1)/2$ compact smooth manifold. To precise the situation we describe explicitly a local parameterization. At a point L the set $\mathbb{L}^n$ can be parameterized explicitly by the space Sym_n of symmetric $n \times n$ complex matrices. Indeed, if $(v_1, \ldots, v_n)$ and $(v_1', \ldots, v_n')$ are orthonormal basis of respectively L and $L^\perp$ (for the usual scalar product on $\mathbb{C}^{2n}$ and $L^\perp$ the orthogonal subspace of L for this scalar product) then the map

$$(32) \qquad \begin{aligned} \mathrm{Sym}_n &\longrightarrow \mathbb{L}^n \\ Q &\longmapsto \mathrm{Vect}\{v_i + \textstyle\sum_j Q_{i,j} v_j'\}_{i=1\ldots n} \end{aligned}$$

defines a local set of coordinates. Indeed, it is easy to check that the subspace $\mathrm{Vect}\{v_i + \sum_j Q_{i,j} v'_j\}_{i=1\ldots n}$ is Lagrangian and that any Lagrangian subspace in a neighborhood of L can be represented in such a form.

Let $(e_1, \ldots, e_n, e'_1, \ldots e'_n)$ denote the canonical basis of $\mathbb{C}^{2n}$. We denote by $\mathcal{P}(\wedge^n \mathbb{C}^{2n})$ the projective space associated with $\wedge^n \mathbb{C}^{2n}$ and by $\pi : \wedge^n \mathbb{C}^{2n} \to \mathcal{P}(\wedge^n \mathbb{C}^{2n})$ the canonical surjection. Classically, the manifold $\mathbb{L}^n$ can be embedded in the projective space $\mathcal{P}(\wedge^n \mathbb{C}^{2n})$ by the Plücker embedding

$$(33) \qquad \begin{aligned} \mathbb{L}^n &\longrightarrow \mathcal{P}(\wedge^n \mathbb{C}^{2n}) \\ L = \mathrm{Vect}\{x_1, \ldots, x_n\} &\longmapsto \pi(x_1 \wedge \cdots \wedge x_n). \end{aligned}$$

We come back now to the Grassmann algebra introduced in section 2.2.1. Take $n = |F|$, the subalgebra $\mathcal{A}$ can be easily identified with $\wedge^{|F|} \mathbb{C}^{2|F|}$ by the isomorphism defined on monomials by:

$$(34) \qquad \begin{aligned} \wedge^{|F|} \mathbb{C}^{2|F|} &\longrightarrow \mathcal{A} \\ e_{i_1} \wedge \cdots \wedge e_{i_{|F|-k}} \wedge e'_{j_1} \wedge \cdots \wedge e'_{j_k} &\longmapsto (-1)^{\sum_{p=1}^{|F|-k} i_p - p} \overline{\eta}_{\widehat{i}_1} \eta_{j_1} \cdots \overline{\eta}_{\widehat{i}_k} \eta_{j_k}, \end{aligned}$$

where $i_1 < \cdots < i_{|F|-k}$, $j_1 < \cdots < j_k$ and $\widehat{i}_1, \ldots, \widehat{i}_k$ is defined by $\widehat{i}_1 < \cdots < \widehat{i}_k$ and $\{\widehat{i}_1, \ldots, \widehat{i}_k, i_1, \ldots, i_{|F|-k}\} = \{1, \ldots, |F|\}$.

Thanks to the embedding (33) and the isomorphism (34) the manifold $\mathbb{L}^{|F|}$ can be considered as a smooth subvariety of $\mathcal{P}(\mathcal{A})$. It is easy to see that by the isomorphism (34) the point

$$\wedge_{i=1}^{|F|} (e_i + \sum Q_{i,j} e_j)$$

is sent to the point $\exp(\overline{\eta} Q \eta)$ and thus that $\pi(\exp(\overline{\eta} Q \eta))$ is in $\mathbb{L}^{|F|}$. Hence we deduce the following

PROPOSITION 2.12. — *The application $Q \mapsto \pi(\exp \overline{\eta} Q \eta)$ defines an embedding of the space Sym_F of $F \times F$ symmetric matrices into the smooth projective subvariety $\mathbb{L}^{|F|}$.*

More precisely, the set Sym_F is sent onto the subset $\mathbb{L}^{|F|} \smallsetminus \pi\{X \in \mathcal{A}, \langle X, 1 \rangle = 0\}$. Hence, the closure of the set of points of the type $\pi(\exp \overline{\eta} Q \eta)$ is equal to $\mathbb{L}^{|F|}$.

REMARK 2.13. — Therefore the set $\mathbb{L}^{|F|}$ defines a compactification of Sym_F. There are many different compactifications of Sym_F (for example, in section 4.4 we consider the compactification by a projective space) but this one seems to be the best-suited to our problem.

We will need some results on the dimension of the cohomology groups of $\mathbb{L}^{|F|}$. We recall from [**42**] that the first and second betti numbers are given by:

$$(35) \qquad b_1 = \dim(H^1(\mathbb{L}^{|F|}, \mathbb{C})) = 0, \quad b_2 = \dim(H^2(\mathbb{L}^{|F|}, \mathbb{C})) = 1.$$

The manifold $\mathbb{L}^{|F|}$ is a Kähler manifold, as a smooth projective subvariety. A natural Kähler form on $\mathbb{L}^{|F|}$ is the restriction of the Fubini-Study form on $\mathcal{P}(\mathcal{A})$ (*cf.* appendix A.5). By definition, the Kähler form is in $H^{1,1}(\mathbb{L}^{|F|})$ the $(1,1)$ Dolbeault cohomology

group of $\mathbb{L}^{|F|}$ (which coincides for Kähler manifold with the subspace of $H^2(\mathbb{L}^{|F|}, \mathbb{C})$ generated by the forms of the type $(1,1)$, *cf.* appendix C). Thus we have by (35) and general results on Kähler manifold that $\dim(H^{2,0}(\mathbb{L}^{|F|})) = \dim(H^{0,2}(\mathbb{L}^{|F|})) = 0$ and $\dim(H^{1,1}(\mathbb{L}^{|F|})) = 1$.

G-invariant Lagrangian Grassmannian. — We suppose now given a finite group G acting on F. We denote by $\mathbb{L}^G$ the closure in $\mathbb{L}^{|F|}$ of the subset Sym^G, the space of G-invariant complex $F \times F$ matrices. As we shall see in Appendix E, $\mathbb{L}^G$ is a smooth projective variety, whose structure can be explicitly described. It is also clear that for the isomorphism (34) the submanifold $\mathbb{L}^G$ is the closure in $\mathcal{P}(\mathcal{A})$ of the points of the type $\pi(\exp \overline{\eta} Q \eta)$ for Q in Sym^G.

REMARK 2.14. — Assume that F' is a subset of F, invariant by the group G. The element $\pi(R_{F \to F'}(\exp \overline{\eta} Q \eta))$ is then in the G-invariant Lagrangian Grassmannian associated with F'. Moreover, remark that formula (30) of proposition (2.8) remains valid for Q_0 in Sym^G if we let Q run in Sym^G instead of Sym_F. Indeed, using (ii) of proposition 2.8, we know that for B real symmetric positive, G-invariant

$$\dim \ker^{ND}(Q_0) = \mathrm{ord}(\lambda \longmapsto R_{F \to F'}(\exp \overline{\eta}(Q_0 - \lambda B)\eta), 0)$$
$$\geqslant \mathrm{ord}(Q \longmapsto R_{F \to F'}(\exp(\overline{\eta} Q \eta)), Q_0)$$

for Q running in Sym^G. This last expression is bounded from below by the same expression for Q running in Sym_F instead of Sym^G, which is equal to $\dim \ker^{ND}(Q_0)$ by proposition 2.8, (i).

2.3. Trace of a Dirichlet form in the continuous situation

We recall here some results from [**38**] that will be useful for the continuous case. Let X be a locally compact denumerable metric space and m a finite positive Radon measure on X such that $\mathrm{supp}(m) = X$.

Let $(a, \mathcal{D})$ be a regular Dirichlet form on $L^2(X, m)$ such that:

(i) a is irreducible (*i.e.* $a(f) = 0$ implies that f is constant).
(ii) $(a, \mathcal{D})$ has a compact resolvent.
(iii) There exists $c > 0$ such that $\mathrm{cap}_1(\{x\}) \geqslant c$ for all $x \in X$.

N.B.: $\mathrm{cap}_1(\{x\})$ stands for the 1-capacity of the point $\{x\}$ (*cf.* [**17**], section 2).

The assumption (iii) implies in particular that the functions of the domain have a continuous modification, so that the value at one point can be defined (*cf.* [**17**], theorem 2.1.3). A second implication of assumption (iii) is that the resolvent R_λ is trace-class (*cf.* [**38**]).

Let F be a finite subset of X. The regularity of the form and assumption (iii) imply that for any $f \in \mathbb{R}^F$ there exists $g \in \mathcal{D}$ such that $g_{|F} = f$.

We define the trace of $(a, \mathcal{D})$ on the subset F as the bilinear form on $\mathbb{R}^F$ defined by:

$$(36) \qquad a_F(f, f) = \inf\{a(g, g) \mid g \in \mathcal{D}, \; g_{|F} = f\}, \quad \forall f \in \mathbb{R}^F.$$

The irreducibility of $(a, \mathcal{D})$ implies that the infimum in (36) is reached at unique point called the harmonic continuation of f with respect to a.

If F is endowed with a positive measure b with full support then $(a_F, \mathbb{R}^F)$ is a regular, irreducible Dirichlet form on $L^2(F, b)$ (the process associated with a_F and b on states space F can be represented by a time changed of the initial process associated with $(a, \mathcal{D})$ on $L^2(X, m)$ (*cf.* [**17**], theorem 6.2.1).

For $\lambda \geqslant 0$, let $a_\lambda(f) = a(f) + \lambda \int_X f^2 dm$ for $f \in \mathcal{D}$. The bilinear form a_λ is a regular irreducible Dirichlet form satisfying (i), (ii) and (iii). We denote by $A_{(\lambda)}$ the $F \times F$ symmetric matrix given by $\langle A_{(\lambda)} \cdot, \cdot \rangle = (a_\lambda)_F(\cdot, \cdot)$ where $\langle \cdot, \cdot \rangle$ is the usual scalar product on $\mathbb{R}^F$, and by $H_\lambda f$ the harmonic continuation of $f \in \mathbb{R}^F$ with respect to a_λ, so that $\langle A_{(\lambda)}(f), f \rangle = a_\lambda(H_\lambda f, H_\lambda f)$.

Set $\mathcal{D}^- = \{f \in \mathcal{D} \mid f_{|F} = 0\}$ (N.B.: $\mathcal{D}^-$ is the domain with Dirichlet boundary conditions on F; $(a, \mathcal{D}^-)$ is a regular Dirichlet form on $L^2(X \smallsetminus \{F\}, m)$).

We denote by $0 > \lambda_1^+ \geqslant \cdots \geqslant \lambda_k^+ \geqslant \cdots$ the negative eigenvalues of the infinitesimal generator associated with $(a, \mathcal{D})$ and by σ_0 the multiplicity of the eigenvalue 0 (which can be 0 or 1, indeed $\sigma_0 = 1$ if $1 \in \mathcal{D}$ and 0 otherwise).

We also denote by $0 > \lambda_1^- \geqslant \cdots \lambda_k^- \geqslant \cdots$ the eigenvalues of the infinitesimal generator of $(a, \mathcal{D}^-)$ (in this case 0 is not eigenvalue because of the boundary condition and assumption (i)). Let $f_1^-, \ldots, f_k^-, \ldots$ be an orthonormal basis of eigenfunctions associated with the preceding eigenvalues.

LEMMA 2.15 ([**38**], lemma 2.1). — *For any* $f \in \mathbb{R}^F$, $\lambda \geqslant 0$:

$$(37) \qquad \langle A_{(\lambda)}(f), f \rangle = \langle A_{(0)}(f), f \rangle + \lambda \int (H_0 f)^2 dm - \lambda^2 \sum_{k=1}^\infty \frac{(\int H_0 f f_k^- dm)^2}{\lambda - \lambda_k^-}.$$

In particular $A_{(\lambda)}$ *is meromorphic on* $\mathbb{C}$ *with at worst simple poles at the points* $\{\lambda_1^-, \ldots, \lambda_k^- \ldots\}$.

We define some infinite dimensional determinants by the following formula: for $\lambda \in \mathbb{C}$ we set

$$(38) \qquad d^+(\lambda) = \lambda^{\sigma_0} \prod_{k=1}^\infty \left(1 - \frac{\lambda}{\lambda_k^+}\right),$$

$$(39) \qquad d^-(\lambda) = \prod_{k=1}^\infty \left(1 - \frac{\lambda}{\lambda_k^-}\right).$$

The existence of these functions comes from the fact that the resolvent of $(a, \mathcal{D})$ and $(a, \mathcal{D}^-)$ are trace class.

LEMMA 2.16. — *There exists a constant $C > 0$ such that*

$$(40) \qquad \forall\, \lambda \in \mathbb{C} \smallsetminus \{\lambda_1^-, \ldots, \lambda_k^-, \ldots\}, \quad \det(A_{(\lambda)}) = C \frac{d^+(\lambda)}{d^-(\lambda)}.$$

A proof of this result is given in [**38**]. A more general version of this result, but valid only for differential operators on $\mathbb{R}^d$, can be found in [**15**].

CHAPTER 3

THE RENORMALIZATION MAP. EXPRESSION OF
THE DENSITY OF STATES

3.1. Construction of the renormalization map

In section 3 we will only be interested in the density of states. Therefore, since the counting measures $\nu_{\langle n \rangle}^{\pm}$ do not depend on the particular blow-up, we suppose to simplify notations that $\omega_k = 1$ for all k.

3.1.1. On the set of symmetric $F \times F$ matrices. —
We come back to the situation described in section 1 and denote by Sym_F the space symmetric $F \times F$ matrices. We denote by Sym^G the subspaces of Sym_F of G-invariant matrices, *i.e.* of symmetric matrices Q satisfying:

$$g \cdot Qf = Q(g \cdot f), \quad \forall\, g \in G, \forall\, f \in \mathbb{C}^F.$$

Starting from Q in Sym_F we can construct a $F_{\langle 1 \rangle} \times F_{\langle 1 \rangle}$ symmetric matrix $Q_{\langle 1 \rangle}$ by:

$$Q_{\langle 1 \rangle}(f) = \sum_i \alpha_1 \alpha_i^{-1} Q_{\langle 1 \rangle, i},$$

where $Q_{\langle 1 \rangle, i}$ is a copy of Q on the cell $F_{\langle 1 \rangle, i}$, *i.e.* it is a $F_{\langle 1 \rangle} \times F_{\langle 1 \rangle}$ symmetric matrix defined by

$$\begin{cases} (Q_{\langle 1 \rangle, i} f)_{|F_{\langle 1 \rangle, i}} = Q(f_{|F_{\langle 1 \rangle, i}}), \\ Q_{\langle 1 \rangle, i} f(x) = 0, \ \text{ if } x \notin F_{\langle 1 \rangle, i}, \end{cases}$$

for all f in $\mathbb{R}^{F_{\langle 1 \rangle}}$.

On the set $\{\det(Q_{\langle 1 \rangle})_{|F_{\langle 1 \rangle} \smallsetminus \partial F_{\langle 1 \rangle}} \neq 0\}$ we consider the trace on $\partial F_{\langle 1 \rangle}$ of $Q_{\langle 1 \rangle}$, which is an element of Sym_F. So we define:

$$(41) \qquad \begin{aligned} T \;:\; &\mathrm{Sym}_F \longrightarrow \mathrm{Sym}_F \\ &Q \longmapsto (Q_{\langle 1 \rangle})_{\partial F_{\langle 1 \rangle}}. \end{aligned}$$

Considering the symmetries of the structure we see that TQ is G-invariant if Q is G-invariant, *i.e.* $T(\mathrm{Sym}^G) \subset \mathrm{Sym}^G$. In all the following we will rather consider T as

a map on Sym^G than on Sym_F. Using proposition (2.3) we know that the map T is rational with poles included in the set $\{\det(Q_{\langle 1\rangle})_{|F_{\langle 1\rangle}\smallsetminus\partial F_{\langle 1\rangle}} = 0\}$.

Let S_+ denote the set of complex symmetric $F \times F$ matrices with positive definite imaginary part:

(42) $S_+ = \{Q \text{ complex symmetric } F \times F \text{ matrix} \mid \mathrm{Im}(Q) \text{ is positive definite}\}$,

and set $S_+^G = S_+ \cap \mathrm{Sym}^G$. Usually, S_+ is called the Siegel upper half-space (*cf.* [**46**], [**50**]). In the next lemma we prove a key property of T, which is that S_+ is left invariant by T. We also give some estimates, useful in lemma 3.9, to bound the speed at which the iterates of T approach the boundary of S_+. In appendix D, we present a different approach which avoid all explicit estimates, but only uses properties of contraction of holomorphic maps on S_+.

For a matrix Q we denote by $\underline{\rho}(Q)$ and $\overline{\rho}(Q)$ the minimal and maximal characteristic root of Q (*i.e.* the minimal and maximal eigenvalues of $\sqrt{Q^*Q}$).

LEMMA 3.1. — *The map T is well-defined and holomorphic on S_+ (resp. on S_+^G) and S_+ (resp S_+^G) is T-invariant. Moreover for all Q in S_+ we have*

(43) $\underline{\rho}(\mathrm{Im}(TQ)) \geqslant \alpha_1\overline{\alpha}^{-1}\underline{\rho}(\mathrm{Im}Q)$,

(44) $\underline{\rho}(\mathrm{Im}((TQ)^{-1})) \geqslant \alpha_1^{-1}\underline{\alpha}\underline{\rho}(\mathrm{Im}(Q^{-1}))$,

where $\underline{\alpha} = \inf\{\alpha_i\}$, $\overline{\alpha} = \sup\{\alpha_i\}$.

Proof. — It is clear by proposition (2.3) that T is well-defined on S_+. We first derive an expression for $\mathrm{Im}(TQ)$. Let $Q = \mathrm{Re}Q + i\mathrm{Im}Q$ be in S_+. Let f be a real function on F (which can also be considered as a real function on $\partial F_{\langle 1\rangle}$), and Hf its harmonic prolongation with respect to $Q_{\langle 1\rangle}$ (which can be defined for Q in S_+ using proposition (2.3)). We have:

$$\langle TQf, f\rangle = \langle Q_{\langle 1\rangle}Hf, Hf\rangle$$
$$= \langle Q_{\langle 1\rangle}Hf, \mathrm{Re}Hf\rangle$$
$$= \langle \mathrm{Re}Q_{\langle 1\rangle}\mathrm{Re}Hf, \mathrm{Re}Hf\rangle + i\langle \mathrm{Re}Q_{\langle 1\rangle}\mathrm{Im}Hf, \mathrm{Re}Hf\rangle$$
$$+ i\langle \mathrm{Im}Q_{\langle 1\rangle}\mathrm{Re}Hf, \mathrm{Re}Hf\rangle - \langle \mathrm{Im}Q_{\langle 1\rangle}\mathrm{Im}Hf, \mathrm{Re}Hf\rangle$$

Considering that $\langle Q_{\langle 1\rangle}Hf, \mathrm{Im}Hf\rangle = 0$ we deduce the following identities:

$$\langle \mathrm{Re}Q_{\langle 1\rangle}\mathrm{Re}Hf, \mathrm{Im}Hf\rangle = \langle \mathrm{Im}Q_{\langle 1\rangle}\mathrm{Im}Hf, \mathrm{Im}Hf\rangle,$$
$$\langle \mathrm{Im}Q_{\langle 1\rangle}\mathrm{Re}Hf, \mathrm{Im}Hf\rangle = -\langle \mathrm{Re}Q_{\langle 1\rangle}\mathrm{Im}Hf, \mathrm{Im}Hf\rangle.$$

Replacing in the expression of $\langle TQf, f\rangle$ we get

$$\langle \mathrm{Im}TQf, f\rangle = \langle \mathrm{Im}Q_{\langle 1\rangle}\mathrm{Re}Hf, \mathrm{Re}Hf\rangle + \langle \mathrm{Im}Q_{\langle 1\rangle}\mathrm{Im}Hf, \mathrm{Im}Hf\rangle$$
$$\geqslant \sum_{i=1}^{N} \alpha_1\alpha_i^{-1}\left\langle \mathrm{Im}Q\mathrm{Re}Hf_{|F_{\langle 1\rangle,i}}, \mathrm{Re}Hf_{|F_{\langle 1\rangle,i}}\right\rangle.$$

Since $\mathrm{Im}(Q)$ is positive definite we deduce from the last expression that $\langle \mathrm{Im}(TQ)f, f \rangle > 0$ since $\mathrm{Re}(Hf)$ is not zero, and thus that TQ is indeed in S_+. From the last inequality we can also deduce

$$\langle \mathrm{Im}TQf, f \rangle \geqslant \underline{\rho}(\mathrm{Im}Q)\alpha_1\overline{\alpha}^{-1}\sum_{i=1}^{N}\left\langle \mathrm{Re}Hf_{|F_{\langle 1 \rangle, i}}, \mathrm{Re}Hf_{|F_{\langle 1 \rangle, i}} \right\rangle$$

$$\geqslant \underline{\rho}(\mathrm{Im}Q)\alpha_1\overline{\alpha}^{-1}\|f\|^2.$$

Thus formula (43) is proved.

To prove equation (44) we need to express $Q_{\langle 1 \rangle}^{-1}$ in terms of Q^{-1} by a kind of harmonic prolongation (just like TQ is expressed in terms of Q by a harmonic prolongation). This is done in [**37**] in a slightly less general context. We state without proof the following lemma (the proof is a simple algebraic manipulation and essentially similar to the proof of proposition 1.1 of [**37**]. Note that this technical lemma can be avoided, *cf.* appendix D.)

LEMMA 3.2. — *Let ν be in $\mathbb{C}^{F_{\langle 1 \rangle}}$ and $\mathcal{D}_\nu$ be the set*

$$\mathcal{D}_\nu = \{(\nu_1, \ldots, \nu_N) \in \mathbb{C}^{F_{\langle 1 \rangle, 1}} \times \cdots \times \mathbb{C}^{F_{\langle 1 \rangle, N}} \mid \textstyle\sum_{i=1}^{N} \nu_i = \nu\}.$$

If Q is invertible then there exists a unique $(\nu_1, \ldots, \nu_N)$ in $\mathcal{D}_\nu$ such that

$$\sum_{i=1}^{N}\alpha_1^{-1}\alpha_i\left\langle Q^{-1}\nu_i, \widetilde{\nu}_i\right\rangle = 0$$

for all $(\widetilde{\nu}_1, \ldots, \widetilde{\nu}_N)$ in $\mathcal{D}_0$, and we have:

$$((Q_{\langle 1 \rangle})^{-1}\nu)_{|F_{\langle 1 \rangle, i}} = \alpha_1^{-1}\alpha_i Q^{-1}\nu_i.$$

Let Q be in S_+. We have $\underline{\rho}(\mathrm{Im}((TQ)^{-1})) \geqslant \underline{\rho}(\mathrm{Im}((Q_{\langle 1 \rangle})^{-1}))$. Let ν be in $\mathbb{R}^{F_{\langle 1 \rangle}}$. Proceeding just as previously we can prove:

$$\langle \mathrm{Im}((Q_{\langle 1 \rangle})^{-1})\nu, \nu \rangle = \sum_{i=1}^{N}\alpha_1^{-1}\alpha_i\big(\langle \mathrm{Im}(Q^{-1})\mathrm{Re}\nu_i, \mathrm{Re}\nu_i \rangle + \langle \mathrm{Im}(Q^{-1})\mathrm{Im}\nu_i, \mathrm{Im}\nu_i \rangle\big),$$

$$\geqslant \underline{\rho}(\mathrm{Im}(Q^{-1}))\sum_{i=1}^{N}\alpha_1^{-1}\alpha_i\big(\langle \mathrm{Re}\nu_i, \mathrm{Re}\nu_i \rangle + \langle \mathrm{Im}\nu_i, \mathrm{Im}\nu_i \rangle\big)$$

$$\geqslant \underline{\rho}(\mathrm{Im}(Q^{-1}))\alpha_1^{-1}\underline{\alpha}\|\nu\|^2,$$

where $(\nu_1, \ldots, \nu_N)$ is the element of $\mathcal{D}_\nu$ obtained from lemma (3.2). Thus we have proved formula (44). $\qquad\square$

3.1.2. The map R defined on the Grassmann algebra. — Consider the Grassmann algebra with generators $\{\overline{\eta}_x, \eta_x\}_{x \in F}$ and $\mathcal{A}$ its subspace generated by the monomials containing the same number of variables η and $\overline{\eta}$ (*cf.* section 2.2). We denote by $\mathcal{A}_{\langle 1 \rangle}$ the counterpart for the set $F_{\langle 1 \rangle}$. Remind that the elements of $\mathcal{A}$ and $\mathcal{A}_{\langle 1 \rangle}$ commute since they contain only monomials of even degrees. The canonical injections

$s_i \; : \; F \to F_{\langle 1 \rangle}$ given by $s_i(x) = (i, x)$ naturally induce the morphism $s_i \; : \; \mathcal{A} \to \mathcal{A}_{\langle 1 \rangle}$ defined on the generators by: $(\overline{\eta}_x, \eta_x) \mapsto (\overline{\eta}_{s_i(x)}, \eta_{s_i(x)})$.

For a real $\beta > 0$ we denote by τ_β the linear map defined on monomials by:

$$
\tau_\beta \; : \quad \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ \overline{\eta}_{i_1} \cdots \overline{\eta}_{i_k} \eta_{j_1} \cdots \eta_{j_k} & \longmapsto & \beta^k \overline{\eta}_{i_1} \cdots \overline{\eta}_{i_k} \eta_{j_1} \cdots \eta_{j_k}. \end{array}
$$

Then we set

$$
\tau_{\beta_1, \ldots, \beta_N} \; : \; \mathcal{A} \longrightarrow \mathcal{A}_{\langle 1 \rangle}
$$
$$
X \longmapsto s_1(\tau_{\beta_1}(X)) \cdots s_N(\tau_{\beta_N}(X)).
$$

Using the commutativity of the subalgebra $\mathcal{A}_{\langle 1 \rangle}$, with these definitions, we get:

$$
\exp \overline{\eta} Q_{\langle 1 \rangle} \eta = \prod_{i=1}^{N} \exp(\alpha_1 \alpha_i^{-1} \overline{\eta} Q_{\langle 1 \rangle, i} \eta) = \tau_{\alpha_1 \alpha_1^{-1}, \ldots, \alpha_1 \alpha_N^{-1}}(\exp \overline{\eta} Q \eta)
$$

Finally, using the construction of section 2, we define the map $R : \mathcal{A} \to \mathcal{A}$ by

$$
R = R_{F_{\langle 1 \rangle} \mapsto \partial F_{\langle 1 \rangle}} \circ \tau_{\alpha_1 \alpha_1^{-1}, \ldots, \alpha_1 \alpha_N^{-1}}.
$$

For Q in Sym_F, we denote by $Q_{\langle n \rangle}$, the $F_{\langle n \rangle} \times F_{\langle n \rangle}$, G-invariant, symmetric matrix defined, as in formula (11), by

$$
Q_{\langle n \rangle} = \sum_{i_1, \ldots, i_n = 1}^{N} \alpha_1^n \alpha_{i_1}^{-1} \cdots \alpha_{i_n}^{-1} Q_{\langle n \rangle, i_1, \ldots, i_n},
$$

where $Q_{\langle n \rangle, i_1, \ldots, i_n}$ is the copy of Q on $F_{\langle n \rangle, i_1, \ldots, i_n}$, as in section 1.2.1 (N.B.: remind that here we suppose $\omega_i = 1$ for all i). Similarly $Q_{\langle n \rangle, k}$ denotes the $F_{\langle n \rangle, k} \times F_{\langle n \rangle, k}$ matrix defined by

$$
Q_{\langle n \rangle, k} = \sum_{i_2, \ldots, i_n = 1}^{N} \alpha_1^{n-1} \alpha_{i_2}^{-1} \cdots \alpha_{i_n}^{-1} Q_{\langle n \rangle, k, i_1, \ldots, i_n}.
$$

PROPOSITION 3.3

(i) *The map R is polynomial homogeneous of degree N.*

(ii) *We have the following relation:*

$$
(45) \qquad R^n(\exp \overline{\eta} Q \eta) = C_{\langle n \rangle} \det \left((Q_{\langle n \rangle})_{| \overset{\circ}{F}_{\langle n \rangle}} \right) \exp(\overline{\eta} T^n Q \eta),
$$

where $C_{\langle n \rangle}$ is a constant depending only on the α_i's

$$
C_{\langle n \rangle} = \Big(\prod_{k=1}^{N} \alpha_1 \alpha_k^{-1} \Big)^{\sum_{i=0}^{n-1} | \overset{\circ}{F}_{\langle i \rangle} | N^{n-1-j}}.
$$

N.B.: For any matrix Q in Sym^G, $Q_{\langle n \rangle}$ is the G-invariant, symmetric $F_{\langle n \rangle} \times F_{\langle n \rangle}$ matrix with complex coefficients defined by formula (11), where A is replaced by Q.

Proof

(i) The map $\tau_{\beta_1, \ldots, \beta_N}(X)$ is obviously homogeneous polynomial of degree N in the coefficients of X. The map $R_{F_{\langle 1 \rangle} \mapsto \partial F_{\langle 1 \rangle}}$ is linear.

(ii) We clearly have, using, for example, the variational formula of proposition 2.3 that $T^n Q$ is the trace on $\partial F_{\langle n \rangle}$ of $Q_{\langle n \rangle}$, *i.e.*

$$(46) \qquad T^n Q = (Q_{\langle n \rangle})_{\partial F_{\langle n \rangle}}$$

(where as usual we identify $\partial F_{\langle n \rangle}$ and F). Iterating the map R we have

$$(47) \qquad R^n(\exp(\overline{\eta} Q \eta)) = C_{\langle n \rangle} i_{\prod_{x \in \overset{\circ}{F}_{\langle n \rangle}} \overline{\eta}_x \eta_x} \left(\exp(\overline{\eta} Q_{\langle n \rangle} \eta) \right),$$

where as usual we identify the points of $\partial F_{\langle n \rangle}$ with the points of F. Let us prove the last formula by recurrence. Suppose that for $n > 0$

$$R^{n-1}(\exp(\overline{\eta} Q \eta)) = C_{\langle n-1 \rangle} i_{\prod_{x \in \overset{\circ}{F}_{\langle n-1 \rangle}} \overline{\eta}_x \eta_x} \left(\exp(\overline{\eta} Q_{\langle n-1 \rangle} \eta) \right).$$

By definition, for all k in $\{1, \dots, N\}$, if we identify $F_{\langle 1 \rangle, k}$ with $\partial F_{\langle n \rangle, k}$ we get

$$(\alpha_1 \alpha_k^{-1})^{|\overset{\circ}{F}_{\langle n-1 \rangle}|} s_k(\tau_{\alpha_1 \alpha_k^{-1}}(R^{n-1}(\exp \overline{\eta} Q \eta)))$$
$$= C_{\langle n-1 \rangle} i_{\prod_{x \in \overset{\circ}{F}_{\langle n \rangle, k}} \overline{\eta}_x \eta_x} \left(\exp(\alpha_1 \alpha_k^{-1} \overline{\eta} Q_{\langle n \rangle, k} \eta) \right).$$

Considering that the term $\exp \overline{\eta} Q_{\langle n \rangle, k} \eta$ does not contain any term $\overline{\eta}_x, \eta_x$ for x in $\overset{\circ}{F}_{\langle n \rangle, k'}$ and $k' \neq k$, we see that

$$\prod_{k=1}^{N} \left(i_{\prod_{x \in \overset{\circ}{F}_{\langle n \rangle, k}} \overline{\eta}_x \eta_x} \left(\exp \alpha_1 \alpha_k^{-1} \overline{\eta} Q_{\langle n \rangle, k} \eta \right) \right) = i_{\prod_{k=1}^{N} \prod_{x \in \overset{\circ}{F}_{\langle n \rangle, k}} \overline{\eta}_x \eta_x} (\exp \overline{\eta} Q_{\langle n \rangle} \eta)$$

since $Q_{\langle n \rangle} = \sum_{k=1}^{N} \alpha_1 \alpha_k^{-1} Q_{\langle n \rangle, k}$. Formula (47) follows directly this last equality since $C_{\langle n \rangle} = C_{\langle n-1 \rangle}^N (\prod_k \alpha_1^{-1} \alpha_k)^{|\overset{\circ}{F}_{\langle n-1 \rangle}|}$. We see then that formula (45) is a direct consequence of formula (47), (46) and proposition (2.7). $\qquad \square$

We denote by $\mathcal{P}(\mathcal{A})$ the projective space associated with $\mathcal{A}$ and by $\pi : \mathcal{A} \to \mathcal{P}(\mathcal{A})$ the canonical projection. We denote by $\mathbb{L}^G$ the closure in $\mathcal{P}(\mathcal{A})$ of the elements of the type $\pi(\exp(\overline{\eta} Q \eta))$ for $Q \in \mathrm{Sym}^G$. We know from section 2.2.2 that $\mathbb{L}^G$ is a smooth subvariety of the Lagrangian Grassmannian $\mathbb{L}^{|F|}$ with the dimension of Sym^G. We remark from (45) that $\pi^{-1}(\mathbb{L}^G) \cup \{0\}$ is invariant by R.

3.2. The main theorem in the lattice case

We define the Green function (*cf.* appendix B) associated with R as the map $G : \mathcal{A} \to \mathbb{R} \cup \{-\infty\}$ given by

$$G(x) = \lim_{n \to \infty} \frac{1}{N^n} \ln \|R^n(x)\|, \quad x \in \mathcal{A}.$$

This limit always exists and is a plurisubharmonic function (*cf.* Appendix A & B). This function is related to the dynamics of the map on $\mathcal{P}(\mathcal{A})$ induced by R (*cf.* appendix B).

For $x \in \mathbb{L}^G$ we denote by $\rho_n(x)$ the order of vanishing at x of the restriction of the map R^n to the submanifold $\mathbb{L}^G$: precisely if $s : U \to \mathcal{A}$ is a local holomorphic section of the projection π on an open subset $U \subset \mathbb{L}^G$ containing x, then we denote by $\rho_n(x) = \operatorname{ord}(R^n \circ s, x)$ the order of vanishing of the function $R^n \circ s$ at the point x (*cf.* section 2.2). The functions ρ_n satisfy:

$$\rho_{n+1}(x) \geqslant N\rho_n(x).$$

since R is homogeneous of degree N. Thus the limit

$$\rho_\infty(x) = \lim_{n \to \infty} \frac{1}{N^n} \rho_n(x),$$

exists since ρ_n is bounded by N^n, the degree of R^n.

Remind that in section 1.2.1 we fixed a difference operator A with real coefficients and a positive measure b on F. We denote by $I_{b_{\langle n \rangle}}$ (and simply $I_b = I_{b_{\langle 0 \rangle}}$ for $n = 0$) the diagonal $F_{\langle n \rangle} \times F_{\langle n \rangle}$ matrix with diagonal terms $(I_{b_{\langle n \rangle}})_{x,x} = b_{\langle n \rangle}(x)$. We denote by $\phi : \mathbb{C} \to \mathcal{A}$ the map

(48) $$\phi(\lambda) = \exp \overline{\eta}(A - \lambda I_b)\eta, \quad \lambda \in \mathbb{C}.$$

REMARK 3.4. — Remark that the map ϕ is polynomial (*cf.* formula (22)).

PROPOSITION 3.5

(i) *The Neumann and Dirichlet spectrum are related to the map R by the following formulas:*

(49) $$\nu_{\langle n \rangle}^- = \frac{1}{2\pi} \Delta \ln |\langle R^n \circ \phi(\lambda), 1 \rangle|$$

(50) $$\nu_{\langle n \rangle}^+ = \frac{1}{2\pi} \Delta \ln \left| \left\langle R^n \circ \phi(\lambda), \prod_{x \in F} \overline{\eta}_x \eta_x \right\rangle \right|$$

where Δ denotes the distributional Laplacian.

(ii) *The Neumann-Dirichlet spectrum is related to the zeroes of R^n by the following formula:*

(51) $$\nu_{\langle n \rangle}^{ND} = \sum_\lambda \rho_n(\pi(\phi(\lambda)))\delta_\lambda.$$

N.B.: δ_λ is the Dirac mass at λ. The terms in the sum (51) are non null only for a finite set of points λ.

N.B.: we recall that $\langle \cdot, \cdot \rangle$ appearing in formulas (49), (50) is the scalar product on $\mathcal{A}$ defined in section 2.2.

Proof. — Using (47) and proposition (2.7) we get

$$C_{\langle n \rangle} \det((A_{\langle n \rangle} - \lambda I_{b_{\langle n \rangle}})_{|\overset{\circ}{F}_{\langle n \rangle}}) = \langle R^n \circ \phi(\lambda), 1 \rangle,$$

$$C_{\langle n \rangle} \det(A_{\langle n \rangle} - \lambda I_{b_{\langle n \rangle}}) = \left\langle R^n \circ \phi(\lambda), \prod_{x \in F} \overline{\eta}_x \eta_x \right\rangle.$$

Hence, formulas (49) and (50) come from the classical $\frac{1}{2\pi}\Delta\ln|\lambda| = \delta_0$.

(ii) The map $Q \mapsto \pi(\exp(\overline{\eta}Q\eta))$ is locally invertible from Sym^G to $\mathbb{L}^G$. Hence, for any Q_0 in Sym^G, we have

$$\rho_n(\pi(\exp\overline{\eta}Q_0\eta)) = \mathrm{ord}(Q \longmapsto R^n(\exp(\overline{\eta}Q\eta)), Q_0),$$

for Q running in Sym^G. By formula (47) we have

$$R^n(\exp\overline{\eta}Q\eta) = C_{\langle n\rangle}R_{F_{\langle n\rangle}\to\partial F_{\langle n\rangle}}(\exp\overline{\eta}Q_{\langle n\rangle}\eta),$$

for any Q in Sym^G. This implies that

$$\rho_n(\pi(\exp\overline{\eta}Q_0\eta)) \geqslant \mathrm{ord}(Q \longmapsto R_{F_{\langle n\rangle}\to\partial F_{\langle n\rangle}}(\exp\overline{\eta}Q\eta), (Q_0)_{\langle n\rangle}),$$

where in the right-hand side Q runs in the set of $F_{\langle n\rangle} \times F_{\langle n\rangle}$, complex symmetric matrices. For Q_0 real, the right-hand side equals $\dim\ker^{ND}((Q_0)_{\langle n\rangle})$ by proposition 2.8. On the other hand, we have

$$\rho_n(\pi(\exp\overline{\eta}Q_0\eta)) \leqslant \mathrm{ord}(\lambda \longmapsto R^n(\exp\overline{\eta}(Q_0 - \lambda I_b)\eta), 0)$$
$$= \mathrm{ord}(\lambda \longmapsto R_{F_{\langle n\rangle}\to\partial F_{\langle n\rangle}}(\exp\overline{\eta}((Q_0)_{\langle n\rangle} - \lambda I_{b_{\langle n\rangle}})), 0)$$

and this last expression equals $\dim\ker^{ND}((Q_0)_{\langle n\rangle})$ when Q_0 is real, by proposition 2.8, (ii). Thus we proved that for all Q_0 real we have

(52)
$$\rho_n(\pi(\exp\overline{\eta}Q_0\eta)) = \dim\ker^{ND}((Q_0)_{\langle n\rangle}).$$

Formula (51) is a direct consequence of this last formula since for any λ_0 in $\mathbb{R}$ we have

$$\nu_{\langle n\rangle}^{ND}(\{\lambda_0\}) = \dim\ker^{ND}((A - \lambda_0 I_b)_{\langle n\rangle}) = \rho_n(\pi(\phi(\lambda_0))). \qquad \square$$

THEOREM 3.6

(i) *The density of states is given by the following formula*

(53)
$$\mu = \frac{1}{2\pi}\Delta(G \circ \phi).$$

(ii) *The density of Neumann-Dirichlet eigenvalues is given by*

(54)
$$\mu^{ND} = \sum_{\lambda} \rho_\infty(\pi(\phi(\lambda)))\delta_\lambda.$$

REMARK 3.7. — Remark from formula (48) that ϕ is holomorphic, thus $G \circ \phi$ is subharmonic and $\Delta(G \circ \phi)$ defines a positive measure.

REMARK 3.8. — By construction, we have $\mathrm{supp}\,\mu \subset \mathbb{R}$. This implies that $G \circ \phi$ is harmonic on $\mathbb{C} \smallsetminus \mathbb{R}$. This property can be seen directly from the dynamics of g. We know from lemma 3.9 that g is holomorphic on the Siegel upper-half space S^+, and that S^+ is left invariant by g. On the other hand, S^+ is hyperbolic in the sense of Kobayashi, *cf.* for example, [**45**], definition 2.1. This implies that S_+^G is in the Fatou set of g, that G is pluriharmonic on S_+^G, and thus that $G \circ \phi$ is harmonic on $\{\lambda,\ \mathrm{Im}(\lambda) > 0\}$, since $\phi(\lambda) \in S_+^G$ for $\mathrm{Im}(\lambda) < 0$.

Proof. — The proof of (ii) is a direct application of proposition (3.5), (ii).

(i) By general results on subharmonic functions (*cf.* for example, [**24**]) we know from formulas (49) and (50) that the weak convergence $\frac{1}{N^n}\nu^{\pm}_{\langle n\rangle} \to \frac{1}{2\pi}\Delta(G\circ\phi)$ would be implied by the following convergence in $L^1_{\mathrm{loc}}(\mathbb{C})$:

$$(55) \qquad\qquad \lim_{n\to\infty}\frac{1}{N^n}\ln|\langle R^n\circ\phi, 1\rangle| = G\circ\phi,$$

$$(56) \qquad\qquad \lim_{n\to\infty}\frac{1}{N^n}\ln\left|\left\langle R^n\circ\phi, \prod_{x\in F}\overline{\eta}_x\eta_x\right\rangle\right| = G\circ\phi.$$

This will follow from

LEMMA 3.9. — *For all Q in S_+*

$$(57) \qquad\qquad \lim_{n\to\infty}\frac{1}{N^n}\ln\left|\left\langle\frac{R^n(\exp\overline{\eta}Q\eta)}{\|R^n(\exp\overline{\eta}Q\eta)\|}, 1\right\rangle\right| = 0,$$

$$(58) \qquad\qquad \lim_{n\to\infty}\frac{1}{N^n}\ln\left|\left\langle\frac{R^n(\exp\overline{\eta}Q\eta)}{\|R^n(\exp\overline{\eta}Q\eta)\|}, \prod_{x\in F}\overline{\eta}_x\eta_x\right\rangle\right| = 0.$$

Let us finish the proof of the theorem before starting the proof of lemma (3.9). We only prove (55), the proof of (56) is strictly identical. We write:

$$\frac{1}{N^n}\ln|\langle R^n\circ\phi(\lambda), 1\rangle| = \frac{1}{N^n}\ln\left|\left\langle\frac{R^n\circ\phi(\lambda)}{\|R^n\circ\phi(\lambda)\|}, 1\right\rangle\right| + \frac{1}{N^n}\ln\|R^n\circ\phi(\lambda)\|.$$

The first term of the right-hand side is negative since $\left|\left\langle\frac{R^n\circ\phi(\lambda)}{\|R^n\circ\phi(\lambda)\|}, 1\right\rangle\right| \leqslant 1$, and converges to 0 for $\lambda \in \mathbb{C}\smallsetminus\mathbb{R}$ since $A - \lambda I_\omega$ is in S_+ or in $-S_+$ for λ in $\mathbb{C}\smallsetminus\mathbb{R}$, using lemma (3.9). Therefore, we know that the sequence of psh functions $\frac{1}{N^n}\log|<R^n\circ\phi, 1>|$ is uniformly bounded from above and converges pointwise to $G\circ\phi$ for λ in $\mathbb{C}\smallsetminus\mathbb{R}$. Using proposition (A.1) of appendix (or proposition 3.2.12 of [**24**]) we know that (55) is true for the convergence in $L^1_{\mathrm{loc}}(\mathbb{C})$. $\qquad\square$

Proof of lemma 3.9. — We first remark that the terms of the sequences in formulas (57) and (58) are non-positive. By proposition (3.3) and lemma (2.5) we have:

$$\left|\left\langle\frac{R^n(\exp\overline{\eta}Q\eta)}{\|R^n(\exp\overline{\eta}Q\eta)\|}, 1\right\rangle\right| = \left|\left\langle\frac{\exp\overline{\eta}T^nQ\eta}{\|\exp\overline{\eta}T^nQ\eta\|}, 1\right\rangle\right|$$

$$= \frac{1}{\|\exp\overline{\eta}T^nQ\eta\|}$$

$$\geqslant \frac{1}{(1+(\overline{\rho}(T^nQ))^2)^{|F|/2}} = \left(\frac{\left(\underline{\rho}((T^nQ)^{-1})\right)^2}{1+\left(\underline{\rho}((T^nQ)^{-1})\right)^2}\right)^{|F|/2}.$$

where $\overline{\rho}(T^nQ)$ is the maximal characteristic root of T^nQ and $\underline{\rho}((T^nQ)^{-1})$ the minimal characteristic root of $(T^n(Q))^{-1}$. Since $(T^nQ)^{-1}$ is symmetric we have

$$\underline{\rho}((T^nQ)^{-1}) \geqslant \underline{\rho}(\mathrm{Im}((T^nQ)^{-1})) \geqslant \alpha_1^n\underline{\alpha}^{-n}\underline{\rho}(\mathrm{Im}Q^{-1}),$$

using lemma (3.1) in the last inequality. This is enough to prove formula (57).

The proof of formula (58) works similarly. Using proposition (3.3) and lemma (2.5) we get:

$$\left|\left\langle \frac{R^n(\exp \overline{\eta}Q\eta)}{\|R^n(\exp \overline{\eta}Q\eta)\|}, \prod_{x \in F} \overline{\eta}_x \eta_x \right\rangle\right| = \frac{|\det(T^n Q)|}{\|\exp \overline{\eta}T^n Q\eta\|} \geqslant \frac{1}{(1 + (\overline{\rho}((T^n Q)^{-1}))^2)^{|F|/2}}$$

$$= \left(\frac{\left(\underline{\rho}(T^n Q)\right)^2}{1 + \left(\underline{\rho}(T^n Q)\right)^2} \right)^{|F|/2},$$

and we conclude similarly using lemma (3.1). □

3.3. The continuous case

We know that the Dirichlet forms $(a_{\langle n \rangle}, \mathcal{D}_{\langle n \rangle}^{\pm})$ introduced in section 1.2.2 satisfy the conditions (i), (ii), (iii) of section 2.3 (*cf.* [**36**]). For $\lambda \geqslant 0$, we denote by $a_{\langle n \rangle, \lambda}$ the Dirichlet form defined by $a_{\langle n \rangle, \lambda}(f, g) = a_{\langle n \rangle}(f, g) + \lambda \int fg \, dm_{\langle n \rangle}$, and by $A_{\langle n \rangle, (\lambda)}$ the symmetric $F_{\langle n \rangle} \times F_{\langle n \rangle}$ matrix defined by

$$\left\langle A_{\langle n \rangle, (\lambda)} \cdot, \cdot \right\rangle = (a_{\langle n \rangle, \lambda})_{F_{\langle n \rangle}}(\cdot, \cdot).$$

We simply write $A_{(\lambda)} = A_{\langle 0 \rangle, (\lambda)}$. Using lemma (2.15) we see that $A_{\langle n \rangle, (\lambda)}$ can be extended to a meromorphic function on $\mathbb{C}$ with poles included in the spectrum of $a_{\langle n \rangle, \lambda}$ with Dirichlet condition on $F_{\langle n \rangle}$, *i.e.* in the spectrum of $(a, \mathcal{D}^-)$ (indeed, $X_{\langle n \rangle} \smallsetminus F_{\langle n \rangle}$ is the disjoint union of N^n copies of $\overset{\circ}{X}$).

We see that

(59) $$A_{\langle n \rangle, (\lambda)} = (A_{(\lambda)})_{\langle n \rangle},$$

with $(A_{(\lambda)})_{\langle n \rangle}$ defined by formula (11) (where A is replaced by $A_{(\lambda)}$).

PROPOSITION 3.10. — *For all $\lambda \in \mathbb{C}$ the following equality is satisfied (when the terms are defined)*

$$\alpha_1^{-1} T(A_{(\lambda)}) = A_{(\gamma\lambda)}.$$

REMARK 3.11. — This means that, at least locally, $A_{(\lambda)}$ is a holomorphic curve invariant by the map $\alpha_1^{-1}T$. We remark that $A_{(0)}$ is a fixed point of $\alpha_1^{-1}T$ (in general the existence of this fixed point is equivalent to the existence of a self-similar diffusion on the fractal, *cf.* [**36**]) and that the direction $(\frac{d}{d\lambda}A_{(\lambda)})_{\lambda=0}$ is an unstable direction of $\alpha_1^{-1}T$ since $\gamma > 1$.

Proof. — For any f in $\mathbb{R}^{\partial F_{\langle 1 \rangle}} \sim \mathbb{R}^F$ we have:

$$\alpha_1^{-1} T(A_{(\lambda)})(f) = \alpha_1^{-1}(A_{\langle 1 \rangle, (\lambda)})_{\partial F_{\langle 1 \rangle}}(f)$$
$$= \alpha_1^{-1}((a_{\langle 1 \rangle, \lambda})_{F_{\langle 1 \rangle}})_{\partial F_{\langle 1 \rangle}}(f)$$
$$= \alpha_1^{-1}(a_{\langle 1 \rangle, \lambda})_{\partial F_{\langle 1 \rangle}}(f)$$
$$= (a_{\gamma\lambda})_F(f),$$

where the last relation comes from the scaling relation $a_{\langle 1 \rangle, \lambda} = \alpha_1 a_{\gamma \lambda}$.   □

Denote by $d_{\langle n \rangle}^{\pm}$ the infinite dimensional determinant of $(a_{\langle n \rangle}, \mathcal{D}_{\langle n \rangle}^{\pm})$, as in section 2.3, and simply $d^{\pm} = d_{\langle 0 \rangle}^{\pm}$. The infinite dimensional determinant of $a_{\langle n \rangle}$ with Dirichlet boundary conditions on $F_{\langle n \rangle}$ is equal, up to a constant, to $(d^{-})^{N^n}$ (indeed, $X_{\langle n \rangle} \smallsetminus F_{\langle n \rangle}$ is the disjoint union of N^n copies of $\overset{\circ}{X}$). Hence, if we apply lemma (2.16) to $X_{\langle n \rangle}$ and $F_{\langle n \rangle}$ we see that

$$(60) \qquad \det(A_{\langle n \rangle, (\lambda)}) = c_{\langle n \rangle}^{+} \frac{d_{\langle n \rangle}^{+}(\lambda)}{d^{-}(\lambda)^{N^n}},$$

$$(61) \qquad \det((A_{\langle n \rangle, (\lambda)})_{|\overset{\circ}{F}_{\langle n \rangle}}) = c_{\langle n \rangle}^{-} \frac{d_{\langle n \rangle}^{-}(\lambda)}{d^{-}(\lambda)^{N^n}},$$

for some constants $c_{\langle n \rangle}^{\pm}$.

PROPOSITION 3.12. — *Formula (49), (50) and (51) of proposition (3.5) are true in the continuous case on the open ball $B(0, |\lambda_1^{-}|)$ with center 0 and radius $|\lambda_1^{-}|$ and when the function $\phi : B(0, |\lambda_1^{-}|) \to \mathcal{A}$ is replaced by*

$$(62) \qquad \phi(\lambda) = \exp(\overline{\eta} A_{(\lambda)} \eta).$$

N.B.: Remind that λ_1^{-} is the first Dirichlet eigenvalue of a, and that $A_{(\lambda)}$ is well-defined on $B(0, |\lambda_1^{-}|)$.

Proof. — On $B(0, |\lambda_1^{-}|)$ we have, using formula (60), and proposition (2.7),

$$\begin{aligned}
(\nu_{\langle n \rangle}^{+})_{|B(0,|\lambda_1^{-}|)} &= (\tfrac{1}{2\pi} \Delta \ln |d_{\langle n \rangle}^{+}|)_{|B(0,|\lambda_1^{-}|)} \\
&= (\tfrac{1}{2\pi} \Delta \ln |\det A_{\langle n \rangle, (\cdot)}| + N^n \nu_{\langle 0 \rangle}^{-})_{B(0,|\lambda_1^{-}|)} \\
&= \left(\tfrac{1}{2\pi} \Delta \ln \left| \left\langle R^n \circ \phi(\lambda), \prod_{x \in F} \overline{\eta}_x \eta_x \right\rangle \right| \right)_{|B(0,|\lambda_1^{-}|)}.
\end{aligned}$$

This proves formula (50). The proof of formula (49) is similar. To prove formula (51) we first prove that

$$(63) \qquad \dim E_{\langle n \rangle, \lambda}^{ND} = \dim \ker^{ND}(A_{\langle n \rangle, (\lambda)}) = \dim \ker^{ND}((A_{(\lambda)})_{\langle n \rangle})$$

for $\lambda_1^{-} < \lambda \leqslant 0$, where $E_{\langle n \rangle, \lambda}^{ND}$ denotes the vector space generated by the N-D eigenvalues of $a_{\langle n \rangle}$ with eigenvalues larger that λ. When this formula is proved then formula (51) is a direct consequence of formula (52). Remark that

$$E_{\langle n \rangle, \lambda}^{ND} \cap \{f \in \mathcal{D}_{\langle n \rangle}, \ f_{|F_{\langle n \rangle}} = 0\} = \{0\}$$

(otherwise, there would be a Dirichlet eigenvalue of $(a, \mathcal{D}^{-})$ with absolute value smaller than $|\lambda_1^{-}|$). Hence, if f is in $E_{\langle n \rangle, \lambda}^{ND}$ then $f_{|F_{\langle n \rangle}}$ is non-null and in $\ker^{ND}(A_{\langle n \rangle, (\lambda)})$. Conversely, consider g in $\ker^{ND}(A_{\langle n \rangle, (\lambda)})$, its harmonic prolongation with respect to $a_{\langle n \rangle, \lambda}$ is well-defined and is in $E_{\langle n \rangle, \lambda}^{ND}$. This means that the map $f \mapsto f_{|F_{\langle n \rangle}}$ is a

bijection from $E^{ND}_{\langle n \rangle, \lambda}$ to $\ker^{ND}(A_{\langle n \rangle, (\lambda)})$, thus this proves the first equality of (63). The second equality is given by relation (59). $\square$

THEOREM 3.13. — *All formulas of theorem (3.6) are true in the continuous case on the ball $B(0, |\lambda_1^-|)$ and when ϕ is given by (62) as in proposition (3.12).*

Proof. — It is similar to the lattice case, using lemma (3.9). We just have to check that $A_{(\lambda)}$ is in S_+ for $\mathrm{Im}(\lambda) < 0$. This follows the classical relation

$$\mathrm{Im}(A_{(\lambda)}(f, f)) = -\mathrm{Im}(\lambda) \int |H_\lambda f|^2 dm,$$

for $f \in \mathbb{R}^F$ and $H_\lambda f$ the harmonic prolongation of f with respect to a_λ (we remark from the explicit expression of $H_\lambda f$ given in the proof of lemma 2.1 of [**38**] that $H_\lambda f$ admits an analytic prolongation for $\mathrm{Im}\lambda < 0$). $\square$

CHAPTER 4

ANALYSIS OF THE PSH FUNCTION $G_{|\pi^{-1}(\mathbb{L}^G)}$

4.1. The dichotomy theorem

In this part we analyze the structure of the Green function G restricted to the subvariety $\pi^{-1}(\mathbb{L}^G)$, or equivalently the current on $\mathbb{L}^G$ with potential $G_{|\pi^{-1}(\mathbb{L}^G)}$. This will give some information on the measures μ^{ND} and μ. The structure of this current is related to the dynamics of the restriction to $\mathbb{L}^G$ of the map induced by R on $\mathcal{P}(\mathcal{A})$ (indeed as seen in section 3, the subvariety $\pi^{-1}(\mathbb{L}^G)$ is invariant by R). We first describe precisely this map. As in appendix C, we define the meromorphic map $g : \mathbb{L}^G \to \mathbb{L}^G$ and its iterates g^n by their graph $\Gamma_{g^n} \subset \mathbb{L}^G \times \mathbb{L}^G$ constructed as the closure of the graph

$$(64) \qquad \Gamma^0_{g^n} = \{(\pi(x), \pi(R^n(x))) \mid x \in \pi^{-1}(\mathbb{L}^G) \smallsetminus \{R^n(x) = 0\}\}.$$

We denote by $\pi_1, \pi_2 : \mathbb{L}^G \times \mathbb{L}^G \to \mathbb{L}^G$ the projection on the first and second coordinates. The set of indeterminacy points of g^n, denoted I_{g^n} is defined (*cf.* appendix C) as the set of points where $\pi_1 : \Gamma_{g^n} \to \mathbb{L}^G$ is not a local biholomorphism. On $\mathbb{L}^G \smallsetminus I_{g^n}$ the map is defined by $g^n(x) = \pi_2(\pi_1^{-1}(x))$. The codimension of I_{g^n} is at least 2 and it will be useful to describe the structure of the set I_{g^n} in terms of the map R^n. This can be done locally: let x be a point in $\mathbb{L}^G$ and U an open subset of $\mathbb{L}^G$ containing x, identified with a subset of $\mathbb{C}^{\dim \mathbb{L}^G}$ by a local set of coordinates. If s is a section of the projection π on the subset U then we can find holomorphic functions $f_1, \ldots, f_k$ on U and positive integers $c_1, \ldots, c_k$ such that

   – we can write

$$(65) \qquad\qquad\qquad R^n \circ s = f_1^{c_1} \cdots f_k^{c_k} \widetilde{R}_n,$$

where $\widetilde{R}_n$ is a holomorphic function from U to $\mathcal{A}$ such that the set $\{\widetilde{R}_n = 0\}$ is at least of codimension 2.

   – The analytic sets

$$(66) \qquad\qquad\qquad Z_i = \{f_i = 0\}$$

are irreducible and f_i is a generator of the ideal

$$\mathcal{I}(Z_i) = \{f \text{ holomorphic on } U \mid f = 0 \text{ on } Z_i\}.$$

Then the set of indeterminacy points $I_{g^n} \cap U$ is exactly the set $\{\widetilde{R}_n = 0\}$ and on the set $\{\widetilde{R}_n \neq 0\}$ the map g^n is given by $g^n(x) = \pi(\widetilde{R}_n(x))$. This gives us the opportunity to introduce the divisor associated with the hypersurfaces of zeroes of the restriction of R^n to $\mathbb{L}^G$. Precisely, we call divisor a formal sum $\sum_Z c_Z Z$ where the sum runs over the set of irreducible subvarieties of codimension 1 of $\mathbb{L}^G$ and the coefficients c_Z are integers and null except for a finite set of indices. We define $D_n = \sum c_Z Z$ as the divisor on $\mathbb{L}^G$ such that for any subset $U \subset \mathbb{L}^G$ we have

(67) $$(D_n)_{|U} = c_1 Z_1 + \cdots + c_k Z_k$$

where Z_i and c_i are defined by (65) and (66) (the restriction to U of an irreducible subvariety Z can be decomposed as a sum of irreducible hypervarieties of U: this naturally allows us to represent $D_n \cap U$ as a divisor on U *i.e.* as a formal positive sum of irreducible hypervarieties of U).

Since R is of degree N it is easy to see that

(68) $$D_{n+1} \geqslant N D_n$$

for the natural ordering on divisors.

We denote by S (resp. S_n) the closed positive $(1,1)$ current on $\mathbb{L}^G$ with potential $G_{|\pi^{-1}(\mathbb{L}^G)}$ (resp. $(G_n)_{|\pi^{-1}(\mathbb{L}^G)}$ where $G_n = \ln \|R^n\|$) *i.e.* S and S_n are defined locally on an open subset U of $\mathbb{L}^G$ by

$$S = dd^c G \circ s, \quad S_n = dd^c \log \|R^n \circ s\|,$$

if s is a holomorphic section of the projection π on U. By definition the current S_0 is the restriction to $\mathbb{L}^G$ of the Fubini-Study form defined on the projective space $\mathcal{P}(\mathcal{A})$ and is therefore a Kähler form on $\mathbb{L}^G$ (*cf.* Appendix A.5). The currents S_n are well-defined for all n since G_n is not equal to $-\infty$ on $\mathbb{L}^G$ (indeed, we know that $R^n(\exp(\overline{\eta}Q\eta))$ is non-null for Q in S_+, *cf.* lemma (3.1)). Since $\frac{1}{N^n}G_n$ converges pointwise in $\mathbb{R} \cup \{-\infty\}$ to G we know that

(69) $$\lim_{n \to \infty} \frac{1}{N^n} S_n = S$$

for the topology of current.

In Appendix C.2 we define the pull-back of a positive closed $(1,1)$ current by a meromorphic map. With this definition we have the following result.

PROPOSITION 4.1. — *For all integer n we have*

(70) $$S_n = (g^n)^* S_0 + [D_n].$$

N.B.: For a divisor $D = \sum c_Z Z$ we denote by $[D]$ the current $[D] = \sum c_Z [Z]$ where $[Z]$ is the current of integration on the hypervariety Z (*cf.* appendix A).

Proof. — Let U be an open subset of $\mathbb{L}^G$ identified with an open subset of $\mathbb{C}^{\dim \mathbb{L}^G}$ thanks to a local chart. If s is a holomorphic section on U of the projection π then $R^n \circ s$ can be written (*cf.* formula (65))

$$R^n \circ s = f_1^{c_1} \cdots f_k^{c_k} \widetilde{R}_n$$

and the map g^n is defined on the set $\{\widetilde{R}_n \neq 0\}$ by

$$g^n(x) = \pi(\widetilde{R}_n(x)).$$

The current S_n is defined on U by

$$\begin{aligned} S_n &= dd^c G_n \circ s \\ &= dd^c \ln \|\widetilde{R}_n\| + c_1 dd^c \ln |f_1| + \cdots + c_k dd^c \ln |f_k|, \end{aligned}$$

with notations of formula (65) and (66). By Lelong-Poincaré formula (*cf.* appendix A) $dd^c \ln |f_i| = [Z_i]$ and we will show that $dd^c \ln \|\widetilde{R}_n\| = (g^n)^* S_0$ on $U \smallsetminus \{\widetilde{R}_n = 0\}$, which implies the equality on all U since by Siu theorem (*cf.* for example, [**11**]), a $(1,1)$ positive closed current cannot charge analytic subset of codimension strictly larger than 1. Let x_0 be in $U \smallsetminus \{\widetilde{R}_n = 0\}$ and set $w_0 = g(x_0)$. Let $r > 0$ and $r_1 > 0$ be such that $g(B(x_0, r_1)) \subset B(w_0, r)$. Let $\widetilde{s}$ be a holomorphic section of the projection π on $B(w_0, r)$. We can write $\widetilde{R}_n(z) = j(z)\widetilde{s} \circ g^n(z)$ on $B(x_0, r_1)$ where j is a holomorphic function which does not take the value 0 on the set $\{\widetilde{R}_n \neq 0\}$. This implies by definition $dd^c \log \|\widetilde{R}_n\| = (g^n)^* S_0$ on $B(x_0, r_1) \smallsetminus \{\widetilde{R}_n = 0\}$. $\qquad\square$

Considering equation (68) and relation (69) we know that the limit of $\frac{1}{N^n}[D_n]$ exists and that the limit of $\frac{1}{N^n}(g^n)^* S_0$ exists. The question is now to know whether these limits can be non null at the same time. The following proposition answers the question.

PROPOSITION 4.2. — *If $D_n \neq 0$ for an integer n then*

$$\lim_{n \to \infty} \frac{1}{N^n} (g^n)^* S_0 = 0.$$

The proof of this proposition relies on considerations on the cohomology classes of the currents S_n and is sent to section 4.2. Actually, this result would be straightforward if $\mathbb{L}^G$ was a projective space : it would be obtained by simple considerations on the degree as in the proof of theorem (B.5) of Appendix B. We deduce from the last proposition the following results.

THEOREM 4.3

(i) *If $D_n \neq 0$ for an integer n then*

$$S = \lim_{n \to \infty} \frac{1}{N^n} [D_n]$$

and S is a countable sum of current of integration on hypersurfaces of $\mathbb{L}^G$.

For both the continuous and the lattice case (i.e. for any choice of (A, b) in the discrete case) we have:

$$\mu^{ND} = \mu, \tag{71}$$

in particular, for almost all blow-up the spectrum of the operator is pure point and the eigenfunctions have compact support.

(ii) *If $D_n = 0$ for all n then the map g is algebraically stable (cf. appendix C),*

$$S = \lim_{n \to \infty} \frac{1}{N^n} (g^n)^* S_0$$

and S is the Green current of g. In particular, the current S does not charge hypersurfaces and the support of S is included in the Julia set of g.

Moreover, in the lattice case for a generic choice of (A, b) we have

$$\mu^{ND} = 0.$$

N.B.: For a generic choice of (A, b) means for any (A, b) in the complement of a proper analytic subset.

REMARK 4.4. — The interesting information in this result is that a dichotomy appears between situations where either the N-D eigenvalues contribute for all the density of states or generically do not exist.

REMARK 4.5. — In proposition (4.14) we will relate this dichotomy theorem with an asymptotic degree associated with g.

Proof

(i) Since $\frac{1}{N^n}[D_n]$ is increasing it is obvious that its limit will be a countable sum of currents of integration. It is equal to S by proposition (4.2).

By proposition (3.5), (ii) we know that $(\pi \circ \phi)^*[D_n] \leqslant \mu^{ND}_{\langle n \rangle}$ ($(\pi \circ \phi)^*[D_n]$ is the pull-back of the current $[D_n]$ as defined in appendix A.6, *i.e.* on an open subset U the current $[D_n]$ has potential $\sum c_i \log |f_i|$ where the c_i's and f_i's come from formula (65), the pull-back is then defined by $(\pi \circ \phi)^*[D_n] = dd^c \sum c_i \log |f_i \circ \pi \circ \phi|$). There is not equality a priori since it may happen that the curve ϕ meets some component of $\{R^n = 0\}$ of codimension larger than 2 which do not appear in $[D_n]$. This implies that $(\pi \circ \phi)^* S \leqslant \mu^{ND}$, but $(\pi \circ \phi)^* S = \mu$ by theorem (3.6), so $\mu^{ND} = \mu$.

(ii) We will see in the next section that S_0 satisfies an equation in homology $N^n \{S_0\} = (g^n)^*(\{S_0\})$. Hence, S is by definition the Green current of g as defined in appendix C.4 and theorem (C.6). It only remains to prove that for a generic choice of (A, b), $\mu^{ND} = 0$. Suppose that $D_n = 0$ for all n. We want to prove that for a generic choice of (A, b) the line $A + \lambda I_b$, $\lambda \in \mathbb{C}$, does not meet the set $\{R^n = 0\}$. This is equivalent to prove that for any choice of A as in section 1.2.1, and D positive

diagonal, G-invariant, with Trace$D = 1$, in the complement of an analytic subset of codimension at least 2, the line $A + \lambda D$ does not meet the set $R^n = 0$. But the map

$$j \; : \; \{A \text{ as in section 1.2.1}\} \times \{D \text{ diag.} \geqslant 0, \; G\text{-inv., and } \mathrm{Tr}D = 1\} \times \mathbb{C} \to \mathrm{Sym}^G$$

$$(A, D, \lambda) \mapsto A + \lambda D$$

is a local diffeomorphism for $\lambda \neq 0$. Therefore the subset $j^{-1}((R^n)^{-1}\{0\}) \cap \{\lambda \neq 0\}$ is of codimension at least 2. The set $j^{-1}((R^n)^{-1}\{0\}) \cap \{\lambda = 0\}$ is also of codimension at least 2 since R^n is not identically null on $j(\{\lambda = 0\})$. This implies that the projection of $j^{-1}((R^n)^{-1}\{0\})$ on the first 2 components (A, D) is of codimension at least 1. $\quad\square$

4.2. Proof of Proposition (4.2). Structure of the variety $\mathbb{L}^G$

4.2.1. Notations and preliminary results. — We denote by $H^{(1,1)}(\mathbb{L}^G)$ the $(1,1)$ Dolbeault cohomology group of $\mathbb{L}^G$ (*cf.* Appendix C, $H^{(1,1)}$ is equal to the subspace of $H^2(\mathbb{L}^G, \mathbb{C})$ generated by the forms of type $(1,1)$). If α is a $(1,1)$ closed form we denote by $\{\alpha\}$ its cohomology class. As explained in the appendix the cohomology class of a current ω can also be defined and is denoted $\{\omega\}$.

We remind from the appendix that if α is a positive closed $(1,1)$ current then its pull-back $(g^n)^*\alpha$ is well defined and that if α is a smooth differential form then

   – $(g^n)^*\alpha$ is in $L^1_{\mathrm{loc}}(\mathbb{L}^G)$,
   – $(g^n)^*\alpha$ is smooth on $\mathbb{L}^G \smallsetminus I_{g^n}$.

Moreover, the pull back $(g^n)^*$ induces a pull-back on cohomology group $H^{(1,1)}(\mathbb{L}^G)$.

We first prove the following result.

LEMMA 4.6. — *For all integer n then:*

(72) $$\{S_n\} = N^n\{S_0\} = (g^n)^*\{S_0\} + \{[D_n]\}.$$

Proof. — The equality between $\{S_n\}$ and the last term of the expression is immediately deduced from proposition (4.1). We only have to prove that $\{S_n\} = N^n\{S_0\}$. Remember that S_n has potential $G_n = \log\|R^n\|$ on $\pi^{-1}(\mathbb{L}^G)$, and G_n satisfies the following homogeneity relation:

$$G_n(\lambda x) = N^n \log|\lambda| + G_n(x).$$

This immediately implies that $\{S_n\} = N^n\{S_0\}$: indeed, the function $u(z) = \log\|R^n(z)\| - N^n \log\|z\|$ is well defined on $\mathbb{L}^G$. Thus $S_n - N^n S_0 = dd^c u$ and $\{dd^c u\} = 0$ by definition. $\quad\blacksquare$

4.2.2. Description of the structure of $\mathbb{L}^G$. — To go further we need to describe the structure of the cohomology group $H^{1,1}(\mathbb{L}^G)$. In Appendix E, we describe the topological structure of $\mathbb{L}^G$. The main point is that $\mathbb{L}^G$ is isomorphic to the product $\mathcal{L}_0 \times \cdots \times \mathcal{L}_r$, where $\mathcal{L}_i$ is a smooth projective variety with Betti numbers $b_1 = \dim H^1(\mathcal{L}_i) = 0$ and $b_2 = \dim H^2(\mathcal{L}_i) = 1$. When G is the trivial group then $\mathbb{L}^G = $

$\mathbb{L}^{|F|}$, and the situation is considerably simpler. We treat this case separately, for convenience, and the ready can restrict to this simpler situation, at a first reading.

Let us be more precise. The structure of $\mathbb{L}^G$ depends on the decomposition of $\mathbb{R}^F$ into real irreducible representations of G. The space $\mathbb{R}^F$ can be decomposed into isotopic representation as

$$(73) \qquad \mathbb{R}^F = V_0 \oplus \cdots \oplus V_r,$$

where each V_i is an isotopic representation equal to the sum of n_i representations isomorphic to a single representation W_i: we write $V_i = n_i W_i$ for simplicity. We denote by $V_i^{\mathbb{C}}$ and $W_i^{\mathbb{C}}$ the complexifications of V_i and W_i. The representation W_i can be of one of the following 3 types (mutually exclusive). If $W_i^{\mathbb{C}}$ is $\mathbb{C}$-irreducible we say that W_i is of type 2. Otherwise $W_i^{\mathbb{C}} = U_i \oplus \overline{U}_i$, where U_i and its complex conjugate, $\overline{U}_i$, are $\mathbb{C}$-irreducible. If the character of U_i (and hence of $\overline{U}_i$) is not real, then we say that W_i is of type 1. If the character of U_i (and hence of $\overline{U}_i$) is real, then we say that W_i is of type 3 (*cf.* Appendix E for details and justification).

To state the result of appendix E, we need to introduce three types of Grassmannians. We denote by $\mathbb{G}^{n,2n}$ the Grassmannian of n-dimensional subspaces of $\mathbb{C}^{2n}$. The Lagrangian Grassmannian $\mathbb{L}^n$ has been defined in section 2. Finally we define the orthogonal Grassmannian as follows: $\mathbb{O}^n$ is the set of n-dimensional isotropic subspaces of $\mathbb{C}^{2n}$ for the non-degenerate symmetric bilinear form $(\cdot, K_n \cdot)$, where

$$K_n = \begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}.$$

The set $\mathbb{O}^n$ is a smooth subvariety of $\mathbb{G}^{n,2n}$, and has 2 connected components. We denote by $\mathbb{SO}^n$ the connected component which contains the isotropic subspace $\mathbb{C}^n \oplus 0$.

These three types of Grassmannian are homogeneous spaces associated respectively with the classical groups $Gl(2n, \mathbb{C})$, $Sp(n, \mathbb{C})$ and $SO(2n, \mathbb{C})$ (*cf.* Appendix E for details).

In appendix E, we proved that

$$\mathbb{L}^G \simeq \mathcal{L}_0 \times \cdots \times \mathcal{L}_r,$$

where

- $\mathcal{L}_i \simeq \mathbb{G}^{n_i, 2n_i}$ if W_i is of type 1; the dimension of $\mathcal{L}_i$ is n_i^2.
- $\mathcal{L}_i \simeq \mathbb{L}^{n_i}$ if W_i is of type 2; the dimension of $\mathcal{L}_i$ is $n_i(n_i + 1)/2$.
- $\mathcal{L}_i \simeq \mathbb{SO}^{2n_i}$ if W_i is of type 3; the dimension of $\mathcal{L}_i$ is $2n_i^2 - n_i$.

REMARK 4.7. — Since the subspaces $V_i^{\mathbb{C}}$ are orthogonal for the canonical symmetric bilinear form on $\mathbb{C}^F$, the space $\mathrm{Sym}^G(\mathbb{C})$ is isomorphic to the product $\mathrm{Sym}^G(V_0^{\mathbb{C}}) \times \cdots \times \mathrm{Sym}^G(V_r)$. The tangent space to $\mathbb{L}^G$ is $\mathrm{Sym}^G(\mathbb{C})$, and the dimension of each $\mathcal{L}_i$ corresponds to the dimension of $\mathrm{Sym}^G(V_i^{\mathbb{C}})$.

REMARK 4.8. — In particular, when all the irreducible representations of $\mathbb{C}^F$ are realizable over $\mathbb{R}$, then $\mathbb{L}^G$ is a product of Lagrangian Grassmannians. This is the case in all the examples we are going to consider.

The main point is that the first and second Betti numbers of $\mathcal{L}_i$ do not depend on the type of $\mathcal{L}_i$ and are equal to

$$b_1 = \dim(H^1(\mathcal{L}_i, \mathbb{C})) = 0, \quad b_2 = \dim(H^2(\mathcal{L}_i, \mathbb{C})) = 1$$

(*cf.* for example [**22**] for the case of $\mathbb{G}^{n,2n}$, and [**42**], for the case of $\mathbb{L}^n$ and $\mathbb{SO}^n$). Since $\mathcal{L}_i$ is a smooth projective variety, and hence a Kähler manifold, it implies that $H^{1,0}(\mathcal{L}_i) = H^{0,1}(\mathcal{L}_i) = \{0\}$, $H^{2,0}(\mathcal{L}_i) = H^{0,2}(\mathcal{L}_i) = \{0\}$ and $H^{1,1}(\mathcal{L}_i) \simeq \mathbb{C}$ (*cf.* appendix A.5). Thus,

$$(74) \qquad H^{1,1}(\mathbb{L}^G) = H^{1,1}(\mathcal{L}_0) \oplus \cdots \oplus H^{1,1}(\mathcal{L}_r),$$

and $H^{1,1}(\mathbb{L}^G)$ is of dimension $r + 1$ (hence, we see that we are in the situation described in appendix C.3). Each of these Grassmannians have a canonical embedding in a projective space $\mathbb{P}^k$: indeed, the Grassmannian $\mathbb{G}^{n,2n}$ is naturally embedded in the projective space $\mathbb{P}(\wedge^n \mathbb{C}^{2n})$ by the Plücker embedding, and $\mathbb{L}^n$ and $\mathbb{SO}^n$ are subvarieties of $\mathbb{G}^{n,2n}$. We call canonical Kähler form on $\mathbb{G}^{n,2n}$, $\mathbb{L}^n$ and $\mathbb{SO}^n$, the restriction of the Fubiny-Study form on the projective space $\mathbb{P}(\wedge^n \mathbb{C}^{2n})$, renormalized to be a generator of the integral cohomology $H_{\mathbb{Z}}^{1,1}$ (*cf.* appendix A.5 and C.3). We denote by ν_i the canonical Kähler form on each $\mathcal{L}_i$. By abuse of notations we also denote by ν_i the pull-back of ν_i by the canonical projection $p_i : \mathbb{L}^G \to \mathcal{L}_i$. A natural basis for $H^{1,1}(\mathbb{L}^G)$ is given by $(\{\nu_0\}, \ldots, \{\nu_r\})$ where $\{\nu_i\}$ is the cohomology class of ν_i. In this basis the pull-back $(g^n)^*$ on $H^{1,1}(\mathbb{L}^G)$ is given by a $(r+1) \times (r+1)$ matrix $d_n = (d_{n,i,j})_{0 \leqslant i,j \leqslant r}$ defined by

$$(75) \qquad (g^n)^*\{\nu_j\} = \sum_{i=0}^{r} d_{n,i,j}\{\nu_i\}.$$

We know, *cf.* appendix C proposition (C.4), that (d_n) has non-negative integer coefficients. As explained in the appendix the matrix d_n plays the same role as the degree in the case of maps on projective spaces (*i.e.* if $\mathbb{L}^G$ was a projective space $\mathbb{P}^k$ then d_n would be scalar (since $\dim H^{1,1}(\mathbb{P}^k) = 1$) and would be the degree of the map g^n as defined in appendix B). Before going further we detail two particular cases where this notion is easier to handle.

The case $\mathbb{L}^G = \mathbb{L}^{|F|}$. — This means that $\mathbb{C}^F$ is the sum of $|F|$ times the trivial representation W_0. This happens if and only if G acts trivially on F. The cohomology group $H^{1,1}(\mathbb{L}^G)$ is then 1-dimensional and the map $(g^n)^*$ is scalar and represented by the positive integer d_n. In this situation the proof of proposition (4.2) is significantly simpler and we give it for the convenience of the reader. Considering relation (72) we have

$$N^n\{S_0\} = d_n\{S_0\} + \{[D_n]\},$$

thus $D_n \neq 0$ implies that $d_n < N^n$. The sequence d_n being submultiplicative (*cf.* proposition (C.4)) this implies $\lim_{n\to\infty} \frac{1}{N^n} d_n = 0$ and then

$$\lim_{n\to\infty} \frac{1}{N^n} (g^n)^*(S_0) = 0$$

since the total mass of the current $\frac{1}{N^n}(g^n)^*(S_0)$ goes to zero. $\qquad\square$

The case $\mathbb{L}^G \sim \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ *(r + 1 times).* — This occurs when $n_i = 1$ for all i. In [**37**], we gave a sufficient condition for this to happen.

PROPOSITION 4.9 ([**37**], Théorème 3.2). — *The following properties are equivalent:*

 i) For all (x, y) in F^2 there exists $h \in G$ such that $h \cdot x = y$ and $h \cdot y = x$.

 ii) The representation $\mathbb{C}^F$ can be decomposed into $r + 1$ distinct $\mathbb{C}$-irreducible representations and this decomposition can be realized in $\mathbb{R}$ (i.e. $n_i = 1$ for all i and the representations W_i are realizable in $\mathbb{R}$).

REMARK 4.10. — It means that all the representations of $\mathbb{C}^F$ are of type 2. Hence $\mathrm{Sym}^G \simeq \mathbb{C}^{r+1}$.

REMARK 4.11. — In particular this is true for nested fractals (*cf.* examples, section 1.1.4, and [**31**]).

As explained in the appendix, when $\mathbb{L}^G = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ then g^n can be lifted to a polynomial map on $\mathbb{C}^{2(r+1)}$ and the matrix of degrees is equal to the degrees of the polynomials involved in this map. In the example of the Sierpinski gasket we will use this polynomial representation to compute the map g and analyze its dynamics.

4.2.3. Proof of proposition (4.2).— The proof of the proposition relies on the following lemma.

LEMMA 4.12. — *For any n the matrix d_n is primitive. More precisely, for any n and any $j = 0, \ldots, r$, $d_{n,j,0} > 0$ and $d_{n,0,j} > 0$.*

N.B.: Primitive means that there exists a power with positive coefficients. Here, we see that $(d_n)^2$ has positive coefficients.

Suppose this lemma proved then the proof of proposition (4.2) runs very much like in the simpler case $\mathbb{L}^G = \mathbb{L}^{|F|}$. Let us first prove that $\{S_0\}$ has non-negative coordinates in the basis $(\{\nu_0\}, \ldots, \{\nu_r\})$. This is a direct consequence of the fact that S_0 is a Kähler form, as the restriction of the Fubini-Study form to $\mathbb{L}^G$ (*cf.* appendix A.5). Indeed the real

$$\int_{\mathbb{L}^G} \nu_i \wedge S_0^{\dim \mathbb{L}^G - 1},$$

is called the mass of ν_i and is a positive real when S_0 is Kähler. Thus if $\{S_0\} = \sum_i c_i \{\nu_i\}$ then

$$c_{i_0} = \frac{\int_{\mathbb{L}^G} S_0^{\dim \mathbb{L}^G}}{\int_{\mathbb{L}^G} \nu_i \wedge S_0^{\dim \mathbb{L}^G - 1}},$$

and $\int_{\mathbb{L}^G} S_0^{\dim \mathbb{L}^G}$ is positive. (Actually, the coefficients c_i are even integral, since S_0 is an integral class, *cf.* [**6**], section 6.4.2, iii)).

Then Equation (70) reads as follows:

$$N^n\{S_0\} = d_n\{S_0\} + \{[D_n]\},$$

where $\{S_0\}$ and $\{[D_n]\}$ are considered as vectors of coordinates in the basis $(\{\nu_0\}, \ldots \{\nu_r\})$. Denote by ℓ_n the largest eigenvalue of the non-negative matrix d_n. Suppose that $D_{n_0} \neq 0$, the primitivity of d_{n_0} immediately implies that $\ell_{n_0} < N^{n_0}$. Indeed, $(d_{n_0})^2$ has positive coefficients, thus

$$(N^{n_0} - d_{n_0})(d_{n_0}^2\{S_0\}) = (d_{n_0})^2((N^{n_0} - d_{n_0})\{S_0\})$$

has positive coefficients which implies that $\ell_{n_0} < N^{n_0}$, *cf.* proof of theorem 1.1 of [**43**]). Thus, since d_n is submultiplicative (*cf.* proposition (C.4)) we have $\lim_{n \to \infty} \frac{1}{N^n}(g^n)^* S_0 = 0$. $\qquad \square$

At any point $x \in \mathbb{L}^G$ the tangent space $T_x(\mathbb{L}^G)$ is isomorphic to $\mathrm{Sym}^G(V_0^{\mathbb{C}}) \times \cdots \times \mathrm{Sym}^G(V_r^{\mathbb{C}})$. Therefore, at any point $x \in \mathbb{L}^G \smallsetminus I_{g^n}$ where g^n is smooth the differential $dg^n(x)$ of g^n can be decomposed into blocks

$$(76) \qquad B_{i,j}^n(x) \; : \; \mathrm{Sym}^G(V_i^{\mathbb{C}}) \longrightarrow \mathrm{Sym}^G(V_j^{\mathbb{C}}), \quad 0 \leqslant i, j \leqslant r.$$

We first prove the following lemma.

LEMMA 4.13. — *The coefficient $d_{n,i,j}$ is positive if there exists x in $\mathbb{L}^G \smallsetminus I_{g^n}$ such that $B_{i,j}^n(x) \neq 0$.*

Proof. — Let us first remark that when $\mathbb{L}^G \sim \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ this lemma is trivial. Indeed, in this case the map g^n can be represented in homogeneous coordinates by

$$(77) \qquad g([x_0 : y_0], \ldots, [x_r : y_r]) = ([P_0^n : Q_0^n], \ldots, [P_r^n : Q_r^n])$$

where $P_i^n(x_0, y_0, \ldots, x_r, y_r)$ and $Q_i^n(x_0, y_0, \ldots, x_r, y_r)$ are homogeneous polynomials of same degree in the variables (x_j, y_j) (and P_i and Q_i are prime). As explained in the appendix, the degree of (P_i^n, Q_i^n) in the variables (x_j, y_j) is equal to $d_{n,i,j}$. Therefore $d_{n,i,j} > 0$ means that P_i^n and Q_i^n are not constant in (x_j, y_j) and therefore that $B_{i,j}^n$ is not the constant 0.

Let us now prove the lemma in the general case. The volume of $\mathcal{L}_i$, $\int_{\mathcal{L}_i} \nu_i^{\dim \mathcal{L}_i}$, is positive and we set

$$C = \int_{\mathbb{L}^G} \nu_0^{\dim \mathcal{L}_0} \wedge \cdots \wedge \nu_r^{\dim \mathcal{L}_r} = \int_{\mathcal{L}_0} \nu_0^{\dim \mathcal{L}_0} \cdots \int_{\mathcal{L}_r} \nu_r^{\dim \mathcal{L}_r}.$$

Remark that if ω is a closed current of bidegree $(1,1)$ and α a smooth closed form of bidimension $(1,1)$, then $\int_{\mathbb{L}^G} \omega \wedge \alpha$ depends only on the cohomology class of ω.

Therefore if ω is a closed current of bidegree $(1,1)$ with cohomology class $\{\omega\} = c_0\{\nu_0\} + \cdots c_r\{\nu_r\}$, then

$$(78) \qquad \int_{\mathbb{L}^G} \omega \wedge \nu_0^{\dim \mathcal{L}_0} \wedge \cdots \wedge \nu_i^{\dim \mathcal{L}_i - 1} \wedge \cdots \wedge \nu_r^{\dim \mathcal{L}_r} = C c_i.$$

Applying this formula to $(g^n)^*\nu_j$ we get

$$C d_{n,i,j} = \int_{\mathbb{L}^G} (g^n)^*\nu_j \wedge \nu_0^{\dim \mathcal{L}_0} \wedge \cdots \wedge \nu_i^{\dim \mathcal{L}_i - 1} \wedge \cdots \wedge \nu_r^{\dim \mathcal{L}_r}.$$

Since $(g^n)^*\nu_j$ is a positive closed $(1,1)$ current it does not charge the analytic subset of codimension bigger than 1. Thus, we have:

$$(79) \qquad C d_{n,i,j} = \int_{\mathbb{L}^G \smallsetminus I_{g^n}} (g^n)^*\nu_j \wedge \nu_0^{\dim \mathcal{L}_0} \wedge \cdots \wedge \nu_i^{\dim \mathcal{L}_i - 1} \wedge \cdots \wedge \nu_r^{\dim \mathcal{L}_r}.$$

As explained in the appendix the current, $(g^n)^*\nu_j$ is smooth on $\mathbb{L}^G \smallsetminus I(g^n)$ and we can write at any point $x \in \mathbb{L}^G \smallsetminus I_{g^n}$

$$(80) \qquad (g^n)^*\nu_j \wedge \nu_0^{\dim \mathcal{L}_0} \wedge \cdots \wedge \nu_i^{\dim \mathcal{L}_i - 1} \wedge \cdots \wedge \nu_r^{\dim \mathcal{L}_r}(x) = e_{i,j}^n(x) v(x),$$

where $e_{i,j}^n(x)$ is a smooth positive function on $\mathbb{L}^G \smallsetminus I_{g^n}$ and v the volume form $v = \nu_0^{\dim \mathcal{L}_0} \wedge \cdots \wedge \nu_r^{\dim \mathcal{L}_r}$. To prove that $d_{n,i,j} > 0$ it is enough to prove that $e_{i,j}^n(x) > 0$ for at least one point x in $\mathbb{L}^G \smallsetminus I_{g^n}$. Let $x = (x_0, \ldots, x_r)$ be in $\mathbb{L}^G \smallsetminus I_{g^n}$ and set $g^n(x) = w = (w_0, \ldots, w_r)$. We denote by $B_{\mathcal{L}_i}(x, \varepsilon)$ the ball of $\mathcal{L}_i$ with center x and radius ε. We choose ε and $\widetilde{\varepsilon}$ such that

$$g^n(B_{\mathcal{L}_0}(x_0, \varepsilon) \times \cdots \times B_{\mathcal{L}_r}(x_r, \varepsilon)) \subset B_{\mathcal{L}_0}(w_0, \widetilde{\varepsilon}) \times \cdots \times B_{\mathcal{L}_r}(w_r, \widetilde{\varepsilon}),$$

and holomorphic coordinates $z_1^i, \ldots, z_{\dim \mathcal{L}_i}^i$ and $\widetilde{z}_1^i, \ldots, \widetilde{z}_{\dim \mathcal{L}_i}^i$ on $B_{\mathcal{L}_i}(x_i, \varepsilon)$ and $B_{\mathcal{L}_i}(w_i, \widetilde{\varepsilon})$. If the Kähler form ν_j has locally a psh potential u_j on $B_{\mathcal{L}_j}(w_j, \widetilde{\varepsilon})$ then we can write

$$(81) \qquad (g^n)^*\nu_j = \sum_{l,l'=0}^{r} \sum_{k=1}^{\dim \mathcal{L}_l} \sum_{k'=1}^{\dim \mathcal{L}_{l'}} C_{k,k'}^{l,l'} i dz_k^l \wedge d\overline{z}_{k'}^{l'},$$

where

$$\left(C_{k,k'}^{l,l'}(z) \right)_{\substack{k \leqslant \dim \mathcal{L}_l \\ k' \leqslant \dim \mathcal{L}_{l'}}} = \left(\frac{\partial}{\partial z_k^l} \frac{\partial}{\partial \overline{z}_{k'}^{l'}} (u_j \circ g^n)(z) \right)_{\substack{k \leqslant \dim \mathcal{L}_l \\ k' \leqslant \dim \mathcal{L}_{l'}}}$$

$$(82) \qquad = (B_{l,j}^n(z)) \left(\frac{\partial^2 u_j}{\partial \widetilde{z}_m^j \partial \widetilde{z}_{m'}^j} (g^n(z)) \right)_{m,m' \leqslant \dim \mathcal{L}_j} \left(B_{l',j}^n(z) \right)^*,$$

where we wrote $B_{l,j}^n(z)$ for the matrix of the block differential $B_{l,j}^n$ at the point z, for the local coordinates (z_k^l) and $(\widetilde{z}_k^j)$. Since ν_l are kähler forms on $\mathcal{L}_l$ we can find $c > 0$ such that

$$\nu_l \geqslant c \sum_{k=1}^{\dim \mathcal{L}_l} i dz_k^l \wedge d\overline{z}_k^l,$$

on $B(x_l, \varepsilon)$. This implies that

$$(g^n)^* \nu_j \wedge \nu_0^{\dim \mathcal{L}_0} \wedge \cdots \wedge \nu_i^{\dim \mathcal{L}_i - 1} \wedge \cdots \wedge \nu_r^{\dim \mathcal{L}_r}$$

$$\geqslant \Big(\sum_k C_{k,k}^i \Big) \prod_{l=0}^{r} \prod_{k=1}^{\dim \mathcal{L}_l} c(idz_k^l \wedge d\overline{z}_k^l).$$

Thus $e_{i,j}^n(x) = 0$ implies that $\sum_k C_{k,k}^{i,i}(x) = 0$, but the matrix $(C_{k,k'}^{i,i}(x))_{k,k'}$ is positive so this implies that the matrix $(C_{k,k'}^{i,i}(x))_{k,k'}$ is null. From formula (82), and since the matrix $\left(\frac{\partial u_j}{\partial \widetilde{z}_m^j \partial \overline{\widetilde{z}}_{m'}^j}(g^n(x)) \right)$ is of maximal rank, since ν_j is a Kähler form, we have $B_{i,j}^n(x) = 0$. $\square$

Proof of lemma 4.12. — We know that the map $Q \mapsto \pi(\exp \overline{\eta} Q \eta)$ defines an embedding of Sym^G into $\mathbb{L}^G$. On Sym^G the map g is given by the map T (*cf.* proposition (3.3)). Firstly, we compute the differential dT at a point Q where Q is real and positive definite. For f in $\mathbb{R}^F$ we denote by $H_Q(f)$ the harmonic prolongation of f with respect to $Q_{\langle 1 \rangle}$. We can easily see from formula (21) that the differential dT_Q of T at the point Q satisfies:

$$(83) \qquad \langle dT_Q(Y)(f), f \rangle = \langle Y_{\langle 1 \rangle}(H_Q(f)), H_Q(f) \rangle,$$

for any Y in Sym^G. At the point $Q = \mathrm{Id}$ the harmonic prolongation $H_{\mathrm{Id}}(f)$ equals $\widetilde{f}$ where:

$$\widetilde{f} = \begin{cases} f & \text{on } \partial F_{\langle 1 \rangle} \\ 0 & \text{on } F_{\langle 1 \rangle} \smallsetminus \partial F_{\langle 1 \rangle} \end{cases}$$

We want to prove that $d_{n,j,0} > 0$ and $d_{n,0,j} > 0$. From lemma (4.13) it is enough to prove that $B_{0,j}(\mathrm{Id})$ and $B_{j,0}(\mathrm{Id})$ are non-null. Let us prove that $B_{0,j}(\mathrm{Id}) \neq 0$. We choose a convenient Y in $\mathrm{Sym}^G(V_0)$: we choose $Y = p_{V_0}$, the projection on the subspace V_0 for the decomposition (73). The projection on the component $\mathrm{Sym}^G(V_j)$ of $dT_{\mathrm{Id}}(p_{V_0})$ is just given by the restriction to the subspace V_j. To prove that $B_{0,j}(\mathrm{Id}) \neq 0$ it is enough to find f in V_j such that

$$\langle f, dT_{\mathrm{Id}}(p_{V_0})(f) \rangle > 0.$$

But

$$\langle f, dT_{\mathrm{Id}}(p_{V_0})(f) \rangle = \sum_{x \in F} \alpha_x^{-1} \left\langle \widetilde{f}_{|F_{\langle 1 \rangle}, x}, p_{V_0} \widetilde{f}_{F_{\langle 1 \rangle}, x} \right\rangle,$$

and $\widetilde{f}_{|F_{\langle 1 \rangle}, x}(z) = \delta_x(z) f(x)$ where δ_x is the Dirac function at the point x. Thus the last term equals, since p_{V_0} is an orthogonal projector,

$$\sum_{x \in F} \alpha_x^{-1} |f(x)|^2 \langle p_{V_0} \delta_x, p_{V_0} \delta_x \rangle.$$

Since W_0 is the trivial representation of G, the subspace V_0 is the subspace of functions invariant by G. Thus $\langle p_{V_0} \delta_x, p_{V_0} \delta_x \rangle > 0$ for all x in F. (Indeed, the function $\frac{1}{|G|} \sum_{g \in G} \delta_{g \cdot x}$ is non null and contained in V_0. This implies that δ_x cannot be in the orthogonal complement of V_0.) This implies that the block $0, j$ of the matrix dT_{Id} is

non-null, *i.e.* that $B_{0,j}(\text{Id})$ is non-null. To prove that $B_{j,0}(\text{Id})$ is non-null we proceed similarly. We consider $Y = p_{V_j}$ and $f = 1 \in V_0$. We have:

$$\langle 1, dT_{\text{Id}}(p_{V_j})(1) \rangle = \sum_{x \in F} \alpha_x^{-1} \langle \delta_x, p_{V_j}(\delta_x) \rangle.$$

But $\langle p_{V_j}(\delta_x), \delta_x \rangle$ cannot be null for all x since $(\delta_x)_{x \in F}$ generates the space $\mathbb{R}^F$. □

4.3. Asymptotic degree of g^n

We denote by l_n the maximal eigenvalue of d_n. Using the submultiplicativity of d_n we set

$$(84) \qquad d_\infty = \lim_{n \to \infty} \frac{1}{n} \log l_n.$$

Considering formula (72) we see that $d_\infty \leqslant N$. If we denote by $\|d_n\|$ the L_∞ norm of d_n, *i.e.* $\|d_n\| = \sup_i \sum_{j=0}^r (d_n)_{i,j}$, we can easily check that

$$(85) \qquad d_\infty = \lim_{n \to \infty} \frac{1}{n} \log \|d_n\|.$$

(Indeed, classically $\|d_n\| \geqslant l_n$ and for all n_0 we can find a constant $K > 0$ such that for all $p \geqslant 0$ and $k = 0 \ldots n-1$, $\|d_{n_0 p + k}\| \leqslant \|(d_{n_0})^p d_k\| \leqslant K(l_{n_0})^p$, which immediately gives the other inequality $\lim \frac{1}{n} \|d_n\| \leqslant d_\infty$).

PROPOSITION 4.14

(i) *We are in the case (i) of theorem (4.3) if and only if $d_\infty < N$.*

(ii) *In the lattice case for all choice of (A, b)*

$$(86) \qquad d_\infty \geqslant \limsup_{n \to \infty} \frac{1}{n} \log |\nu_{\langle n \rangle}^\pm - \nu_{\langle n \rangle}^{ND}|,$$

where $|\nu_{\langle n \rangle}^\pm - \nu_{\langle n \rangle}^{ND}|$ is the total mass of the measure. Furthermore, for a generic choice of (A, b) we have the equality

$$(87) \qquad d_\infty = \lim_{n \to \infty} \frac{1}{n} \log |\nu_{\langle n \rangle}^\pm - \nu_{\langle n \rangle}^{ND}|.$$

Proof

(i) Indeed, we remarked in the proof of proposition (4.2) that $D_n \neq \varnothing$ is equivalent to $l_n < N$.

(ii) Let $V^\pm$ be the analytic subsets defined by

$$V^- = \{ X \in \pi^{-1}(\mathbb{L}^G) \mid \langle X, 1 \rangle = 0 \},$$
$$V^+ = \{ X \in \pi^{-1}(\mathbb{L}^G) \mid \langle X, \prod_{x \in F} \overline{\eta}_x \eta_x \rangle = 0 \}.$$

Proceeding exactly as in proposition (4.1) we know that $dd^c \ln |\langle R^n, 1 \rangle|$ is a potential for $(g^n)^*[V^-] + [D_n]$ (and idem for $\langle R^n, \prod_{x \in F} \overline{\eta}_x \eta_x \rangle$ and V^+). Hence, by proposition (3.5) we have

$$(88) \qquad \nu_{\langle n \rangle}^\pm = (\pi \circ \phi)^* ((g^n)^*[V^\pm] + [D_n]),$$

where $(\pi \circ \phi)^*$ is the pull-back of $\pi \circ \phi$ as defined in appendix A.6. By proposition (3.5) we know that the N-D spectrum corresponds to the zeroes of R^n and we deduce

$$(89) \qquad \nu_{\langle n \rangle}^{\pm} - \nu_{\langle n \rangle}^{ND} \leqslant (\pi \circ \phi)^*((g^n)^*[V^{\pm}]),$$

with equality when $\phi(\lambda)$ does not meet the indeterminacy points of g^n (which is the case for a generic choice of (A, b), as seen in the proof of theorem (4.3)). The map ϕ can be extended into a holomorphic function on $\mathbb{P}^1$ by setting $\phi(\infty) = \pi(\exp(\overline{\eta} I_b \eta))$. Remark that $\phi(\infty)$ does not belong to V^+ or V^-, so that formula (88) and (89) remain valid when $(\pi \circ \phi)^*$ is the pull-back of the extension of $\pi \circ \phi$ to $\mathbb{P}^1$. In $\mathbb{P}^1$ the cohomology class of a positive $(1, 1)$-current (*i.e.* a positive measure) corresponds to its total mass, more precisely, we have $\{\nu\} = |\nu|\{\delta_0\}$ where ν is a positive measure and $|\nu|$ its integral on $\mathbb{P}^1$. One can easily check that the coordinates of the linear map $(\pi \circ \phi)^* : H^{1,1}(\mathbb{L}^G) \to H^{1,1}(\mathbb{P}^1)$ in the basis $(\{\nu_0\}, \dots, \{\nu_r\})$ and $\{\delta_0\}$ are positive as well as the coordinates of $\{[V^{\pm}]\}$ in the basis $(\{\nu_0\}, \dots, \{\nu_r\})$. Considering equation (89) in homology we get

$$|\nu_{\langle n \rangle}^{\pm} - \nu_{\langle n \rangle}^{ND}| \leqslant (\pi \circ \phi)^*(d_n \{[V^{\pm}]\}),$$

with equality for a generic choice of (A, b). Thus we have

$$|\nu_{\langle n \rangle}^{\pm} - \nu_{\langle n \rangle}^{ND}| \leqslant \|(\pi \circ \phi)^*\| \, \|d_n\| \, \|\{[V^{\pm}]\}\|$$

where $\|\pi \circ \phi)^*\|$ and $\|d_n\|$ are the L^∞ matrix norm and $\|\{[V^{\pm}]\}\|$ the L^∞ norm of the vector $\{[V^{\pm}]\}$. Thus (86) easily follows form (85). On the other hand, since $\{[V^{\pm}]\}$ and $(\pi \circ \phi)^*$ have positive coordinates we can find $K > 0$ such that $|\nu_{\langle n \rangle}^{\pm} - \nu_{\langle n \rangle}^{ND}| \geqslant K\|d_n\|$ when equality in (89) is satisfied. Thus, for a generic choice of (A, b), equality (87) is satisfied. $\qquad \square$

4.4. Regularity of the density of states

We first state a conjecture and show how it can be related to results on Lelong numbers of the Green current.

CONJECTURE 4.15. — *The measure $\mu - \mu^{ND}$ is continuous, i.e. it does not charge any point.*

We introduce the notion of Lelong numbers of a psh function. Let u be a psh function in a neighborhood $U \subset \mathbb{C}^n$ of 0. Then the Lelong number of u at 0 is defined by

$$\nu(u, 0) = \max\{c \geqslant 0 \mid u(z) \leqslant c \log \|z\| + O(1)\}.$$

The Lelong number of a positive closed $(1, 1)$ current T is defined by $\nu(T, p) = \nu(u, p)$ for any local psh potential u of T, $T = dd^c u$. On $\mathbb{C}$ it is easy to see that the Lelong number at p of a positive closed $(1, 1)$ current (*i.e.* a positive measure) is the mass of the point p.

Coming back to our situation, we suppose that we are in the lattice case and that $d_\infty = N$ (otherwise $\mu - \mu^{ND} = 0$). We remark that the Lelong number of $dd^c \ln \|R^n \circ \phi\|$ at λ is the order of vanishing $\mathrm{ord}(R^n \circ \phi, \lambda)$. By proposition (2.8), (ii), we know that this order of vanishing is equal to $\nu^{ND}_{\langle n \rangle}(\{\lambda\})$. Hence, we see that the conjecture is equivalent to

$$\lim_{n \to \infty} \frac{1}{N^n} \nu(dd^c \ln \|R^n \circ \phi\|, \lambda) = \nu(dd^c G \circ \phi, \lambda).$$

Since the current $S = dd^c G$ is the limit of the sequence of currents $\frac{1}{N^n} S_n$ with potential $\frac{1}{N^n} dd^c \ln \|R^n\|$ the question is equivalent to know whether the limit of the Lelong numbers of the restriction of the currents S_n to a curve is equal to the Lelong numbers of the restriction of the limit S. There is, at the present time, only one result on Lelong numbers of Green currents: it says, *cf.* [**13**] theorem 2.4.6, that the Lelong numbers of the Green current are null on the complement of the indeterminacy points of the iterates: with our notations it means that the Lelong number of S is null when the Lelong numbers of the S_n are null. Unfortunately, it cannot be applied easily to our case. Firstly, because it does not state that the Lelong numbers of S is the limit of the Lelong numbers of $\frac{1}{N^n} S_n$ at any point, and secondly, because we need information on the restriction of the Green current of S to the curve $\phi(\lambda)$, which can be strictly bigger than the Lelong number of S. Nevertheless, it could certainly be possible, with a little more work, to prove a result in the generic case, but we prefer to leave the general result as a conjecture.

The following result gives a criterion for the regularity in a much stronger sense.

PROPOSITION 4.16. — *When $d_\infty = N$, the integrated density of states $\mu(\lambda) = \int_\lambda^0 d\mu$ is locally Hölder continuous on the set of λ's such that there exist open subsets $U \subset \mathbb{C}$ and $V \subset \mathbb{L}^G$ such that $\lambda \in U$, $\cup_{n=1}^\infty I_{g^n} \subset V$ and $g^n \circ \phi(U) \cap V = \varnothing$ for all n.*

N.B.: By locally Hölder continuous we mean that for any relatively compact open set $U \subset \mathbb{C}$, we can find $\alpha_0 > 0$ and $C > 0$ such that $|\mu(\lambda) - \mu(\lambda')| \leqslant C|\lambda - \lambda'|^{\alpha_0}$ on U.

Proof. — By proposition VI.3.9 of [**8**] we know that the Hölder regularity of $\mu(\lambda)$ is equivalent to the Hölder regularity of $G \circ \phi(\lambda)$. Thus the result is a direct adaptation of Theorem 7.1 of [**45**] (*cf.* appendix, theorem (B.3) (iii)) which states that the Green function is locally Hölder continuous in the set of normal points: a careful reading of the proof shows that it can be straightforwardly modified to prove the Hölder regularity of $G \circ \phi$ under the milder conditions of proposition (4.16). $\qquad\square$

4.5. Some related rational maps

As explained in the introduction, the map g defined on the projective variety $\mathbb{L}^G$ is from the theoretical point of view the best-suited to our problem, but from the computational point of view it is not easy to handle. In this section we introduce

related rational maps defined on projective spaces. The map $\widehat{g}$ we introduce is the map considered in the initial work of Rammal in the case of the Sierpinski gasket, *cf.* [**34**], as well as in previous work of the author, *cf.* [**38**].

The map T is 1-homogeneous, hence it can be written

$$(90) \qquad\qquad T = \frac{\widehat{R}}{p},$$

where $\widehat{R} : \mathrm{Sym}^G \to \mathrm{Sym}^G$ is a homogeneous polynomial map and p a homogeneous polynomial, prime with $\widehat{R}$. We define $\widetilde{R} : \mathrm{Sym}^G \times \mathbb{C} \to \mathrm{Sym}^G \times \mathbb{C}$ by

$$\widetilde{R}(Q, z) = (\widehat{R}(Q), p(Q)z).$$

The map $\widetilde{R}$ has no common factor and induces a rational map $\widetilde{g}$ on $\mathbb{P}^{\dim \mathrm{Sym}^G}$. We denote by $\widetilde{d}_n$ the degree of the iterate $\widetilde{g}^n$ (*cf.* appendix B).

LEMMA 4.17. — *The map $\widetilde{g}$ is birationally equivalent to the map g, i.e. there exists a rational map $h : \mathbb{P}^{\dim \mathrm{Sym}^G} \to \mathbb{L}^G$ and two analytic subsets $E \subset \mathbb{L}^G$ and $\widetilde{E} \subset \mathbb{P}^{\dim \mathrm{Sym}^G}$ of codimension at least 1, such that h is a biholomorphism from $\mathbb{P}^{\dim \mathrm{Sym}^G} \smallsetminus \widetilde{E}$ onto $\mathbb{L}^G \smallsetminus E$ and such that*

$$g = h \circ \widetilde{g} \circ h^{-1}.$$

Proof. — This is clear from formula ((45) since $h : \mathrm{Sym}^G \to \mathbb{L}^G$ given by $h(Q) = \pi(\exp(\overline{\eta}Q\eta))$ is biholomorphic from Sym^G onto $\pi\{X \in \pi^{-1}(\mathbb{L}^G) \mid \langle X, 1 \rangle = 0\}$. Hence, the map g and $\widetilde{g}$ extends the map T to 2 different compactifications of Sym^G. $\qquad\square$

Using the 1-homogeneity of T, one can introduce a rational map on $\mathbb{P}^{\dim \mathrm{Sym}^G - 1}$ which contains most of the information on $\widetilde{g}$. Precisely, for all n one can write

$$\widehat{R}^n = h_n \widehat{R}_n,$$

for a homogeneous polynomial map $\widehat{R}_n$ with no common factor and a homogeneous polynomial h_n. This induces a map $\widehat{g}^n$ on $\mathbb{P}^{\dim \mathrm{Sym}^G - 1}$ with degree $\widehat{d}_n = \mathrm{degree}(\widehat{R}_n)$. Set

$$\widehat{d}_\infty = \lim_{n \to \infty} \frac{1}{n} \ln \widehat{d}_n, \quad \widetilde{d}_\infty = \lim_{n \to \infty} \frac{1}{n} \ln \widetilde{d}_n,$$

which are well-defined thanks to the subadditivity of $\ln \widehat{d}_n$ and $\ln \widetilde{d}_n$.

REMARK 4.18. — As we shall see in the examples, the map $\widehat{g}$ is the renormalization map that was previously considered, for example, in [**34**], [**20**], [**38**]. The map $\widehat{g}$ is 1-dimensional when the space Sym^G is of dimension 2. Hence, the property of spectral decimation is related to the fact that $\dim \mathrm{Sym}^G = 2$.

PROPOSITION 4.19. — *We have the equality $d_\infty = \widetilde{d}_\infty = \widehat{d}_\infty$.*

Proof. — To prove that $\widetilde{d}_\infty = d_\infty$ we proceed as in [**12**]: we can write in homology $(g^n)^* = h_*(\widetilde{g}^n)^* h^*$ and $(\widetilde{g}^n)^* = h^*(g^n)^* h_*$. If $\|\|$ denotes the L^∞ norm then we have $\|d_n\| \leqslant \widetilde{d}_n \|h_*\| \|h^*\|$, from which we get $d_\infty \leqslant \widetilde{d}_\infty$ (N.B: note that $\widetilde{d}_n$ is a scalar, *i.e.* $\widetilde{d}_n = \|\widetilde{d}_n\|$). Conversely, $\widetilde{d}_n \leqslant \|h^*\| \|h_*\| \|d_n\|$, from which we deduce $\widetilde{d}_\infty \leqslant d_\infty$.

To prove that $\widehat{d}_\infty = \widetilde{d}_\infty$ we write

$$\widetilde{R}^n(Q,z) = (\widehat{R}^n(Q), p \circ \widehat{R}^{n-1}(Q) \cdots p(Q)z)$$
$$= \big(h_n \widehat{R}_n, \big(\textstyle\prod_{j=0}^{n-1} h_j^{\mathrm{degree}(p)}(Q)\big) p \circ \widehat{R}_{n-1}(Q) \cdots p(Q)z\big).$$

But $h_{n-1}^{\mathrm{degree}(p)+1}$ divides h_n (since $\mathrm{degree}(\widehat{R}) = \mathrm{degree}(p) + 1$), thus $\prod_{j=0}^{n-1} h_j^{\mathrm{degree}(p)}$ divides h_n and $\widetilde{d}_n \leqslant \mathrm{degree}(p)(\widehat{d}_{n-1} + \cdots \widehat{d}_1 + 1) + 1$. It immediately follows that $\widehat{d}_\infty = \widetilde{d}_\infty$. $\qquad\square$

Remark that from the proof we see that $\widetilde{R}$ is algebraically stable if and only if $\mathrm{degree}(h_n) = 0$ for all n and that in this case $\widetilde{G}(Q,z) = \widehat{G}(Q)$ if $\widetilde{G}$ and $\widehat{G}$ are the Green functions of respectively $\widetilde{R}$ and $\widehat{R}$. But this result is not really useful since in general the maps $\widehat{g}$ and $\widetilde{g}$ are not algebraically stable. To convince the reader of this fact we explicitly compute the degrees $\widetilde{d}_n$ when $d_\infty = N$. Set $p_{\langle n \rangle}^-(Q) = \det((Q_{\langle n \rangle})_{|F_{\langle n \rangle} \smallsetminus \partial F_{\langle n \rangle}})$. We claim that when $d_\infty = N$ (*i.e.* when we are in the case (ii) of theorem (4.3)) then

$$\widetilde{R}^n(Q,z) = (p_{\langle n \rangle}^-(Q)T^n(Q), p_{\langle n \rangle}^-(Q)z).$$

Indeed, we know from formula (45) that $p_{\langle n \rangle}^-$ simplifies the singularities of T^n, and we know from the same formula that $p_{\langle n \rangle}^-$ and $p_{\langle n \rangle}^- T^n$ cannot have a common factor because otherwise it would imply that R^n is null on a hypersurface of $\mathbb{L}^G$, which is impossible since we are in case (ii) of theorem (4.3). Thus,

$$\widetilde{d}_n = \mathrm{degree}(p_{\langle n \rangle}^-) + 1$$
$$= |F_{\langle n \rangle} \smallsetminus \partial F_{\langle n \rangle}| + 1 = \frac{N^n - 1}{N - 1}|F_{\langle 1 \rangle} \smallsetminus \partial F_{\langle 1 \rangle}| + 1.$$

So, we remark that $\widetilde{d}_n$ grows like $aN^n + b$ for some a and b. In particular, we see that $\widetilde{d}_n$ cannot be equal to $d_\infty^n = N^n$ except in the case where $N - 1 = |F_{\langle 1 \rangle} \smallsetminus \partial F_{\langle 1 \rangle}|$. In general it is not the case. However, in the example of the interval we have $N = 2$ and $|F_{\langle 1 \rangle} \smallsetminus \partial F_{\langle 1 \rangle}| = 1$ so there is a priori no incompatibility between $d_\infty = N$ and the algebraic stability of $\widehat{g}$ and $\widetilde{g}$. As we shall see, in this particular case $\widetilde{R}$ and $\widehat{R}$ are indeed algebraically stable with degree 2 and one can express the density of states in terms of the Green function $\widehat{G}$ (this was shown directly in [**38**]). But this is an exceptional situation: indeed, it is not difficult to see that the equality $N - 1 = |F_{\langle 1 \rangle} \smallsetminus \partial F_{\langle 1 \rangle}|$ can occur only when the lattice is based on the unit interval of $\mathbb{R}$, *i.e.* when we are, up to isomorphism, in the following situation: the self-similar set X is the interval $[0,1]$ and $\Psi_1([0,1]),\ldots,\Psi_N([0,1])$ are N subintervals of $[0,1]$.

REMARK 4.20. — From the formula we derived for $\widetilde{d}_n$, we see that the map $\widetilde{g}$ is not algebraically stable (and none of its iterates is) in the case $N = d_\infty$, at the exception of the unit interval (indeed, otherwise $\widetilde{d}_n$ would be equal to N^n after a certain level). This means that there is no natural way to define a non degenerate Green current associated with $\widetilde{R}$ and $\widetilde{g}$. This clearly confirms the essential role played by the map g defined on the algebraic subvariety $\mathbb{L}^G$. It is quite interesting to note that this is very coherent with the general philosophy that seems to emerge in the study of iteration of rational map which is roughly that when a map is not algebraically stable then one must seek for a birational transformation that makes it algebraically stable (*cf.* [**12**], theorem 0.1, where a result in this direction is proved for a particular case). In our case, the birational transformation has some "physical" meaning, since the complex Lagrangian Grassmannian is known to be a natural compactification of the space of complex symmetric matrices (*cf.* for example, [**9**]).

CHAPTER 5

EXAMPLES

5.1. The Sierpinski Gasket

The maps T and g. — For the Sierpinski Gasket the group of symmetries is $G = S_3$ and $\mathbb{C}^F$ can be decomposed into a sum of 2 irreducible representations, realizable in $\mathbb{R}$, $\mathbb{C}^F = W_0 \oplus W_1$, where W_0 is the subspace of constant functions and W_1 its orthogonal complement (for the usual scalar product on $\mathbb{C}^F$). Hence, any Q in Sym^G can be written

$$(91) \qquad Q = u_0 p_{|W_0} + u_1 p_{|W_1},$$

where (u_0, u_1) are in $\mathbb{C}^2$ and $p_{|W_0}$ and $p_{|W_1}$ are the orthogonal projections on W_0 and W_1 respectively. We denote by Q^{u_0, u_1} the element (91) and we have $\mathrm{Sym}^G \sim \mathbb{C}^2$. As in section 4.2.2, $\mathbb{L}^G \sim \mathbb{P}^1 \times \mathbb{P}^1$. A point in $\mathbb{P}^1 \times \mathbb{P}^1$ will be represented in homogeneous coordinates by

$$([u_0 : v_0], [u_1 : v_1]).$$

We denote by $\pi \times \pi : (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2 \to \mathbb{P}^1 \times \mathbb{P}^1$ the canonical projection. The space Sym^G is embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ by the injection $Q^{u_0, u_1} \to ([u_0 : 1], [u_1 : 1])$.

An easy computation shows that with the isomorphism $\mathrm{Sym}^G \sim \mathbb{C}^2$ the map T is given by

$$(92) \qquad T(u_0, u_1) = 3 \left(\frac{u_0 u_1}{2u_0 + u_1}, \frac{u_1(u_0 + u_1)}{5u_1 + u_0} \right).$$

Thus, in homogeneous coordinates the maps g is given by

$$(93) \quad g\left([u_0 : v_0], [u_1 : v_1]\right)$$
$$= \left([3u_0 u_1 : 2u_0 v_1 + u_1 v_0], [3u_1(u_0 v_1 + u_1 v_0) : 5u_1 v_0 v_1 + u_0 v_1^2]\right).$$

The matrix of degrees is

$$(94) \qquad d_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

The map $\widehat{g}$ on $\mathbb{P}^1$. — As shown in section 4.4 the 1-homogeneity of T naturally induces a map on $\mathbb{P}^1$. Indeed, if we set $z = u_0/u_1$ and $\widetilde{z} = \widetilde{u}_0/\widetilde{u}_1$ where $(\widetilde{u}_0, \widetilde{u}_1) = T(u_0, u_1)$ then we have $\widetilde{z} = \widehat{g}(z)$ where

$$(95) \qquad \widehat{g}(z) = \frac{z(z+5)}{(2z+1)(z+1)}.$$

In homogeneous coordinates in $\mathbb{P}^1$, we see that $\widehat{g}$ is given by

$$(96) \qquad \widehat{g}([z_0 : z_1]) = [z_0(5z_1 + z_0) : (2z_0 + z_1)(z_0 + z_1)]$$

(*i.e.*, in (95), $\widehat{g}$ is given in coordinates $[z, 1]$). More formally, if we denote by $\widehat{s}$ the rational map $\widehat{s} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ given by

$$\widehat{s}([u_0 : v_0], [u_1 : v_1]) = [u_0 v_1 : u_1 v_0]$$

then the following diagram is commutative

$$
\begin{array}{ccc}
\mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\ g\ } & \mathbb{P}^1 \times \mathbb{P}^1 \\
\downarrow{\widehat{s}} & & \downarrow{\widehat{s}} \\
\mathbb{P}^1 & \xrightarrow{\ \widehat{g}\ } & \mathbb{P}^1
\end{array}
$$

We set $C_z = \widehat{s}^{-1}(\{z\})$ for z in $\mathbb{P}^1$. We see that

$$g(C_z) \subset C_{\widehat{g}(z)}.$$

REMARK 5.1. — At this point we already know, by proposition (4.19), that the asymptotic degree d_∞ is equal to the degree of $\widehat{g}$, i.e is equal to 2. So we know that we are in case (ii) of theorem (4.3), thus that we have $\mu^{ND} = \mu$ and that for almost all blow-up the spectrum is pure point with compactly supported eigenfunctions.

The map R. — At this point we are not yet in a position to describe the current S_n. Indeed, we computed the map g on $\mathbb{L}^G$ but not the map R on $\pi^{-1}(\mathbb{L}^G)$ (and to describe the sequence of currents S_n we need to compute the map g and the hypersurface D_1 of zeroes of R). In general, it is not easy to compute the map R on $\mathcal{A}$, since $\mathcal{A}$ has quite large dimension. To overcome this difficulty we will lift the restriction of R to $\pi^{-1}(\mathbb{L}^G)$ to a polynomial map on $\mathbb{C}^2 \times \mathbb{C}^2$. We first describe more precisely the isomorphism $\mathbb{L}^G \sim \mathbb{P}^1 \times \mathbb{P}^1$. Let ψ_0 be an orthonormal vector of W_0 and ψ_1, ψ_1' be an orthonormal basis of W_1. We consider the homogeneous polynomial map $s : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathcal{A}$ given by

$$s\left((u_0, v_0), (u_1, v_1)\right) = (v_0 + u_0\overline{\psi}_0\psi_0)(v_1 + u_1\overline{\psi}_1\psi_1)(v_1 + u_1\overline{\psi}_1'\psi_1').$$

We see that

$$s\left((u_0, 1), (u_1, 1)\right) = \exp(u_0\overline{\psi}_0\psi_0 + u_1(\overline{\psi}_1\psi_1 + \overline{\psi}_1'\psi_1')) = \exp\overline{\eta}Q^{u_0, u_1}\eta,$$

and that s is $(1, 2)$ homogeneous, *i.e.* that

$$s\left(\beta(u_0, v_0), \beta'(u_1, v_1)\right) = \beta(\beta')^2 s\left((u_0, v_0), (u_1, v_1)\right).$$

Hence, the map s takes values in $\pi^{-1}(\mathbb{L}^G)$ and the isomorphism $\mathbb{L}^G \sim \mathbb{P}^1 \times \mathbb{P}^1$ is represented by the commutation of the following diagram

$$
\begin{array}{ccc}
(\mathbb{C}^2)^* \times (\mathbb{C}^2)^* & \xrightarrow{\pi \times \pi} & \mathbb{P}^1 \times \mathbb{P}^1 \\
\Big\downarrow{\widehat{s}} & & \Big\downarrow{\sim} \\
\pi^{-1}(\mathbb{L}^G) & \xrightarrow{\pi} & \mathbb{L}^G
\end{array}
$$

Remind from formula (45) that

$$
R(\exp(\overline{\eta} Q \eta)) = \det((Q_{\langle 1 \rangle})_{|F_{\langle 1 \rangle} \smallsetminus \partial F_{\langle 1 \rangle}}) \exp(\overline{\eta} T Q \eta).
$$

An easy computation gives

$$
\det((Q_{\langle 1 \rangle}^{u_0, u_1})_{|F_{\langle 1 \rangle} \smallsetminus \partial F_{\langle 1 \rangle}}) = 4(2u_0 + u_1)(u_0 + 5u_1)^2.
$$

By homogeneity, we get

$$
R\big(s((u_0, v_0), (u_1, v_1))\big) = (v_0 v_1^2)^3 R\big(s((u_0/v_0, 1), (u_1/v_1, 1))\big).
$$

Hence, we see that

$$
\langle R(s\,((u_0, v_0), (u_1, v_1))), 1 \rangle = 4v_1^3(2u_0 v_1 + u_1 v_0)(u_0 v_1 + 5u_1 v_0)^2,
$$

and that the value of $R(s\,((u_0, v_0), (u_1, v_1)))$ on the monomials of degree 1 of $\mathcal{A}$ is

$$
12 u_0 u_1 v_1^3 (u_0 v_1 + 5u_1 v_0)^2 \left(\overline{\eta} p_{|W_0} \eta\right)
$$
$$
+ 12 u_1 v_1^2 (u_0 v_1 + v_1 u_0)(2u_0 v_1 + u_1 v_0)(u_0 v_1 + 5u_1 v_0) \left(\overline{\eta} p_{|W_1} \eta\right).
$$

Thus, if we denote by $R : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2$ the homogeneous polynomial map given by (we adopt the same notation for this map and for the map on $\mathcal{A}$, since they are different representations of the same map)

$$
(97) \quad R\,((u_0, v_0), (u_1, v_1))
$$
$$
= \big((3u_0 u_1 v_1, 2u_0 v_1^2 + v_0 u_1 v_1), (6u_1(u_0 v_1 + u_1 v_0), 2(5u_1 v_0 v_1 + u_0 v_1^2))\big),
$$

then we see that the following diagram commutes

$$
\begin{array}{ccc}
\mathbb{C}^2 \times \mathbb{C}^2 & \xrightarrow{R} & \mathbb{C}^2 \times \mathbb{C}^2 \\
\Big\downarrow{s} & & \Big\downarrow{s} \\
\pi^{-1}(\mathbb{L}^G) \cup \{0\} & \xrightarrow{R} & \pi^{-1}(\mathbb{L}^G) \cup \{0\}
\end{array}
$$

(Indeed, by the previous computation we see that $R(s((u_0, v_0), (u_1, v_1)))$ and $s(R((u_0, v_0), (u_1, v_1)))$ coincide on the unit 1 of $\mathcal{A}$ and on the monomials of degree 1 of $\mathcal{A}$. Since they are also elements of $\pi^{-1}(\mathbb{L}^G) \cup \{0\}$ they are equal.)

REMARK 5.2. — This means that we are able to lift the map R on $\pi^{-1}(\mathbb{L}^G) \cup \{0\}$ to a polynomial map on $\mathbb{C}^2 \times \mathbb{C}^2$. This will be useful for computation. It is clear that it would be possible to do the same thing when $\mathbb{L}^G$ is a product of $\mathbb{P}^1$, *i.e.* for example, in the case of nested fractals.

Hence, we see that in $\mathbb{P}^1 \times \mathbb{P}^1$ the analytic set D_1 is given by

$$D_1 = \pi \times \pi \{v_1 = 0\} = \mathbb{P}^1 \times [1, 0].$$

Thanks to (97) we will be able to describe S_n and S. Indeed $\|s((u_0, v_0), (u_1, v_1))\|^2 = (u_0^2 + v_0^2)(u_1^2 + v_1^2)^2$, thus S_0 is the current with potential

$$\ln \|(u_0, v_0)\| + 2 \ln \|(u_1, v_1)\|$$

on $\mathbb{C}^2 \times \mathbb{C}^2$ (*i.e.* this means that we have $(\pi \times \pi)^* S_0 = dd^c \ln \|(u_0, v_0)\| + 2dd^c \ln \|(u_1, v_1)\|$). In particular, we remark that $\{S_0\} = \binom{1}{2}$ for the canonical basis of $H^{1,1}(\mathbb{P}^1 \times \mathbb{P}^1)$. Since $\{[D_1]\} = \binom{0}{1}$ we see that equation (70) is indeed verified for $n = 1$. The current S_n will by defined by its potential on $\mathbb{C}^2 \times \mathbb{C}^2$, $\ln \|(R^n)_0\| + 2 \ln \|(R^n)_1\|$ where $(R^n)_0$ and $(R^n)_1$ are the coordinates of R^n on the first and second components of $\mathbb{C}^2 \times \mathbb{C}^2$.

The Green current S. — We are now in a position to describe the iterates R^n and the currents S_n. We set $z_0 = u_0 v_1$, $z_1 = u_1 v_0$. We remark that equation (97) can be rewritten

$$R((u_0, v_0), (u_1, v_1)) = ((u_1 P_1(z_0, z_1), v_1 P_2(z_0, z_1)), (u_1 P_3(z_0, z_1), v_1 P_4(z_0, z_1))),$$

where

$$P_1(z_0, z_1) = 3z_0, \qquad P_2(z_0, z_1) = 2z_0 + z_1,$$
$$P_3(z_0, z_1) = 6(z_0 + z_1), \qquad P_4(z_0, z_1) = 2(z_0 + 5z_1).$$

We define $\widehat{R}$ by

$$\widehat{R}(z_0, z_1) = (P_1(z_0, z_1)P_4(z_0, z_1), P_2(z_0, z_1)P_3(z_0, z_1)).$$

Note that $\widehat{R}$ is a lift on $\mathbb{C}^2$ of $\widehat{g}$. For all $k \geqslant 0$ we set

$$P_{3,k} = P_3 P_3 \circ \widehat{R} \cdots P_3 \circ \widehat{R}^k,$$
$$P_{4,k} = P_4 P_4 \circ \widehat{R} \cdots P_4 \circ \widehat{R}^k.$$

An easy computation shows

$$R^n((u_0, v_0), (u_1, v_1)) = (u_1 v_1)^{3^{n-2}} \prod_{k=0}^{n-3} (u_1 v_1 P_{3,k} P_{4,k})^{3^{n-k-3}}$$
$$\cdot \left((P_{3,n-2} P_1 \circ \widehat{R}^{n-1}, P_{4,n-2} P_2 \circ \widehat{R}^{n-1}), (P_{3,n-1}, P_{4,n-1}) \right).$$

We remark that $P_3(z_0, z_1)$ and $P_4(z_0, z_1)$ are null respectively on C_{-1} and C_{-5}, thus we have

$$[D_n] \geqslant 3 \frac{3^{n-1} - 1}{2} ([u_1 = 0] + [v_1 = 0])$$
$$+ \sum_{k=0}^{n-3} 3 \frac{3^{n-k-2} - 1}{2} \Big(\sum_{z, \widehat{g}^k z = -1} [C_z] + \sum_{z, \widehat{g}^k z = -5} [C_z] \Big).$$

Since $S = \lim_{n\to\infty} \frac{1}{3^n} S_n$ we see that

$$(98) \qquad S \geqslant \tfrac{1}{2}([u_1 = 0] + [v_1 = 0]) + \sum_{k=0}^{\infty} \frac{3^{-k-1}}{2} \Big(\sum_{z,\widehat{g}^k z = -1} [C_z] + \sum_{z,\widehat{g}^k z = -5} [C_z] \Big).$$

To prove that there is actually equality in the last formula it is enough to check that there is equality in homology. It is easy if we remark that $\{S\} = \{S_0\} = \binom{1}{2}$ and $\{[C_z]\} = \binom{1}{1}$. Finally we sum-up our results in the next theorem

THEOREM 5.3. — *For the Sierpinski gasket we are in the case (ii) of theorem (4.3). The asymptotic degree is 2 and the Green current S is given by the right hand term of (98).*

Description of $\mu = \mu^{ND}$ in the discrete case. — It is useful to make the change of variable $v = 3z/(1-z) = 3u_0/(u_1 - u_0)$ in (95). With this change of variable $\widehat{g}$ is conjugated to the polynomial

$$(99) \qquad\qquad\qquad \widehat{p}(v) = v(5 + 2v).$$

REMARK 5.4. — This polynomial is, up to a change of variable the polynomial that appears in the initial work of Rammal, [**34**], and in subsequent works, [**19**], [**49**].

With the choice made in section 1.2.3 we see that the coordinates of A are $(0, 3)$ in $\mathbb{C}^2 \sim \mathrm{Sym}^G$. We take for b the uniform measure on F. In $\mathbb{L}^G \sim \mathbb{P}^1 \times \mathbb{P}^1$, $\pi(\phi(\lambda))$ has homogeneous coordinates $([\lambda : 1], [3 + \lambda : 1])$, which means that in coordinates v it corresponds to $v(\lambda) = \frac{3\lambda}{(3+\lambda)-\lambda} = \lambda$. Applying theorem (3.6) and theorem (5.3) we get

THEOREM 5.5. — *For the Sierpinski gasket in the discrete case we have*

$$\mu = \mu^{ND} = \tfrac{1}{2}\delta_{-3} + \sum_{k=0}^{\infty} \frac{3^{-k-1}}{2} \Big(\sum_{\lambda,\widehat{p}^k \lambda = -3/2} \delta_\lambda + \sum_{\lambda,\widehat{p}^k \lambda = -5/2} \delta_\lambda \Big).$$

One can remark that $[-5/2, 0]$ is backward invariant by $\widehat{p}$, hence the Julia set $\mathcal{J}$ of $\widehat{p}$ is included in $[-5/2, 0]$ and it is not difficult to see that $\mathcal{J}$ is a Cantor subset of $[-5/2, 0]$.

Since $\widehat{p}(-5/2) = 0$ we remark that $-5/2$ is in the Julia set of $\widehat{p}$, hence the Dirac masses obtained by preimages of $-5/2$ are accumulating points of μ. Iterating $\widehat{p}$ we see that $-3/2$ is in the Fatou set of $\widehat{p}$ and hence the Dirac masses at the preimages of $-3/2$ are isolated in μ. The point -3 is in the complement of the Julia set and hence is an isolated mass in μ.

5.2. The unit interval

We will show that for this particular case, the density of states can be expressed thanks to a rational map simpler than the map g, namely we express the density of states in terms of the Green current of $\widehat{g}$, defined on $\mathbb{P}^2$, which has been introduced in section 4.5. This relates the present work with our previous work [**38**]. But firstly, we illustrate some of the notions we introduced in the text by an explicit computation of the map R on this example.

In this case, $F = \{0,1\}$ and G is the trivial group so $\mathrm{Sym}^G = \mathrm{Sym}_F$ can be identified with $\mathbb{C}^3$: Q in Sym_F is represented by the point of coordinates (a,d,q) if

$$Q = \begin{pmatrix} a & q \\ q & d \end{pmatrix}.$$

In the Grassmann algebra generated by $\{\overline{\eta}_0, \eta_0, \overline{\eta}_1, \eta_1\}$ we have:

(100) $\quad \exp(\overline{\eta}Q\eta) = 1 + a\overline{\eta}_0\eta_0 + d\overline{\eta}_1\eta_1 + q(\overline{\eta}_0\eta_1 + \overline{\eta}_1\eta_0) + (ad - q^2)\overline{\eta}_0\eta_0\overline{\eta}_1\eta_1.$

Since G is the trivial group $\mathbb{L}^G = \mathbb{L}^2$ and from the last formula we easily deduce that

(101) $\quad \mathbb{L}^2 = \pi\{Z1 + a\overline{\eta}_0\eta_0 + d\overline{\eta}_1\eta_1 + q(\overline{\eta}_0\eta_1 + \overline{\eta}_1\eta_0) + D\overline{\eta}_0\eta_0\overline{\eta}_1\eta_1, \ ad - q^2 = DZ\}.$

We set $\delta = \alpha/(1-\alpha)$. An easy computation shows that

(102) $\quad R(Z1 + a\overline{\eta}_0\eta_0 + d\overline{\eta}_1\eta_1 + q(\overline{\eta}_0\eta_1 + \overline{\eta}_1\eta_0) + D\overline{\eta}_0\eta_0\overline{\eta}_1\eta_1)$

$$= \widetilde{Z}1 + \widetilde{a}\overline{\eta}_0\eta_0 + \widetilde{d}\overline{\eta}_1\eta_1 + \widetilde{q}(\overline{\eta}_0\eta_1 + \overline{\eta}_1\eta_0) + \widetilde{D}\overline{\eta}_0\eta_0\overline{\eta}_1\eta_1,$$

where

$$\widetilde{Z} = \delta(a + \delta^{-1}d)Z, \quad \widetilde{D} = \delta^2(a + \delta^{-1}d)D,$$
$$\widetilde{a} = \delta(a^2 + \delta^{-1}DZ), \quad \widetilde{d} = \delta(d^2 + \delta DZ),$$
$$\widetilde{q} = -\delta q^2$$

On $\pi^{-1}(\mathbb{L}^G)$, considering formula (101), we see that $\widetilde{a}$ and $\widetilde{d}$ are also equal to

$$\widetilde{a} = \delta(a(a + \delta^{-1}d) - \delta^{-1}q^2), \quad \widetilde{d} = \delta(\delta d(a + \delta^{-1}d) - \delta q^2).$$

It is then clear that

$$\pi\{R^n = 0\} \cap \mathbb{L}^2 = \mathbb{L}^2 \cap \cup_{k=1}^{n}\pi\{\alpha^k a + (1-\alpha)^k d = 0 \mid q = 0\}.$$

(hence the zeroes of R^n are of codimension 2 in $\mathbb{L}^2$, so $D_n = \varnothing$ for all n and we are in case (ii) of theorem (4.3)). In the lattice case we can easily describe the function $\phi(\lambda)$. Indeed, A is the usual discrete Laplace operator with coordinates $(a = 1, d = 1, q = -1)$. If b is the measure that gives weights m_0 and m_1 to the points 0 and 1 then we see that $\phi(\lambda) = \exp(\overline{\eta}(A + \lambda I_b)\eta)$ is a point of the form (100) with $a = 1 + \lambda m_0$, $d = 1 + \lambda m_1$ and $q = -1$. In particular we see that $\phi(\lambda)$ does not meet the zeroes of R^n so that $\mu^{ND} = 0$ (actually, it is very easy to see directly that there cannot be any Neumann-Dirichlet eigenfunction for this 1-dimensional Laplace operator).

The maps T and $\widehat{g}$. — For this example the easiest way to describe the density of states is to consider the simpler map $\widehat{g}$ introduced in section 4.5. A simple computation or an application of formula (102) gives in coordinates (a, d, q):

$$T(a, d, q) = \frac{1}{a + \delta^{-1}d}(a(a + \delta^{-1}d) - \delta^{-1}q^2, \delta d(a + \delta^{-1}d) - \delta q^2, -q^2).$$

Set $p(Q) = \det((Q_{\langle 1 \rangle})_{|F_{\langle 1 \rangle} \smallsetminus \partial F_{\langle 1 \rangle}}) = \delta(a + \delta^{-1}d)$. As in section 4.5. we define the map $\widehat{R} : \mathbb{C}^3 \to \mathbb{C}^3$ obtained from T by simplifying the singularities:

$$\begin{aligned}
\widehat{R}(Q) &= p(Q)TQ \\
&= \delta(a(a + \delta^{-1}d) - \delta^{-1}q^2, \delta d(a + \delta^{-1}d) - \delta q^2, -q^2).
\end{aligned}$$

We denote by $\widehat{g}$ the rational map on $\mathbb{P}^2$ induced by $\widehat{R}$. The map $\widehat{g}$ has a simple indeterminacy point $[1, -\delta, 0]$ and we easily see that $\widehat{g}$ is algebraically stable (*cf.* [**38**], for more details). We denote by $\widehat{G}$ the Green function $\widehat{G} = \lim_{n \to \infty} \frac{1}{2^n} \log \|\widehat{R}^n\|$. In the lattice case we set $\widehat{\phi}(\lambda) = A - \lambda I_b$ and in the continuous case $\widehat{\phi}(\lambda) = A_{(\lambda)}$.

THEOREM 5.6. — *In the lattice case*

$$\mu = \frac{1}{2\pi}\Delta\widehat{G} \circ \widehat{\phi}.$$

In the continuous case the same formula holds on the ball $B(0, |\lambda_1^-|)$.

REMARK 5.7. — This result was proved directly in theorem 3.1 of [**38**]. As shown in [**38**], this formula is related to the classical Thouless formula, and we can relate the Green function $\widehat{G} \circ \widehat{\phi}$ with the Lyapounov exponent of the propagator of the PDE associated with our second order differential operator. In [**38**] we used this formula to prove that the density of states is continuous and supported by a Cantor subset of $\mathbb{R}$ for $\alpha \neq \frac{1}{2}$ (for $\alpha = \frac{1}{2}$ we are in the situation of the classical Laplacian and everything is well-known). We also proved the Hölder regularity of μ for some values of the parameter α. In [**41**] we go further and describe the spectral type of the operator, depending on α and on the blow-up ω.

Proof. — Thanks to the expressions of R and $\widehat{R}$ we can easily deduce that $\widehat{G}(Q) \leqslant G(\exp \overline{\eta}Q\eta)$ for any Q in Sym_F. On the other hand for Q in S_+ we know that (*cf.* lemma (3.9))

$$G(\exp \overline{\eta}Q\eta) = \lim_{n \to \infty} \frac{1}{N^n} \log |\langle R^n(\exp \overline{\eta}Q\eta), 1 \rangle|,$$

and by direct computation we see that $\langle R^n(\exp \overline{\eta}Q\eta), 1 \rangle = \prod_{k=0}^{n-1} p \circ \widehat{R}^k(Q)$. Hence we see that for any Q in S_+ we have $\widehat{G}(Q) \geqslant G(\exp \overline{\eta}Q\eta)$. The equality $\widehat{G} \circ \widehat{\phi}(\lambda) = G \circ \phi(\lambda)$ is satisfied for λ in $\mathbb{C} \smallsetminus \mathbb{R}$, so in L^1_{loc}. Theorem (5.6) is proved. $\qquad\square$

5.3. Nested fractals

For nested fractals we shall prove that we are in the case (ii) of theorem (4.3). (The existence of N-D eigenvalues was proved initially in [**3**]. The fact that $\mu^{ND} = \mu$ was proved directly in [**39**]). Denote by $W_0, \ldots, W_r, W_{r+1}, \ldots, W_{r'}$ the list of irreducible representations of the group G, and assume that the representations $W_0, \ldots, W_r$ are contained in $\mathbb{R}^F$. In [**39**], proposition 2.3, we proved that $r' > r$, *i.e.* that there exists at least one irreducible representation which is not contained in $\mathbb{R}^F$. The space $\mathbb{R}^{F_{\langle n \rangle}}$ can be decomposed in

$$\mathbb{R}^{F_{\langle n \rangle}} = V_{\langle n \rangle, 0} \oplus \cdots \oplus V_{\langle n \rangle, r'},$$

where $V_{\langle n \rangle, j}$ is the isotopic representation associated with W_j. It is easy to check that for n large enough, $V_{\langle n \rangle, j} \neq \varnothing$ for all j (indeed, for n large enough, there is at least one point x in $F_{\langle n \rangle}$ such that $g \mapsto g \cdot x$ is injective; thus $\mathbb{R}^{F_{\langle n \rangle}}$ contains the representation $\mathbb{R}^G$ and since the representation $\mathbb{R}^G$ contains at least once each irreducible representation, *cf.* [**44**], we know that $V_{\langle n \rangle, j} \neq \varnothing$ for all j). For Q in Sym^G the operator $Q_{\langle n \rangle}$ can be decomposed in blocks on $V_{\langle n \rangle, 0}, \ldots, V_{\langle n \rangle, r'}$. Consider, for example, $(Q_{\langle n \rangle})_{|V_{\langle n \rangle, r'}}$: clearly, we have $\ker((Q_{\langle n \rangle})_{|V_{\langle n \rangle, r'}}) \subset \ker^{ND}(Q_{\langle n \rangle})$ since $V_{\langle n \rangle, r'} \subset \mathcal{D}_{\langle n \rangle}^-$. Hence $R^n(\exp \overline{\eta} Q \eta)$ is vanishing on $\{\det((Q_{\langle n \rangle})_{|V_{\langle n \rangle, r'}}) = 0\}$, which is an analytic set of codimension 1 in Sym^G. This implies that $D_n \neq \varnothing$ and that we are in case (ii) of theorem (4.3).

Remark that this implies that $d_\infty < N$ but that we have no more information on the value of d_∞.

CHAPTER 6

REMARKS, QUESTIONS AND CONJECTURE

The main open problem is to understand the almost sure Lebesgue decomposition of the operator $H_{\langle\infty\rangle}$. Let us be more precise. Consider a typical blow-up ω, in particular for which $\partial F_{\langle\infty\rangle} = \varnothing$. The Hilbert space $\mathcal{D}_{\langle\infty\rangle}$ can be decomposed into three parts $\mathcal{H}_{\mathrm{ac}}, \mathcal{H}_{\mathrm{sc}}, \mathcal{H}_{\mathrm{pp}}$ such that the restriction of $H_{\langle\infty\rangle}$ to these subspaces is respectively absolutely continuous, singular continuous or purely punctual (*i.e.* this means that the spectral measure of any function in these subspaces is resp. absolutely continuous, singular continuous or purely punctual). The spectrum of $H_{\langle\infty\rangle}$ and the spectrum of the restriction of $H_{\langle\infty\rangle}$ to $\mathcal{H}_{\mathrm{ac}}, \mathcal{H}_{\mathrm{sc}}, \mathcal{H}_{\mathrm{pp}}$ are respectively denoted by $\Sigma(\omega), \Sigma_{\mathrm{ac}}(\omega), \Sigma_{\mathrm{sc}}(\omega), \Sigma_{\mathrm{pp}}(\omega)$ (they a priori depend on ω). Proposition 2 of [40] states that these sets are almost surely constant in ω, *i.e.* equal to deterministic sets $\Sigma, \Sigma_{\mathrm{ac}}, \Sigma_{\mathrm{sc}}, \Sigma_{\mathrm{pp}}$ for almost all blow-up ω. One can be more precise and split the Hilbert space $\mathcal{H}_{\mathrm{pp}}$ into two parts: the first, that we denote $\mathcal{H}_{ND}$, is the subspace generated by the Neumann-Dirichlet eigenfunctions, *i.e.* by the eigenfunctions of $H_{\langle\infty\rangle}$ with compact support, the second, $\widetilde{\mathcal{H}}_{\mathrm{pp}}$, is defined as its orthogonal complement in $\mathcal{H}_{\mathrm{pp}}$. We denote by Σ_{ND} and $\widetilde{\Sigma}_{\mathrm{pp}}$ the spectrum of the restriction of $H_{\langle\infty\rangle}$ to resp. $\mathcal{H}_{ND}$ and $\widetilde{\mathcal{H}}_{\mathrm{pp}}$. It is clear that $\Sigma^{ND} = \operatorname{supp} \mu^{ND}$, and we proved in proposition 2 of [40] that $\widetilde{\Sigma}_{\mathrm{pp}}(\omega)$ is also determined almost surely in ω. We also know that $\Sigma = \operatorname{supp} \mu$ from proposition 1 of [40]. In this text we showed that μ and μ^{ND} can be computed from the Green function of the renormalization map R and the order of vanishing of R. The natural question is then whether it is possible to characterize the other parts of the Lebesgue decomposition of the spectrum $\Sigma_{\mathrm{ac}}, \Sigma_{\mathrm{sc}}, \Sigma_{\mathrm{pp}}, \widetilde{\Sigma}_{\mathrm{pp}}$ in terms of the renormalization map R. We can even ask more: in [40], we introduced several measures $\mu^{\mathrm{ac}}, \mu^{\mathrm{sc}}, \mu^{\mathrm{pp}}, \widetilde{\mu}^{\mathrm{pp}}$ which split μ in several parts corresponding to the different components of the spectrum by $\Sigma_\cdot = \operatorname{supp} \mu_\cdot$. The question is whether it is possible to compute these measures from some characteristics of the map R, as we computed μ and μ^{ND} from its Green function and the asymptotic multiplicities of its zeroes. In particular, it would be interesting to understand at which condition

there may be a pure point component $\widetilde{\Sigma}_{\mathrm{pp}}$ not created by the Neumann-Dirichlet eigenfunctions (*i.e.* at which conditions there are L^2 eigenfunctions which are not generated by compactly supported eigenfunctions). But all these questions seem to be very difficult.

Let us now make few remarks and a conjecture. When $N > d_\infty$ we know that the spectrum is pure point with compactly supported eigenfunctions, *i.e.* that $\Sigma_{\mathrm{ac}} = \Sigma_{\mathrm{sc}} = \widetilde{\Sigma}_{\mathrm{pp}} = \varnothing$. Consider now the case where $N = d_\infty$. We showed in this text that the N-D spectrum is related to the zeroes of R, *i.e.* to the indeterminacy points of g. Considering that eigenfunctions of $H_{\langle\infty\rangle}$ which are not with compact support are in some sense approximated by functions with compact support which are not far from being eigenfunctions we can ask the following very imprecise question: does the existence of a pure point component in the spectrum different from the N-D component, *i.e.* $\widetilde{\Sigma}_{\mathrm{pp}} \neq \varnothing$ imply that the iterates of g on $\phi(\lambda)$ approach the indeterminacy points, in a sense to be made precise. This leads to propose the following conjecture.

CONJECTURE 6.1. — *Consider the case $d_\infty = N$. Assume that the condition of proposition 4.16 is satisfied for all $\lambda \in \mathbb{R}$, i.e. that for any $\lambda \in \mathbb{R}$ there exists two open subsets $U \subset \mathbb{C}$ and $V \subset \mathbb{L}^G$ such that $\lambda \in U$, $\cup_{n=1}^\infty I_{g^n} \subset V$ and $g^n(\phi(U)) \cap V = \varnothing$. Is it true that in this case $\Sigma_{\mathrm{pp}} = \varnothing$?*

REMARK 6.2. — It is already known from theorem 3.6 that under these conditions $\Sigma_{ND} = \varnothing$.

One of the main problem is the lack of examples where computations are possible. There are very few examples where the spectral type of the operator can be analyzed. The case of the Sierpinski gasket is now well understood and computations in this example are easy. In this case $N > d_\infty$ and so the Neumann-Dirichlet eigenfunctions are complete. The case of nested fractal is also understood, in this case also $N > d_\infty$. We present the spectral analysis of the self-similar Sturm-Liouville operator on the real line in [**41**]. In this case we can prove that $\Sigma_{\mathrm{pp}} = \varnothing$.

But we have no example where $\widetilde{\Sigma}_{\mathrm{pp}} \neq \varnothing$, *i.e.* where there is a pure point component not induced by the N-D spectrum. We think it could be interesting to understand the following situation (at least by numerical computations), which could be a candidate for $\widetilde{\Sigma}_{\mathrm{pp}} \neq \varnothing$. Consider the Sierpinski gasket and take $G = \{\mathrm{Id}\}$, the trivial group as group of symmetries of the picture. In this case the renormalization map T is defined on the bigger space Sym_F of symmetric matrices on $F = \{1, 2, 3\}$. The subvariety $\mathbb{L}^G$ is the Lagrangian Grassmannian $\mathbb{L}^3$. There is then no reason that $N > d_\infty$. We would bet, on the contrary, that $N = d_\infty$ in this case. The question is the following: what happens if we take for the initial operator $A_{\langle0\rangle}$ and initial measure $b_{\langle0\rangle}$ a small perturbation of the usual discrete Laplace operator and of the uniform measure? Does the spectrum remains pure point or not?

APPENDIX

For the convenience of the reader we review several notions of pluricomplex analysis and pluricomplex dynamics. Our point of view is very partial and only motivated by the notions we need in the main text. The reader is strongly advised to refer to the relevant literature for a better understanding. In appendix A.1 and A.2 we introduce the notion of plurisubharmonic functions and of currents on complex manifolds. Most of the material is taken from the appendix of review texts of Fornaess and Sibony (*cf.* [**45**], [**16**] and related literature as, for example, [**6**], [**22**], [**24**]). In B.1 we give a short introduction to basic notions that appear in relation with iteration of rational maps of the projective spaces. All the material comes from [**45**], [**16**]. In B.2 we introduce some notions on iteration of meromorphic maps of compact complex manifolds. The material comes from works of Favre, Diller-Favre and Guedj-Favre (*cf.* [**13**], *cf.* [**12**], [**14**]).

A. Plurisubharmonic functions and positive currents

A.1. Plurisubharmonic functions. — Let Ω be a domain of $\mathbb{C}^m$. A function $f : \Omega \to \mathbb{R} \cup \{-\infty\}$ is said to be plurisubharmonic (resp. pluriharmonic) if

  – f is upper-semi continuous,
  – f is not constant equal to $-\infty$,
  – the restriction of f to any complex line is subharmonic (resp. harmonic).

Plurisubharmonic (psh for short) functions are in L^1_{loc} and can be characterized by the following property: a function v in $L^1_{\text{loc}}(\Omega)$ is almost surely equal to a psh function if and only if for all vectors $w \in \mathbb{C}^m$

$$(103) \qquad \sum_{j,k} \frac{\partial^2 v}{\partial z_j \overline{z}_k} w_j \overline{w}_k \geqslant 0$$

in the sense of distributions, *i.e.* the first term is a positive measure.

Examples and basic properties

(1) If f is a holomorphic function in Ω then $\log|f|$ is a psh function. Moreover $\log|f|$ is pluriharmonic on $\Omega \smallsetminus \{f = 0\}$.

(2) The function $u = \log\|z\|$ is psh in $\mathbb{C}^m$.

(3) If $g : \Omega \to \Omega'$ is a holomorphic map between an open subset of $\mathbb{C}^m$ to an open subset of $\mathbb{C}^{m'}$ and if u is psh in Ω' then $u \circ g$ is psh or equal to $-\infty$ (note that if g has generic maximal rank m' then $u \circ g$ is not identically $-\infty$). In particular the notion of psh functions can be extended to complex manifolds.

PROPOSITION A.1. — *Let v_j be a sequence of psh functions in a domain Ω of $\mathbb{C}^m$. Suppose that v_j is uniformly bounded from above in any compact subset of Ω, then*

(i) *either v_j converges to $-\infty$ on compacts or there exists a subsequence v_{j_k} which is convergent in L^1_{loc} to a subharmonic function.*

(ii) *If v is subharmonic and $v_j \to v$ in L^1_{loc} then for any compact $K \subset \Omega$ and any continuous function f:*

$$\limsup_{j \to \infty} \sup_K (v_j - f) \leqslant \sup_K (v - f).$$

N.B.: This result is a corollary of the same statement for subharmonic functions on $\mathbb{R}^n$ (*cf.* [**45**]), as psh functions on $\mathbb{C}^m$ are subharmonic on $\mathbb{R}^{2m}$.

A.2. Currents. — We denote by $D_{(p,q)}(\Omega)$ the space of C^∞ differential forms with compact support on Ω and of bidegree (p,q), *i.e.* of the type

$$\phi = \sum_{\substack{|I|=p \\ |J|=q}} \phi_{I,J} dz_I \wedge d\overline{z}_J,$$

where $I = (i_1, \ldots, i_p)$, $J = (j_1, \ldots, j_q)$ and

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \quad d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}.$$

We denote by $D'_{(p,q)}(\Omega)$ the space of currents of bidimension (p,q), *i.e.* the space of continuous linear forms on $D_{(p,q)}(\Omega)$. One can also talk about currents of bidegree $(m-p, m-q)$ since a current S can be represented as a $(m-p, m-q)$ differential form with distributional coefficients:

$$S = \sum_{\substack{|I'|=m-p \\ |J'|=m-q}} S_{I',J'} dz_{I'} \wedge d\overline{z}_{J'}.$$

The differential d is defined by duality by

$$\langle dS, \phi \rangle = (-1)^{p+q+1} \langle S, d\phi \rangle,$$

or equivalently on coefficients by

$$dS = \sum_j \sum_{I',J'} \Big(\frac{\partial S_{I',J'}}{\partial z_j} dz_j + \frac{\partial S_{I',J'}}{\partial \overline{z}_j} d\overline{z}_j \Big) \wedge dz_{I'} \wedge d\overline{z}_{J'}.$$

It can be decomposed in $d = \partial + \overline{\partial}$ where

$$\partial S = \sum_j \sum_{I',J'} \frac{\partial S_{I',J'}}{\partial z_j} dz_j \wedge dz_{I'} \wedge d\overline{z}_{J'}, \quad \overline{\partial}S = \sum_j \sum_{I',J'} \frac{\partial S_{I',J'}}{\partial \overline{z}_j} d\overline{z}_j \wedge dz_{I'} \wedge d\overline{z}_{J'}.$$

We write $d^c = \frac{i}{2\pi}(\partial - \overline{\partial})$ so that we have $dd^c = \frac{i}{\pi}\partial\overline{\partial}$.

A.3. Positive currents. — A current of bidimension (p,p) (or of bidegree $(m-p, m-p)$) is positive if $\langle S, \phi \rangle \geqslant 0$ for all ϕ of the form

$$\phi = i\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge i\alpha_p \wedge \overline{\alpha}_p,$$

with $\alpha_i \in D_{(1,0)}(\Omega)$. A current S of bidegree $(1,1)$ can be written

(104) $$S = \sum_{j,k} S_{j,k} i dz_j \wedge d\overline{z}_k,$$

and is positive if for any w in $\mathbb{C}^m$ the distribution

$$\sum_{j,k} S_{j,k} w_j \overline{w}_k$$

is a positive measure.

   With formula (103) we see that a function u in $L^1_{\mathrm{loc}}(\Omega)$ is almost surely equal to a psh function if and only if $dd^c u$ is a positive current. We have the following converse result.

PROPOSITION A.2. — *Let S be a $(1,1)$ positive closed current on an open ball of $\mathbb{C}^m$ then there exists a psh function u such that $S = dd^c u$. We say that u is a potential of S.*

Examples and basic properties

   (i) Let Z be an analytic subset of Ω with pure dimension p. Let $\mathrm{Reg}(Z)$ be the subset of regular points of Z (i.e the subset of points where Z is locally a complex manifold of dimension p). We define $[Z]$ as the current of bidimension (p,p) defined by

$$\langle [Z], \phi \rangle = \int_{\mathrm{Reg}(Z)} \phi, \quad \forall \phi \in D_{(p,p)}(\Omega).$$

This current is positive of bidimension (p,p) and in fact closed, as shown by Lelong.

   (ii) Let f be a holomorphic function on Ω. We call divisor a formal sum of irreducible analytic hypersurfaces of Ω. The divisor of f is the divisor $\sum m_i Z_i$, where the Z_i's are irreducible analytic hypersurfaces, such that f can be written

$$f = g \prod_i f_i^{m_i},$$

for a holomorphic function g which does not take the value 0, and holomorphic functions f_i such that $Z_i = \{f_i = 0\}$ and f_i is a generator of the ideal $V(Z_i) =$

$\{f$ holomorphic on Ω, $f(z) = 0$ on $Z_i\}$ (*cf.* for example, [**22**] section 2.1). Then the Lelong-Poincaré equation states

(105)
$$dd^c \log|f| = \sum_i m_i [Z_i].$$

(iii) All these definitions can be extended to complex manifolds using a local chart.

A.4. Currents of bidegree $(1,1)$ **on** $\mathbb{P}^k$**.** — Let $\mathbb{P}^k$ be the complex projective space of dimension k and $\pi : \mathbb{C}^{k+1} \smallsetminus \{0\} \to \mathbb{P}^k$ the canonical projection. Let $\mathcal{P}$ be the convex cone of psh functions u on $\mathbb{C}^{k+1}$ such that for a real $c > 0$

(106)
$$u(\lambda z) = c \ln |\lambda| + u(z), \quad \lambda \in \mathbb{C}, \ z \in \mathbb{C}^{k+1}.$$

To any u in $\mathcal{P}$ we can associate a positive closed current of bidegree $(1,1)$ on $\mathbb{P}^k$: let s be a holomorphic section of π on an open subset $U \subset \mathbb{P}^k$, then $dd^c(u \circ s)$ defines a positive closed current S of bidegree $(1,1)$ on U. If s' is another section of π then $s' = j \cdot s$ for a holomorphic function j which does not take the value 0. Hence $dd^c(u \circ s) = dd^c(u \circ s')$ so S does not depend on the particular section and defines a positive closed current on all $\mathbb{P}^k$. We denote by $L : \mathcal{P} \to D'_{k-1,k-1}$ the operator defined by $L(u) = S$.

PROPOSITION A.3 ([**45**], theorem A.5.1). — *For any positive closed current S of bidegree $(1,1)$ on $\mathbb{P}^k$ there exists a unique (up to an additive constant) function $u \in \mathcal{P}$ such that $L(u) = S$. The function u is called a potential of S.*

For example, if P is an irreducible homogeneous polynomial of degree d on $\mathbb{C}^{k+1}$ then

$$\log|P(\lambda z)| = d \log |\lambda| + \log|P(z)|,$$

and by the Lelong-Poincaré formula (105) the current $[P = 0]$ has potential $\log|P|$.

A.5. The Fubini-Study form, Kähler forms. — Consider on $\mathbb{P}^k$ the closed positive form ω of bidegree $(1,1)$ with potential $\log \|z\|$ on $\mathbb{C}^{k+1}$. We take homogeneous coordinates

$$[z_0 : z_1 : \cdots : z_k]$$

on $\mathbb{P}^k$ (*i.e.* the point $[z_0 : \cdots : z_k]$ represents the image by π of the point $(z_0, \dots, z_k)$ of $\mathbb{C}^{k+1}$). The space $\mathbb{C}^k$ is identified with $\mathbb{P}^k \smallsetminus \pi(\{z_0 = 0\})$ taking for coordinates $w_i = z_i/z_0$. On $\mathbb{C}^k$ the form ω is given by (*cf.* for example, [**22**], page 30):

$$\omega = \frac{i}{2\pi} \left(\frac{\sum dw_i \wedge d\overline{w}_i}{1 + w_i \overline{w}_i} - \frac{(\sum \overline{w}_i dw_i) \wedge (\sum w_i d\overline{w}_i)}{(1 + \sum w_i \overline{w}_i)^2} \right).$$

The form ω is called the Fubini-Study form on $\mathbb{P}^k$ and has the following properties: it is smooth, closed, and at any point the coefficients $(S_{j,k})$ defined by equation (104) defines a positive definite matrix. In general, on a complex manifold a $(1,1)$ form with these properties is called a Kähler form and a complex manifold with such a form is called a Kähler manifold. We do not want to enter into the details of this notion

(*cf.* for example, [**22**] or [**6**]) but we just want to point out that if X is a smooth analytic subvariety of $\mathbb{P}^k$ then the restriction of the Fubini-Study form ω to X defines a Kähler form on X (which is canonical for the embedding $X \subset \mathbb{P}^k$). The volume of X defined as

$$\int_X (\omega_X)^{\dim X}$$

is finite and actually equal to the degree of X.

If S is a positive closed current on $\mathbb{P}^k$ of bidegree (p, p) on $\mathbb{P}^k$ then the total mass of S is defined as

$$\|S\| = \int_{\mathbb{P}^k} S \wedge \omega^{k-p}.$$

If S is of bidegree $(1, 1)$ and has potential u then $\|S\| = c$ where c is the homogeneity constant appearing in formula (106). Indeed, the function $v = u - c \log \| \cdot \|$ is well defined on $\mathbb{P}^k$, hence $\|S\| = c \int w^k + \int dd^c v \wedge \omega^{k-1} = c \int \omega^k$ since ω^{k-1} is closed, and $\int \omega^k = 1$. Finally, we mention that if (S_n) is a sequence of positive currents such that the total mass converges to 0 then S_n converges to 0 in the sense of currents (*cf.* [**45**]).

In the same way, if S is a current on X (or on any compact Kähler manifold) then $\|S\|$ is defined by $\int_X S \wedge (\omega_{|X})^{\dim X - p}$.

A.6. Pull-back, push-forward of a current. — Let $f : \Omega \to \Omega'$ be a holomorphic map between open subsets of $\mathbb{C}^m$ and $\mathbb{C}^{m'}$. The pull-back $f^*\alpha$ of a smooth form α in $D_{(p,q)}(\Omega')$ is well-defined as an element of $D_{(p,q)}(\Omega)$. Thus we can define the push-forward f_*S of a current S of bidimension (p, q) by duality, *i.e.*

$$\langle f_*S, \phi \rangle = \langle S, f^*\phi \rangle, \quad \forall \phi \in D_{(p,q)}(\Omega').$$

For some particular class of maps we can define the push-forward of differential forms and then the pull-back of currents (*cf.* [**45**], A3) but we do not want to enter into details since we will only consider the case of $(1, 1)$ positive closed currents for which the situation is simpler. Suppose that the map f is dominating, *i.e.* that its differential is generically surjective. Let S be a positive closed current of bidegree $(1, 1)$ on Ω'. Let z_0 be in Ω and set $w_0 = f(z_0)$. For $r > 0$ we can write $S = dd^c u$ for a psh function u on $B(w_0, r)$. Choose r_1 such that $f(B(z_0, r_1)) \subset B(w_0, r)$. The function $u \circ f$ is psh (indeed, it is not equal to $-\infty$ since f is dominating). This definition does not depend on the choice of the potential u since if u_1 and u_2 are 2 potential of S then $u_1 - u_2$, and hence $(u_1 - u_2) \circ f$ are pluriharmonic. Then the pull-back f^*S is defined locally by $f^*S = dd^c u \circ f$. In [**45**] it is proved that the pull-back is continuous on the set of positive closed currents. Remark that when f is not dominating the pull-back can be defined similarly, as soon as $f(\Omega)$ is not included in the set where the potential of S is $-\infty$ (when f is not dominating the pull-back is a priori not continuous). We will see in next sections that actually the pull-back can be defined for meromorphic maps on compact complex manifolds.

B. Dynamics of rational maps on the projective space $\mathbb{P}^k$

B.1. Definitions, indeterminacy points. — Let $\mathbb{P}^k$ be the complex projective space of dimension k and $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ the canonical projection. A point z in $\mathbb{P}^k$ can be represented in homogeneous coordinates by

$$z = [z_0 : \cdots : z_k],$$

where $[z_0 : \cdots : z_k]$ denotes the point $\pi(z_0, \ldots, z_k)$. Consider now a homogeneous polynomial map $R : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ of degree d, *i.e.* $R = (R_0, \ldots, R_k)$ where the R_i's are homogeneous polynomials of degree d in the variables $(z_0, \ldots, z_k)$. Suppose that the R_i have no common factor. It is natural to associate a map f on the projective space $\mathbb{P}^k$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{C}^{k+1} & \xrightarrow{\ R\ } & \mathbb{C}^{k+1} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\mathbb{P}^k & \xrightarrow{\ f\ } & \mathbb{P}^k
\end{array}
$$

The map f can be defined only on the set $\mathbb{P}^k \setminus I$ where $I = \pi\{z,\ R(z) = 0\}$. The set I is called the set of indeterminacy points of f and is an analytic subset of codimension at least 2 (indeed if all the R_i's vanish on an analytic hypersurface then a polynomial can be factorized in R). Usually one writes f in homogeneous coordinates as

$$f = [R_0, \ldots, R_k].$$

The function f is a rational map of $\mathbb{P}^k$ (a precise meaning to this notion is given at the beginning of appendix C) and the polynomial map R, which is called the lift of f to $\mathbb{C}^{k+1}$, exists and is unique (up to a multiplicative constant) for any rational map of $\mathbb{P}^k$. The degree of f is defined as the degree of R, *i.e.* d. The map f is holomorphic on $\mathbb{P}^k$ if and only if $I = \varnothing$. The map f is said to be dominating if its differential is generically surjective. If f is dominating and if S is a positive closed current of bidegree $(1, 1)$ with potential u on $\mathbb{C}^{k+1}$ we define the pull-back f^*S as the current with potential $u \circ R$.

At a point $p \in I$ the image of f can be defined as a subset of $\mathbb{P}^k$: let $\mathcal{B}_p$ be the subset of $\mathbb{P}^k$ defined by

$$\mathcal{B}_p = \cap_{\varepsilon > 0} \overline{f(B(p, \varepsilon) \setminus I)}$$

where $B(p, \varepsilon)$ is the ball of center p and radius ε. Then $\mathcal{B}_p$ is an analytic subset of $\mathbb{P}^k$ called the blow-up of f at p (as we will see in appendix C, the blow-up can also be defined using the graph of f). On the other hand an irreducible subvariety of dimension p can be sent into a subvariety of dimension strictly smaller. These phenomenons are new compared to the situation of 1-dimensional complex dynamics. This results in the fact that the degree of the iterates of f do not necessarily grow like d^n. Let us explain this clearly: suppose, for example, that a hypersurface V is sent by f on a point of indeterminacy. Then the map R^2 is null on V and a hence a

polynomial can be factorized in R^2. The degree of f^2 is then smaller than $\text{degree}(f)^2$ since f^2 is lifted to a polynomial map with degree smaller than that of R^2. In general, we can always write

$$R^n = h_n R_n,$$

where h_n is a homogeneous polynomial and R_n a homogeneous polynomial map, with no common factor, of degree $d_n = d^n - \deg(h_n)$. The map R_n is a reduced lift of f^n, thus the degree of f^n is d_n. We say that there is decreaseness in the degree if $d_n < d^n$.

PROPOSITION B.1 ([**45**], proposition 4.3). — *Let f and g be rational maps of $\mathbb{P}^k$ of degree d and d'. The degree of $f \circ g$ is smaller than dd' and equal to dd' if and only if there does not exist any hypersurface V such that $g(V \smallsetminus I_g) \subset I_f$.*

DEFINITION-PROPOSITION B.2. — *The map f is algebraically stable if there does not exist an integer n and a hypersurface V such that*

$$f^n(V \smallsetminus I) \subset I.$$

If f is algebraically stable the degree of f^n is d^n, but in general we only have the inequality

$$d_{n+m} \leqslant d_n d_m.$$

We define the dynamical degree as the limit

$$d_\infty = \lim_{n \to \infty} \frac{1}{n} \log d_n.$$

The map f is algebraically stable if and only if $d_\infty = d$.

Proof. — The proof of this result is clear from proposition (B.1). Indeed $d_n = d^n$ for all n if and only if f is algebraically stable and if $d_n < d^n$ for a $n > 0$ then clearly the limit d_∞ (which exists by subadditivity) is smaller than d. $\qquad\square$

B.2. Green function and Green current. The algebraically stable case

Let f be a rational map, algebraically stable, with degree d. We denote by I_n the set of indeterminacy points of f^n, we have $I_n \subset I_m$ for $n \leqslant m$, and we set $I_\infty = \cup_{n=1}^\infty I_n$. We recall the following definitions.

  – A point p in $\mathbb{P}^k \smallsetminus I_\infty$ is in the Fatou set of f if there exists a neighborhood U of p such that $f^n_{|U}$ is equicontinuous.

  – The Julia set is the complement of the Fatou set.

  – A point p is normal if there exist neighborhoods U of p and V of I such that $f^n(U) \cap V = \varnothing$ for all n. We denote by N the set of normal points.

Denote by $G_n : \mathbb{C}^{k+1} \to \mathbb{R} \cup \{-\infty\}$ the psh function

(107) $$G_n(z) = \log \|R^n(z)\|, \quad z \in \mathbb{C}^{k+1}.$$

Remark that G_0 is by definition the potential of the Fubini-Study form ω on $\mathbb{P}^k$ and that G_n is the potential of the pull-back $S_n = (f^n)^*\omega$.

THEOREM B.3 ([**45**], Théorème 6.1, 6.5, 7.1). — *Let f be a rational map, algebraically stable, with degree d. The sequence $\frac{1}{d^n}G_n$ converges pointwise and in L^1_{loc} to a psh function G satisfying:*

$$\begin{cases} G(\lambda z) = \log|\lambda| + G(z) \\ G(R(z)) = dG(z) \end{cases}$$

The function G is called the Green function of f. The current S with potential G is called the Green current of f. It is the limit of the sequence $\frac{1}{d^n}(f^n)^*\omega$ and satisfies

$$f^*S = dS.$$

(i) *The current S does not charge hypersurfaces.*

(ii) *The support of S is contained in the Julia set of f. The set $N \cap (\mathbb{P}^k \smallsetminus \mathrm{supp}\, S)$ is contained in the Fatou set (in particular, when $N = \mathbb{P}^k$, the Julia set equals the support of S).*

(iii) *The Green function is Hölder continuous in the set of normal points $\pi^{-1}(N)$.*

REMARK B.4. — The question of whether the sequence of currents $\frac{1}{d^n}(f^n)^*S_0$ converges to S when we start from a particular current S_0 is not easy in general. In particular, it is interesting to consider the preimages of the current $[V]$ of integration on a hypersurface V. In this case the limit $\frac{1}{d^n}(f^n)^*[V]$ represents the asymptotic repartitions of the preimage of V. Generically, the limit is S (*cf.* [**45**]), but for a particular V the problem to know whether the limit is S is a priori not easy. This is more or less the problem we encounter in a particular case to prove theorem (3.6). There are general results for this problem in the case of birational maps on compact Kähler manifold, *cf.* [**12**].

B.3. The non algebraically stable case. — When f is not algebraically stable then we can write for n large enough

$$R^n = h_n R_n,$$

where h_n is a homogeneous polynomial of positive degree, R_n a homogeneous polynomial map with non common factor. We can still define G_n by equation (107) and S_n as the current with potential G_n. Remark that in this case S_n is not equal to the pull-back $(f^n)^*\omega$ since the map R^n is not reduced (*i.e.* its components have common factors). Precisely, we have

$$S_n = (f^n)^*\omega + [h_n = 0].$$

Indeed, we see that

$$dd^c \log\|R^n\| = dd^c \log\|R_n\| + dd^c \log|h_n|.$$

The first term equals $(f^n)^*\omega$ since R_n is a reduced lift for f^n and $\log|h_n|$ is a potential of $[h_n = 0]$ (counting multiplicities, *i.e.* $[h_n = 0]$ stands for the current of integration on the divisor of h_n) by the Lelong-Poincaré equation. Remark also that h_{n+1} divides h_n^d so that we have $[h_{n+1} = 0] \geqslant d[h_n = 0]$ (counting multiplicities). This implies that $\frac{1}{d^n}[h_n = 0]$ converges towards a current with support contained in $\cup_n\{h_n = 0\}$. On the other hand, we remark that the total mass of $(f^n)^*\omega$ is equal to the degree of R_n, *i.e.* d_n, thus $\frac{1}{d^n}(f^n)^*\omega$ converges to 0. This sums-up in the following result.

THEOREM B.5 (*cf.* [**45**], Théorème 9.1). — *When f is not algebraically stable then*

(108)
$$S = \lim_{n \to \infty} \frac{1}{d^n}[h_n = 0]$$

and the current S is supported by a countable union of hypersurfaces.

REMARK B.6. — We see that in this case the current S does not contain much information about the dynamics of f, but just about the distribution of the hypersurfaces going to indeterminacy points.

C. Iteration of meromorphic maps on compact complex manifolds

C.1. Definitions, indeterminacy points. — Let X and Y be compact complex manifolds. Denote by $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ the projection on the first and second components of $X \times Y$. A meromorphic function $f : X \to Y$ is defined by its graph $\Gamma_f \subset X \times Y$, an irreducible subvariety of $X \times Y$ for which the first projection is a proper modification, *i.e.* such that there exists a proper subvariety $V \subset X$ such that π_1 is a biholomorphism from $\Gamma_f \smallsetminus \pi_1^{-1}(V)$ to $X \smallsetminus V$. We denote by $I_f \subset X$ the set of points of indeterminacy, *i.e.* the set of points where π_1 has no local inverse. The set I_f is an analytic subset of codimension at least 2. Of course, at a point $x \in X \smallsetminus I_f$ the image by f is defined by $f(x) = \pi_2(\pi_1^{-1}(x))$. When x is an indeterminacy point the image by f is an analytic set defined by $f(x) = \pi_2(\pi_1^{-1}(\{x\}))$. The map f is holomorphic on $X \smallsetminus I_f$. The map is said to be dominating if π_2 is surjective.

If $g : Y \to Z$ is another meromorphic map then the graph $\Gamma_{g \circ f}$ is defined as the closure:

$$\Gamma_{g \circ f} = \overline{\{(x, g(f(x))) \mid x \in X \smallsetminus I_f,\ f(x) \in Y \smallsetminus I_g\}}$$

One can also consider the graph

$$\Gamma_g \circ \Gamma_f = \{(x, z) \mid \exists\, y \in Y \text{ such that } (x, y) \in \Gamma_f \text{ and } (y, z) \in \Gamma_g\}.$$

PROPOSITION C.1 (*cf.* [**12**], proposition 1.5). — *The equality $\Gamma_{g \circ f} = \Gamma_g \circ \Gamma_f$ is satisfied if and only if $\Gamma_g \circ \Gamma_f$ is irreducible and this is true if and only if there is no hypersurface $V \subset X$ such that $f(V) \subset I_g$.*

Definition C.2. — A dominating meromorphic map $f : X \to X$ is said to be analytically stable if $\Gamma_f \circ \Gamma_{f^n} = \Gamma_{f^{n+1}}$, *i.e.* there does not exist an hypersurface $V \subset X$ and an integer n such that $f^n(V) \subset I_f$.

Remark C.3. — When X is a smooth algebraic projective variety (*i.e.* a smooth irreducible analytic subset of a projective space) the terminology meromorphic map and analytically stable are replaced by rational map and algebraically stable (due to the algebraicity of the manifold). In the case of a rational map on the projective space defined as in appendix B the reader can check that the definitions are consistent.

C.2. Action on cohomology groups. — We suppose now that X is a smooth algebraic variety, *i.e.* that it can be embedded as a smooth irreducible analytic subset of a projective space (actually, we could only suppose that X is a Kähler manifold). The counterpart of the degree in the case of projective spaces will come from the action of f on the cohomology groups of X. We first need to introduce some notions and notations. We denote by $H^r(X, \mathbb{C})$ the De-Rham r-cohomology group of X, *i.e.* the quotient of closed r-differential forms by exact r-differential forms. The cohomology class of a current can also be defined since there is identification between $H^r(X, \mathbb{C})$ and the quotient of closed currents of degree r by exact currents of degree r. We denote by $\{\alpha\}$ the class of a differential form (or current) α. For us the Dolbeault cohomology group $H^{(p,q)}(X)$ will be the subspace of $H^{p+q}(X, \mathbb{C})$ of classes of differential forms of bidegree (p, q) (in general the Dolbeault cohomology groups are not defined in this way but in the case of Kähler manifolds there is identification).

Let $f : X \to X$ be a rational map. Using a desingularization of the graph Γ_f (we do not want to enter into the definition of this notion here) it is possible to consider the pull-back of π_2 in the sense of forms and the push-forward of π_1 in the sense of currents. This allows us to define the pull-back $f^*\alpha = \pi_{1*}(\pi_2^*\alpha)$ for all smooth form $\alpha \in D_{p,q}(X)$ as a current on X. The push-forward f_* can be defined similarly. We do not want to enter into details of the construction here (*cf.* [**12**] or [**13**]), but we just want to point out that f^* and f_* have the following properties

- $f^*\alpha$ is smooth on $X \smallsetminus I_f$,
- $f^*\alpha$ is in L^1_{loc},
- f^* and f_* commutes with d.

By the third property we see that the operators f^* and f^* induce linear operators on the Dolbeault cohomology groups $H^{(p,q)}(X)$ by

$$f^*\{\alpha\} = \{f^*\alpha\}, \quad \forall \, \alpha \in D_{p,q}(X).$$

In general, it is not possible to define the pull-back f^* on the space of currents in a continuous way (*cf.* [**13**]) but when S is a closed positive $(1, 1)$ current we can define $f^*S = \pi_{1*}\pi_2^*S$ where the pull-back π_2^*S is defined as in appendix A.6 using locally

a psh potential for S and the push forward is defined in the sense of current. The pull-back f^* has the following properties (*cf.* [**13**], proposition 2.2.8):

- f^* is continuous on the cone of closed positive $(1,1)$ currents.
- For any hypersurface $V \subset X$

$$f^*[V] = [f^{-1}(V)].$$

- $f^*\{S\} = \{f^*S\}$ for any closed positive $(1,1)$ current S.

C.3. The matrix of degrees. — Suppose now that

(109)
$$X = X_1 \times \cdots \times X_r.$$

where the X_i's are smooth algebraic varieties, simply connected, and such that $\dim_{\mathbb{C}} H^{1,1}(X_i) = 1$ for all i in $1, \ldots, r$. Since X is Kähler and simply connected we have $H^{1,0}(X_i) = H^{0,1}(X_i) = 0$ and thus $H^{1,1}(X) = H^{1,1}(X_1) \oplus \cdots \oplus H^{1,1}(X_r)$ is of dimension r. Consider ν_i, the Kähler form on X_i obtained by restriction of the Fubini-Study form to X_i for an embedding of X_i in a projective space, renormalized to be a generator of the $\mathbb{Z}$-cohomology. By abuse of notations we also denote by ν_i the form on X obtained as the pull-back of the form ν_i on X_i by the canonical projection on the i-th coordinate of the right term of (109). The family $(\{\nu_1\}, \ldots, \{\nu_r\})$ gives a natural basis of $H^{1,1}(X)$. In this basis the linear operator $(f^n)^*$ on $H^{1,1}(X)$ is represented by a matrix $d_n = (d_{n,i,j})$ defined by

$$(f^n)^*\{\nu_j\} = \sum_i d_{n,i,j}\{\nu_i\}.$$

PROPOSITION C.4

(i) *The matrix d_n has non-negative integer coefficients.*

(ii) *The sequence of matrices (d_n) is submultiplicative, i.e. $d_{n+m} \leqslant d_n d_m$ with equality for all n, m if and only if f is algebraically stable.*

Proof

(i) Consider the group of integral classes $H^{1,1}_{\mathbb{Z}}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ where $H^2(X, \mathbb{Z})$ is the cohomology group with values in $\mathbb{Z}$ (actually a class α is integral if its integral along any 1-simplex is an integer). The element $\{\nu_i\}$ is a generator of the group $H^{1,1}_{\mathbb{Z}}(X_i)$, be definition. Therefore $(\{\nu_1\}, \ldots, \{\nu_r\})$ generates $H^{1,1}_{\mathbb{Z}}(X)$ and the coefficients of d_n are integers since the linear operator f^* leaves invariant the lattice $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ (*cf.* [**12**], proposition 1.11). The positivity of coefficients comes from the following fact: let $H^{1,1}_{\mathrm{psef}}$ be the cone of classes $\{S\}$ generated by closed positive currents. In our case it is easy to see that $H^{1,1}_{\mathrm{psef}}$ is the cone of elements with positive coordinates in the basis $(\{\nu_1\}, \ldots, \{\nu_r\})$: indeed, suppose that $\{S\} \in H^{1,1}_{\mathrm{psef}}$ then $\langle S, \nu_{i_1} \wedge \cdots \wedge \nu_{i_{n-1}} \rangle \geqslant 0$ for all choice $(i_1, \ldots, i_{n-1})$. Thus, if $\{S\} = \sum_{i=1}^n s_i\{\nu_i\}$, we can choose $(i_1, \ldots, i_{n-1})$ such that $\langle S, \nu_{i_1} \wedge \cdots \wedge \nu_{i_{n-1}} \rangle = s_i$. Finally, we conclude

the proof using the fact that the cone $H^{1,1}_{\mathrm{psef}}$ is left invariant by f^*, *cf.* [**12**], proposition 1.11.

(ii) This is a direct application of proposition 1.13 of [**12**]. □

REMARK C.5. — In our case a considerable simplification comes from the fact that we assumed (109). This implies that the cones $H^{1,1}_{\mathrm{psef}}$ and $H^{1,1}_{\mathrm{nef}}$ considered in [**12**] are equal and coincide with the cone $\mathbb{R}^r_+$ of classes which have positive coordinates in the basis $(\{\nu_1\}, \ldots, \{\nu_r\})$.

C.4. Green currents. — We take from [**13**], [**45**] the following result.

THEOREM C.6. — *Let f be a dominating meromorphic map, algebraically stable. Let α be a smooth closed positive form of bidegree $(1,1)$ such that the cohomology class satisfies $f^*\{\alpha\} = \rho\{\alpha\}$ for a positive real $\rho > 1$.*

(i) The sequence of currents $\rho^{-n}(f^n)^*\alpha$ converges towards a positive closed $(1,1)$ current S such that

$$(110) \qquad\qquad\qquad\qquad f^*S = \rho S.$$

The current S depends only on the cohomology class of α, i.e. if α' is a smooth differential form cohomologous to α then the limit is the same.

(ii) The support of S is included in the Julia set of f.

REMARK C.7. — When the matrix d_1 is primitive (*i.e.* when d_1 admits a power with strictly positive coefficients) there exists a unique (up to a multiplicative constant) class with positive coordinates which satisfies equation (110). Thus, the current S defined in this way is unique (up to a constant) and it is natural to call it the Green current of f.

C.5. Examples

The projective spaces. — Let $X = \mathbb{P}^k$. Since $\dim_{\mathbb{C}} H^{1,1}(\mathbb{P}^k) = 1$ the matrix d_n is scalar and actually equals the degree of the map f as defined in appendix B. Indeed, if ω is the Fubini-study form on $\mathbb{P}^k$ then the potential $\log \|R_n\|$ of $(f^n)^*\omega$ has the following homogeneity

$$\log \|R_n(\lambda z)\| = \deg(f^n) \log|\lambda| + \log \|R_n(z)\|.$$

This implies that $u(z) = \log \|R_n(z)\| - \deg(f^n) \log \|z\|$ is defined globally on $\mathbb{P}^k$ and thus that $\deg(f^n)\omega - (f^n)^*\omega = dd^c u$. Thus $d_n\{\omega\} = \{(f^n)^*\omega\} = \deg(f^n)\{\omega\}$.

$X = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ *(r times).* — A point z in X can be represented in homogeneous coordinates by

$$(111) \qquad\qquad\qquad z = ([x_1 : y_1], \ldots, [x_r : y_r]).$$

A map $f : X \to X$ can be represented in homogeneous coordinates by

$$f = ([P_1 : Q_1], \ldots, [P_r : Q_r]),$$

where $P_j(x_1, y_1, \ldots, x_r, y_r)$ and $Q_j(x_1, y_1, \ldots, x_r, y_r)$ are polynomials homogeneous in the variables (x_i, y_i) with same degree and no common factor, *i.e.* we have

$$P_j(x_1, y_1, \ldots, \lambda x_i, \lambda y_i, \ldots, x_r, y_r) = \lambda^{d_{i,j}} P_j(x_1, y_1, \ldots, x_r, y_r)$$

(and idem for Q_j) if the degree of (P_j, Q_j) in the variables (x_i, y_i) is $d_{i,j}$. It happens that the degrees $(d_{i,j})$ coincide with the matrix of degrees of f defined previously (*cf.* [**21**]).

D. A different proof of lemma 3.9

We present here a different proof of lemma 3.9. This proof does not involve any direct estimate but just the fact that S_+ is T-invariant and general properties of holomorphic maps on S_+.

On S_+ we consider the hyperbolic metric given by (*cf.* [**46**], [**50**])

$$ds^2 = \operatorname{trace}(V^{-1}dQ V^{-1}\overline{dQ}), \quad \text{for } Q = U + iV \in S_+.$$

It is well known that S_+ can be identified with $sp(n, \mathbb{R}) \backslash U(n)$, and that ds^2 is invariant under the action of the symplectic group $sp(2n, \mathbb{R})$ (*cf.* [**46**], page 3, *cf.* [**50**]). The geodesic distance induced by ds^2 on S_+ is given by (*cf.* [**46**])

$$(d_{S_+}(Q, Q_1))^2 = \sum_{i=1}^{|F|} \left(\log^2 \left(\frac{1 + r_k^{1/2}}{1 - r_k^{1/2}} \right) \right),$$

where $0 \leqslant r_1 \leqslant \cdots \leqslant r_{|F|} < 1$ are the characteristic roots of the cross ratio $R(Q, Q_1)$ of Q, Q_1 given by

$$R(Q, Q_1) = (Q - Q_1)(Q - \overline{Q}_1)^{-1}(\overline{Q} - \overline{Q}_1)(\overline{Q} - Q_1)^{-1}$$

(*i.e.* this means that the r_k's are the eigenvalues of $\sqrt{R(Q, Q_1)R(Q, Q_1)^*}$. These values r_k satisfy $0 \leqslant r_k \leqslant 1$, *cf.* [**46**], page 16).

The upper-half plane S_+ can be mapped holomorphically onto the matrix ball

$$\{\mathcal{E} \text{ complex sym} \mid I - \mathcal{E}\overline{\mathcal{E}} > 0\}$$

by the Cayley transform

(112) $$Q \longmapsto \mathcal{E} = (Q - i\mathrm{Id})(Q + i\mathrm{Id})^{-1}.$$

Inverting this relation we get

(113) $$Q = i(\mathrm{Id} + \mathcal{E})(\mathrm{Id} - \mathcal{E})^{-1}.$$

With these notations it is clear that the cross ratio $R(Q, i\mathrm{Id})$ is given by

(114) $$R(Q, i\mathrm{Id}) = \mathcal{E}\overline{\mathcal{E}}.$$

The key property we are going to use is the following generalization of the Schwartz-Pick lemma (*cf.* [**29**]). Let f be a holomorphic map from S_+ to S_+, then for any x, y in S_+

$$(115) \qquad\qquad d_{S_+}(f(x), f(y)) \leqslant \sqrt{|F|}\, d_{S_+}(x, y),$$

($|F|$ is the rank of the symmetric space S_+. The exact value of the constant does not matter, the important point is that it does not depend on the function f).

Then we prove the following lemma instead of lemma 3.1.

LEMMA D.1

(i) *The map T is holomorphic on S_+ (resp. on S_+^G) and S_+ (resp. S_+^G) is T-invariant. Moreover, for any Q in S_+ and n we have*

$$(116) \qquad d_{S_+}(i\mathrm{Id}, T^n Q) \leqslant \sqrt{|F|}\, (d_{S_+}(i\mathrm{Id}, Q) + n d_{S_+}(i\mathrm{Id}, T(i\mathrm{Id}))).$$

(ii) *For any Q in S_+ we have the following inequality*

$$\det(\mathrm{Id} + \overline{Q}Q) \leqslant 2^{|F|} \exp(2|F| d_{S_+}(i\mathrm{Id}, Q)).$$

Proof. — The fact that S_+ is T-invariant is proved in lemma 3.1. Using (115) for each of the iterates T^n (indeed, each T^n is a holomorphic map from S_+ to S_+) we get

$$\begin{aligned}
d_{S_+}(i\mathrm{Id}, T^n Q) &\leqslant d_{S_+}(i\mathrm{Id}, T^n(i\mathrm{Id})) + d_{S_+}(T^n(i\mathrm{Id}), T^n Q) \\
&\leqslant \sum_{k=0}^{n-1} d_{S_+}(T^k(i\mathrm{Id}), T^{k+1}(i\mathrm{Id})) + \sqrt{|F|}\, d_{S_+}(i\mathrm{Id}, Q) \\
&\leqslant \sqrt{|F|}\, (n d_{S_+}(i\mathrm{Id}, T(i\mathrm{Id})) + d_{S_+}(i\mathrm{Id}, Q)).
\end{aligned}$$

(ii) Let $\mathcal{E}$ be the image of Q by the Cayley transform (112). Let $\overline{\rho}$ be the largest eigenvalue of $\mathcal{E}\overline{\mathcal{E}}$. We have $\overline{\rho} < 1$ and we deduce from (114) that

$$d_{S_+}(i\mathrm{Id}, Q) \geqslant \log\left(\frac{1 + \overline{\rho}^{1/2}}{1 - \overline{\rho}^{1/2}} \right)$$

Considering now relation (113) we deduce

$$Q\overline{Q} \leqslant \left(\frac{1 + \overline{\rho}^{1/2}}{1 - \overline{\rho}^{1/2}} \right)^2,$$

which immediately implies

$$\det(\mathrm{Id} + Q\overline{Q}) \leqslant 2^{|F|} \exp(2|F| d_{S_+}(Q, i\mathrm{Id})). \qquad \square$$

Proof of lemma 3.9. — This will be a consequence of the estimates we proved in lemma (D.1). We first remark that the terms of the sequences in formulas (57) and

(58) are non-positive. By proposition (3.3), lemma (2.5) and lemma (D.1) we have:

$$
\left| \left\langle \frac{R^n(\exp \overline{\eta} Q \eta)}{\| R^n(\exp \overline{\eta} Q \eta) \|}, 1 \right\rangle \right| = \left| \left\langle \frac{\exp \overline{\eta} T^n Q \eta}{\| \exp \overline{\eta} T^n Q \eta \|}, 1 \right\rangle \right|
$$

$$
= \frac{1}{\| \exp \overline{\eta} T^n Q \eta \|}
$$

$$
= \left(\frac{1}{\det(\mathrm{Id} + T^n Q \overline{T^n Q})} \right)^{1/2}
$$

$$
\geqslant 2^{-\frac{1}{2}|F|} \exp(-|F| d_{S_+}(T^n Q, i\mathrm{Id}))
$$

$$
\geqslant 2^{-\frac{1}{2}|F|} \exp(-|F| \sqrt{|F|} \left(d_{S^+}(Q, i\mathrm{Id}) + n d_{S^+}(i\mathrm{Id}, T(i\mathrm{Id})) \right)).
$$

This immediately implies equality (57). The proof of formula (58) works similarly. Remark first that for any Q in S_+, $d_{S^+}(i\mathrm{Id}, Q) = d_{S^+}(i\mathrm{Id}, -Q^{-1})$ (indeed, $Q \mapsto -Q^{-1}$ is an isometry of S_+ fixing $i\mathrm{Id}$). Using proposition (3.3) and lemma (2.5) we get:

$$
\left| \left\langle \frac{R^n(\exp \overline{\eta} Q \eta)}{\| R^n(\exp \overline{\eta} Q \eta) \|}, \prod_{x \in F} \overline{\eta}_x \eta_x \right\rangle \right| = \frac{|\det(T^n Q)|}{\| \exp \overline{\eta} T^n Q \eta \|}
$$

$$
= \left(\frac{1}{\det(\mathrm{Id} + (T^n Q)^{-1} \overline{(T^n Q)^{-1}})} \right)^{1/2}
$$

and we conclude similarly using lemma (D.1). $\square$

REMARK D.2. — We proved in theorem 3.6 the convergence of the counting measure for the Neumann and Dirichlet boundary condition. The technic presented in this appendix allows, with a little extra effort, to prove the convergence of the counting measure for any boundary condition. Indeed, for each n let B_n be a real symmetric operator on $\mathbb{R}^{F_{\langle n \rangle}}$ supported on $\partial F_{\langle n \rangle}$ (i.e. $B_n f = 0$ if $f_{|\partial F_{\langle n \rangle}} = 0$) (we can thus also consider B_n as an operator on $\mathbb{R}^F$ by identifying $\partial F_{\langle n \rangle}$ with F). We consider the boundary condition induced by B_n, i.e. we denote by $\nu_{\langle n \rangle}^{B_n}$ the counting measure of the eigenvalues of $A_{\langle n \rangle} - B_n$. Proceeding as for the Neumann and Dirichlet boundary condition, we can prove that $\nu_{\langle n \rangle}^{B_n}$ is given by

$$
\nu_{\langle n \rangle}^{B_n} = \frac{1}{2\pi} \Delta \ln \left| \left\langle \exp(-\overline{\eta} B_n \eta) \cdot R^n(\phi(\lambda)), \prod_{x \in F} \overline{\eta}_x \eta_x \right\rangle \right|.
$$

We can as well replace the last expression by

$$
\frac{1}{2\pi} \Delta \ln \left| \left\langle \frac{\exp(-\overline{\eta} B_n \eta)}{\| \exp(\overline{\eta} B_n \eta) \|} R^n(\phi(\lambda)), \prod_{x \in F} \overline{\eta}_x \eta_x \right\rangle \right|.
$$

By the same strategy as before we see that the convergence of $\nu_{\langle n \rangle}^{B_n}$ would be implied by the following convergence for Q in S_+

$$
\lim_{n \to \infty} \frac{1}{N^n} \ln \left| \left\langle \frac{\exp(-\overline{\eta} B_n \eta)}{\| \exp(-\overline{\eta} B_n \eta) \|} \frac{R^n(\exp \overline{\eta} Q \eta)}{\| R^n(\exp \overline{\eta} Q \eta) \|}, \prod_{x \in F} \overline{\eta}_x \eta_x \right\rangle \right| = 0.
$$

We see that as before the term in the logarithm is bounded from above by 1. The term inside the logarithm is equal to

$$(117) \quad \left| \left\langle \frac{\exp \overline{\eta}(T^n Q - B_n)\eta}{\| \exp(-\overline{\eta}B_n\eta)\| \| \exp \overline{\eta}T^n Q\eta\|}, \prod_{x \in F} \overline{\eta}_x \eta_x \right\rangle \right|$$

$$= \frac{|\det(T^n Q - B_n)|}{\left(\det(\mathrm{Id} + \overline{T^n Q}T^n Q) \det(\mathrm{Id} + B_n^2)\right)^{1/2}}$$

The key point is that the change of boundary condition can be viewed as a "rotation" in S_+ (by a rotation in S_+ we mean the image by the Cayley transform of a rotation in the unit ball). It is actually easier to map everything in the unit ball since rotations have a simple expression. Let us first derive some simple formula. Let Q_1 and Q_2 and $\mathcal{E}_{Q_1}$ and $\mathcal{E}_{Q_2}$ there images by the Cayley transform. Then a direct computation gives

$$\frac{|\det(Q_1)|^2}{\det(\mathrm{Id} + \overline{Q}_1 Q_1)} = \frac{|\det(\mathrm{Id} + \mathcal{E}_{Q_1})|^2}{\det(2(\mathrm{Id} + \overline{\mathcal{E}}_{Q_1} \mathcal{E}_{Q_1}))},$$

$$\frac{|\det(Q_1 - Q_2)|^2}{\det(\mathrm{Id} + \overline{Q}_1 Q_1) \det(\mathrm{Id} + \overline{Q}_2 Q_2)} = \frac{|\det(2(\mathcal{E}_{Q_1} - \mathcal{E}_{Q_2}))|^2}{\det(2(\mathrm{Id} + \overline{\mathcal{E}}_{Q_1} \mathcal{E}_{Q_1})) \det(2(\mathrm{Id} + \overline{\mathcal{E}}_{Q_2} \mathcal{E}_{Q_2}))}.$$

In particular in the last expression if $Q_2 = B$ is a real matrix then $\mathcal{E}_B$ is unitary and we get

$$\frac{|\det(Q_1 - B)|^2}{\det(\mathrm{Id} + \overline{Q}_1 Q_1) \det(\mathrm{Id} + B^2)} = \frac{|\det(\mathcal{E}_{Q_1} - \mathcal{E}_B)|^2}{\det(2(\mathrm{Id} + \overline{\mathcal{E}}_{Q_1} \mathcal{E}_{Q_1}))}.$$

If we apply this to (117) we get

$$\frac{|\det(\mathcal{E}_{T^n Q} - \mathcal{E}_{B_n})|}{\det(2(\mathrm{Id} + \overline{\mathcal{E}}_{T^n Q} \mathcal{E}_{T^n Q}))^{1/2}}.$$

The isometries of the unit ball that fixes the point 0 (the rotations) are exactly the maps $\mathcal{E} \mapsto U^t \mathcal{E} U$ for U unitary (*cf.* [**46**], page 11). If we take any U_n that sends $\mathcal{E}_{B_n}$ to $-\mathrm{Id}$ and if we denote by $\tau_n(T^n Q)$ the point of S_+ such that $\mathcal{E}_{\tau_n(T^n Q)} = U_n^t \mathcal{E}_{T^n Q} U_n$, then we have $d_{S_+}(i\mathrm{Id}, \tau(T^n Q)) = d_{S_+}(i\mathrm{Id}, T^n Q)$ and the last expression equals

$$\frac{|\det(\mathrm{Id} + \mathcal{E}_{\tau(T^n Q)})|}{\det(2(\mathrm{Id} + \overline{\mathcal{E}}_{\tau(T^n Q)} \mathcal{E}_{\tau(T^n Q)}))^{1/2}} = \frac{|\det(\tau_n(T^n Q))|}{\left(\det(\mathrm{Id} + \overline{\tau_n(T^n Q)}\tau_n(T^n Q))\right)^{1/2}}.$$

Then we can apply as previously the estimates of lemma D.1 to conclude.

E. G-Lagrangian Grassmannian

Let us first recall the classification of complex irreducible representations, and the Frobenius-Schur Theorem. Let G be a finite group and U an irreducible representation of G over $\mathbb{C}$. We denote be χ its character. The representation U is said to be

  – of type 1 if the character χ is not real.

– of type 2 if the character of χ is real, and the representation U is realizable over $\mathbb{R}$ (*i.e.*, U can be realized as the complexification of an irreducible representation of G over $\mathbb{R}$).

– of type 3 if the character χ is real, but U is not realizable over $\mathbb{R}$.

Let us now recall a consequence of the Frobenius-Schur theorem (*cf.* [**44**], proposition 38).

PROPOSITION E.1

  i) If G does not have a non-zero invariant bilinear form on U, then U is of type 1.

  ii) If such a form exists, it is unique up to homothety, is non-degenerate, and is either symmetric or skew-symmetric. If it is symmetric, then U is of type 2, and if it is skew-symmetric, then U is of type 3.

Let us now consider an irreducible representation W of G over $\mathbb{R}$, and denote by $W^{\mathbb{C}}$ its complexification. Then, there are three possible cases (mutually exclusive). If $W^{\mathbb{C}}$ is irreducible in $\mathbb{C}$, then $W^{\mathbb{C}}$ is of type 2, and by extension we say that W is of type 2. If $W^{\mathbb{C}}$ is not irreducible in $\mathbb{C}$, then $W = U \oplus \overline{U}$, where U and $\overline{U}$ (the complex conjugate of U) are irreducible in $\mathbb{C}$. In this case U is necessarily of type 1 or 3. If U is of type 1, then so is $\overline{U}$, and U and $\overline{U}$ are not isomorphic. By extension, in this case, we say that W is of type 1. If U is of type 3, then so is $\overline{U}$, and U and $\overline{U}$ are isomorphic. By extension, in this case, we say that W is of type 3.

Let us now introduce some definitions. Let n be an integer. We consider $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$. We denote by $(\,,\,)$ the canonical symmetric bilinear form on $\mathbb{C}^{2n}$, $(X,Y) = X^t Y$, and by $\langle\,,\,\rangle$ the canonical hermitian scalar product, $\langle X, Y \rangle = (X, \overline{Y})$. We denote Id_n the $n \times n$ identity matrix and by J_n the $2n \times 2n$ antisymmetric matrix defined by

$$(118) \qquad\qquad \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}.$$

We define three types of Grassmannian on $\mathbb{C}^{2n}$. We first denote by $\mathbb{G}^{n,2n}$, the Grassmannian of n dimensional subspaces of $\mathbb{C}^{2n}$. The group $Gl(2n, \mathbb{C})$ acts transitively on $\mathbb{G}^{n,2n}$, and $\mathbb{G}^{n,2n}$ is isomorphic to the homogeneous space $Gl(2n, \mathbb{C})/P_n$, where

$$P_n = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in Gl(2n, \mathbb{C}) \mid C = 0 \right\}.$$

In particular, the tangent space of $\mathbb{G}^{n,2n}$ is isomorphic to $M_n(\mathbb{C})$.

We recall that we defined $\mathbb{L}^n$ as the set of Lagrangian subspaces of $\mathbb{C}^{2n}$, *i.e.* as the set of n-dimensional isotropic subspaces for the canonical symplectic form $\omega(X,Y) = (X, J_n Y)$. We denote by $Sp(n, \mathbb{C})$ the linear symplectic group, *i.e.* the group of complex $2n \times 2n$ matrices S, such that $S^t J_n S = J_n$. The group $Sp(n, \mathbb{C})$ acts transitively on the set of Lagrangian subspaces of $\mathbb{C}^{2n}$, and $\mathbb{L}^n$ is isomorphic to the homogeneous spaces $Sp(n, \mathbb{C})/P_n$, where P_n is the parabolic subgroup

$$P_n = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in Sp(n, \mathbb{C}) \mid C = 0 \right\}.$$

In particular, the tangent space of $\mathbb{L}^n$ is isomorphic to $\mathrm{Sym}_n(\mathbb{C})$, the space of complex symmetric $n \times n$ matrices.

We consider now the non degenerate symmetric bilinear form $(X, K_n Y)$ on $\mathbb{C}^{2n}$, where K_n is the symmetric matrix

(119)
$$\begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}.$$

We denote by $\mathbb{O}^n$ the set of maximal isotropic subspaces, for the symmetric bilinear form $(\cdot, K_n \cdot)$, *i.e.*, the set of n-dimensional isotropic subspaces of $\mathbb{C}^{2n}$. The group $O(\mathbb{C}^{2n}, K_n)$ of linear transformation which preserve the bilinear form $(\cdot, K_n \cdot)$ is isomorphic to the classical orthogonal group $O(2n, \mathbb{C})$ (indeed, $(\cdot, K_n \cdot)$ is symmetric and non degenerate) and acts transitively on $\mathbb{O}^n$. Hence, $\mathbb{O}^n$ is isomorphic to $O(\mathbb{C}^{2n}, K_n)/P_n$, where P_n is the parabolic subgroup which leaves invariant the isotropic subspace $\mathbb{C}^n \oplus 0$,

(120)
$$P_n = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(2n, \mathbb{C}) \mid C = 0 \right\}.$$

In particular, the tangent space of $\mathbb{O}^n$ is isomorphic to the space of $n \times n$ complex skew symmetric matrices. We denote by $SO(\mathbb{C}^{2n}, K_n)$ the subgroup of $O(\mathbb{C}^{2n}, K_n)$ of element of determinant 1. Since $P_n \subset SO(\mathbb{C}^{2n}, K_n)$, $\mathbb{O}^n$ has two connected components, $SO(\mathbb{C}^{2n}, K_n)/P_n$ and $\xi SO(\mathbb{C}^{2n}, K_n)/\xi P_n$, where ξ is any element of $O(\mathbb{C}^{2n}, K_n)$ with determinant -1. We denote by $\mathbb{SO}^n = SO(\mathbb{C}^{2n}, K_n)/P_n$, the connected component which contains the $\mathrm{Id} \cdot P_n$.

We suppose now that k is an integer and that $\mathbb{R}^k$ is an orthogonal representation of a finite group G, *i.e.* we consider G as a finite subgroup of the orthogonal group $O(k, \mathbb{R})$. The vector space $\mathbb{C}^k$ is a complex representation of G, and we consider the diagonal action of G on $\mathbb{C}^{2k} = \mathbb{C}^k \oplus \mathbb{C}^k$, *i.e.* G is the subgroup of $O(2k)$ of elements of the form

$$\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \quad \forall g \in G.$$

We consider the Lagrangian Grassmannian $\mathbb{L}^k \simeq Sp(k, \mathbb{C})/P_k$. We denote by $\widehat{\mathbb{L}}^G \subset \mathbb{L}^k$ the subvariety of G-invariant Lagrangian subspaces. We know that $\mathrm{Sym}_k(\mathbb{C})$, the space of $k \times k$ symmetric complex matrices, is embedded in $\mathbb{L}^k$ as follows

$$\mathrm{Sym}_k(\mathbb{C}) \longrightarrow \mathbb{L}^k$$
$$Q \longrightarrow L_Q = \mathrm{vect}\left\{ e_i + \sum_{j=1}^k Q_{i,j} e'_j \right\},$$

where $\{e_1, \ldots, e_k, e'_1, \ldots, e'_k\}$ is the canonical basis of $\mathbb{C}^{2k}$. We denote by $\mathrm{Sym}^G(\mathbb{C})$ the subspace of $\mathrm{Sym}_k(\mathbb{C})$ of Q which commutes with G. It is clear that the Lagrangian subspace L_Q is invariant under G if and only if $Q \in \mathrm{Sym}^G(\mathbb{C})$. We denote by $\mathbb{L}^G$ the closure in $\mathbb{L}^k$ of $\mathrm{Sym}^G(\mathbb{C})$. It is clear that $\mathbb{L}^G \subset \widehat{\mathbb{L}}^G$.

We suppose that $\mathbb{R}^k$ is decomposed into isotopic representations as follows,

(121)
$$\mathbb{R}^k = V_0 \oplus \cdots \oplus V_r,$$

where V_i is an isotopic representation equal to the direct sum of n_i representations iso-morphic to a single $\mathbb{R}$-irreducible representation W_i (and the W_i are not isomorphic). This means that $V_i = W_{i,1} \oplus \cdots \oplus W_{i,n_i}$, where the $W_{i,j}$ are isomorphic to W_i and we can choose the $W_{i,j}$ orthogonal. For simplicity, we write sometimes $V_i = n_i W_i$.

THEOREM E.2. — *The subvariety $\mathbb{L}^G$ is the connected component of $\widehat{\mathbb{L}}^G$ which con-tains the Lagrangian subspace $\mathbb{C}^k \oplus 0$, and is isomorphic to*

$$\mathcal{L}_0 \times \cdots \times \mathcal{L}_r,$$

where

- $\mathcal{L}_i \simeq \mathbb{G}^{n_i, 2n_i}$ *if W_i is of type 1; the dimension of $\mathcal{L}_i$ is n_i^2.*
- $\mathcal{L}_i \simeq \mathbb{L}^{n_i}$ *if W_i is of type 2; the dimension of $\mathcal{L}_i$ is $n_i(n_i + 1)/2$.*
- $\mathcal{L}_i \simeq \mathbb{SO}^{2n_i}$ *if W_i is of type 3; the dimension of $\mathcal{L}_i$ is $2n_i^2 - n_i$.*

Proof. — Let us first introduce some notations. If $M = (M_{i,j})$ is a $n \times n$ matrix and N a $m \times m$ matrix, we denote by $M \otimes N$, the $mn \times mn$ matrix of the tensor product, given by blocks by

$$\left(\begin{array}{c} M_{i,j}N \end{array} \right)_{i,j=1,\ldots,n}.$$

If U is a subspace of $\mathbb{C}^k$, then we will denote by U and U' the subspaces of $\mathbb{C}^{2k}$, equal respectively to the copy of U on the first and the second component of $\mathbb{C}^k \oplus \mathbb{C}^k$, so that we have $U' = J_k U$. Similarly, if f is a vector of $\mathbb{C}^k$, we denote by f and f', respectively the copy of f on the first and second component of $\mathbb{C}^k \oplus \mathbb{C}^k$, so that $f' = Jf$. We denote by $V_i^{\mathbb{C}}$ the complexification of V_i. Hence, with the previous notations we have

(122) $$\mathbb{C}^{2k} = V_0^{\mathbb{C}} \oplus \cdots \oplus V_r^{\mathbb{C}} \oplus (V_0^{\mathbb{C}})' \oplus \cdots \oplus (V_r^{\mathbb{C}})'.$$

Let us first prove that $\mathbb{L}^G$ is a connected component of $\widehat{\mathbb{L}}^G$. The space $\mathrm{Sym}_k(\mathbb{C})$ is embedded in $\mathbb{L}^k$ and, with this embedding,

$$\mathrm{Sym}_k(\mathbb{C}) = \mathbb{L}^k \smallsetminus \{L \in \mathbb{L}^k \mid L \cap (0 \oplus \mathbb{C}^k) \neq \{0\}\}.$$

As we already remarked, we know that a Lagrangian subspace of the type L_Q is in $\widehat{\mathbb{L}}^G$ if and only if Q is in $\mathrm{Sym}^G(\mathbb{C})$. Otherwise stated, it means that

$$\mathrm{Sym}^G(\mathbb{C}) = \widehat{\mathbb{L}}^G \smallsetminus \{L \in \widehat{\mathbb{L}}^G \mid L \cap (0 \oplus \mathbb{C}^k) \neq \{0\}\}.$$

Since $\mathrm{Sym}^G(\mathbb{C})$ is connected, it means that $\mathbb{L}^G$ is a connected component of $\widehat{\mathbb{L}}^G$. We set

$$Sp^G(\mathbb{C}) = \{S \in Sp(k, \mathbb{C}) \mid gS = Sg, \ \forall g \in G\},$$

and $P^G = P_k \cap Sp^G$. Let us now prove that $\mathbb{L}^G$ is isomorphic to the connected component of Sp^G/P^G which contains $\mathrm{Id} \cdot P^G$. Indeed, Sp^G/P^G is isomorphic to the

subset of $\mathbb{L}^k \simeq Sp(k, \mathbb{C})/P_k$ of Lagrangian subspaces L such that there exists S in $Sp^G(\mathbb{C})$, such that

$$S(\mathbb{C}^k \oplus 0) = L.$$

Hence $Sp^G/P^G \subset \widehat{\mathbb{L}}^G$. But for Q in $\mathrm{Sym}^G(\mathbb{C})$,

$$S = \begin{pmatrix} \mathrm{Id}_k & 0 \\ Q & \mathrm{Id}_k \end{pmatrix},$$

is in $Sp^G(\mathbb{C})$ and $S(\mathbb{C}^k \oplus 0) = L_Q$. Hence, $\mathrm{Sym}^G(\mathbb{C}) \subset Sp^G/P^G$, and thus $\mathbb{L}^G \subset Sp^G/P^G$, since Sp^G/P^G is compact. This implies that $\mathbb{L}^G$ is a connected component of $Sp^G(\mathbb{C})/P^G$.

REMARK E.3. — It is not true in general that $\widehat{\mathbb{L}}^G \simeq Sp^G/P^G$. Actually, we have the inclusions $\mathbb{L}^G \subset Sp^G/P^G \subset \widehat{\mathbb{L}}^G$, and each of these inclusions can be strict. We will try to clarify this point in a subsequent work.

Let us first prove that we can reduce the problem to the case where $\mathbb{R}^k$ contains a unique type of irreducible representation, *i.e.* to the case where $r = 0$. Since the subspaces V_i, V_i' are real orthogonal, the subspaces $V_i^{\mathbb{C}} \oplus (V_i^{\mathbb{C}})'$ are ω-orthogonal and the restriction of the symplectic form ω to $V_i^{\mathbb{C}} \oplus (V_i^{\mathbb{C}})'$ is non degenerated, hence is a symplectic form. This implies that

$$Sp^G(\mathbb{C}) \simeq Sp^G(V_0^{\mathbb{C}} \oplus (V_0^{\mathbb{C}})') \times \cdots \times Sp^G(V_r^{\mathbb{C}} \oplus (V_r^{\mathbb{C}})'),$$

and

$$P^G \simeq P^G(V_0^{\mathbb{C}} \oplus (V_0^{\mathbb{C}})') \times \cdots \times P^G(V_r^{\mathbb{C}} \oplus (V_r^{\mathbb{C}})'),$$

where $Sp^G(V_i^{\mathbb{C}} \oplus (V_i^{\mathbb{C}})')$ is the group of G-invariant symplectic transformation on $V_i^{\mathbb{C}} \oplus (V_i^{\mathbb{C}})'$, and $P^G(V_i^{\mathbb{C}} \oplus (V_i^{\mathbb{C}})')$ the subgroup which leaves invariant $V_i^{\mathbb{C}}$. Hence, $\mathbb{L}^G \simeq \mathcal{L}_0 \times \cdots \times \mathcal{L}_r$, where $\mathcal{L}_i$ is the connected component of

$$Sp^G(V_i^{\mathbb{C}} \oplus (V_i^{\mathbb{C}})')/P^G(V_i^{\mathbb{C}} \oplus (V_i^{\mathbb{C}})'),$$

which contains $\mathrm{Id} \cdot P^G(V_i^{\mathbb{C}} \oplus (V_i^{\mathbb{C}})')$.

Hence, we suppose now that $\mathbb{R}^k$ contains a unique type of irreducible representations, *i.e.* that $\mathbb{R}^k = nW = W_1 \oplus \cdots \oplus W_n$. This means that we have the decomposition

(123) $$\mathbb{C}^{2k} = W_1^{\mathbb{C}} \oplus \cdots \oplus W_n^{\mathbb{C}} \oplus (W_1^{\mathbb{C}})' \oplus \cdots \oplus (W_n^{\mathbb{C}})'.$$

If W if of type 2. — This is the simplest situation. In this case $W_j^{\mathbb{C}}$ and $(W_j^{\mathbb{C}})'$ are irreducible over $\mathbb{C}$. Let us set $p = \dim W$, and choose real orthonormal basis $(g_{1,j}, \ldots, g_{p,j})$ of W_j, which realize the isomorphism $W_j \simeq W_{j'}$. We denote by $(g_{i,j}')$ the corresponding basis of W_j'. By Schur lemma, we know that, in this base, any element of $\mathrm{End}^G(\mathbb{C}^k \oplus \mathbb{C}^k)$ (where End^G is the space of endomorphism commuting with G), is of the form

$$M \otimes \mathrm{Id}_p,$$

where $M \in M_{2n}(\mathbb{C})$. Since the change of base is orthogonal, the matrix of the symplectic form ω remains equal to

$$J_k = J_n \otimes \mathrm{Id}_p.$$

This implies that, by this change of bases, $Sp^G(\mathbb{C})$ becomes the space of matrix of the type $M \otimes \mathrm{Id}_p$ such that

$$(M \otimes \mathrm{Id}_p)^t (J_n \otimes \mathrm{Id}_p)(M \otimes \mathrm{Id}_p) = J_n \otimes \mathrm{Id}_p,$$

which is equivalent to $M^t J_n M = J_n$. Hence $Sp^G(\mathbb{C}) \simeq Sp(n, \mathbb{C})$. Similarly, P^G is isomorphic to P_n and $\mathbb{L}^G \simeq \mathbb{L}^n$, since $Sp(n, \mathbb{C})/P_n$ is connected.

If W is of type 1. — In this case, $W^{\mathbb{C}} = U \oplus \overline{U}$, and U and $\overline{U}$ are $\mathbb{C}$-irreducible, not isomorphic, and orthogonal for the hermitian scalar product $\langle\,,\,\rangle$. Let $p = \dim W$, and choose an orthonormal basis $(g_1, \ldots, g_p)$ of W (for the hermitian scalar product). The family $(\overline{g}_1, \ldots, \overline{g}_p)$ is an orthonormal basis of $\overline{U}$.

By isomorphism, we have the corresponding decomposition $W_j^{\mathbb{C}} = U_j \oplus \overline{U}_j$, $(W_j^{\mathbb{C}})' = U_j' \oplus \overline{U}_j'$, and the corresponding basis $(g_{1,j}, \ldots, g_{p,j})$, $(\overline{g}_{1,j}, \ldots, \overline{g}_{p,j})$, and $(g_{1,j}', \ldots, g_{p,j}')$, $(\overline{g}_{1,j}', \ldots, \overline{g}_{p,j}')$. We rewrite now the decomposition of $\mathbb{C}^{2k}$ as

$$\mathbb{C}^{2k} = U_1 \oplus \cdots \oplus U_n \oplus U_1' \oplus \cdots \oplus U_n' \oplus \overline{U}_1 \oplus \cdots \oplus \overline{U}_n \oplus \overline{U}_1' \oplus \cdots \oplus \overline{U}_n',$$

and we denote by $\mathcal{B}$ the corresponding basis (*i.e.* we endow each component with the basis $(g_{i,j})$, $(g_{i,j}')$, $\cdots$, we just described).

By Schur lemma, in this basis, any element of $\mathrm{End}^G(\mathbb{C}^k \oplus \mathbb{C}^k)$ has the form

$$(124) \qquad Z = \begin{pmatrix} Z_1 \otimes \mathrm{Id}_p & 0 \\ 0 & Z_2 \otimes \mathrm{Id}_p \end{pmatrix},$$

where Z_1 and Z_2 are $2n \times 2n$ complex matrices. Let us now compute the matrix of the symplectic form ω in this new basis (the change of basis, from the canonical basis of $\mathbb{C}^{2k}$ to $\mathcal{B}$ is unitary, but not orthogonal). Since $\omega(X, Y) = (X, JY) = \langle X, \overline{JY} \rangle$, we see that ω is null on all term except on the $U_j \times \overline{U}_j'$, $U_j' \times \overline{U}_j$, and the symmetric terms. On $U_j \times \overline{U}_j'$, we have

$$\omega(g_{i,j}, \overline{g}_{i',j}') = -\langle g_{i,j}, g_{i',j} \rangle = -\delta_{i,i'}.$$

On $U_j' \times \overline{U}_j$, we have $\omega(g_{i,j}', \overline{g}_{i',j}) = -\overline{\omega(g_{i',j}, g_{i,j}')} = \delta_{i,i'}$. Hence, the matrix of ω in the base $\mathcal{B}$ is

$$\widehat{J} = \begin{pmatrix} 0 & J_n \otimes \mathrm{Id}_p \\ J_n \otimes \mathrm{Id}_p & 0 \end{pmatrix}.$$

Hence, by this change of basis, $Sp^G(\mathbb{C})$ is isomorphic to the group of matrices Z of the form (124), which satisfies $Z^t \widehat{J} Z = \widehat{J}$. Hence $Sp^G(\mathbb{C})$ is isomorphic to

$$(125) \qquad \left\{ Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \mid Z_1, Z_2 \in M_{2n}(\mathbb{C}), \ Z_2^t J_n Z_1 = J_n \right\}.$$

Similarly, P^G is isomorphic to the subgroup of element Z of (125), such that Z_1 and Z_2 are of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Since Z_2 is determined by Z_1 in (125) by $Z_2 = -J_n(Z_1^t)^{-1}$, we see that $Sp^G(\mathbb{C})/P^G$ is isomorphic to

$$Gl(2n, \mathbb{C})/\big\{Z \in Gl(2n, \mathbb{C}) \mid Z \text{ of the form } \big(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\big)\big\} \simeq \mathbb{G}^{n,2n}.$$

W is of type 3.— In this case $W = U \oplus \overline{U}$, where U and $\overline{U}$ are irreducible and isomorphic. Let us first remark that U and $\overline{U}$ are necessarily orthogonal for the hermitian scalar product $\langle\,,\,\rangle$. Indeed, by proposition E.1, we know that the symmetric scalar product $(\,,\,)$ is null on U, thus $\langle x, y \rangle = (x, \overline{y}) = 0$ for $x \in U$ and $y \in \overline{U}$. Let us now describe an explicit isomorphism between U and $\overline{U}$. By proposition E.1, we know that there exists a non degenerate G-invariant skew symmetric bilinear form B on U. Thus, for all x in U, there exists $\phi(x) \in U$ such that

$$B(x, y) = \overline{\langle \phi(x), y \rangle}, \quad \forall y \in U.$$

The map ϕ is antilinear (*i.e.* $\phi(\lambda x + y) = \overline{\lambda}\phi(x) + \phi(y)$) and bijective, since B is non degenerate. Thus, $\psi = \overline{\phi} : U \to \overline{U}$, is an isomorphism, commuting with the action of G. Since B is skew-symmetric and non degenerate, $\dim U = 2p$, and there exists a symplectic basis $(g_1, \ldots, g_{2p})$ of U, *i.e.* a basis such that

$$B(g_i, g_j) = \delta_{i,j+p} - \delta_{i,j-p},$$

(*i.e.*, the matrix of B in this basis is J_p). We set $f_i = \psi(g_i)$. The family $(f_1, \ldots, f_{2p})$ is a basis of $\overline{U}$, which realizes the isomorphism $U \simeq \overline{U}$.

Let us come back to $\mathbb{C}^{2k}$. In each term $W_j^{\mathbb{C}}$ (resp. $(W_j^{\mathbb{C}})'$) of the decomposition (123) we make the corresponding decomposition $W_j^{\mathbb{C}} = U_j \oplus \overline{U}_j$ (resp. $(W_j^{\mathbb{C}})' = U_j' \oplus \overline{U}_j'$), and we define the corresponding basis $(g_{1,j}, \ldots, g_{2p,j})$ and $(f_{1,j}, \ldots, f_{2p,j})$ (resp. $(g_{1,j}', \ldots, g_{2p,j}')$ and $(f_{1,j}', \ldots, f_{2p,j}')$).

We rewrite the decomposition of $\mathbb{C}^{2k}$ in

$$\mathbb{C}^{2k} = U_1 \oplus \cdots \oplus U_n \oplus \overline{U}_1 \oplus \cdots \oplus \overline{U}_n \oplus U_1' \oplus \cdots \oplus U_n' \oplus \overline{U}_1' \oplus \cdots \oplus \overline{U}_n',$$

and we denote by $\mathcal{B}$ the corresponding basis (*i.e.*, we endow each component by the basis $(g_{1,j}, \ldots, g_{2p,j})$, $(f_{1,j}, \ldots, f_{2p,j})$, $\ldots$, we just described). By Schur lemma, the matrix of any element of $\mathrm{End}^G(\mathbb{C}^{2k})$ in the base $\mathcal{B}$, is of the type $M \otimes \mathrm{Id}_{2p}$, where $M \in M_{4n}(\mathbb{C})$.

Let us now compute the matrix of ω in the basis $\mathcal{B}$. Clearly, since $\omega(X, Y) = (X, \overline{JY})$, we see that ω is null on all component except of $\mathbb{C}^{2k} \times \mathbb{C}^{2k}$ except on components of the type $U_j \times \overline{U}_j'$, $\overline{U}_j \times U_j'$, and the symmetric components. On

$U_j \times \overline{U}'_j$, we have

$$\begin{aligned}
\omega(g_{i,j}, f'_{i,j'}) &= -\big\langle g_{i,j}, \overline{f}_{i',j} \big\rangle \\
&= -\overline{\big\langle \phi(g_{i',j}), g_{i,j} \big\rangle} \\
&= -B(g_{i',j}, g_{i,j}) = \delta_{i,i'+p} - \delta_{i,i'-p}.
\end{aligned}$$

Similarly, $\omega(f_{i,j}, g'_{i',j}) = -(\delta_{i,i'+p} - \delta_{i,i'-p})$. Thus, the matrix of ω in $\mathcal{B}$ is

$$\widehat{J} = \begin{pmatrix} 0 & -J_n \otimes J_p \\ J_n \otimes J_p & 0 \end{pmatrix} = \widehat{K} \otimes J_p,$$

where $\widehat{K}$ is the symmetric matrix

$$\widehat{K} = \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix}.$$

Hence, we see that $Sp^G(\mathbb{C})$ is isomorphic to the set of matrices of the form $M \otimes \mathrm{Id}_{2p}$ such that

$$(M \otimes \mathrm{Id}_{2p})^t (\widehat{K} \otimes J_p)(M \otimes \mathrm{Id}_{2p}) = \widehat{K} \otimes J_p.$$

Since the first term is equal to $(M^t \widehat{K} M) \otimes J_p$, we see that the condition becomes

$$M^t \widehat{K} M = \widehat{K}.$$

By the orthogonal change of base given by

$$O = \begin{pmatrix} \mathrm{Id}_{2n} & 0 \\ 0 & J_n \end{pmatrix},$$

we see that $\widehat{K}$ becomes

$$O^t \widehat{K} O = K_{2n},$$

defined in formula (119). Thus $Sp^G(\mathbb{C})$ is isomorphic to $O(\mathbb{C}^{4n}, K_{2n})$. Similarly, P^G is isomorphic to P_{2n} defined in formula (120). Thus, $Sp^G(\mathbb{C})/P^G \simeq \mathbb{O}^{2n}$ and $\mathbb{L}^G \simeq \mathbb{SO}^{2n}$. $\qquad\square$

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