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WEYL LAW FOR SEMI-CLASSICAL RESONANCES WITH RANDOMLY PERTURBED POTENTIALS

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WEYL LAW FOR SEMI-CLASSICAL RESONANCES WITH RANDOMLY PERTURBED POTENTIALS

Johannes Sjöstrand

Abstract. — We consider semi-classical Schrödinger operators with potentials supported in a bounded strictly convex subset $\mathcal{O}$ of $\mathbb{R}^n$ with smooth boundary. Letting h denote the semi-classical parameter, we consider classes of small random perturbations and show that with probability very close to 1, the number of resonances in rectangles $[a,b]-i[0,ch^{\frac{2}{3}}[$, is equal to the number of eigenvalues in $[a,b]$ of the Dirichlet realization of the unperturbed operator in $\mathcal{O}$ up to a small remainder.

***Résumé* (Loi de Weyl pour des résonances semi-classiques associées aux potentiels avec perturbations aléatoires)**

On considère des opérateurs de Schrödinger dont les potentiels ont leur supports dans un ensemble strictement convexe à bord lisse $\mathcal{O} \Subset \mathbb{R}^n$. En désignant par h le paramètre semi-classique, nous considérons des classes de petites perturbations aléatoires et montrons qu'avec une probabilité très proche de 1, le nombre de résonances dans des rectangles $[a,b]-i[0,ch^{\frac{2}{3}}[$ est égal (à un petit reste près) au nombre de valeurs propres dans $[a,b]$ de la réalisation de Dirichlet de l'opérateur dans $\mathcal{O}$.

CONTENTS

CHAPTER 1

INTRODUCTION

There is now a very large literature about the distribution of scattering poles (resonances) often using methods from non-self-adjoint spectral theory and microlocal analysis, including many results about upper and lower bounds on the density of resonances. See for instance [**34**], [**6**] and the references given there. Less is known about actual asymptotics for the number of resonances in various domains. In this paper we shall give such a result for the semi-classical Schrödinger operator

$$P = -h^2\Delta + V(x), \tag{1.1}$$

on $\mathbb{R}^n$ where $V \in L^\infty(\mathbb{R}^n;\mathbb{R})$ has compact support.

Recall that the resonances or scattering poles of the operator (1.1) can be defined as the poles of the meromorphic extension of the resolvent

$$(P-z)^{-1} : C_0^\infty(\mathbb{R}^n) \longrightarrow H^2_{\mathrm{loc}}(\mathbb{R}^n)$$

across the positive real axis, to the logarithmic covering space of $\mathbb{C}\setminus\{0\}$ when n is even and to the double covering when n is odd. Alternatively we can continue $(P-k^2)^{-1}$ from the upper half-plane across $\mathbb{R}\setminus\{0\}$ which gives a meromorphic function on $\mathbb{C}$ when n is odd. Using the second definition, we can introduce the number $N(r)$ of resonances in the disc $D(0,r)$ when n is odd.

In one dimension and for $h=1$, M. Zworski [**38**] showed that if $[a,b]$ is the convex hull of the support of V, then

$$N(r) = \frac{2(b-a)}{\pi} r + o(r), \quad r \to \infty, \tag{1.2}$$

which is 2 times the asymptotic number of eigenvalues $\leq r^2$ of the Dirichlet realization of $-\Delta + V$ on $[a,b]$, the factor 2 being explained by the fact that the resonances are symmetric around the imaginary axis. He also showed that most of these concentrate to narrow sectors around the real axis. This extended an earlier result of T. Regge [**20**]. Subsequently, B. Simon [**21**] gave a different proof, inspired by the work of R. Froese [**12**], who got similar results for potentials that do not necessarily have compact support but are very small near infinity. See also the recent works [**8**], [**7**], [**10**] about Weyl and non-Weyl asymptotics for graphs.

In higher odd dimensions, M. Zworski [**40**] considered the case of radial potentials of the form

$$V(x) = f\big(|x|\big)$$

with support in $\overline{B(0,a)}$ where $f \in C^2([0,a])$, $a > 0$, $f(a) \neq 0$ and obtained a Weyl type asympotics (still with $h = 1$),

$$N(r) = K_n a^n r^n + o(r^n), \quad r \to +\infty, \tag{1.3}$$

where $K_n > 0$. Recall also that Zworski [**39**] gave an upper bound in the non-radial case with the correct power of r and using his analysis, P. Stefanov [**34**], gave an explicit formula for the constant $K_n a^n$ in the radial case and showed that the right hand side of (1.3) is up to $o(r^n)$ the sum of 2 times the number of eigenvalues $\leq r^2$ for the interior Dirichlet problem in the ball $B(0,a)$ and the number of scattering poles for the exterior Dirichlet Laplacian in $\mathbb{R}^n \setminus B(0,a)$. (See also G. Vodev [**35**].) He also showed (as a corollary of a more general result for operators with black box) that if we drop the radiality assumption and only assume that $V \in L^\infty(\mathbb{R}^n;\mathbb{R})$ has its support in $\overline{B(0,a)}$, then we have the upper bound

$$N(r) \leq K_n a^n r^n + o(r^n), \quad r \to +\infty. \tag{1.4}$$

T. Christiansen [**6**] introduced the set $\mathfrak{M}_a$ of L^∞ potentials V with support in $\overline{B(0,a)}$ for which we have (1.3) and gave the leading asymptotics, of the form Cr^n, for the number of resonances in sectors in the lower half-plane intersected with the disc $D(0,r)$. These formulas were implicit in [**40**], [**34**] in the case of the radial potentials considered there. In particular, when considering smaller and smaller sectors adjacent to $\mathbb{R}_+$ or $\mathbb{R}_-$ we can see, using Lemma 3.3 of [**6**] and some wellknown formulas for the Γ function and the volume of the unit ball, that the constant C converges to the one we get in the leading Weyl asymptotics for the number of Dirichlet eigenvalues for the Laplacian in $B(0,a)$. In the theorems 1.2, 1.3 of the same paper the author gives interesting extensions "for most values of z" to the case of potentials $V(x,z)$ depending holomorphically on a parameter z with $\operatorname{supp} V(\,.\,,z) \subset \overline{B(0,a)}$ such that $V(.,z_0)$ belongs to $\mathfrak{M}_a$ for at least one value of z_0. Such results remain significant also after restriction to real-valued potentials. (See also earlier results of the same author, cited in [**6**].) In the recent work [**9**] (which appeared after the submission of the present work), T.-C. Dinh and D.-V. Vu obtain sharper results, namely that for holomorphic families of potentials, if one element is in a sharpened version of the class $\mathfrak{M}_a$, then so do all elements away from a pluri-polar set.

The main result of this paper has some relations to the above mentioned ones. We work in the semi-classical limit ($h \to 0$) and the ball $B(0,a)$ is replaced by a more general strictly convex set. Our is result does not make use of any class of the type $\mathfrak{M}_a$ and the conclusion concerns the number of resonances in a thin rectangle. Nevertheless it is very interesting to note the similarities of the results, and there are also similarities in the proofs at least on some ideological level.

We next proceed with a rough description of our result and leave the precise statements to the next section. Let $\mathcal{O} \Subset \mathbb{R}^n$ be open strictly convex with smooth boundary and let $V_0 \in C^\infty(\overline{\mathcal{O}};\mathbb{R})$ vanish to the order $v_0 > 0$ on the boundary. By V_0 we also denote the extension to all of $\mathbb{R}^n$ which vanishes outside $\mathcal{O}$ and we consider the potential

$$V(x) = V_0(x) + \delta \widetilde{q}_\omega(x)$$

where $\delta > 0$ is a small parameter > 0 and $\widetilde{q}_\omega$ a random perturbation whose properties will be specified in the next section. A possible choice of δ is a high power of h. Our main result, Theorem 2.2 then states that if $0 < a < b < \infty$ and if $C > 0$ is large enough so that the exterior Dirichlet problem for $-h^2\Delta$ has no resonances in the rectangle $[a,b] + ih^{\frac{2}{3}}[-C^{-1},0]$, then with probability very close to 1, the number of resonances of

$$P = -h^2\Delta + V$$

in the rectangle $[a,b] + ih^{\frac{2}{3}}[-C^{-1},0]$ is equal to the number $N_0([a,b])$ of eigenvalues in $[a,b]$ of the Dirichlet realization of $h^2\Delta + V_0$ in $\mathcal{O}$ plus two "errors". The first error is a term that can be bounded by a positive power of h times h^{-n}. The second error is bounded by a constant times

$$N_0\big([a-\rho,a+\rho]\big) + N_0\big([b-\rho,b+\rho]\big)$$

where $\rho = h^{\frac{2}{3}-\delta}$ for any fixed $\delta > 0$. As will be stated more explicitly in the theorems 2.1 and 2.5, we can choose our random perturbations to be concentrated to a ball of radius h^N in the Sobolev space H^s for arbitrarily large N and s.

In the case of a deterministic potential with a potential well in an island, one can count resonances in rectangles closer to the real axis. Such results can be found in the appendix of [**19**] and in Section 9 of [**16**]. The phenomen is now a little different however, due to the potential barrier, and the reference asymptotics of eigenvalues now depends on the behaviour of the operator near the potential well.

The motivation for this work was to apply recent results and techniques for proving Weyl asymptotics for non-self-adjoint differential operators with small random perturbations either in the semi-classical limit or in the limit of large eigenvalues [**25**], [**27**], [**4**], to the problem of resonances.

Indeed, using some version of complex scaling or its microlocal versions, this can be viewed as an eigenvalue problem for a non-self-adjoint operator.

The new difficulty here is however that if we want to keep a realistic problem we should apply the random perturbation first and use complex scaling only outside the support of the perturbation. If we let $p(x,\xi)$ denote the leading semi-classical symbol of the scaled operator, and we let z vary in a complex domain like a thin recatngle along the real axis, then as soon as z is not real, the set $p^{-1}(z)$ must belong to the part of phase space which corresponds to the scaled region (since the original unscaled symbol is real valued) and hence the support of the random perturbation is away from the x-space projection of this set. This leads to a difficulty since the method in [**25**], [**27**] is based on the study of the random matrix $(\widetilde{q}_\omega e_j \mid \bar{e}_k)$, where $e_1,\dots,e_N$ is an orthonormal family of eigenfunctions of $(P-z)^*(P-z)$ corresponding to the small eigenvalues and where we let P denote the scaled operator. Now, the e_j will be concentrated to the projection of $p^{-1}(z)$ which sits outside the obstacle, hence away from the support of the random perturbation. Our random matrix will therefore tend to be small which is a serious problem in the approach of [**25**] and [**27**]. In order to make the distance smaller, one could try to make the distorsion very important already very close to the support of the perturbation, but that leads to the use of very exotic symbols and after some attempts in that direction we decided to follow a different less intuitive approach. In the next section we formulate the result and in Chapter 3 we give an outline of the proof.

It would be interesting to have related statements about almost sure Weyl asymptotics of large resonances in certain parabolic neighborhoods of the real axis in the non-semi-classical case ($h=1$). It is quite possible that such a result can be obtained from the present paper along the same lines as the corresponding result for large eigenvalues by W. Bordeaux Montrieux and the author [**4**].

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We thank J.M. Bouclet for having pointed out the reference [**5**] where the idea of differentiating several times to reach trace class operators is clearly present (cf. Chapter 4). We also thank T. Christiansen for helpful comments about [**6**], A. Voros for indicating references about the complex WKB-method and V. Ivrii and L. Zielinski for references and information about Weyl laws for the eigenvalues of semi-classical Schrödinger operators with potentials of limited regularity. Discussions with M. Zworski and M. Hitrik around other joint works and projects have been helpful when preparing the sections 5, 9. Comments by V. Petkov and M. Zworski led to the correction of some errors.

We also thank the referee for his many pertinent remarks and even for sending us some numerical calculations in dimension 1 which would deserve to be available to a larger audience.

CHAPTER 2

THE RESULT

We start with a concrete case of our main result (Theorem 2.1). After that we give the full formulation (Theorem 2.2) which includes a description of the probability measures that are involved. After that we give a simplified and partially generalized version of the main result (Theorem 2.5) which combined with a result of V. Ivrii [**17**] gives Theorem 2.1.

Let $\mathcal{O} \Subset \mathbb{R}^n$ be open, strictly convex with smooth boundary. Let $\kappa > 0$ be the geometric constant in (2.14) below and let $\zeta_1 > 0$ be the smallest zero of the Airy function $\mathrm{Ai}(-t)$. The concrete version of the main result is then

Theorem 2.1. — *Let $s > \frac{1}{2}n$, $\beta > 0$ and*

$$N = \min\big( ]\tfrac{1}{2}(n-1), +\infty[\cap \mathbb{Z}\big), \quad \widetilde{s} > \max\big(\tfrac{1}{2}n + 3, 2N + \tfrac{1}{2}n\big).$$

Then there exists a probability measure μ on $H^s(\overline{\mathcal{O}})$ with support in the ball

$$\big\{W \in H^s(\overline{\mathcal{O}});\ \|W\|_{H^s} \leq h^\beta\big\}$$

such that the following holds. Let $0 < c_1 < c_2 < 2\,(\frac{1}{2})^{\frac{2}{3}}\kappa\zeta_1$. There exists a constant $C > 0$ such that if $\frac{1}{2} \leq a < b \leq 2$, $c_1 \leq c \leq c_2$, $\widetilde{\epsilon} \geq Ch(\ln 1/h)^2$ and $V_0 \in H^{\widetilde{s}}(\overline{\mathcal{O}})$, then for

$$P = -h^2\Delta + V_0 + W, \quad W \in H^s(\overline{\mathcal{O}}),$$

we have with probability (with respect to the random term W)

$$\geq 1 - \mathcal{O}(1)\frac{h(\ln 1/h)^2}{h^{N_7}}\,\mathrm{e}^{-\widetilde{\epsilon}/(Ch(\ln 1/h)^2)}, \tag{2.1}$$

that for the set $\sigma(P)$ *of resonances of* P*, counted with their algebraic multiplicity,*

$$\left|\#\big(\sigma(P)\cap\big([a,b]+ih^{\frac{2}{3}}c[-1,0]\big)\big)-\frac{1}{(2\pi h)^n}\iint_{a\leq\xi^2+V_0(x)\leq b}\mathrm{d}x\mathrm{d}\xi\right| \tag{2.2}$$
$$\leq\mathcal{O}(1)h^{-\frac{2}{3}-n}\widetilde{\epsilon}.$$

Here we also assume that $n\geq 3$ *or that neither* a *nor* b *is a critical value of* V_0*.*

The constant N_7 *is independent of the other parameters, while the constants* $\mathcal{O}(1)$ *in* (2.1), (2.2) *depend on* $c_1, c_2, \beta, \widetilde{s}, s$ *and on an upper bound on* $\|V_0\|_{H^{\widetilde{s}}(\overline{\mathcal{O}})}$.

We now start to formulate the more complete result. Our unperturbed operator will be

$$P_0=-h^2\Delta+V_0 : L^2(\mathbb{R}^n)\longrightarrow L^2(\mathbb{R}^n), \tag{2.3}$$

where $V_0\in C^\infty(\overline{\mathcal{O}})$ and we identify V_0 with its zero extension. We also assume

$$\text{On } \partial\mathcal{O} \text{ we have } V_0(x)=0 \text{ and } \partial_\nu V_0\leq 0, \tag{2.4}$$

where ν denotes the exterior unit normal.

The result concerns the distribution of resonances of

$$P=P_\delta=P_0+\delta\Theta(x)q_\omega(x), \tag{2.5}$$

where $\Theta(x)\in C^\infty(\overline{\mathcal{O}})$ satisfies

$$0<\Theta(x)\asymp \mathrm{dist}(x,\partial\mathcal{O})^{v_0},\quad x\in\mathcal{O}\setminus\partial\mathcal{O},\quad v_0\in\big]\tfrac{1}{2}(n-1),+\infty\big[\cap\mathbb{N}. \tag{2.6}$$

As in (2.3), Θ also denotes the 0-extension to all of $\mathbb{R}^n$. It belongs to $C_0^k(\mathbb{R}^n)$ if $v_0>k$. It would be interesting to be able to work with a profile in C_0^∞.

As in [**25**], [**27**], we choose the random function q_ω of the form

$$q_\omega(x)=\sum_{0<\mu_k\leq L}\alpha_k(\omega)\epsilon_k(x),\qquad |\alpha|_{\mathbb{R}^D}\leq R, \tag{2.7}$$

where ϵ_k is an orthonormal basis of real eigenfunctions of $h^2\widetilde{R}$, where $\widetilde{R}$ is an h-independent real positive elliptic 2nd order operator on X with smooth coefficients. Here X is a smooth compact manifold of dimension n containing $\overline{\mathcal{O}}$ (in the sense that we have some diffeomorphism from a neighborhood of $\overline{\mathcal{O}}$ onto an open set in X and we identify $\overline{\mathcal{O}}$ with its image). For instance, we can let X be an n-dimensional torus and choose $-\widetilde{R}$ to be the Laplacian.

Moreover, $h^2\widetilde{R}\epsilon_k = \mu_k^2\epsilon_k$, $\mu_k > 0$. We choose $L = L(h)$, $R = R(h)$ in the following intervals where $s \in \left]\frac{1}{2}n, v_0 + \frac{1}{2}\right[$, $\epsilon \in \left]0, s - \frac{1}{2}n\right[$, $\theta \in \left]0, \frac{1}{2}\right[$ are fixed:

$$\text{(2.8)}\qquad \begin{cases} h^{-M_{\min}} \ll L \leq Ch^{-M}, & M \geq M_{\min} := \dfrac{v_0 + (\frac{1}{3} + n)/(1 - 2\theta)}{s - \frac{1}{2}n - \epsilon}, \\ h^{-\widetilde{M}_{\min}} \leq R \leq h^{-\widetilde{M}}, & \widetilde{M} \geq \widetilde{M}_{\min} := (\dfrac{1}{2}n + \epsilon)M_{\min} + 1 + \dfrac{3}{2}n + v_0, \end{cases}$$

and we shall denote by $L_{\min}$ and $R_{\min}$ the lower bounds for L and R in these estimates. By Weyl's law for the large eigenvalues of elliptic self-adjoint operators, the dimension D is of the order of magnitude $(L/h)^n$. We introduce the small parameter

$$\text{(2.9)}\qquad \delta = \tau_0 h^\alpha / C, \quad \tau_0 \in \left]0, h^{\frac{5}{3}}\right], \quad \alpha \geq \alpha(n, v_0, s, \epsilon, \theta, M, \widetilde{M}),$$

where an explicit (and not very nice) expression for $\alpha(n, v_0, s, \epsilon, \theta, M, \widetilde{M})$ can be deduced from the proof.

The random variables $\alpha_j(\omega)$ will have a joint probability distribution

$$\text{(2.10)}\qquad P(\mathrm{d}\alpha) = C(h)\mathrm{e}^{\Phi(\alpha;h)}L(\mathrm{d}\alpha),$$

where for some $N_4 > 0$,

$$\text{(2.11)}\qquad |\nabla_\alpha \Phi| = \mathcal{O}(h^{-N_4}),$$

and $L(\mathrm{d}\alpha)$ is the Lebesgue measure on $\mathbb{R}^D$. ($C(h)$ is the norming constant.)

We need the parameter

$$\text{(2.12)}\qquad \epsilon_0(h) = h\Big(\Big(\ln\frac{1}{h}\Big)^2 + \ln\frac{1}{\tau_0}\Big)$$

and assume that $\tau_0 = \tau_0(h)$ is not too small, so that $\epsilon_0(h)$ is small.

It was shown by T. Hargé and G. Lebeau [**15**], see also [**30**], that the exterior Dirichlet problem for $-h^2\Delta$ on $\mathbb{R}^n \setminus \mathcal{O}$ has no resonances in the set

$$\text{(2.13)}\qquad \Im z \geq -2(h\Re z)^{\frac{2}{3}}\kappa\zeta_1 + Ch, \quad \frac{1}{2} \leq \Re z \leq 2,$$

if C is large enough, where

$$\text{(2.14)}\qquad \kappa = 2^{-\frac{1}{3}}\cos\frac{\pi}{6}\min_{S\partial\mathcal{O}} Q^{\frac{2}{3}},$$

Q is the second fundamental form on $\partial\mathcal{O}$ and $\zeta_1 > 0$ is the smallest zero of $\mathrm{Ai}(-t)$ with Ai denoting the Airy function which spans the space of solutions to $(-\partial_t^2 + t)u = 0$ that are exponentially subdominant on the positive real axis.

For technical reasons, we shall restrict the attention to rectangles of the form $R = [a, b] + ih^{\frac{2}{3}}c[-1, 0]$, $\frac{1}{2} \leq a < b \leq 2$, $c > 0$ with c small enough so that R is contained in the domain (2.13). Thus we will assume that $c < 2(\frac{1}{2})^{\frac{2}{3}}\kappa\zeta_1$. (We could replace the bounds $\frac{1}{2}$ and 2 by any other positive bounds $0 < b_1 < b_2$.)

Let $P_{\rm in}^0$ denote the Dirichlet realization of P_0 in $\mathcal{O}$ and let $N_0(\lambda)$ denote the number of eigenvalues of $P_{\rm in}^0$ in the interval $]-\infty,\lambda]$, counted with their multiplicity. Similarly, if $I \subset \mathbb{R}$ we let $N_0(I)$ denote the number of such eigenvalues in I. The main result of this work is:

Theorem 2.2. — *Let $\sigma(P_\delta)$ denote the set of resonances of P_δ. Let $0 < c_1 < c_2 < 2(\frac{1}{2})^{\frac{2}{3}}\kappa\zeta_1$, $\rho = h^{-\delta_0+\frac{2}{3}}$, where $\delta_0 > 0$ is arbitrarily small but fixed. Then there exists a constant $C > 0$ such that for $\frac{1}{2} \le a < b \le 2$, $c_1 \le c \le c_2$ and $\widetilde{\epsilon} \ge C\epsilon_0(h)$, we have with probability*

$$\ge 1 - \mathcal{O}(1)\frac{\epsilon_0(h)}{h^{n+N_6+\frac{2}{3}}}\,\mathrm{e}^{-\widetilde{\epsilon}/C\epsilon_0(h)}, \tag{2.15}$$

where the constant $\mathcal{O}(1)$ is independent of $a, b, c, \widetilde{\epsilon}, h$, that

$$\begin{aligned}\Big|\#\big(\sigma(P_\delta)\cap\big([a,b]+ih^{\frac{2}{3}}c[-1,0]\big)\big) - N_0\big([a,b]\big)\Big| \\ \le \mathcal{O}(1)\Big(\sum_{w=a,b} N_0\big([w-\rho,w+\rho]\big)\Big) + h^{-\frac{2}{3}-n}\widetilde{\epsilon}.\end{aligned} \tag{2.16}$$

Here $N_6 = \max(N_3, N_5)$, where $N_3 = n(M+1)$, $N_5 = N_4 + \widetilde{M}$.

Remark 2.3. — As in [**25**], [**27**] and in an earlier work with M. Hager cited there, with probability

$$\ge 1 - \mathcal{O}(1)\frac{\epsilon_0(h)}{h^{n+N_6+\frac{4}{3}}}\,\mathrm{e}^{-\widetilde{\epsilon}/C\epsilon_0(h)}, \tag{2.17}$$

we have (2.16) simultaneously for $\frac{1}{2} \le a < b \le 2$ and $c_1 \le c \le c_2$.

As we point out in Remark 15.1, for a general perturbation $W = \delta\Theta q_\omega$ as in Theorem 2.2, we have

$$\|W\|_{H_h^{\widetilde{s}}(\mathbb{R}^n)} \le \mathcal{O}(\delta)L^{\widetilde{s}}R,$$

provided that $\frac{1}{2}n < \widetilde{s} < v_0 + \frac{1}{2}$. Here $H_h^{\widetilde{s}}$ is the standard Sobolev space equipped with its natural semi-classical norm (see Chapter 6). By playing with the parameters, the perturbations in Theorem 2.2 can be chosen to be bounded by arbitrarily high powers of h in Sobolev spaces with arbitrarily high regularity exponents.

We also have:

PROPOSITION 2.4. — *The conclusion in Theorem 2.2 remains valid if we change V_0 by adding an h-independent potential $W_0 \in L^\infty(\mathcal{O})$ such that $W_0 = \mathcal{O}(\operatorname{dist}(x, \partial\mathcal{O})^3)$, $\partial^\alpha W_0 \in L^\infty$ for $|\alpha| \leq 2N$ and $W_0 \in H^s(\overline{\mathcal{O}})$. Here N is the smallest integer in $]\frac{1}{2}(n-1), +\infty[$ and $s > \frac{1}{2}n$ is the parameter in Theorem 2.2.*

Recall that $H^s(\overline{\mathcal{O}}) = \{v \in H^s(\mathbb{R}^n); \operatorname{supp} v \subset \overline{\mathcal{O}}\}$. Combining the remark and Theorem 2.2, we get the following less detailed but perhaps more transparent version of our main result, where our unperturbed potential is $V_0 = W_0$.

THEOREM 2.5. — *Let $s > \frac{1}{2}n$, $\beta > 0$ and let*

$$N = \min\big(\,]\tfrac{1}{2}(n-1), +\infty[\,\cap \mathbb{Z}\big), \quad \widetilde{s} > \max\big(\tfrac{1}{2}n + 3, 2N + \tfrac{1}{2}n\big).$$

Then there exists a probability measure μ on $H^s(\overline{\mathcal{O}})$ with support in the ball

$$\big\{W \in H^s(\overline{\mathcal{O}}); \|W\|_{H^s} \leq h^\beta\big\}$$

such that the following holds. Let $0 < c_1 < c_2 < 2(\frac{1}{2})^{\frac{2}{3}}\kappa\zeta_1$, $\rho = h^{-\delta_0 + \frac{2}{3}}$, where $\delta_0 > 0$ is arbitrarily small but fixed. There exists a constant $C > 0$ such that if $\frac{1}{2} \leq a < b \leq 2$, $c_1 \leq c \leq c_2$, $\widetilde{\epsilon} \geq Ch(\ln 1/h)^2$ and $V_0 \in H^{\widetilde{s}}(\overline{\mathcal{O}})$, then for

$$P = -h^2\Delta + V_0 + W, \quad W \in H^s(\overline{\mathcal{O}}),$$

we have with probability (with respect to the random term W)

$$\geq 1 - \mathcal{O}(1)\frac{h(\ln 1/h)^2}{h^{N_7}}\,\mathrm{e}^{-\widetilde{\epsilon}/Ch(\ln 1/h)^2}, \tag{2.18}$$

that for the set $\sigma(P)$ of resonances of P,

$$\begin{aligned}\Big|\#\big(\sigma(P) \cap \big([a,b] + ih^{\frac{2}{3}}c[-1,0]\big)\big) - N_0\big([a,b]\big)\Big| \\ \leq \mathcal{O}(1)\Big(\sum_{w=a,b} N_0\big([w-\rho, w+\rho]\big)\Big) + h^{-\frac{2}{3}-n}\widetilde{\epsilon}.\end{aligned} \tag{2.19}$$

Here N_7 (equal to $n + N_6 + \frac{2}{3}$ as in Theorem 2.2, with $M = M_{\min}$, $\widetilde{M} = \widetilde{M}_{\min}$) is independent of the other parameters, while the constants $\mathcal{O}(1)$ in (2.18), (2.19) depend on $c_1, c_2, \beta, \widetilde{s}, s$ and on an upper bound on $\|V_0\|_{H^{\widetilde{s}}(\overline{\mathcal{O}})}$.

Indeed, it suffices to apply Proposition 2.4 with $V_0 = W_0$ and to observe:

- ▷ V_0 is of class C^3 with support in $\partial\overline{\mathcal{O}}$ and therefore $V_0 = \mathcal{O}(\operatorname{dist}(x, \partial\mathcal{O})^3)$,
- ▷ It suffices to choose the perturbation $W = \delta\Theta q_\omega$ as in (2.5)–(2.9) with $M = M_{\min}$, $\widetilde{M} = \widetilde{M}_{\min}$, $\tau_0 = h^{5/3}$ and the parameters v_0 and α sufficiently large.

▷ We can choose the probability μ to be "P" in (2.10), with $\Phi = 0$ (so that $N_4 = 0$), but any other choice as in (2.10), (2.11) is OK.

We end the section by explaining how Theorem 2.1 follows from Theorem 2.5. It suffices to apply the following result of V. Ivrii [**17**], Theorem 2.1. (See also related results by L. Zieliński [**37**] in the case without boundary.)

Consider the semi-classical Schrödinger operator $P = -h^2\Delta + V(x)$ on the open set $X \Subset \mathbb{R}^n$ with smooth (C^∞) boundary. We assume that ∇V is continuous with modulus of continuity $\nu(t) = \mathcal{O}(|\ln t|^{-1})$. We equip P with Dirichlet boundary conditions. When $n = 1, 2$ we assume the micro-hyperbolicity property that $|\nabla V| \neq 0$ when $V = E$, uniformly for E in some compact interval J in $]0, +\infty[$. Then, uniformly for E in J, $0 < h \leq 1$, the number of eigenvalues in $]-\infty, E]$ is equal to the standard Weyl term $(2\pi h)^{-1}\mathrm{vol}\,(\{(x,\xi) \in T^*X;\, \xi^2 + V(x) \leq E\})$ plus a remainder which is $\mathcal{O}(h^{1-n})$ for $n \geq 2$ and $\mathcal{O}(\ln 1/h)$ for $n = 1$.

CHAPTER 3

SOME ELEMENTS OF THE PROOF

We will introduce a distorsion $\Gamma \subset \mathbb{C}^n$ of $\mathbb{R}^n$ which concides with $\mathbb{R}^n$ along $\mathcal{O}$ and with an exterior dilation of $\mathbb{R}^n$ outside $\mathcal{O}$ as in [**29**], [**30**], [**31**] and [**15**]. Let $P = P_\Gamma$ be the corresponding dilation of $-h^2\Delta + V$, $V = V_0 + \delta\Theta(x)q_\omega(x)$. Then (see for instance [**28**]) $P = P_\Gamma$ has discrete spectrum in an angle $-\theta_0 < \arg z \leq 0$ and the eigenvalues there coincide with the resonances.

Let P_{ext} be the Dirichlet realization of P on $\Gamma \setminus \mathcal{O}$, so that the spectrum of P_{ext} in the above angle coincides with the set of resonances for the exterior Dirichlet problem for $-h^2\Delta$ (recalling that $\operatorname{supp} V \subset \overline{\mathcal{O}}$). As we recalled in Chapter 2, there are no such resonances in $[\frac{1}{2}, 2] + ih^{\frac{2}{3}}c_0[-1, 0]$ if we fix

$$0 < c_0 < 2\left(\tfrac{1}{2}\right)^{\frac{2}{3}}\kappa\zeta_1. \tag{3.1}$$

Restricting z to the domain

$$\frac{1}{2} < \Re z < 2, \quad \Im z > -c_0 h^{\frac{2}{3}}, \tag{3.2}$$

we can therefore introduce the

- ▷ Green operator $G_{\text{ext}}(z) : H^0(\Gamma \setminus \mathcal{O}) \to H^2(\Gamma \setminus \mathcal{O})$ and
- ▷ the Poisson operator $K_{\text{ext}} : H^{\frac{3}{2}}(\partial\mathcal{O}) \to H^2(\Gamma \setminus \mathcal{O})$

so that the exterior Dirichlet operator

$$\mathcal{P}_{\text{ext}}(z) = \begin{pmatrix} P - z \\ h^{\frac{1}{2}}\gamma \end{pmatrix} = H^2(\Gamma \setminus \mathcal{O}) \longrightarrow H^0(\Gamma) \times H^{\frac{3}{2}}(\Gamma \setminus \partial\mathcal{O}) \tag{3.3}$$

has the bounded inverse

$$\mathcal{E}_{\text{ext}}(z) = \begin{pmatrix} G_{\text{ext}} & h^{-\frac{1}{2}}K_{\text{ext}}(z) \end{pmatrix} : H^0(\Gamma \setminus \mathcal{O}) \times H^{\frac{3}{2}}(\partial\mathcal{O}) \longrightarrow H^2(\Gamma \setminus \mathcal{O}). \tag{3.4}$$

Here γ is the operator of restriction to $\partial\mathcal{O}$. Let

$$\mathcal{N}_{\text{ext}} = \gamma h D_\nu K_{\text{ext}}$$

denote the exterior Dirichlet to Neumann operator, where $D_\nu = \frac{1}{i}\frac{\partial}{\partial\nu}$ and ν denotes the exterior unit normal. Introduce

$$B = \gamma h D_\nu - \mathcal{N}_{\mathrm{ext}}\gamma : H^2(\mathcal{O}) \longrightarrow H^{\frac{1}{2}}(\partial\mathcal{O}), \tag{3.5}$$

$$\mathcal{P}_{\mathrm{out}}(z) = \begin{pmatrix} P-z \\ h^{\frac{1}{2}}B \end{pmatrix} : H^2(\mathcal{O}) \longrightarrow H^0(\mathcal{O}) \times H^{\frac{1}{2}}(\partial\mathcal{O}). \tag{3.6}$$

For z in the domain (3.2) we shall see, by considering the continuity conditions at $\partial\mathcal{O}$, that z is a resonance (i.e. belongs to the spectrum of P_Γ) if and only if $\mathcal{P}_{\mathrm{out}}(z)$ is non-bijective, or equivalently if $0 \in \sigma(P_{\mathrm{out}}(z))$ where $P_{\mathrm{out}}(z) = P - z : H^0(\mathcal{O}) \to H^0(0)$ is the closed unbounded operator whose domain is the "outgoing" space: $\mathcal{D}(P_{\mathrm{out}}(z)) = \{u \in H^2(\mathcal{O});\ B(z)u = 0\}$.

Let

$$\mathcal{P}_{\mathrm{in}}(z) = \begin{pmatrix} P-z \\ h^{\frac{1}{2}}\gamma \end{pmatrix} : H^2(\mathcal{O}) \longrightarrow H^0(\mathcal{O}) \times H^{\frac{3}{2}}(\partial\mathcal{O}), \tag{3.7}$$

which is bijective precisely when z is not a (real) eigenvalue of the Dirichlet realization of P in $\mathcal{O}$. Away from the Dirichlet spectrum we introduce the inverse

$$\mathcal{E}_{\mathrm{in}}(z) = \big(G_{\mathrm{in}}(z), h^{-\frac{1}{2}}K_{\mathrm{in}}(z)\big) : H^0(\mathcal{O}) \times H^{\frac{3}{2}}(\partial\mathcal{O}) \longrightarrow H^2(\mathcal{O})$$

and notice (cf. (7.18), (7.19)) that

$$\mathcal{P}_{\mathrm{out}}(z) = \begin{pmatrix} 1 & 0 \\ h^{\frac{1}{2}}BG_{\mathrm{in}} & \mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}} \end{pmatrix} \mathcal{P}_{\mathrm{in}}(z). \tag{3.8}$$

Here $\mathcal{N}_{\mathrm{in}} = \gamma h D_\nu K_{\mathrm{in}}$ is the interior Dirichlet to Neumann map. Thus for z away from the Dirichlet spectrum, z is a resonance precisely when 0 belongs to the spectrum of $\mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}} : H^{\frac{3}{2}}(\partial\mathcal{O}) \to H^{\frac{1}{2}}(\partial\mathcal{O})$.

In Chapter 4 we show how to define — up to some non-vanishing factor — $\det A(z)$ for certain holomorphic or meromorphic families of operators that are not necessarily Schatten class perturbations of the identity. With this extended notion of the determinant we get from (3.8) that

$$\det \mathcal{P}_{\mathrm{out}}(z) = \det \mathcal{P}_{\mathrm{in}} \det(\mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}}). \tag{3.9}$$

A rather substantial part of the paper is devoted to the study of $\mathcal{N}_{\mathrm{in}}$, $\mathcal{N}_{\mathrm{ext}}$, in the regions $|\Im z| \geq h^{\frac{2}{3}}/\widetilde{C}$ and $\Im z \geq -c_0 h^{\frac{2}{3}}$ respectively, where $\widetilde{C}$ is an arbitrarily large constant. Many such studies have already been done (see for instance [**31**]), but as is often the case, we found it necessary to make a new one for the needs of this paper. From this study we get somewhat roughly,

$$\ln\big|\det(\mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}})\big| \leq \mathcal{O}(h^{1-n}). \tag{3.10}$$

for

$$\Re z \in \left]\tfrac{1}{2}, 2\right[, \quad |\Im z| \asymp h^{\frac{2}{3}}, \quad \Im z \geq -h^{\frac{2}{3}} c_0. \tag{3.11}$$

The exponent in (3.10) reflects the fact that we have made a reduction to the $n-1$ dimensional manifold $\partial\mathcal{O}$.

In view of (3.9) this gives a precise upper bound on $\ln|\det \mathcal{P}_{\mathrm{out}}(z)|$ for z in the region (3.11). Combined with a rough polynomial upper bound on $\ln|\det \mathcal{P}_{\mathrm{out}}(z)|$ in the full region $|\Im z| \leq h^{\frac{2}{3}}/C$ and the maximum principle, we get the upper bound

$$\ln\left|\det \mathcal{P}_{\mathrm{out}}(z)\right| \leq \Phi_{\mathrm{in}}(z) + \mathcal{O}(h^{1-n}) \tag{3.12}$$

in the rectangle (3.11), where $\Phi_{\mathrm{in}}(z)$ coincides with $\ln|\det \mathcal{P}_{\mathrm{in}}(z)|$ for $|\Im z| \geq h^{\frac{2}{3}}/\widetilde{C}$ and is extended (suitably) as a harmonic function inside $|\Im z| < h^{\frac{2}{3}}/\widetilde{C}$.

A last and quite substantial part of the paper is to show (in the spirit of [**25**], [**27**]) that for every z with $h^{\frac{2}{3}}/\widetilde{C} \leq |\Im z| \leq c_0 h^{\frac{2}{3}}$, $\frac{1}{2} < \Re z < 2$, we also have a lower bound on $\ln|\det(\mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}})|$ almost as sharp as the upper bound (3.10) with probability very close to 1.

With these upper and lower bounds at our disposal, the main result follows by applying Theorem 1.2 of [**26**] to the holomorphic function $\det \mathcal{P}_{\mathrm{out}}(z)$, whose zeros are the resonances.

CHAPTER 4

GRUSHIN PROBLEMS AND DETERMINANTS

The results in the first three sections below are not new, see [**3**], [**13**], but we thought that a short and self-contained presentation can be useful.

4.1. Gaussian elimination

We review some standard material, see for instance [**32**]. Let $\mathcal{H}_j$, $\mathcal{G}_j$, $j = 1, 2$, be complex Hilbert spaces[(1)]. Consider a bounded linear operator

$$\mathcal{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{G}_1 \times \mathcal{G}_2. \tag{4.1}$$

When $\mathcal{P}$ is bijective (with bounded inverse) we denote the inverse by

$$\mathcal{P}^{-1} = \mathcal{E} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}. \tag{4.2}$$

Proposition 4.1. — 1) *Assume that P_{11} is bijective. Then by Gaussian elimination we have the standard factorization into lower and upper triangular matrices:*

$$\mathcal{P} = \begin{pmatrix} P_{11} & 0 \\ P_{21} & 1 \end{pmatrix} \begin{pmatrix} 1 & P_{11}^{-1} P_{12} \\ 0 & P_{22} - P_{21} P_{11}^{-1} P_{12} \end{pmatrix}. \tag{4.3}$$

The first factor is bijective since P_{11} is, so the bijectivity of $\mathcal{P}$ is equivalent to that of the second factor, which in turn is equivalent to that of $P_{22} - P_{21} P_{11}^{-1} P_{12}$. When $\mathcal{P}$ is bijective, we have the formula,

$$\mathcal{P}^{-1} = \begin{pmatrix} 1 & a \\ 0 & (P_{22} - P_{21} P_{11}^{-1} P_{12})^{-1} \end{pmatrix} \begin{pmatrix} P_{11}^{-1} & 0 \\ b & 1 \end{pmatrix} =: \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} =: \mathcal{E}, \tag{4.4}$$

(1) All Hilbert spaces in this work are assumed to be separable.

where $a = -P_{11}^{-1}P_{12}(P_{22} - P_{21}P_{11}^{-1}P_{12})^{-1}$, $b = -P_{21}P_{11}^{-1}$ *and in particular,*

$$E_{22} = (P_{22} - P_{21}P_{11}^{-1}P_{12})^{-1}. \tag{4.5}$$

2) *Now assume that* $\mathcal{P}$ *is bijective. Then* P_{11} *is bijective precisely when* E_{22} *is, and when that bijectivity holds we have*

$$E_{22}^{-1} = P_{22} - P_{21}P_{11}^{-1}P_{12}, \quad P_{11}^{-1} = E_{11} - E_{12}E_{22}^{-1}E_{21} \tag{4.6}$$

The first statement is clear. The second statement is more standard and also quite simple to verify.

4.2. Generalized determinants for holomorphic Fredholm families

Let $\Omega \subset \mathbb{C}$ be open connected, let $\mathcal{H}_1$, $\mathcal{H}_2$ be two complex Hilbert spaces and let

$$P : \Omega \longrightarrow \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$$

be a holomorphic family of Fredholm operators of index 0, such that $P(z)$ is bijective for at least one $z \in \Omega$. Then by analytic Fredholm theory (see for instance the appendix in [**16**]) we know that the set $\sigma(P) \subset \Omega$ where $P(z)$ is not bijective, is discrete. Let $z_0 \in \sigma(P)$. Then we can find $N \in \mathbb{N}$ and operators $R_+ : \mathcal{H}_1 \longrightarrow \mathbb{C}^N$, $R_- : \mathbb{C}^N \to \mathcal{H}_2$ such that

$$\mathcal{P}(z) := \begin{pmatrix} P(z) & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H}_1 \times \mathbb{C}^N \to \mathcal{H}_2 \times \mathbb{C}^N \tag{4.7}$$

is bijective for $z \in \mathrm{neigh}(z_0, \Omega)$ (*i.e.* for z in some neighborhood of z_0 in Ω). Let

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} : \mathcal{H}_2 \times \mathbb{C}^N \longrightarrow \mathcal{H}_1 \times \mathbb{C}^N \tag{4.8}$$

denote the inverse, depending holomorphically on z.

Working in a small neighborhood of z_0 disjoint from $\sigma(P) \setminus \{z_0\}$, we apply the following standard computations and arguments (see [**18**], [**32**]) where the first formula is already in (4.6):

$$\begin{aligned} P(z)^{-1} &= E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z), \\ P^{-1}\partial_z P &= E(z)\partial_z P - E_+(z)E_{-+}(z)^{-1}E_-(z)\partial_z P, \end{aligned}$$

writing $\partial = \partial_z = \partial/\partial z$. Here the first term to the right is holomorphic and the second term is of finite rank with a finite pole at $z = z_0$. Let γ be the oriented boundary of the open disc $D(z_0,\epsilon)$ with center z_0 and with radius $\epsilon>0$ small enough. Integrating along γ, we get

$$\frac{1}{2\pi i}\int_\gamma P^{-1}\partial_z P \mathrm{d}z = -\frac{1}{2\pi i}\int_\gamma E_+E_{-+}^{-1}E_-\partial_z P\mathrm{d}z.$$

The integrand to the right is of trace class, so the left hand side is of trace class and we get

$$\operatorname{tr}\frac{1}{2\pi i}\int_\gamma P^{-1}\partial P \mathrm{d}z = -\frac{1}{2\pi i}\int_\gamma \operatorname{tr} E_+E_{-+}^{-1}E_-\partial P\mathrm{d}z. \tag{4.9}$$

The relation $\mathcal{E}\mathcal{P}=1$ implies

$$E_-P+E_{-+}R_+=0,\quad E_-R_-=1, \tag{4.10}$$

and differentiating the relation $\mathcal{P}\mathcal{E}=1$ gives

$$(\partial P)E_+ + P\partial E_+ + R_-\partial E_{-+}=0. \tag{4.11}$$

Combining this with the cyclicity of the trace, we have

$$\begin{aligned}-\operatorname{tr} E_+E_{-+}^{-1}E_-\partial P &= -\operatorname{tr} E_{-+}^{-1}E_-(\partial P)E_+\\ &= \operatorname{tr} E_{-+}^{-1}E_-P\partial E_+ + \operatorname{tr} E_{-+}^{-1}E_-R_-\partial E_{-+}\\ &= -\operatorname{tr} E_{-+}^{-1}E_{-+}R_+\partial E_+ + \operatorname{tr} E_{-+}^{-1}\partial E_{-+}\\ &= -\operatorname{tr} R_+\partial E_+ + \operatorname{tr} E_{-+}^{-1}\partial E_{-+}.\end{aligned}$$

The first term in the last expression vanishes since $R_+\partial E_+ = \partial(R_+E_+) = \partial(1)=0$, so (4.9) becomes

$$\begin{aligned}\operatorname{tr}\frac{1}{2\pi i}\int_\gamma P(z)^{-1}\partial P(z)\mathrm{d}z &= \frac{1}{2\pi i}\int_\gamma \operatorname{tr} E_{-+}^{-1}\partial E_{-+}\mathrm{d}z\\ &= \frac{1}{2\pi}\operatorname{var}\arg_\gamma(\ln\det E_{-+}) = m(z_0,\det E_{-+}),\end{aligned} \tag{4.12}$$

where $m(z_0,\det E_{-+})$ denotes the multiplicity of z_0 as a zero of $\det E_{-+}(z)$.

Remark 4.2. — From the cyclicity of the trace in the beginning of the calculations we see that $\int_\gamma(\partial_z P)P^{-1}\mathrm{d}z$ is of trace class and has the same trace as $\int_\gamma P^{-1}\partial_z P\mathrm{d}z$.

A more elegant presentation of the above discussion could be based on (4.3):

$$\mathcal{P} = \begin{pmatrix} P(z) & 0\\ * & 1\end{pmatrix}\begin{pmatrix} 1 & *\\ 0 & E_{-+}^{-1}\end{pmatrix} =: \mathcal{A}\mathcal{B},$$

which at least formally leads to

$$0 = \operatorname{tr}\int_\gamma \mathcal{P}^{-1}\partial\mathcal{P}\mathrm{d}z = \operatorname{tr}\int_\gamma \mathcal{A}^{-1}\partial\mathcal{A}\mathrm{d}z + \operatorname{tr}\int_\gamma \mathcal{B}^{-1}\partial\mathcal{B}\mathrm{d}z \\ = \operatorname{tr}\int_\gamma P^{-1}\partial P\mathrm{d}z - \operatorname{tr}\int_\gamma E_{-+}^{-1}\partial E_{-+}\mathrm{d}z. \tag{4.13}$$

Definition 4.3. — *By* $\det P = \det_\Omega P$ *we denote any holomorphic function* f *on* Ω *with* $f^{-1}(0) = \sigma(P)$ *for which*

$$m(z_0, f) = \operatorname{tr}\frac{1}{2\pi i}\int_{\partial D(z_0,r)} P(z)^{-1}\partial P(z)\mathrm{d}z, \text{ for all } z_0 \in \sigma(P). \tag{4.14}$$

Here $r > 0$ *is small enough so that* $\sigma(P) \cap D(z_0, r) = \{z_0\}$.

By Mittag-Leffler's theorem such a holomorphic function exists and it is unique up to a non-vanishing holomorphic factor.

Proposition 4.4. — *Let* $Q : \Omega \to \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ *have the same general properties as* $P(z)$. *Then the determinants of* P, Q, QP *can be defined as above so that*

$$\det\big(Q(z)P(z)\big) = \big(\det Q(z)\big)\big(\det P(z)\big). \tag{4.15}$$

Proof. — We clearly have

$$\sigma(QP) = \sigma(Q) \cup \sigma(P)$$

as sets, and we have to prove that

$$m\big(z_0, \det(QP)\big) = m(z_0, \det P) + m(z_0, \det Q), \tag{4.16}$$

for every $z_0 \in \Omega$, where $m(z_0, \det P)$ is defined to be zero when $z \notin \sigma(P)$ and otherwise as in (4.14).

Let $z_0 \in \sigma(P) \cup \sigma(Q)$ and let $z_0 \neq z \in \operatorname{neigh}(z_0)$. We have at z,

$$(QP)^{-1}\partial(QP) = P^{-1}Q^{-1}(\partial Q)P + P^{-1}\partial P. \tag{4.17}$$

Here the first term to the right needs to be transformed. For each of the operators $A = P^{-1}$, $B = Q^{-1}(\partial Q)P$ we make a decomposition

$$A = A_{\mathrm{hol}} + A_{\mathrm{sing}}$$

where A_{hol} is holomorphic in a full neighborhood of z_0 and A_{sing} has a pole at z_0 but is of finite rank and hence of trace class. Now write

$$AB - BA = (A_{\mathrm{hol}}B_{\mathrm{hol}} - B_{\mathrm{hol}}A_{\mathrm{hol}}) + (A_{\mathrm{hol}}B_{\mathrm{sing}} - B_{\mathrm{sing}}A_{\mathrm{hol}}) \\ +(A_{\mathrm{sing}}B_{\mathrm{hol}} - B_{\mathrm{hol}}A_{\mathrm{sing}}) + (A_{\mathrm{sing}}B_{\mathrm{sing}} - B_{\mathrm{sing}}A_{\mathrm{sing}}). \tag{4.18}$$

The first term to the right is holomorphic near z_0, while the other three are of trace class with vanishing trace. Thus if $\gamma = \partial D(z_0, r)$ with $0 < r$ small enough, $\int_\gamma (AB - BA)\mathrm{d}z$ is of trace class and with trace 0.

Applying this to the first term to the right in (4.17), we see that

$$\int_\gamma \big(P^{-1}Q^{-1}(\partial Q)P - Q^{-1}\partial Q\big)\,\mathrm{d}z$$

is of trace class and has trace 0. It follows that $(2\pi i)^{-1}\int_\gamma P^{-1}Q^{-1}(\partial Q)P\mathrm{d}z$ is of trace class and has the same trace as $(2\pi i)^{-1}\int_\gamma Q^{-1}\partial Q\mathrm{d}z$ and we get

$$\operatorname{tr}\frac{1}{2\pi i}\int_\gamma (QP)^{-1}\partial(QP)\mathrm{d}z = \operatorname{tr}\frac{1}{2\pi i}\int_\gamma Q^{-1}\partial Q\mathrm{d}z + \operatorname{tr}\frac{1}{2\pi i}\int_\gamma P^{-1}\partial P\mathrm{d}z,$$

which amounts to (4.16). □

4.3. Extension to meromorphic families

In this section we essentially follow [**13**], see also [**3**]. Let Ω be open and connected. Let $P : \Omega \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be meromorphic with the poles $z_1, z_2, \dots$. Here $\mathcal{H}_j$ are complex Hilbert spaces.

Definition 4.5. — *We say that $P(z)$ is a* meromorphic Fredholm function (*or* Fredholm family) *if the following hold:*

- ▷ *$P(z)$ is Fredholm of index 0 on $\Omega \setminus \{z_1, z_2, \dots\}$ and bijective for at least one z in that set.*
- ▷ *Let z_0 be any pole and write the Laurent series at z_0 as*

$$P(z) = \sum_1^{N_0} (z - z_0)^{-j} P_j + B(z), \quad z \in \operatorname{neigh}(z_0),$$

with $B(z)$ holomorphic. Then P_j are of finite rank (implying that $B(z)$ is Fredholm of index zero for $z \neq z_0$). Moreover, $B(z_0)$ is a Fredholm operator of index 0.

The motivation for introducing this class is that if $Q(z)$ is a holomorphic family of Fredholm operators on Ω, bijective for at least one $z \in \Omega$, then $P(z) = Q(z)^{-1}$ is a meromorphic Fredholm function.

If $P^j(z)$, $j = 1, 2$ are meromorphic Fredholm families on Ω, then $P^1(z)P^2(z)$ is also such a family. In fact, the first property in the definition is easy to verify and if z_0 is a pole for one or both factors, we write

$$P^j(z) = \sum_1^{N_j} (z - z_0)^{-k} P_k^j + B^j(z)$$

and check that

$$P^1(z)P^2(z) = \sum_1^{N_1+N_2} (z - z_0)^{-k} P_k + B(z)$$

where P_k are of finite rank and $B(z_0) = B^1(z_0)B^2(z_0) + K$, where K is of finite rank.

We shall show that the class of meromorphic Fredholm functions on Ω is closed under inversion and introduce the notion of meromorphic determinant for such families. The key will be a well chosen Grushin problem.

We pause to recollect the condition for the well-posedness of a Grushin problem

$$Pu + R_-u_- = v, \quad R_+u = v_+, \tag{4.19}$$

when $P : \mathcal{H}_1 \to \mathcal{H}_2$ is a fixed Fredholm operator of index 0 and $R_+ : \mathcal{H}_1 \to \mathbb{C}^N$ and $R_- : \mathbb{C}^N \to \mathcal{H}_2$ are of rank N. Since (4.19) defines an operator

$$\mathcal{P} = \begin{pmatrix} P & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H}_1 \times \mathbb{C}^N \longrightarrow \mathcal{H}_2 \times \mathbb{C}^N$$

of index 0, it is bijective precisely when it is injective, so it suffices to review when (4.19) is injective. The necessary and sufficent condition for that is

$$u \in \mathcal{N}(R_+) \text{ and } Pu \in \mathcal{R}(R_-) \implies u = 0, \tag{4.20}$$

where $\mathcal{N}$ indicates the null space and $\mathcal{R}$ the range. Now let $P(z)$ be a meromorphic Fredholm function with a pole at z_0. We look for $R_\pm$ as above (independent of z) such that the problem

$$\Big(\sum_1^{N_0} (z - z_0)^{-j} P_j + B(z)\Big)u + R_-u_- = v, \quad R_+u = v_+ \tag{4.21}$$

is well-posed for all z in a pointed neighborhood of z_0.

Since the P_j are finitely many operators of finite rank, we can choose R_+ with N large enough, so that

$$P_{j_{|\mathcal{N}(R_+)}} = 0, \quad \mathcal{N}(R_+) \subset \mathcal{N}\big(B(z_0)\big)^\perp.$$

Then $B(z_0)(\mathcal{N}(R_+))$ is a closed subspace of $\mathcal{H}_2$ of codimension N, and we choose R_- of rank N such that $B(z_0)(\mathcal{N}(R_+)) \cap \mathcal{R}(R_-) = 0$, *i.e.*

$$\mathcal{H}_2 = B(z_0)\big(\mathcal{N}(R_+)\big) \oplus \mathcal{R}(R_-). \tag{4.22}$$

Then the problem

$$B(z_0)u + R_-u_- = v, \quad R_+u = v_+$$

is well-posed and we check that (4.21) has the same property. Indeed, $P(z) = B(z)$ on $\mathcal{N}(R_+)$ and hence this restriction is injective for z close to z_0, and $P(z)(\mathcal{N}(R_+)) \oplus \mathcal{R}(R_-) = \mathcal{H}_2$.

Let us also analyze the structure of the solution operator to the problem (4.21). Let $\widetilde{E}_+$ be a right inverse of R_+ so that a general $u \in \mathcal{H}_1$ has the direct sum decomposition

$$u = u' + \widetilde{E}_+\widetilde{v}_+, \quad u' \in \mathcal{N}(R_+), \quad \widetilde{v}_+ \in \mathbb{C}^N. \tag{4.23}$$

Then the second equation of (4.21) holds precisely when $\widetilde{v}_+ = v_+$. Let Π', Π'' be the projections on the first and second summands in the direct sum decomposition (4.22) and write $\mathcal{H}_2 \ni v = \Pi'v + \Pi''v = v' + v''$.

Since $P_ju' = 0$, the first equation in (4.21) becomes

$$B(z)u' + R_-u_- = v - \sum_1^{N_0}(z-z_0)^{-j}P_j\widetilde{E}_+v_+ - B(z)\widetilde{E}_+v_+$$

and we determine u' and u_- by applying Π' and Π'' respectively, using that $\Pi'B(z)_{|\mathcal{N}(R_+)} = \Pi'B(z_0)_{|\mathcal{N}(R_+)} + \mathcal{O}(z-z_0)$ is bijective: $\mathcal{N}(R_+) \to B(z_0)(\mathcal{N}(R_+))$, that $\Pi''R_- = R_-$ and that $R_- : \mathbb{C}^N \to R_-(\mathbb{C}^N)$ is bijective. If $\widetilde{E}_-$ is a left inverse of R_-, we get

$$\Pi'B(z)u' = v' - \sum_1^{N_0}(z-z_0)^{-j}\Pi'P_j\widetilde{E}_+v_+ - \Pi'B(z)\widetilde{E}_+v_+,$$

$$u' = \big(\Pi'B(z)_{|_{\mathcal{N}(R_+)}}\big)^{-1}\Big(v' - \sum_1^{N_0}(z-z_0)^{-j}\Pi'P_j\widetilde{E}_+v_+ - \Pi'B(z)\widetilde{E}_+v_+\Big),$$

and

$$\begin{aligned} u_- = {} & \widetilde{E}_-\Pi''\Big(v - \sum_1^{N_0}(z-z_0)^{-j}P_j\widetilde{E}_+v_+ - B(z)\widetilde{E}_+v_+\Big) \\ & - \widetilde{E}_-\Pi''\big(B(z) - B(z_0)\big)u'. \end{aligned}$$

As usual, we write the solution of (4.21) in the form

$$u = Ev + E_+v_+, \quad u_- = E_-v + E_{-+}v_+, \tag{4.24}$$

where "explicit" expressions for E, $E_\bullet$ can be obtained from the above computations. We see that

$$E(z) = \left(\Pi' B(z)_{|\mathcal{N}(R_+)}\right)^{-1}\Pi' \tag{4.25}$$

is a holomorphic family of Fredholm operators of index 0, while $E_+(z)$, $E_-(z)$, $E_{-+}(z)$ are meromorphic operator valued functions with singular terms of finite rank. In particular, $E_{-+}(z)$ is a meromorphic function with values in the $N\times N$ matrices which is invertible for $z\neq z_0$, so that $\det E_{-+}$ is meromorphic with a possible pole at z_0, non-vanishing and holomorphic in a pointed neighborhood of that point. Thus E_{-+}^{-1} is also meromorphic and we conclude that

$$P(z)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z)$$

is a meromorphic family of Fredholm operators near z_0. Thus we get

Proposition 4.6. — *If $P(z)$ is a meromorphic Fredholm function, then $P(z)^{-1}$ has the same property.*

We shall next extend the discussion of determinants in Section 4.2. When $R_\pm$ are independent of z and $\mathcal{P} = \left(\begin{smallmatrix} P(z) & R_+ \\ R_- & 0\end{smallmatrix}\right) = \mathcal{H}_1\times\mathbb{C}^N \to \mathcal{H}_2\times\mathbb{C}^N$ is bijective with inverse $\mathcal{E} = \left(\begin{smallmatrix} E & E_- \\ E_+ & E_{-+}\end{smallmatrix}\right)$, we notice that

$$\mathcal{P}^{-1}\partial\mathcal{P} = \begin{pmatrix} E\partial P & 0 \\ E_-\partial P & 0\end{pmatrix}.$$

In the case of our special problem (4.21), $E(z)$ is given in (4.25) and the non-holomorphic part of $E\partial P$ is

$$\left(\Pi' B(z)_{|\mathcal{N}(R_+)}\right)^{-1}\Pi'\partial_z\Big(\sum_1^{N_0}(z-z_0)^{-j}P_j\Big)$$

which is of finite rank and with the same trace as

$$\Pi'\partial_z\Big(\sum_1^{N_0}(z-z_0)^{-j}P_j\Big)\left(\Pi' B(z)_{|\mathcal{N}(R_+)}\right)^{-1}.$$

This operator vanishes, since ${P_j}_{|\mathcal{N}(R_+)} = 0$. Thus $\int_\gamma \mathcal{P}^{-1}\partial\mathcal{P}\mathrm{d}z$ and $\int_\gamma E\partial P\mathrm{d}z$ are of trace class and have the trace 0 if $\gamma = D(z_0,r)$ for $0<r\ll 1$.

As in and around (4.9) we now get

$$\mathrm{tr}\,\frac{1}{2\pi i}\int_\gamma P^{-1}\partial P\mathrm{d}z = -\mathrm{tr}\,\frac{1}{2\pi i}\int_\gamma E_+E_{-+}^{-1}E_-\partial P\mathrm{d}z = \mathrm{tr}\,\frac{1}{2\pi i}\int_\gamma E_{-+}^{-1}\partial E_{-+}\mathrm{d}z,$$

leading to

$$\operatorname{tr}\frac{1}{2\pi i}\int_\gamma P^{-1}\partial P \mathrm{d}z = m(z_0, \det E_{-+}), \tag{4.26}$$

where the integer $m(z_0, \det E_{-+})$ is the order of z_0 as a zero of $\det E_{-+}$ when the latter function is holomorphic near z_0 and when $\det E_{-+}$ has a pole at z_0, then $-m(z_0, \det E_{-+})$ is the order of that pole.

Note for future reference that

$$P^{-1}\partial P = a + b, \tag{4.27}$$

where a is holomorphic near z_0 and b is of finite rank and

$$\operatorname{tr} b = \operatorname{tr}(E_{-+}^{-1}\partial E_{-+}). \tag{4.28}$$

We emphasize that in view of (4.26), $(2\pi i)^{-1}\operatorname{tr}\int_\gamma P^{-1}\partial P\mathrm{d}z$ is an integer, and we can then give the following extension to meromorphic families of the notion of determinant:

Definition 4.7. — *Let $P : \Omega \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be a meromorphic Fredholm function with the poles $z_1, z_2, \dots$ By $\det P = \det_\Omega P$ we denote any meromorphic function $f(z)$ whose restriction to $\Omega \setminus \{z_1, z_2, \dots\}$ is a determinant for P in the sense of Definition* 4.3*, and such that for every pole z_j of P, we have*

$$\operatorname{tr}\frac{1}{2\pi i}\int_{\partial D(z_j,r)} P(z)^{-1}\partial P(z)\mathrm{d}z = m(z_j, f)$$

when $r > 0$ is small enough.

Observe that Proposition 4.4 and its proof extend to the case of meromorphic Fredholm functions.

4.4. Determinants via traces

If $\mathcal{H}$ is a complex Hilbert space and $P = P(z) \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ is a trace class perturbation of the identity, depending holomorphically on the complex parameter z, we can define $D(z) = \ln\det P(z)$ and we have

$$\frac{\mathrm{d}}{\mathrm{d}z}D(z) = \operatorname{tr} P(z)^{-1}\frac{\mathrm{d}P}{\mathrm{d}z}, \tag{4.29}$$

at the points where P is bijective. Now even when P is not a trace class perturbation of the identity, it may happen that $P^{-1}\mathrm{d}P/\mathrm{d}z$ is of trace class, and we can now consider the case when $P(z) \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ for different complex Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$. By integration of (4.29), we may then say that $D(z)$ is well-defined up to a constant as a possibly multivalued function on every connected component of the open set where $P(z)$ is invertible. If $P^{-1}\mathrm{d}P/\mathrm{d}z$ is not of trace class we may differentiate further and hope to reach an expression which is of trace class. Then we would be able to define $D(z)$ up to a polynomial. In this section we carry out such a scheme. The idea of reaching trace class operators by means of differentiation in connection with determinants has been used by G. Carron [**5**].

Let $\Omega \subset \mathbb{C}$ be open and connected, let $\mathcal{H}_j$, $j = 1, 2, 3$, be complex Hilbert spaces. Let $\Sigma = \Sigma(P) \subset \Omega$ be discrete and let $P : \Omega \setminus \Sigma \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be holomorphic and pointwise bijective. Let $C_p = C_p(\mathcal{H}_1, \mathcal{H}_2)$ denote the Schatten class of index $p \in [1, +\infty]$ (see for instance [**14**]). Assume that for some $p \in [1, +\infty[$,

$$\partial_z^k P(z) \in C_{\max(1,p/k)}, \quad 1 \leq k \in \mathbb{N}, \tag{4.30}$$

locally uniformly on Ω. By the Cauchy inequalities, it suffices to check this for $k \leq N$, where $N = N(p)$ is the smallest integer $\geq p$.

Recall that C_p increases with p and that if $C \in C_p(\mathcal{H}_1, \mathcal{H}_2)$ and $D \in C_q(\mathcal{H}_2, \mathcal{H}_3)$, then $DC \in C_r(\mathcal{H}_1, \mathcal{H}_3)$ with $1/r = \min(1, 1/p + 1/q)$. (See [**14**], Prop. 7.2.) In the following, we shall think of bounded operators as being of order $= 0$ and of elements in C_p as being of order $= -1/p$. In all cases we restrict here the order to the interval $[-1, 0]$ and then orders are additive under composition: $\mathrm{ord}(DC) = \max(-1, \mathrm{ord}(D) + \mathrm{ord}(C))$. (We adopt the convention that the order is not unique; if C is of order α and $\alpha \leq \beta \leq 0$, then C is also of order β.)

We also notice that $P(z)^{-1}$ satisfies (4.30).

On the set $\Omega \setminus \Sigma(P)$, we check that

$$\partial_z^{j-1}\big(P(z)^{-1}\partial_z P(z)\big) \in C_{\max(1,p/j)}, \quad j \geq 1, \tag{4.31}$$

i.e. of order $= \max(-1, -j/p)$. Thus, for $p \leq j \in \mathbb{N}$, we can define

$$D_{P,j}(z) = \mathrm{tr}\big(\partial_z^{j-1}(P(z)^{-1}\partial_z P(z))\big), \quad z \in \Omega \setminus \Sigma(P). \tag{4.32}$$

Clearly,

$$\partial_z D_{P,j}(z) = D_{P,j+1}(z).$$

We can now define the determinant of $P(z)$. At the end of the section we show that this new notion coincides with the one for meromorphic families of Fredholm operators of the preceding subsection.

Definition 4.8. — *Let $N = N(p)$ be the smallest integer $\geq p$. We define $D_P(z) = \ln\det P(z)$ to be any multivalued holomorphic function on $\Omega \setminus \Sigma(P)$*

which solves the equation

$$\partial_z^N D_P(z) = \operatorname{tr}\big(\partial_z^{N-1}(P(z)^{-1}\partial_z P(z))\big). \tag{4.33}$$

Thus $D_P(z)$ is well defined (on the universal covering space of $\Omega \setminus \Sigma(P)$) up to a polynomial of degree $N-1$.

Let $Q : \Omega \to \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ be a second family with the same general properties as $P(z)$ and for simplicity with the same p in (the analogue of) (4.30). Then $Q(z)P(z)$ fulfills the same assumptions and we next check the additivity property

$$\ln\det PQ = \ln\det P + \ln\det Q, \quad \text{on } \Omega \setminus \big(\Sigma(P) + \Sigma(Q)\big), \tag{4.34}$$

i.e.

$$\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^N \ln\det PQ = \Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^N \ln\det P + \Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^N \ln\det Q, \tag{4.35}$$

when N is the smallest integer $\geq p$.

When $p = 1 = N$, this is straightforward:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}\ln\det PQ &= \operatorname{tr}(PQ)^{-1}\frac{\mathrm{d}}{\mathrm{d}z}(PQ) \\ &= \operatorname{tr} Q^{-1}P^{-1}\frac{\mathrm{d}P}{\mathrm{d}z}Q + \operatorname{tr} Q^{-1}P^{-1}P\frac{\mathrm{d}Q}{\mathrm{d}z} \\ &= \operatorname{tr} Q^{-1}P^{-1}\frac{\mathrm{d}P}{\mathrm{d}z}Q + \operatorname{tr} Q^{-1}\frac{\mathrm{d}Q}{\mathrm{d}z}. \end{aligned} \tag{4.36}$$

Here we use the cyclicity of the trace to see that the first term in the last expression is equal to $\operatorname{tr}P^{-1}\mathrm{d}P/\mathrm{d}z$ and we thus get (4.35) when $N = 1$.

Recall that the cyclicity of the trace says that $\operatorname{tr}(P_1P_2 - P_2P_1) = 0$, when $P_1 \in C_{p_1}(\mathcal{H}_1, \mathcal{H}_2)$, $P_2 \in C_{p_2}(\mathcal{H}_2, \mathcal{H}_1)$ and $1 = 1/p_1 + 1/p_2$.

Lemma 4.9. — *Let $P_1(z) \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $P_2(z) \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ depend holomorphically on $z \in \Omega$. Then $\mathrm{d}(P_1P_2 - P_2P_1)/\mathrm{d}z$ is a sum of terms of the form $Q_1Q_2 - Q_2Q_1$. More precisely,*

$$(P_1P_2 - P_2P_1)' = [P_1'P_2 - P_2P_1'] + [P_1P_2' - P_2'P_1],$$

where we indicate derivatives with a prime.

Iterating the lemma we see that $(\mathrm{d}/\mathrm{d}z)^N(P_1P_2 - P_2P_1)$ is a linear combination of terms of the form $Q_1Q_2 - Q_2Q_1$, with $Q_j = \partial_z^{N_j}P_j$, $N_1 + N_2 = N$.

Now return to (4.36), or rather the last two equations there that are valid without traces, and write

$$Q^{-1}P^{-1}\frac{\mathrm{d}P}{\mathrm{d}z}Q = P^{-1}\frac{\mathrm{d}P}{\mathrm{d}z} + (P_1P_2 - P_2P_1),$$

with $P_1 = Q^{-1}P^{-1}\mathrm{d}P/\mathrm{d}z$, $P_2 = Q$. The lemma shows that

$$\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^{N-1}\Big(Q^{-1}P^{-1}\frac{\mathrm{d}P}{\mathrm{d}z}Q\Big) = \Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^{N-1}\Big(P^{-1}\frac{\mathrm{d}P}{\mathrm{d}z}\Big)$$
$$+ \text{ a linear combination of terms of the form } Q_1Q_2 - Q_2Q_1$$
$$\text{with } \mathrm{ord}(Q_j) \leq \max(-1, -N_j/p),\ N_1 + N_2 = N.$$

The cyclicity of the trace then implies that

$$\mathrm{tr}\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^{N-1}\Big(Q^{-1}P^{-1}\frac{\mathrm{d}P}{\mathrm{d}z}Q\Big) = \mathrm{tr}\Big(\frac{\mathrm{d}}{\mathrm{d}z}\Big)^{N-1}\Big(P^{-1}\frac{\mathrm{d}P}{\mathrm{d}z}\Big)$$

and we obtain (4.35) for a general N.

As in the case of meromorphic families of Fredholm operators, if $z_0 \in \Sigma(P)$ and $\gamma = \partial D(z_0, r)$ with $r > 0$ small enough, $\int_\gamma P^{-1}\partial P \mathrm{d}z$ is of trace class:

Proposition 4.10. — *With P, p, $N = N(p)$ as in Definition 4.8, let z_0 in $\Sigma(P)$, $\gamma = \partial D(z_0, r)$ with $r > 0$ small enough, so that $D(z_0, r) \cap \Sigma(P) = \{z_0\}$. Then $\int_\gamma P^{-1}\partial P \mathrm{d}z$ is of trace class and we have*

$$\mathrm{tr}\frac{1}{2\pi i}\int_\gamma P^{-1}\partial P \mathrm{d}z = \mathrm{tr}\frac{1}{2\pi i}\int_\gamma \frac{(-z)^{N-1}}{(N-1)!}\partial^{N-1}(P^{-1}\partial P)\mathrm{d}z$$
$$= \frac{1}{2\pi i}\int_\gamma \frac{(-z)^{N-1}}{(N-1)!} D_{P,N}(z)\mathrm{d}z, \tag{4.37}$$

where $z^{N-1}/(N-1)!$ can be replaced by any other polynomial $p(z)$ such that $\partial^{N-1}p(z) = 1$

Proof. — The second equality follows by moving the trace inside the integral and recalling the definition of $D_{P,N}$. The first equality and the fact that $\int_\gamma P^{-1}\partial P \mathrm{d}z$ is of trace class, follows from the corresponding stronger equality without "tr" in front which can be obtained by integration by parts. □

Now, assume in addition that Ω is simply connected and that P is a meromorphic Fredholm function on Ω in the sense of Definition 4.5. Then we know that

$$\mathrm{tr}\,\frac{1}{2\pi i}\int_\gamma P^{-1}\partial P \mathrm{d}z = m(z_0, f) \in \mathbb{Z}, \tag{4.38}$$

where f denotes the meromorphic Fredholm determinant of Definition 4.7. On the other hand, we can do integrations by parts in the last expression in (4.37) and obtain

$$\mathrm{tr}\,\frac{1}{2\pi i}\int_\gamma P^{-1}\partial P \mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \partial_z D_P(z)\mathrm{d}z, \tag{4.39}$$

which, combined with (4.38), says that

$$\mathrm{var}_\gamma D_P = 2\pi i m(z_0, f) \in 2\pi i \mathbb{Z} \tag{4.40}$$

and hence e^{D_P} and its logarithmic derivative ∂D_P are single-valued holomorphic functions on $\Omega \setminus \Sigma$.

So far, this only shows that

$$D_P = \sum_1^\infty (z - z_0)^{-j} a_j + m(z_0, f) \ln(z - z_0) + g(z),$$

where g is holomorphic, so $\mathrm{e}^{D_P} = \mathrm{e}^{g + \sum a_j (z-z_0)^{-j}} (z - z_0)^{m(z_0,f)}$ may have a bad singularity at z_0. We therefore return to the Grushin problem in Section 4.3. The remark (4.27), (4.28) shows that

$$\mathrm{tr}\, \partial^{N-1} P^{-1} \partial P = \mathrm{tr}(\partial^{N-1} a) + \partial^{N-1} \mathrm{tr}(E_{-+}^{-1} \partial E_{-+}),$$

where E_{-+} is a meromorphic finite matrix and $\mathrm{tr}(\partial^{N-1} a)$ is holomorphic in a full neighborhood of z_0. Consequently,

$$\partial D_P = \mathrm{tr}(E_{-+}^{-1} \partial E_{-+}) + \text{holomorphic} = \partial(\ln E_{-+}) + \text{holomorphic},$$

which rules out the bad singularity and we see that $\mathrm{e}^{D_P} = \mathrm{e}^g (z - z_0)^{m(z_0,f)}$ near z_0. Globally $\mathrm{e}^{D_P(z)}$ is indeed a determinant in the sense of Definition 4.7.

PROPOSITION 4.11. — *Let $P(z)$ be a holomorphic family on $\Omega \setminus \Sigma$ as in the beginning of this section and assume in addition that Ω is simply connected and that P is a meromorphic Fredholm function on Ω. Then the determinants* $\det P(z)$ *in the sense of Definition* 4.8 *and in the sense of Definition* 4.7 *coincide up to a non-vanishing holomorphic factor.*

The following complement will be used in Chapter 13.

4.5. Addendum

Consider a Schatten class perturbation of the identity, $Q(z) = 1 - K(z)$, where $K(z) \in C_p$ is holomorphic in some domain in $\mathbb{C}$ and as in (4.30):

$$\partial_z^k K(z) \in C_{\max(1,p/k)}, \quad 1 \le k \in \mathbb{N}. \tag{4.41}$$

This assumption remains valid if we replace p by $N = [p]$, the smallest integer $\ge p$ and then (in view of the mean value property for holomorphic functions) takes the simpler form

$$\partial_z^k K(z) \in C_{N/k}, \quad 1 \le k \le N, \tag{4.42}$$

$$K(z) \in C_N. \tag{4.43}$$

Considering the Taylor expansions (and mimicking the definition of modified determinants for Schatten class perturbations of the identity), we get

$$\begin{cases} Q(z) = A(z)B(z), \\ A(z) = \exp F(z), \quad F(z) = K(z) + \cdots + \dfrac{K(z)^{N-1}}{N-1}, \\ B(z) = \big(1 + R_N(K)K^N\big), \end{cases} \tag{4.44}$$

where $\|R_N(K)\| \leq C(\|K\|)$. Thus

$$\big\|R_N(K)K^N\big\|_{C_1} \leq C\big(\|K\|\big)\|K\|_{C_N}^N,$$

so $\det B(z)$ can be defined as in Section 4.4. The definition coincides with that of determinants of trace class perturbations of the identity and we get

$$\big|\det B(z)\big| \leq \exp\big(C(\|K\|)\|K\|_{C_N}^N\big). \tag{4.45}$$

As for $A(z) = \exp F(z)$, we see that $F(z)$ satisfies (4.42), (4.43). Moreover from applying ∂_z to the differential equation $\partial_t \exp(tF(z)) = F(z)\exp(tF(z))$, we have

$$\partial_z(\mathrm{e}^F) = \int_0^1 \mathrm{e}^{(1-t)F(z)}\big(\partial_z F(z)\big)\,\mathrm{e}^{tF(z)}\,\mathrm{d}t \in C_N$$

and from similar expressions for $\partial_z^k(\mathrm{e}^F)$ we see that $A = \mathrm{e}^F$ satisfies (4.42) and (4.43). Now,

$$\mathrm{e}^{-F}\partial_z\,\mathrm{e}^F = \int_0^1 \mathrm{e}^{-tF}(\partial_z F)\,\mathrm{e}^{tF}\,\mathrm{d}t = \partial_z F + \int_0^1 [\mathrm{e}^{-tF}, (\partial_z F)\,\mathrm{e}^{tF}]\,\mathrm{d}t,$$

so $\operatorname{tr}\partial_z^{N-1}(\mathrm{e}^{-F}\partial_z\,\mathrm{e}^F) = \operatorname{tr}\partial_z^N F$, which is bounded in modulus by

$$\begin{aligned} &\mathcal{O}(1)\sum_{\substack{N_1+\cdots+N_q=N \\ N_q\geq 0,\, q\leq N-1}} \|\partial^{N_1}K\cdots\partial^{N_q}K\|_{C_1} \\ &\qquad \leq \mathcal{O}(1)\sum_{\substack{N_1+\cdots+N_q=N \\ N_q\geq 0,\, q\leq N-1}} \|\partial^{N_1}K\|_{C_{N/N_1}}\cdots\|\partial^{N_q}K\|_{C_{N/N_q}}. \end{aligned} \tag{4.46}$$

Combining this with (4.44), (4.45), we get:

PROPOSITION 4.12. — *Under the above assumptions,*

$$\det Q(z) = \mathrm{I}(z)\,\mathrm{II}(z), \quad \mathrm{I}(z) = \det A(z),\ \mathrm{II}(z) = \det B(z),$$

where $|\mathrm{II}(z)|$ *is bounded by the right hand side of* (4.45) *and* $|\partial_z^N \ln \mathrm{I}(z)|$ *is bounded by the expression* (4.46).

CHAPTER 5

COMPLEX DILATIONS

5.1. Complex dilations and symmetry

We start by reviewing some easy facts for complex distortions (see [**28**], [**29**], [**30**], [**31**], [**33**]) and we shall pay a special attention to symmetry with respect to the natural *bilinear* form. Let $\Gamma \subset \mathbb{C}^n$ be a *maximally totally real* (m.t.r.) simply connected smooth sub-manifold and let $P = \sum_{|\alpha|\le m} a_\alpha D^\alpha$, where $a_\alpha \in C^\infty(\Gamma)$. If $u \in C^\infty(\Gamma)$, we put

$$Pu = (\widetilde{P}\widetilde{u}\,)_{|\Gamma},$$

where $\widetilde{P} = \sum \widetilde{a}_\alpha D^\alpha$ and $\widetilde{a}_\alpha$, $\widetilde{u}$ are almost holomorphic extensions of a_α, u to a neighborhood of Γ.

If $P^{\mathrm{t}} = \sum (-D)^\alpha \circ a_\alpha$ is the formal transpose of P, we can define as above $P^{\mathrm{t}}u \in C^\infty(\Gamma)$ for $u \in C^\infty(\Gamma)$ and if we define

$$\langle u\,|\,v\rangle_\Gamma = \int_\Gamma u(x)v(x)\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n = \int_\Gamma u(x)v(x)\mathrm{d}x, \quad u, v \in C_0^\infty(\Gamma), \tag{5.1}$$

we get from Stokes' formula that

$$\langle Pu\,|\,v\rangle_\Gamma = \langle u\,|\,P^{\mathrm{t}}v\rangle_\Gamma.$$

Now, let $\widehat{\Gamma} \subset \mathbb{C}^n$ be a second maximally totally real smooth manifold and let $\gamma : \widehat{\Gamma} \to \Gamma$ be a smooth diffeomorphism. (For instance, $\widehat{\Gamma}$ can be an open subset of $\mathbb{R}^n$ and γ a "parametrization" of Γ.) We can then define

$$\frac{\partial \gamma}{\partial y} = \Big(\frac{\partial \widetilde{\gamma}_j}{\partial y_k}\Big), \tag{5.2}$$

where $\widetilde{\gamma}(y) = (\widetilde{\gamma}_1(y), \ldots, \widetilde{\gamma}_n(y))$ is an almost holomorphic extension of $\gamma = (\gamma_1, \ldots, \gamma_n)$. Let $f \in C^\infty(\widehat{\Gamma})$ and define $U : C^\infty(\Gamma) \to C^\infty(\widehat{\Gamma})$ by

$$Uu(y) = f(y)u\big(\gamma(y)\big), \quad u \in C_0^\infty(\Gamma). \tag{5.3}$$

If $u, v \in C_0^\infty(\Gamma)$, we get

$$\langle Uu\,|\,Uv\rangle_{\widehat{\Gamma}} = \int_{\widehat{\Gamma}} u\big(\gamma(y)\big)v\big(\gamma(y)\big)f(y)^2\mathrm{d}y,$$

$$\langle u\,|\,v\rangle_{\Gamma} = \int_{\Gamma} u(x)v(x)\mathrm{d}x = \int_{\widehat{\Gamma}} u\big(\gamma(y)\big)v\big(\gamma(y)\big)\det\Big(\frac{\partial\gamma}{\partial y}\Big)\mathrm{d}y.$$

Choose $f = (\det\partial\gamma/\partial y)^{\frac{1}{2}}$ for some fixed continuous branch of the square root (assuming for simplicity that $\widehat{\Gamma}$ is simply connected). Then

$$\langle Uu\,|\,Uv\rangle_{\widehat{\Gamma}} = \langle u\,|\,v\rangle_{\Gamma}, \tag{5.4}$$

so U is orthogonal,

$$U^{\mathrm{t}} = U^{-1}. \tag{5.5}$$

As usual, this imples that the operations of conjugation with U and transposition commute: If P is as above and we define the pull-back

$$\widehat{P} = U\circ P\circ U^{-1} = U\circ P\circ U^{\mathrm{t}},$$

then

$$\widehat{P}^{\mathrm{t}} = UP^{\mathrm{t}}U^{\mathrm{t}}. \tag{5.6}$$

Let now $\widehat{\Gamma}\subset\mathbb{R}^n$. We can use U to define an L^2-inner product on $C_0^\infty(\Gamma)$ by putting

$$(u\,|\,v) = (u\,|\,v)_{\Gamma} = (Uu\,|\,Uv)_{L^2(\widehat{\Gamma})}, \tag{5.7}$$

which is the inner product that makes U formally unitary. More explicitly,

$$(u\,|\,v) = \int_{\widehat{\Gamma}} u\big(\gamma(y)\big)\overline{v\big(\gamma(y)\big)}\,\Big|\det\frac{\partial\gamma}{\partial y}\Big|\mathrm{d}y = \int_{\Gamma} u(x)\overline{v(x)}\theta(x)\mathrm{d}x, \tag{5.8}$$

where

$$\theta(x) = \frac{|\det\partial\gamma/\partial y|}{\det\partial\gamma/\partial y},\quad x=\gamma(y),$$

is the unique unimodular factor for which $\theta(x)\mathrm{d}x$ is a positive density on Γ (and in particular independent of the parametrization γ).

We have

$$(u\,|\,v) = \langle u\,|\,Cv\rangle_{\Gamma},\quad u,v\in C_0^\infty(\Gamma), \tag{5.9}$$

where C is the antilinear involution defined by $Cv = \theta\overline{v}$. The formal adjoint of P for our scalar product on Γ is given by

$$P^* = C^{-1}P^{\mathrm{t}}C = CP^{\mathrm{t}}C. \tag{5.10}$$

5.2. Dilations and convex sets

Let

$$P = -h^2\Delta + V(x), \quad V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n;\mathbb{R}). \tag{5.11}$$

Let first $f : \mathbb{R}^n \to \mathbb{R}$ be smooth, equal to 0 near $\operatorname{supp} V$ and equal to $(\tan\theta)\frac{1}{2}d_0(x)^2$ for large x, where $d_0(x) = |x|$ and $0 < \theta < \frac{1}{2}\pi$. Then we consider the m.t.r. manifold $\Gamma = \Gamma_f$ of $\mathbb{C}^n$, given by

$$x = y + if'(y), \quad y \in \mathbb{R}^n. \tag{5.12}$$

(See [**24**] for a quick review in the semi-classical case.) The bijectivity of the complex Jacobian map $\partial x/\partial y = 1 + if''(y)$ implies indeed that Γ_f is maximally totally real. P_Γ can be computed in the parametrization (5.12) using the formal chain rule:

$$\frac{\partial}{\partial y} = \big(1 + if''(y)\big)\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial x} = \big(1 + if''(y)\big)^{-1}\frac{\partial}{\partial y},$$

and hence away from the support of V we get

$$P_\Gamma = -h^2 \det\big(1+if''(y)\big)^{-1}\Big(\frac{\partial}{\partial y}\Big)^{\mathrm{t}} \det\big(1+if''(y)\big)\big(1+if''(y)\big)^{-2}\Big(\frac{\partial}{\partial y}\Big) \tag{5.13}$$

which has the semi-classical principal symbol

$$\big((1 + if''(y)\big)^{-1}\eta)^2 = \big\langle\big(1 + if''(y)\big)^{-2}\eta, \eta\big\rangle. \tag{5.14}$$

Here $\langle . , . \rangle$ denotes the bilinear scalar product on $\mathbb{R}^n$ and also its bilinear extension to $\mathbb{C}^n$. Since η is real in (5.14), we can write this symbol as

$$\big((1 + if''(y)\big)^{-2}\eta \,|\, \eta\big),$$

where $(.|.)$ is the usual sesquilinear scalar product on $\mathbb{C}^n$.

For large y, we have $f''(y) = (\tan\theta)1$ and here it is convenient to use the equivalent parametrization $x = \mathrm{e}^{i\theta}\widetilde{y}$, where $\widetilde{y}, y \in \mathbb{R}^n$ are related by $y = (\cos\theta)\widetilde{y}$, and get

$$P_\Gamma = \mathrm{e}^{-2i\theta}(-h^2\Delta_{\widetilde{y}}). \tag{5.15}$$

In general we assume

$$f''(y) \geq 0, \tag{5.16}$$

and we shall study the inverse of $(1 + if''(y))^2 = 1 - f''(y)^2 + 2if''(y)$. If C is a complex $n \times n$ matrix, define as usual

$$\Re C = \frac{1}{2}(C + C^*), \quad \Im C = \frac{1}{2i}(C - C^*).$$

PROPOSITION 5.1. — *If* $C = (1 + if''(y))^2$ *for some fixed* $y \in \mathbb{R}^n$*, then under the assumption* (5.16)*, we have :*

1) $\Im C^{-1} \leq 0$.
2) *We have* $\Im C^{-1} < 0$ (*i.e.* C^{-1} *is negative definite*) *iff* $f''(y) > 0$.
3) *The symbol* $(C^{-1}\eta\,|\,\eta)$*,* $\eta \in \mathbb{R}^n$ *is elliptic:* $|(C^{-1}\eta|\eta)| \asymp |\eta|^2$ *and takes its values in a sector* $-\pi + \epsilon \leq \arg(C\eta\,|\,\eta) \leq 0$ *for some* $\epsilon > 0$.
4) *When* $f''(y) > 0$ *it take its values in a sector* $-\pi + \epsilon \leq \arg(C\eta\,|\,\eta) \leq -\epsilon$.

Proof. — We already know that $C : \mathbb{C}^n \to \mathbb{C}^n$ is bijective and a direct calculation shows that

$$\Im C^{-1} = -C^{*-1}(\Im C)C^{-1} = -2C^{*-1}f''(y)C^{-1}, \tag{5.17}$$

$$\Re C^{-1} = C^{*-1}(\Re C)C^{-1} = C^{*-1}(1 - f''(y)^2)C^{-1}. \tag{5.18}$$

Assertions 1) and 2) follow from (5.17). Now look at

$$(C^{-1}\eta\,|\,\eta) = \big((\Re C)C^{-1}\eta\,|\,C^{-1}\eta\big) - i\big((\Im C)C^{-1}\eta\,|\,C^{-1}\eta\big). \tag{5.19}$$

If the imaginary part of this expression (*i.e.* the last term) is zero, then since $\Im C \geq 0$, we conclude that $(\Im C)(C^{-1}\eta) = 0$, *i.e.* $f''(y)C^{-1}\eta = 0$. For such an η the real part of (5.19) becomes

$$\big((\Re C)C^{-1}\eta\,|\,C^{-1}\eta\big) = \big((1 - f''(y)^2)C^{-1}\eta\,|\,C^{-1}\eta\big) = \|C^{-1}\eta\|^2.$$

Assertions 3) and 4) follow. □

The proposition shows that P_Γ is elliptic in the classical sense. Defining the Sobolev spaces $H^s(\Gamma)$ in the usual way and equipping P_Γ with the domain $H^2(\Gamma)$, we see that the essential spectrum of P_Γ is the half-line $\mathrm{e}^{-2i\theta}[0, +\infty[$. As explained for instance in [**28**], [**29**], [**30**], [**31**], [**33**], P_Γ has no spectrum in the open upper half-plane and the eigenvalues in the sector $\mathrm{e}^{-i[0,\theta[}]0, +\infty[$ are precisely the resonances of P there. (For a more complete discussion and further references, see [**28**], [**29**], [**30**], [**31**], [**33**].)

Let $\mathcal{O} \Subset \mathbb{R}^n$ be open with smooth boundary and strictly convex. Then $d(x) := \operatorname{dist}(x, \mathcal{O})$ is smooth on $\mathbb{R}^n \setminus \mathcal{O}$ and we have

$$\partial^\alpha(d - d_0) = \mathcal{O}\big(\langle x\rangle^{-|\alpha|}\big). \tag{5.20}$$

Now assume that

$$\operatorname{supp} V \subset \overline{\mathcal{O}}. \tag{5.21}$$

Outside $\mathcal{O}$ we look for f of the form

$$f(x) = g\big(d(x)\big), \tag{5.22}$$

where $g \in C^\infty(\mathbb{R};\mathbb{R})$ vanishes on the negative half-axis. Then

$$f'(x) = g'\big(d(x)\big)d'(x), \quad f''(x) = g'\big(d(x)\big)d''(x) + g''\big(d(x)\big)d'(x)\otimes d'(x). \tag{5.23}$$

Here $d'(x)$ can be identified with the exterior normal $\nu(\pi(x))$ at the projection $\pi(x) \in \partial\mathcal{O}$ of x. When $x \notin \partial\mathcal{O}$ we also have $d'(x) = (x-\pi(x))/|x-\pi(x)|$. It is further wellknown that $d''(x)$ is positive semi-definite with null-space $\mathbb{R}d'(x)$. Thus we see from (5.23) that $f''(x) \geq 0$ when g', $g'' \geq 0$ and we have $f''(x) > 0$ when g', $g'' > 0$.

Introduce geodesic coordinates: Let $x' : \Omega \to \partial\mathcal{O}$ be a local parametrization of the boundary, where Ω is some open set in $\mathbb{R}^{n-1}$. Then we have local (geodesic) coordinates $(z', z_n) \in \Omega \times]-\epsilon, +\infty[$ on $\mathbb{R}^n$, given by

$$x = x(z') + z_n \nu\big(x(z')\big). \tag{5.24}$$

In these coordinates, if f is as in (5.22), then $\Gamma = \Gamma_f$ is obtained by letting z_n become complex:

$$z' = y',\ z_n = \gamma(y_n), \quad \gamma(y_n) := y_n + ig'(y_n). \tag{5.25}$$

We have (see [**30**], Section 2, also [**31**], Section 3 and [**29**]):

$$P = D_{z_n}^2 + R(z, D_{z'}) + a(z)\partial_{z_n}, \tag{5.26}$$

where

$$R(z, D_{z'}) = R(z', 0, D_{z'}) - z_n Q(z, D_{z'}), \tag{5.27}$$

and R, Q are elliptic second order differential operators with positive principal symbols:

$$r(z, \zeta'),\ q(z, \zeta')\ > 0. \tag{5.28}$$

The coefficients are analytic in z_n and smooth in z. In the parametrization (5.25) for Γ, we get

$$\begin{aligned} P_\Gamma = {} & \Big(\frac{1}{\gamma'(y_n)} D_{y_n}\Big)^2 + R(y', 0; D_{y'}) \\ & -\gamma(y_n) Q\big(y', \gamma(y_n); D_{y'}\big) + a\big(y', \gamma(y')\big)\frac{1}{\gamma'(y_n)}\partial_{y_n}. \end{aligned} \tag{5.29}$$

This formula remains valid if we make a real change of variables in y_n in order to normalize $\gamma'(y_n)$.

If we choose g so that $g(d) = (\tan\theta)d^2$ for large $d \geq r_0 > 0$, then as we have seen, $f'' > 0$ in the corresponding region. Let $\chi \in C_0^\infty(\mathbb{R}^n; [0,1])$ be equal to one in a neighborhood of 0 and put

$$\widetilde{d} = \widetilde{d}_R = \chi\Big(\frac{x}{R}\Big)d(x) + \Big(1 - \chi\Big(\frac{x}{R}\Big)\Big)d_0(x).$$

Then we still have (5.20) if we replace d or d_0 with $\widetilde{d}$ and from this it follows that $\widetilde{f} := (\tan\theta)\widetilde{d}^2$ satisfies $\widetilde{f}''(x) > 0$ for $d(x) \geq r_0$, provided that $R \gg 0$. Summing up we have

Proposition 5.2. — *Let $f(x) = g(d)$ with g as above and assume that $g'(d) > 0$, $g''(d) > 0$ for $d > r_0/2$ where $r_0 > 0$. Then we can find $f = f(x)$ smooth and real-valued such that*

- ▷ *$f(x) = g(d)$ for $d \leq \frac{1}{2}r_0 > 0$,*
- ▷ *$f(x) = \frac{1}{2}(\tan\theta)d_0(x)^2$ near infinity,*
- ▷ *$f''(x) > 0$ for $d(x) \geq \frac{1}{2}r_0$.*

To study the resonances for the exterior Dirichlet problem in $\mathbb{R}^n \setminus \mathcal{O}$ one may use complex scaling with a contour

$$\Gamma_{\mathrm{ext},f}:\ x = y + if'(y), \quad y \in \mathbb{R}^n \setminus \mathcal{O}, \tag{5.30}$$

where $f \in C^\infty(\mathbb{R}^n \setminus \mathcal{O})$ vanishes on $\partial\mathcal{O}$, $f'' > 0$ away from $\partial\mathcal{O}$ and $f(x) = \frac{1}{2}(\tan\theta)d_0(x)^2$ near infinity. One then considers the restriction P_{ext} of $-h^2\Delta$ to this contour with domain $H^2 \cap H_0^1(\Gamma_{\mathrm{ext}})$ and the exterior Dirichlet resonances in the sector $\mathrm{e}^{-i[0,2\theta[}$ coincide with the eigenvalues of this operator. (See [**29**], [**30**], [**31**] and references cited there.) A convenient choice of f near $\partial\mathcal{O}$ is $f(x) = \frac{1}{2}(\tan\theta)d(x)^2$ and according to [**15**] we know that $\theta = \frac{1}{3}\pi$ is in some sense the optimal choice.

In our case it will be convenient to use a Lipschitz contour:

$$f(x) = \begin{cases} 0 \text{ in } \mathcal{O}, \\ \frac{1}{2}(\tan\theta)d(x)^2 \quad \text{near } \partial\mathcal{O} \text{ in } \mathbb{R}^n \setminus \mathcal{O}, \end{cases} \tag{5.31}$$

and as above further away from $\overline{\mathcal{O}}$. Then f is of class $C^{1,1}$ and smooth away from $\partial\mathcal{O}$. Consequently, $\Gamma = \Gamma_f$ is a Lipschitz manifold, smooth away from $\partial\mathcal{O}$ and is naturally decomposed into the interior part $\mathcal{O}$ and the exterior part; $\Gamma_{f,\mathrm{ext}}$. Again, we can define P_Γ as $P_{|\Gamma}$ with the appropriate continuity conditions at $\partial\mathcal{O}$:

$$\mathcal{D}(P_\Gamma) = \big\{u = u_{\mathcal{O}} + u_{\mathrm{ext}};\ u_{\mathcal{O}} \in H^2(\mathcal{O}),\ u_{\mathrm{ext}} \in H^2(\Gamma_{f,\mathrm{ext}}), \\ u_{\mathcal{O}} = u_{\mathrm{ext}},\ \partial_\nu u_{\mathcal{O}} = \partial_\nu u_{\mathrm{ext}} \text{ on } \partial\mathcal{O}\big\}, \tag{5.32}$$

where ν is the exterior unit normal to $\mathcal{O}$. (On the exterior part we identify ∂_ν with $(\partial_\nu)_{\Gamma_{\mathrm{ext}}}$.) It follows from Stokes' formula that P_Γ is symmetric.

Near a point $x_0 \in \partial\mathcal{O}$, the problem

$$\begin{cases} (P-z)u_{\mathcal{O}} = v_{\mathcal{O}}, \\ (P-z)u_{\mathrm{ext}} = v_{ext}, \\ \gamma u_{\mathcal{O}} - \gamma u_{\mathrm{ext}} = v_0, \\ \gamma\partial_\nu u_{\mathcal{O}} - \gamma\partial_\nu u_{\mathrm{ext}} = v_1 \end{cases} \tag{5.33}$$

can be viewed as an elliptic boundary value problem for an operator with matrix valued symbol (after a reflexion so that, near x_0, we consider $u_{\mathcal{O}}$ and u_{ext} to live on the same side of the boundary). Here we take $v_\bullet$ to be in L^2 in a neighborhood of x_0 and make the same starting assumption about $u_{\mathcal{O}}$ and u_{ext}. Then if $v_0 \in H^{\frac{3}{2}}$, $v_1 \in H^{\frac{1}{2}}$, the standard theory tells us that the traces are well-defined and that $u_{\mathcal{O}}$ and u_{ext} actually belong to the spaces $H^2(\mathcal{O})$, $H^2(\mathbb{R}^n \setminus \mathcal{O})$ respectively. Away from the boundary, the usual arguments of complex scaling apply, and we see that $P-z : \mathcal{D}(P) \to L^2$ is a holomorphic family of Fredholm operators of index 0, when $z \in \mathbb{C} \setminus \mathrm{e}^{-2i\theta}[0,+\infty[$.

Proposition 5.3. — *Let Γ be the singular contour above. The spectrum of $P = P_\Gamma$ in the sector $\mathrm{e}^{-i[0,2\theta[}]0,+\infty[$ coincides with the set of resonances for P there.*

We have already recalled that the proposition holds when Γ is a smooth contour, of the same form near infinity. We also recall from [**28**, Section 3] (see also [**24**] for a semi-classical version as well as [**29**], [**30**], [**31**], [**33**]), that one can show directly, using a result on holomorphic extension of null solutions to non-characteristic equations, that P_{Γ_1} and P_{Γ_2} have the same spectrum if Γ_1 and Γ_2 are two smooth contours as above, which coincide near infinity.

The new part of the proof in the case of singular contours will be to show how to extend null-solutions holomorphically near the singular part of Γ, i.e. near $\partial\mathcal{O}$ and in order to do so we need to study holomorphic extensions of the resolvent kernel. Since we are not interested here in how the estimates depend on h, we will take $h = 1$ for simplicity. The arguments below are related with the more abstract method of exterior complex scaling of B. Simon [**22**].

We first consider the free resolvent on $\mathbb{R}^n$ for $\Im z > 0$,

$$R_0(z) = (-\Delta - z)^{-1}.$$

The distribution kernel is of the form $R_0(z)(x,y) = R_0(z)(x-y)$, where

$$R_0(z)(x) = \frac{1}{(2\pi)^n}\int \mathrm{e}^{ix\cdot\xi}\frac{1}{\xi^2 - z}\mathrm{d}\xi. \tag{5.34}$$

As already mentioned, $R_0(z)$ extends holomorphically as an operator $C_0^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ across $]0,+\infty[$ to the double and universal coverings of $\mathbb{C}\setminus\{0\}$, when n is odd and even respectively. Moreover, for x in any compact subset of $\mathbb{R}^n$ and for z in any compact subset of the covering space, there exists a constant $C>0$ such that

$$\big|R_0(z)(x)\big| \leq \begin{cases} C & n=1, \\ C\big(1+\big|\ln|x|\big|\big) & n=2, \\ C|x|^{2-n} & n\geq 3, \end{cases} \tag{5.35}$$

$$\big|\nabla_x R_0(z)(x)\big| \leq \begin{cases} C & n=1, \\ C|x|^{1-n} & n\geq 2. \end{cases} \tag{5.36}$$

More precise results are known of course, see for instance [**35**], but we have a quick proof of (5.35), (5.36) by noticing that we can make an x-dependent complex deformation in the integral (5.34) for large x and obtain

$$R_0(z)(x) = \mathcal{O}(1) + \int_{|\xi|\geq 1} \mathcal{O}(1)\,\mathrm{e}^{-|x|\cdot|\xi|/C}|\xi|^{-2}\mathrm{d}\xi,$$
$$\nabla R_0(z)(x) = \mathcal{O}(1) + \int_{|\xi|\geq 1} \mathcal{O}(1)\,\mathrm{e}^{-|x|\cdot|\xi|/C}|\xi|^{-1}\mathrm{d}\xi,$$

and treating the gradient estimate for $n=1$ separately.

Finally, $R_0(z)$ is rotation invariant; $R_0(z)(Ux) = R_0(z)(x)$ if $U:\mathbb{R}^n\to\mathbb{R}^n$ is orthogonal. See Section 2 of [**23**] as well as further references given there. As explained in that reference, (5.34) remains valid also for z in the covering space, we just have to make a complex deformation of the integration contour in a region where $|\xi|$ is bounded, in order to avoid the zeros ξ^2-z and this has no importance for the local properties of $x\mapsto R_0(z)(x)$ while it does influence the exponential decay or increase near infinity.

We now want to extend (5.34) holomorphically with respect to x. The very first observation is that if $x_0\in\mathbb{R}^n\setminus\{0\}$ then $R_0(z)(x)$ extends holomorphically in x to small neighborhood of x_0, by making the small complex deformation of the integration contour in (5.34) already alluded to.

More generally, assume that $x\in\mathbb{C}^n$ and that $x\cdot x\neq 0$. Write $x=(x\cdot x)^{\frac{1}{2}}f_1$ for some branch of the square root. Then $f_1\cdot f_1 = 1$ and we can find vectors $f_2,\dots,f_n\in\mathbb{C}^n$ such that $f_1,\dots,f_n$ is an orthonormal basis for the bilinear symmetric product $x\cdot y$: $f_j\cdot f_k=\delta_{j,k}$. Let $e_1,\dots,e_n$ be the canonical basis in $\mathbb{R}^n$ and define the complex orthogonal map $U:\mathbb{C}^n\to\mathbb{C}^n$ by

$$Ue_j = f_j. \tag{5.37}$$

Let $\omega = ((x\cdot x)/|x\cdot x|)^{\frac{1}{2}}$ with the same branch of the square root as above. Then $x = \omega U y$, where $y = |x\cdot x|^{\frac{1}{2}} e_1 \in \mathbb{R}^n$ and $y\cdot y = |x\cdot x|$. At least formally, we have

$$R_0(z)(x) =: I(x,z) = \int \mathrm{e}^{ix\cdot\xi} \frac{1}{\xi^2 - z} \frac{\mathrm{d}\xi}{(2\pi)^n} = \int \mathrm{e}^{i\omega U y\cdot\xi} \frac{1}{\xi^2 - z} \frac{\mathrm{d}\xi}{(2\pi)^n}.$$

Choose the integration contour $\xi = \omega^{-1} U\eta$, $\eta \in \mathbb{R}^n$. Then $\mathrm{d}\xi = \omega^{-n}\mathrm{d}\eta$, $\xi^2 = \omega^{-2}\eta^2$ and we get

$$I(x,z) = \int \mathrm{e}^{iy\cdot\eta} \frac{1}{\omega^{-2}\eta^2 - z} \frac{\mathrm{d}\eta}{\omega^n (2\pi)^n} = \frac{1}{\omega^{n-2}} \int \mathrm{e}^{iy\cdot\eta} \frac{1}{\eta^2 - \omega^2 z} \frac{\mathrm{d}\eta}{(2\pi)^n},$$

so at least formally, we have

$$I(x,z) = \omega^{2-n} I(y, \omega^2 z),\ \ \omega = \Big(\frac{x\cdot x}{|x\cdot x|}\Big)^{\frac{1}{2}},\ \ y\in\mathbb{R}^n,\ \ x\cdot x = \omega^2 y\cdot y. \tag{5.38}$$

We can use this formula together with the initial remark about holomorphic extentions to small neighborhoods of real points to define the desired holomorphic extension of $I(x,z)$ from $\mathbb{R}^n_x \setminus \{0\}$. Naturally this will give rise to a ramified (multivalued) function and in order to get some more understanding, let $[0,1] \ni t \mapsto x_t \in \mathbb{C}^n$ be a continuous map starting at a real point $x_0 \in \mathbb{R}^n \setminus \{0\}$ and ending at some given point $x \in \mathbb{C}^n$ with $x\cdot x \neq 0$ such that $x_t\cdot x_t \neq 0$ for all t. Then we can choose $U = U_t$ depending continuously on t with $U_0 = 1$. If we have choosen a branch of $I(y,z)$ for real y, then we get the branch

$$I(x,z) = \omega_1^{2-n} I(y, \omega_1^2 z),$$

obtained by following the curve $[0,1] \ni t \mapsto \omega_t^2 z$ from z to $\omega_1^2 z$. We conclude that $I(x,z)$ is a well-defined multivalued holomorphic function of $x \in \{w\in\mathbb{C}^n;\, w\cdot w \neq 0\}$ and z in the double/universal covering space of $\mathbb{C}\setminus\{0\}$. Moreover for (x,z) in any fixed compact subset of the above domain of definition, we still have (5.35), (5.36).

Now we observe that the singular contour Γ in Proposition 5.3 is of the form $\Gamma = \Gamma_f$: $x = y + if'(y)$, where f is real-valued of class $C^{1,1}(\mathbb{R}^n)$ which is convex and $f(y) = \frac{1}{2}(\tan\theta) d_0(y)^2$ near infinity. If $x_j = y_j + if(y_j)$, $j = 0,1$, are two different points on Γ_f, then

$$f'(y_1) - f'(y_0) = A(y_0,y_1)(y_1 - y_0),$$

where

$$A(y_0,y_1) = \int_0^1 f''(ty_1 + (1-t)y_0)\mathrm{d}t \geq 0,$$

and

$$(x_1 - x_0) \cdot (x_1 - x_0) = \big[(1 - A(y_1, y_0)^2) + 2iA(y_0, y_1)\big](y_1 - y_0) \cdot (y_1 - y_0).$$

The same argument as for the ellipticity of $-\Delta_{\Gamma_f}$ shows that

$$\Gamma_f \times \Gamma_f \ni (x_0, x_1) \longmapsto (x_1 - x_0) \cdot (x_1 - x_0)$$

takes its values in a sector $\mathrm{e}^{i[0,\pi-\epsilon]}[0, +\infty[$ and that

$$\big|(x_1 - x_0) \cdot (x_1 - x_0)\big| \asymp |x_1 - x_0|^2, \quad x_0, x_1 \in \Gamma_f.$$

Combining these facts with the deformation $[0, 1] \ni t \mapsto \Gamma_{tf}$ from $\mathbb{R}^n$ to Γ_f, we see that $R_0(z)(x, y) = R_0(z)(x - y)$ is well-defined on $\Gamma_f \times \Gamma_f$ away from the diagonal, and we can define

$$R_{0,\Gamma} u(x) = \int_\Gamma R_0(z)(x, y)u(y)\mathrm{d}y, \quad x \in \Gamma_f,\ u \in C_0(\Gamma),\ \Gamma = \Gamma_f.$$

This gives a continuous operator $C_0(\Gamma) \to C(\Gamma)$. Let $P_0 = -\Delta$. Using that

$$(-\Delta_x - z)R_0(z)(x, y) = (-\Delta_y^{\mathrm{t}} - z)R_0(z)(x, y) = 0, \quad x \neq y,$$

as well as the bound on the strength of the singularity at $x = y$ described in (5.35), (5.36), we see that in the case when f is smooth, we have

$$\begin{aligned}(P_{0,\Gamma} - z)R_{0,\Gamma}(z)v(x) &= C(x, f)v(x),\\ R_{0,\Gamma}(z)(P_{0,\Gamma} - z)u(x) &= \widetilde{C}(x, f)u(x),\end{aligned}$$

for $x \in \Gamma$, $u, v \in C_0^\infty(\Gamma)$. It is further clear that $C(x, f)$, $\widetilde{C}(x, f)$ only depend on the restriction of f to a small neighborhood of $\Re x$, so we can replace f be a new function $\widetilde{f}$ which is equal to f near $\Re x$ with $\widetilde{f}''$ varying very little and being constant near infinity. We can then determine the constants by letting v, u be suitable Gaussians and possibly after an additional deformation argument, we get $C(x, f) = \widetilde{C}(x, f) = 1$. Thus

$$(P_{0,\Gamma} - z)R_{0,\Gamma}(z)v = v, \tag{5.39}$$

$$R_{0,\Gamma}(z)(P_{0,\Gamma} - z)u = u, \tag{5.40}$$

when $u, v \in C_0^\infty(\Gamma)$, $\Gamma = \Gamma_f$ and f is smooth. To extend this to the general case when f is a convex $C^{1,1}$ function would require first to define the operator $P_{0,\Gamma}$, and we prefer to avoid that work and just consider the case of the special singular contour in Proposition 5.3. Then for $v \in C_0(\Gamma)$ (5.39) still holds away from $\partial\mathcal{O}$.

We also remark that if $v \in C_0(\Gamma)$, then $u := R_{0,\Gamma}v$ is of class C^1 up to the boundary both on $\mathcal{O}$ and on Γ_{ext} and we have

$$\gamma u_\Omega = \gamma u_{\mathrm{ext}}, \quad \gamma\partial_\nu u_\Omega = \gamma\partial_\nu u_{\mathrm{ext}}. \tag{5.41}$$

Using now that (5.33) is an elliptic boundary value problem, we see that $R_{0,\Gamma}v$ belongs locally to $\mathcal{D}(P_\Gamma)$ and this holds more generally for $v \in L^2_{\mathrm{comp}}(\Gamma)$.

If $u \in C_0(\Gamma)$ and $u_{\mathcal{O}}$ and u_{ext} are C^2 up to the boundary and satisfy (5.41), then we can make integrations by parts in

$$R_{0,\Gamma}(P_{0,\Gamma} - z)u(x) = \int R_0(z)(x,y)(-\Delta_\Gamma - z)u(y)\mathrm{d}y$$

after introducing a cutoff around the singularity and passing to the limit and get (5.40) as in the case when f is smooth. By density this extends to the case when $u \in \mathcal{D}(P_\Gamma)$ has compact support.

We can now complete the proof of Proposition 5.3. Let $\Gamma = \Gamma_f$ be the singular contour in that proposition and let $\widetilde{f}$ be smooth, convex, equal to 0 in $\mathcal{O}$ and equal to f outside a small neighborhood of $\overline{\mathcal{O}}$. Let $\widetilde{\Gamma} = \Gamma_{\widetilde{f}}$ be the corresponding smooth contour, so that the spectrum of $\widetilde{P} = P_{\widetilde{\Gamma}}$ in the sector $\mathrm{e}^{-i[0,2\theta[}\,]0,+\infty[$ coincides with the set of resonances there. As in [**28**], it suffices to show the following two facts:

1) If $u \in \mathcal{D}(P_\Gamma)$ and $(P_\Gamma - z)u = 0$, then u has a holomorphic extension to a domain containing

$$\{y + i\big(t\widetilde{f}'(y) + (1-t)f'(y)\big);\ f(y) \neq \widetilde{f}(y),\ 0 \leq t \leq 1\}, \tag{5.42}$$

such that its restriction $\widetilde{u}$ to $\widetilde{\Gamma}$ belongs to $\mathcal{D}(P_{\widetilde{\Gamma}})$ and satisfies $(P_{\widetilde{\Gamma}} - z)\widetilde{u} = 0$.

2) If $\widetilde{u} \in \mathcal{D}(P_{\widetilde{\Gamma}})$ and $(P_{\widetilde{\Gamma}} - z)\widetilde{u} = 0$, then $\widetilde{u}$ has a holomorphic extension to a domain containing the set (5.42) such that its restriction u to Γ belongs to $\mathcal{D}(P_\Gamma)$ and satisfies $(P_\Gamma - z)u = 0$.

Let $\widehat{\chi} \in C_0^\infty(\mathbb{R}^n)$ be equal to one near $\mathrm{supp}\,(f - \widetilde{f}\,)$ and define the cutoffs χ and $\widetilde{\chi}$ on Γ and on $\widetilde{\Gamma}$ respectively by

$$\chi\big(y + if'(y)\big) = \widetilde{\chi}\big(y + i\widetilde{f}'(y)\big) = \widehat{\chi}(y).$$

We first prove 1) and let u be as in that statement. Then

$$(P_\Gamma - z)\chi u = [P_\Gamma, \chi]u, \tag{5.43}$$

where the right hand side has its support in the region where Γ and $\widetilde{\Gamma}$ coincide. We can rewrite (5.43) as

$$(P_{0,\Gamma} - z)\chi u = [P_\Gamma, \chi]u - Vu \tag{5.44}$$

and Vu also has its support where Γ and $\widetilde{\Gamma}$ coincide. Applying (5.40) gives

$$\chi u = R_{0,\Gamma}(z)\big([P_\Gamma, \chi]u - Vu\big). \tag{5.45}$$

From the properties of $R_0(z)$, we see that χu has a holomorphic extension to a domain containing the set (5.42). Its restriction to $\widetilde{\Gamma}$ solves $(P_{\widetilde{\Gamma}} - z)\widetilde{u} = 0$ and $\widetilde{u} = u$ in the regions where Γ and $\widetilde{\Gamma}$ coincide. From elliptic regularity we see that $\widetilde{u}$ is locally in H^2 and hence globally so $\widetilde{u}$ belongs to the domain of $P_{\widetilde{\Gamma}}$. This proves 1).

The proof of 2) works the same way with the small difference that instead of invoking the ellipticity of $P_{\widetilde{\Gamma}}$ on the smooth manifold $\widetilde{\Gamma}$, we invoke the ellipticity of the boundary value problem (5.33). □

CHAPTER 6

SEMI-CLASSICAL SOBOLEV SPACES

This section is a review of some easy facts about Sobolev spaces, see Section 2 in [**25**], [**27**] for more details about the first part. We let

$$H_h^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n), \quad s \in \mathbb{R},$$

denote the semi-classical Sobolev space of order s equipped with the norm $\|\langle hD\rangle^s u\|$ where the norms are the ones in L^2, ℓ^2 or the corresponding operator norms if nothing else is indicated. Here

$$\langle hD\rangle = \big(1 + (hD)^2\big)^{\frac{1}{2}}.$$

PROPOSITION 6.1. — *Let $s > \frac{1}{2}n$. Then there exists a constant $C = C(s)$ such that for all $u, v \in H_h^s(\mathbb{R}^n)$, we have $u \in L^\infty(\mathbb{R}^n)$, $uv \in H_h^s(\mathbb{R}^n)$ and*

$$\|u\|_{L^\infty} \le Ch^{-\frac{1}{2}n}\|u\|_{H_h^s}, \tag{6.1}$$

$$\|uv\|_{H_h^s} \le Ch^{-\frac{1}{2}n}\|u\|_{H_h^s} \cdot \|v\|_{H_h^s}. \tag{6.2}$$

Let X be a compact smooth manifold. We cover X by finitely many coordinate neighborhoods $X_1, \dots, X_p$ and for each X_j, we let $x_1, \dots, x_n$ denote the corresponding local coordinates on X_j. Let $0 \le \chi_j \in C_0^\infty(X_j)$ have the property that $\sum_1^p \chi_j > 0$ on X. Define $H_h^s(X)$ to be the space of all $u \in \mathcal{D}'(X)$ such that

$$\|u\|_{H_h^s}^2 := \sum_1^p \big\|\chi_j \langle hD\rangle^s \chi_j u\big\|^2 < \infty. \tag{6.3}$$

It is standard to show that this definition does not depend on the choice of the coordinate neighborhoods or on χ_j. With different choices of these quantities we get norms in (6.3) which are uniformly equivalent when $h \to 0$. In fact, this follows from the h-pseudodifferential calculus on manifolds with symbols in the Hörmander space $S^m_{1,0}$ that we quickly reviewed in the appendix in [**25**]. An equivalent definition of $H^s_h(X)$ is the following: Let

$$h^2\widetilde{R} = \sum (hD_{x_j})^* r_{j,k}(x) hD_{x_k} \tag{6.4}$$

be a self-adjoint non-negative elliptic operator with smooth coefficients on X, where the star indicates that we take the adjoint with respect to some fixed positive smooth density on X. Then $h^2\widetilde{R}$ is essentially self-adjoint with domain $H^2(X)$, so

$$(1+h^2\widetilde{R})^{\frac{1}{2}s} : L^2 \longrightarrow L^2$$

is a closed densely defined operator for $s \in \mathbb{R}$, which is bounded precisely when $s \leq 0$. Standard methods allow to show that $(1+h^2\widetilde{R})^{\frac{1}{2}s}$ is an h-pseudodifferential operator with symbol in $S^s_{1,0}$ and semi-classical principal symbol given by $(1+r(x,\xi))^{\frac{1}{2}s}$, where

$$r(x,\xi) = \sum_{j,k} r_{j,k}(x)\xi_j\xi_k$$

is the semi-classical principal symbol of $h^2\widetilde{R}$. See the appendix in [**25**]. The h-pseudodifferential calculus gives for every $s \in \mathbb{R}$:

Proposition 6.2. — *The space $H^s_h(X)$ is the space of all $u \in \mathcal{D}'(X)$ such that $(1+h^2\widetilde{R})^{\frac{1}{2}s}u \in L^2$ and the norm $\|u\|_{H^s_h}$ is equivalent to $\|(1+h^2\widetilde{R})^{\frac{1}{2}s}u\|$, uniformly when $h \to 0$.*

Remark 6.3. — From the first definition we see that Proposition 6.1 remains valid if we replace $\mathbb{R}^n$ by a compact n-dimensional manifold X.

Remark 6.4. — We will also consider the case when the manifold X is the disjoint union of a compact part and $\mathbb{R}^n \setminus B(0,R)$ for some $R > 0$. The definition and properties of $H^s_h(X)$ are quite clear.

Of course, $H^s_h(X)$ coincides with the standard Sobolev space $H^s_1(X)$ and the norms are equivalent for each fixed value of h, but not uniformly so with respect to h. We have the following variant (see [**27**], Section 2):

Proposition 6.5. — *Let $s > \frac{1}{2}n$. Then there exists a constant $C = C_s > 0$ such that*

$$\|uv\|_{H^s_h} \leq C\|u\|_{H^s_1} \cdot \|v\|_{H^s_h}, \quad \forall u \in H^s(\mathbb{R}^n),\ v \in H^s_h(\mathbb{R}^n). \tag{6.5}$$

The result remains valid if we replace $\mathbb{R}^n$ by X.

Let $\Omega \Subset \mathbb{R}^n$ be open with smooth boundary. Let $H_h^s(\Omega)$ denote the Banach space of restrictions to Ω of elements in $H_h^s(\mathbb{R}^n)$. It is a standard fact that if $s > \frac{1}{2}$, then the restriction operator $\gamma : u \mapsto u_{|\partial\Omega}$ is bounded:

$$H_1^s(\Omega) \longrightarrow H_1^{s-\frac{1}{2}}(\partial\Omega).$$

The restriction operator γ has a right inverse γ^{-1} which is bounded $H_1^{\widetilde{s}-\frac{1}{2}}(\partial\Omega) \to H_1^{\widetilde{s}}(\Omega)$ for all $\widetilde{s} \in \mathbb{R}$. More generally, if $s > \frac{3}{2}$, then

$$\begin{pmatrix} \gamma \\ \gamma D_\nu \end{pmatrix} : H_1^s(\Omega) \longrightarrow H_1^{s-\frac{1}{2}}(\partial\Omega) \times H_1^{s-\frac{3}{2}}(\partial\Omega)$$

has a right inverse which is $\mathcal{O}(1) : H_1^{\widetilde{s}-\frac{1}{2}} \times H_1^{\widetilde{s}-\frac{3}{2}} \to H_1^{\widetilde{s}}$ for all $\widetilde{s} \in \mathbb{R}$. Here ν is the exterior unit normal and $D_\nu = i^{-1}\partial/\partial\nu$.

In the semi-classical case, we obtain from the same (standard) proofs that

$$\gamma = \mathcal{O}_s(h^{-\frac{1}{2}}) : H_h^s(\Omega) \to H_h^{s-\frac{1}{2}}(\partial\Omega), \quad s > \tfrac{1}{2} \tag{6.6}$$

has a right inverse such that

$$\gamma^{-1} = \mathcal{O}_{\widetilde{s}}(h^{\frac{1}{2}}) : H_h^{\widetilde{s}-\frac{1}{2}}(\partial\Omega) \longrightarrow H_h^{\widetilde{s}}(\Omega), \quad \widetilde{s} \in \mathbb{R}. \tag{6.7}$$

More generally, the operator

$$\begin{pmatrix} \gamma \\ \gamma h D_\nu \end{pmatrix} : H_h^s(\Omega) \longrightarrow H_h^{s-\frac{1}{2}}(\Omega) \times H_h^{s-\frac{3}{2}}(\partial\Omega)$$

has a right inverse which is $\mathcal{O}(h^{\frac{1}{2}}) : H_h^{\widetilde{s}-\frac{1}{2}} \times H_h^{\widetilde{s}-\frac{3}{2}} \to H_h^{\widetilde{s}}$ for all $\widetilde{s} \in \mathbb{R}$.

The following observation can be turned into a proof by reduction to the standard non-semi-classical case: The change of variables $x = h\widetilde{x}$ transforms hD_x into $D_{\widetilde{x}}$ and if $u(x) = \widetilde{u}(\widetilde{x})$, then

$$\|u\|_{H_h^s(\Omega)} = h^{\frac{1}{2}n}\|\widetilde{u}\|_{H_1^s(h^{-1}\Omega)}.$$

Similarly for functions on $\partial\Omega$, we have

$$\|u\|_{H_h^s(\partial\Omega)} = h^{\frac{1}{2}(n-1)}\|\widetilde{u}\|_{H_1^s(h^{-1}\partial\Omega)}.$$

CHAPTER 7

REDUCTIONS TO $\mathcal{O}$ AND TO $\partial\mathcal{O}$

In this section, we let $P = -h^2\Delta + V$ and $\mathcal{O}$ be as in Section 5.2. We choose the contour Γ as there, either singular or smooth. When Γ is smooth, the domain of P_Γ is the space $H_h^2(\Gamma)$, and when Γ has a singularity along the boundary of $\mathcal{O}$, it is given by (5.32). (Later we shall also need to consider the case when Γ is constructed as in the preceeding section but with $\mathcal{O}$ replaced by a slightly larger set $\widetilde{\mathcal{O}}$ with the same properties, containing an h-neighborhood of $\mathcal{O}$.) By abuse of notation we sometimes write $H^2(\Gamma)$ also for $\mathcal{D}(P_\Gamma)$.

The exterior Dirichlet problem is

$$(P - z)u = v \text{ on } \Gamma_{\text{ext}} = \Gamma \setminus \mathcal{O}, \quad u_{|\partial\mathcal{O}} = w, \tag{7.1}$$

for given $v \in L^2(\Gamma \setminus \mathcal{O})$, $w \in H^{\frac{3}{2}}(\partial\mathcal{O})$ with the solution u in $H^2(\Gamma \setminus \mathcal{O})$. Here, $\gamma u = u_{|\partial\mathcal{O}}$. The corresponding closed operator P_{ext} has the domain

$$\mathcal{D}(P_{\text{ext}}) = \big\{u \in H^2(\Gamma \setminus \mathcal{O});\ \gamma u = 0\big\}.$$

The eigenvalues are the resonances for the exterior Dirichlet problem. We restrict the attention to the case when $\frac{1}{2} \leq \Re z \leq 2$, $\Im z \geq -ch^{\frac{2}{3}}$, where $c < 2(\frac{1}{2})^{\frac{2}{3}}\kappa\zeta_1$ (cf. Theorem 2.2). When $z \notin \sigma(P_{\text{ext}})$, we can express the solution of (7.1) as

$$u = G_{\text{ext}}(z)v + K_{\text{ext}}(z)w. \tag{7.2}$$

Put

$$\mathcal{N}_{\text{ext}}w = \gamma h D_\nu K_{\text{ext}} w, \tag{7.3}$$

where γ is the operator of restriction to $\partial\mathcal{O}$ and ν is the exterior unit normal.

Definition 7.1. — *$P_{\text{out}}(z)$ is the operator $-h^2\Delta + V - z$ on $\mathcal{O}$ with domain*

$$\mathcal{D}\big(P_{\text{out}}(z)\big) = \big\{u \in H^2(\mathcal{O});\ (\gamma h D_\nu - \mathcal{N}_{\text{ext}}(z)\gamma)u = 0\big\}. \tag{7.4}$$

Notice that the domain varies with z and this is why we avoid writing "$P_{\rm out} - z$". In the first part of this section we shall show that z is a resonance of P precisely when $0 \in \sigma(P_{\rm out}(z))$, but for technical reasons we will prefer to work with the full problem,

$$P_{\rm out}(z)u = v, \quad h^{\frac{1}{2}}Bu = w, \tag{7.5}$$

where

$$B = \gamma h D_\nu - \mathcal{N}_{\rm ext}\gamma : H^2(\mathcal{O}) \longrightarrow H^{\frac{1}{2}}(\partial\mathcal{O}). \tag{7.6}$$

It is easy to check that this is an elliptic boundary value problem in the classical sense. (The semi-classical structure of $\mathcal{N}_{\rm ext}$ and of (7.5) will require more work below.) The well-posedness of (7.5) is of course equivalent to the bijectivity of

$$\mathcal{P}_{\rm out}(z) = \begin{pmatrix} P - z \\ h^{\frac{1}{2}}B \end{pmatrix} : H^2(\mathcal{O}) \longrightarrow H^0(\mathcal{O}) \times H^{\frac{1}{2}}(\partial\mathcal{O}). \tag{7.7}$$

Here and below we sometimes write H^s instead of H^s_h.

In the following we impose the condition

$$|\Im z| \leq h^{\frac{2}{3}}c_0, \quad \tfrac{1}{2} \leq \Re z \leq 2 \tag{7.8}$$

with c_0 as in (3.1), so that the exterior Dirichlet problem is well-posed. (We could here drop the upper bound on $\Im z$.)

Under the condition (7.8) we shall show that $\mathcal{P}_{\rm out}(z)$ and $P_\Gamma - z$ are "equivalent", and to do so we shall see that $\mathcal{P}_{\rm out}(z)$ appears as the effective Hamiltonian (up to an invertible factor) in a well-posed Grushin problem for $P_\Gamma - z$.

Let $\iota : L^2(\mathcal{O}) \to L^2(\Gamma)$ be the natural zero extension map and let

$$\Pi : H^2(\Gamma) \longrightarrow H^2(\mathcal{O})$$

be the restriction map. Let

$$\widehat{K} = \mathcal{O}(h^{\frac{1}{2}}) : H^{\frac{1}{2}}(\partial\mathcal{O}) \longrightarrow H^2(\mathcal{O})$$

be a right inverse of B (cf. the last observation in Chapter 6). Put

$$\mathcal{P}(z) = \begin{pmatrix} P_\Gamma - z & \iota & 0 \\ \Pi & 0 & \widehat{K} \end{pmatrix} : H^2(\Gamma)\times L^2(\mathcal{O})\times H^{\frac{1}{2}}(\partial\mathcal{O}) \longrightarrow L^2(\Gamma)\times H^2(\mathcal{O}). \tag{7.9}$$

We will view $\mathcal{P}(z)$ as a 2×2 block matrix with the upper left block given by $P_\Gamma - z$. We claim that $\mathcal{P}(z)$ is bijective. This amounts to finding a unique solution $(u, u_-, u'_-) \in H^2(\Gamma) \times L^2(\mathcal{O}) \times H^{\frac{1}{2}}(\partial\mathcal{O})$ of the problem

$$(P_\Gamma - z)u + \iota u_- = v, \quad \Pi u + \widehat{K}u'_- = v_+ \tag{7.10}$$

for every given $(v, v_+) \in L^2(\Gamma) \times H^2(\mathcal{O})$. The exterior part (*i.e.* the restriction to $\Gamma_{\mathrm{ext}} = \Gamma \setminus \mathcal{O}$) of the first equation in (7.10) is (with the natural notation)

$$(P_{\Gamma_{\mathrm{ext}}} - z)u_{\mathrm{ext}} = v_{\mathrm{ext}},$$

which has the general solution

$$u_{\mathrm{ext}} = G_{\mathrm{ext}}(z)v_{\mathrm{ext}} + K_{\mathrm{ext}}(z)g,$$

where $g \in H^{\frac{3}{2}}(\Gamma)$ is arbitrary to start with. Notice that

$$Bu_{\mathrm{ext}} = BG_{\mathrm{ext}}(z)v_{\mathrm{ext}},$$

since $BK_{\mathrm{ext}}(z) = 0$ by the definition of $\mathcal{N}_{\mathrm{ext}}(z)$. Here the continuity condition on u given by (5.32), can be written

$$\gamma u_{\mathrm{int}} = \gamma u_{\mathrm{ext}}, \quad Bu_{\mathrm{int}} = Bu_{\mathrm{ext}}. \tag{7.11}$$

The interior part of (7.10) is

$$(P - z)u_{\mathrm{int}} + u_- = v_{\mathrm{int}}, \quad u_{\mathrm{int}} + \widehat{K}u'_- = v_+ \quad \text{in } \mathcal{O}, \tag{7.12}$$

giving

$$u_{\mathrm{int}} = v_+ - \widehat{K}u'_-, \quad u_- = v_{\mathrm{int}} - (P - z)u_{\mathrm{int}}.$$

The second condition in (7.11) now gives $Bv_+ - u'_- = BG_{\mathrm{ext}}v_{\mathrm{ext}}$, *i.e.*

$$u'_- = Bv_+ - BG_{\mathrm{ext}}v_{\mathrm{ext}}. \tag{7.13}$$

The first part of (7.11) boils down to

$$\gamma v_+ - \gamma\widehat{K}u'_- = g. \tag{7.14}$$

Thus the unique solution of (7.10) is given by $u = u_{\mathrm{int}} + u_{\mathrm{ext}}$, u_-, u'_-, where

$$\begin{aligned} u'_- &= B(v_+ - G_{\mathrm{ext}}v_{\mathrm{ext}}), \\ u_{\mathrm{int}} &= (1 - \widehat{K}B)v_+ + \widehat{K}BG_{\mathrm{ext}}v_{\mathrm{ext}}, \\ u_- &= v_{\mathrm{int}} - (P - z)\widehat{K}BG_{\mathrm{ext}}v_{\mathrm{ext}} - (P - z)(1 - \widehat{K}B)v_+, \\ u_{\mathrm{ext}} &= (1 + K_{\mathrm{ext}}\gamma\widehat{K}B)G_{\mathrm{ext}}v_{\mathrm{ext}} + K_{\mathrm{ext}}\gamma(1 - \widehat{K}B)v_+. \end{aligned}$$

Using the characteristic functions $1_{\mathcal{O}}$ and $1_{\Gamma_{\mathrm{ext}}}$ to indicate the projection to the interior and exterior parts of functions on Γ, we get in matrix form:

$$\mathcal{P}(z)^{-1} = \tag{7.15}$$

$$\begin{pmatrix} 1_{\mathcal{O}}\widehat{K}BG_{\mathrm{ext}}1_{\Gamma_{\mathrm{ext}}} + 1_{\Gamma_{\mathrm{ext}}}(1 + K_{\mathrm{ext}}\gamma\widehat{K}B)G_{\mathrm{ext}}1_{\Gamma_{\mathrm{ext}}} & 1_{\mathcal{O}}(1 - \widehat{K}B) + 1_{\Gamma_{\mathrm{ext}}}K_{\mathrm{ext}}\gamma(1 - \widehat{K}B) \\ 1_{\mathcal{O}} - (P - z)\widehat{K}BG_{\mathrm{ext}}1_{\Gamma_{\mathrm{ext}}} & -(P - z)(1 - \widehat{K}B) \\ -BG_{\mathrm{ext}}1_{\Gamma_{\mathrm{ext}}} & B \end{pmatrix}.$$

As already mentioned we can use block matrix notation and write

$$\mathcal{P}(z) = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

where

$$P_{11} = P_\Gamma - z, \quad P_{12} = (\iota\ 0), \quad P_{21} = \Pi, \quad P_{22} = (0\ \widehat{K}).$$

Then

$$\mathcal{E}(z) := \mathcal{P}(z)^{-1} = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix},$$

where

$$E_{22} = \begin{pmatrix} -(P-z)(1-\widehat{K}B) \\ B \end{pmatrix} = \begin{pmatrix} -1 & h^{-\frac{1}{2}}(P-z)\widehat{K} \\ 0 & h^{-\frac{1}{2}} \end{pmatrix} \mathcal{P}_{\mathrm{out}}(z),$$

and $\mathcal{P}_{\mathrm{out}}(z)$ was defined in (7.7). The upper triangular matrix in the last expression is invertible, so the invertibility of E_{22} is equivalent to that of $\mathcal{P}_{\mathrm{out}}$ and using also the second part of Proposition 4.1, we get

PROPOSITION 7.2. — *For z in the region* (7.8) *we have that $z \in \sigma(P_\Gamma)$ if and only if $0 \in \sigma(\mathcal{P}_{\mathrm{out}}(z))$.*

$P_\Gamma - z$, $\mathcal{P}_{\mathrm{out}}(z)$ are holomorphic families of Fredholm operators of index 0 and combining (4.3) with Proposition 4.4, we see that $\det(P_\Gamma - z)$ and $\det \mathcal{P}_{\mathrm{out}}(z)$ have zeros of the same multiplicity at the points of $\sigma(P_\Gamma)$.

We next discuss a reduction to the boundary when z is not a Dirichlet eigenvalue. Let P_{in} denote the Dirichlet realization of P in $\mathcal{O}$, so that $\mathcal{D}(P_{\mathrm{in}}) = \{u \in H^2(\mathcal{O});\ \gamma u = 0\}$. Let

$$\mathcal{P}_{\mathrm{in}}(z) = \begin{pmatrix} P - z \\ h^{\frac{1}{2}}\gamma \end{pmatrix} : H^2(\mathcal{O}) \longrightarrow H^0(\mathcal{O}) \times H^{\frac{3}{2}}(\partial\mathcal{O}), \tag{7.16}$$

so that $\mathcal{P}_{\mathrm{in}}(z)$ is bijective precisely when z is not a Dirichlet eigenvalue; $z \notin \sigma(P_{\mathrm{in}})$. Let

$$\mathcal{E}_{\mathrm{in}}(z) = \big(G_{\mathrm{in}}(z)\ \ h^{-\frac{1}{2}}K_{\mathrm{in}}(z)\big)$$

be the inverse which is well defined for z away from the spectrum of P_{in}. Then

$$\mathcal{P}_{\mathrm{out}}(z)\mathcal{E}_{\mathrm{in}}(z) = \begin{pmatrix} (P-z)G_{\mathrm{in}} & (P-z)h^{-\frac{1}{2}}K_{\mathrm{in}} \\ h^{\frac{1}{2}}BG_{\mathrm{in}} & BK_{\mathrm{in}} \end{pmatrix}.$$

Here $(P-z)G_{\mathrm{in}} = 1$, $(P-z)K_{\mathrm{in}} = 0$ and

$$BK_{\mathrm{in}} = \gamma h D_\nu K_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}} = \mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}}, \tag{7.17}$$

where the last equility defines $\mathcal{N}_{\mathrm{in}} : H^{\frac{3}{2}}(\partial\mathcal{O}) \to H^{\frac{1}{2}}(\partial\mathcal{O})$ so

$$\mathcal{P}_{\mathrm{out}}(z)\mathcal{E}_{\mathrm{in}}(z) = \begin{pmatrix} 1 & 0 \\ h^{\frac{1}{2}}BG_{\mathrm{in}} & \mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}} \end{pmatrix}. \tag{7.18}$$

Composing with $\mathcal{P}_{\mathrm{in}}$ to the right, we get

$$\mathcal{P}_{\mathrm{out}}(z) = \begin{pmatrix} 1 & 0 \\ h^{\frac{1}{2}}BG_{\mathrm{in}} & \mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}} \end{pmatrix}\mathcal{P}_{\mathrm{in}}(z). \tag{7.19}$$

Notice that this factorization makes sense only when $z \notin \sigma(P_{\mathrm{in}}(z))$ since $\mathcal{N}_{\mathrm{in}}$ is defined only under that assumption. The last factor in the right hand side is of course bijective then, and the first lower triangular factor is bijective precisely when $\mathcal{N}_{\mathrm{in}}(z) - \mathcal{N}_{\mathrm{ext}}(z) : H^{\frac{3}{2}} \to H^{\frac{1}{2}}$ is bijective, or equivalently when 0 is not in the spectrum of this operator, considered as an unbounded operator $H^{\frac{1}{2}} \to H^{\frac{1}{2}}$ with domain $H^{\frac{3}{2}}$.

Proposition 7.3. — *For z in the region* (7.8) *and not in $\sigma(P_{\mathrm{in}})$, we have the equivalence*

$$0 \in \sigma(\mathcal{P}_{\mathrm{out}}(z)) \iff 0 \in \sigma(\mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}}).$$

Again we have holomorphic families of Fredholm operators of index 0 and we have the analogue of the remark after Proposition 7.2.

We end the chapter with a symmetry observation (cf. (5.1)).

Proposition 7.4. — *$P_{\mathrm{out}}(z)$, $\mathcal{N}_{\mathrm{in}}$ and $\mathcal{N}_{\mathrm{ext}}$ are symmetric.*

Proof. — This follows from Green's formula. For $u, v \in H^{\frac{3}{2}}(\partial\mathcal{O})$, we have

$$\begin{aligned}
&\langle \mathcal{N}_{\mathrm{in}}u \,|\, v\rangle_{\partial\mathcal{O}} - \langle u \,|\, \mathcal{N}_{\mathrm{in}}v\rangle_{\partial\mathcal{O}} \\
&\quad = \langle hD_\nu K_{\mathrm{in}}u \,|\, v\rangle_{\partial\mathcal{O}} - \langle u \,|\, hD_\nu K_{\mathrm{in}}v\rangle_{\partial\mathcal{O}} \\
&\quad = \frac{i}{h}\big(\langle -h^2\Delta K_{\mathrm{in}}u \,|\, K_{\mathrm{in}}v\rangle_{\mathcal{O}} - \langle K_{\mathrm{in}}u \,|\, -h^2\Delta K_{\mathrm{in}}v\rangle_{\mathcal{O}}\big) \\
&\quad = \frac{i}{h}\big(\langle (P-z)K_{\mathrm{in}}u \,|\, K_{\mathrm{in}}v\rangle_{\mathcal{O}} - \langle K_{\mathrm{in}}u \,|\, (P-z)K_{\mathrm{in}}v\rangle\big) = 0.
\end{aligned}$$

The symmetry of $\mathcal{N}_{\mathrm{ext}}$ follows in the same way by applying Green's formula on Γ_{ext}. Let $u, v \in \mathcal{D}(P_{\mathrm{out}}(z))$, so that $\gamma hD_\nu u = \mathcal{N}_{\mathrm{ext}}\gamma u$ and similarly for v. Using again Green's formula, we get

$$\begin{aligned}
&\big\langle P_{\mathrm{out}}(z)u\,|\,v\big\rangle_{\mathcal{O}} - \big\langle u\,|\,P_{\mathrm{out}}(z)v\big\rangle_{\mathcal{O}} \\
&\qquad = -h^2\big(\langle \Delta u\,|\,v\rangle_{\mathcal{O}} - \langle u\,|\,\Delta v\rangle_{\mathcal{O}}\big) \\
&\qquad\quad = \frac{h}{i}(\langle hD_\nu u\,|\,v\rangle_{\partial\mathcal{O}} - \langle u\,|\,hD_\nu v\rangle_{\partial\mathcal{O}}) \\
&\qquad\qquad = \frac{h}{i}\big(\langle \mathcal{N}_{\mathrm{ext}} u\,|\,v\rangle_{\partial\mathcal{O}} - \langle u\,|\,\mathcal{N}_{\mathrm{ext}} v\rangle_{\partial\mathcal{O}}\big) = 0,
\end{aligned}$$

where the last equality follows from the symmetry of $\mathcal{N}_{\mathrm{ext}}$. □

CHAPTER 8

SOME ODE PREPARATIONS

In this chapter we make some preparations for the study of the interior and exterior Dirichlet to Neumann maps and some related estimates for the exterior resolvent.

8.1. Nullsolutions and factorizations of 2nd order ODEs

It will be convenient to factorize our equations and we make some extremely elementary and certainly well-known remarks. Let

$$P = \partial_t^2 + a(t)\partial_t + b(t) \tag{8.1}$$

be a differential operator with smooth coefficients on an interval or with holomorphic coefficients on a simply connected open set in $\mathbb{C}$. Let $\mathrm{e}^{-\alpha(t)}$ belong to the kernel of P,

$$P(\mathrm{e}^{-\alpha}) = 0. \tag{8.2}$$

This means that P takes the form $P = (\partial_t + \alpha')^2 + f(t)(\partial_t + \alpha') + g(t)$, where $g \equiv 0$ and we get

$$P = (\partial_t - \beta')(\partial_t + \alpha'), \tag{8.3}$$

where $\beta' = \alpha' - a$,

$$\beta = \alpha - \int^t a \mathrm{d}s. \tag{8.4}$$

Notice that $P^{\mathrm{t}} = (\partial_t - \alpha')(\partial_t + \beta')$, so e^{β} belongs to the kernel of P^{t}. When P is symmetric, $P^{\mathrm{t}} = P$, we have $a = 0$, $\beta = \alpha$.

8.2. Simple turning point analysis

We recall some elements of the complex WKB method and refer to [**36**], [**11**] for more extensive expositions. Let $V = V(x)$ be holomorphic in some simply connected open set $\subset \mathbb{C}$. We consider the equation

$$\big((hD_x)^2 + V(x)\big)u = 0, \tag{8.5}$$

with u holomorphic. The zeros of V are the turning points by definition. Away from those points we can construct formal local solutions of the form

$$u(x) = a(x;h)\,\mathrm{e}^{i\phi(x)/h},\ a(x;h) \sim a_0(x) + ha_1(x) + \cdots, \tag{8.6}$$

where $\phi(x)$ is a solution of the eiconal equation

$$(\phi'(x))^2 + V(x) = 0, \tag{8.7}$$

and a_0, a_1, … solve a sequence of transport equations obtained from

$$\big((\phi'(x) + hD_x)^2 + V(x)\big)a = 0,$$

equivalent to

$$\big(\phi'(x)hD + hD \circ \phi' + (hD)^2\big)a = 0:$$

$$2\phi'(x)\partial a_0 + \phi'' a_0 = 0, \tag{T_0}$$

and for $j \geq 1$:

$$2\phi'(x)\partial a_j + \phi'' a_j = i\partial^2 a_{j-1}. \tag{T_j}$$

We can prescribe $a_0(x_0)$, $a_1(x_0)$, … (if x_0 is not a turning point) and then the formal symbol becomes uniquely determined in a neighborhood of x_0. The so called exact WKB method (see also the appendix) tells us that if $\gamma : [0,1] \to \Omega$ is a C^1 curve with $\gamma(0) = x_0$, avoiding the turning points and with the property that $-\Im\phi(\gamma(t))$ has positive derivative[(1)], then there exists an exact holomorphic solution of (8.5) of the form (8.6) in a neighborhood of $\gamma(]0,1])$ where $a_0(x_0)$, $a_1(x_0)$, … can be arbitrarily prescribed (in the sense that $a(x;h)$ is holomorphic in x with the asymptotic expansion of (8.6) in the space of holomorphic functions in a neighborhood of the range of γ). Moreover, the solution is unique up to a term $\mathcal{O}(h^\infty)\,\mathrm{e}^{-\Im\phi/h}$.

Actually the formal expansion can be improved by using the Ansatz $(\Phi')^{-\frac{1}{2}}\,\mathrm{e}^{i\Phi/h}$, and then determining $\Phi(x;h) \sim \phi(x) + h^2\phi_2(x) + h^4\phi_4(x) + \cdots$ from a Riccati type equation. Notice that the solution of (T_0) is of the form $a_0(x) = C(\phi')^{-\frac{1}{2}} = \widetilde{C}V(x)^{-\frac{1}{4}}$.

We can consider multivalued solutions of (8.7) away from the turning points. A C^1 curve in Ω is called a *Stokes line* if $\Im\phi$ is constant on γ and it is called

(1) So that $\mathrm{e}^{i\phi(x)/h}$ is exponentially growing with increasing t.

an *anti-Stokes line* if $\Re\phi$ is constant. (Sometimes the terminology is reversed.) Locally away from the turning points the Stokes and anti-Stokes lines intersect each other perpendicularly. The curve γ in the above exact WKB remark necessarily intersects the Stokes lines transversally.

A turning point $x_0 \in \Omega$ is called a *simple turning point* if it is a simple zero of V, so that

$$V'(x_0) \neq 0. \tag{8.8}$$

We next consider the singularity of the solution of the eiconal equation near a simple turning point that we assume to be $x_0 = 0$ for simplicity. If the Taylor expansion of $-V$ at $x = 0$ is $-V(x) = a^2x + \mathcal{O}(x^2)$, then $\phi'(x)$ is a double-valued holomorphic function of the form

$$\phi'(x) = ax^{\frac{1}{2}}\big(1 + \mathcal{O}(x)\big),$$

where the last factor is holomorphic in a full neighborhood of $x = 0$. By integration it is clear that ϕ is also double-valued and of the form

$$\phi(x) = \frac{2}{3}ax^{\frac{3}{2}}\big(1 + \mathcal{O}(x)\big),$$

where again the last factor is holomorphic near 0.

The union of the Stokes and anti-Stokes curves reaching the turning point $x = 0$ is contained in

$$\big\{x \in \text{neigh}(0);\ \Im\phi = 0 \text{ or } \Re\phi = 0\big\} = \big\{x \in \text{neigh}(0);\ \Im(\phi^2) = 0\big\}, \tag{8.9}$$

which is also the set of points x solving

$$a^2x^3\big(1 + \mathcal{O}(x)\big) = t^3, \quad t \in \text{neigh}(0, \mathbb{R}),$$

i.e. $a^{\frac{2}{3}}x(1 + \mathcal{O}(x)) = t$, or equivalently

$$x = f(a^{-\frac{2}{3}}t),$$

where f is analytic and $f(0) = 0$, $f'(0) = 1$. Since there are three branches of the cubic root of a we see that the set (8.9) is the union of three smooth curves, γ_j, $j = 0, 1, 2$, that pass through 0 and intersect there at angles $\frac{2}{3}\pi$.

With a suitable orientation, each γ_j is first a Stokes line γ_j^- until it hits 0 and then becomes an anti-Stokes line γ_j^+ on the other side. It will be convenient to let γ_j^- be open in the sense that $0 \notin \gamma_j^-$, $0 \in \gamma_j^+$. The three Stokes lines divide a pointed neighborhood into three "Stokes sectors" Σ_j, $j = 0, 1, 2$, as indicated Figure 1. Each Stokes sector is the union of Stokes lines in addition to the two Stokes lines that make up the boundary. In the figure we draw two such additional lines in each sector.

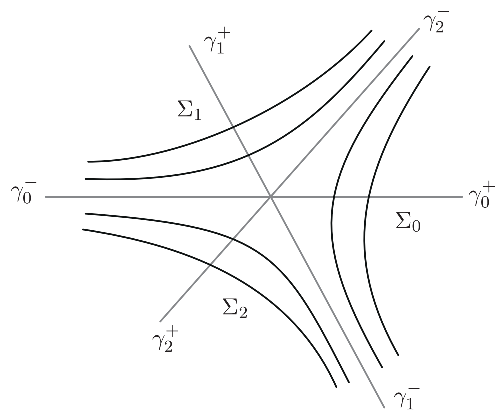


FIGURE 1. Three Stokes sectors

For each $j \in \mathbb{Z}/3\mathbb{Z}$, we choose the branch $\phi = \phi^j$ of the solution of the eiconal equation tending to 0 when $x \to 0$ which has positive imaginary part on the interior of Σ_j and we can extend ϕ holomorphically to $\Omega \setminus \overline{\gamma_j^-}$, so that $\phi^j = -\phi^{j\pm1}$ in $\Sigma_{j\pm1}$. Here Ω is a fixed small open disc centered at 0. The exact WKB method tells us that (8.5) has a solution $u = u_j$ in Ω of the following asymptotic form in $\Omega \setminus \Gamma_j^-$, where Γ_j^- is any fixed neighborhood of $\overline{\gamma_j^-}$:

$$u_j = a^j(x;h)\,\mathrm{e}^{i\phi_j(x)/h}, \quad a^j \sim a_0^j + ha_1^j + \cdots, \quad a_0(x) \neq 0. \tag{8.10}$$

The Wronskian $W(u_j, u_k) := (hDu_j)u_k - u_j hDu_k$ is constant, and can be computed asymptotically for $j \neq k$ at any point on γ_ℓ^- where ℓ is the index different both from j and k. Since $\phi_j = -\phi_k$ there, we get

$$W(u_j, u_k) = 2\phi_j' a_0^j a_0^k + \mathcal{O}(h). \tag{8.11}$$

Also recall that $W(u, u) = 0$.

This can be used to study u_j near γ_j^-. Since the space of solutions of (8.5) is of dimension 2, we have

$$u_j = \sum_{k;\,k\neq j} c_{j,k} u_k, \quad c_{j,k} = c_{j,k}(h) \in \mathbb{C}, \tag{8.12}$$

and if $k \neq j$, we let $\ell = \ell(j,k)$ be the index different both from j and k and get $W(u_j, u_\ell) = c_{j,k} W(u_k, u_\ell)$,

$$c_{j,k} = \frac{W(u_j, u_\ell)}{W(u_k, u_\ell)} \sim c_{j,k}^0 + hc_{j,k}^1 + \cdots, \quad c_{j,k}^0 \neq 0. \tag{8.13}$$

We shall next show that (8.10) extends to $\Omega \setminus (\Gamma_j^- \cup D(0, Ch^{\frac{2}{3}}))$ where now Γ_j^- is a conic neighborhood of γ_j^- and $C \gg 1$, in the sense that the asymptotic expansion for a^j is in powers of $h/x^{\frac{3}{2}}$. Letting j be fixed for a while, we suppress "j" from the notation. Recall that a_0, a_1 are determined by the sequence of transport equations (T_0), (T_1), ... above. Using the eiconal equation for ϕ we get

$$\partial(V^{\frac{1}{4}} a_0) = 0, \quad \partial(V^{\frac{1}{4}} a_k) = \frac{1}{2} V^{-\frac{1}{4}} \partial^2 a_{k-1}. \tag{8.14}$$

Starting with $a_0 = \mathrm{Const.}\, V^{-\frac{1}{4}} = \mathcal{O}(x^{-\frac{1}{4}})$ and using (8.14) and the Cauchy inequalities, we get iteratively that

$$a_k(x) = \mathcal{O}(x^{-\frac{1}{4} - k\frac{3}{2}}), \quad x \to 0. \tag{8.15}$$

Thus, we can give a meaning to

$$\sum_0^\infty a_k h^k = \sum_0^\infty (x^{k\frac{3}{2}} a_k) \Big(\frac{h}{x^{\frac{3}{2}}} \Big)^k,$$

in the region $|x| \gg h^{\frac{2}{3}}$ as an asymptotic sum in powers of the small parameter $h/x^{\frac{3}{2}}$.

In the appendix, we show that the holomorphic function a has this asymptotic expansion in the region $|x| \gg h^{\frac{2}{3}}$.

Proposition 8.1. — *Fix $j \in \mathbb{Z}/3\mathbb{Z}$ and let $u = u_j$ be a solution of* (8.5)*, which has the structure* (8.10) *in a neighborhood of a point $x_0^+ \in \gamma_j^+ \setminus \{0\}$. Then for $r > 0$ small enough, u remains of the form* (8.10) *in*

$$D(0, r) \setminus \big(\Gamma_j^- \cup D(0, Ch^{\frac{2}{3}}) \big),$$

Γ_j is any neighborhood of γ_j^- of the form $\bigcup_{x \in \gamma_j^-} D(x, \epsilon |x|)$ where $C = C_\epsilon > 0$ is large enough. The coefficients a_k^j in (8.10) *satisfy* (8.15) *and the precise meaning of the asymptotics in* (8.10) *is that*

$$a^j - \sum_{k=0}^{N-1} a_k^j h^k = \mathcal{O}\big(x^{-\frac{1}{4}} (h/x^{\frac{3}{2}})^N \big). \tag{8.16}$$

We shall next estimate the region where $u = u_0$ may have its zeros and take $j = 0$ in order to fix the ideas. From Proposition 8.1 it is clear that the zeros have to be close to $\overline{\gamma_0^-}$ and in particular we need to study what happens in an $h^{\frac{2}{3}}$ neighborhood of 0, where we have no asymptotics. If $\gamma : [a, b] \to \mathbb{C}$

is a smooth curve and v, w are holomorphic functions defined near γ, then

$$\int_\gamma vw\,\mathrm{d}x = \int_a^b v_\gamma w_\gamma \,\mathrm{d}t,$$

where we define

$$u_\gamma(t) = \dot\gamma^{\frac12} u\big(\gamma(t)\big).$$

This means that the passage $u \mapsto u_\gamma$ conserves symmetry of differential operators, and more precisely, we check that

$$(Du)_\gamma = \dot\gamma^{-\frac12} D_t \dot\gamma^{-\frac12} u_\gamma,$$

and the equation (8.5) restricted to γ reads

$$\big[(\dot\gamma^{-\frac12} hD_t\dot\gamma^{-\frac12})^2 + V\big(\gamma(t)\big)\big]u_\gamma = 0 \tag{8.17}$$

Here we can rework the first term and put the two D_t together in the center. We get

$$\big(-(h\partial_t)^2 + \dot\gamma^2\widetilde V\big)\dot\gamma^{-1}u_\gamma = 0, \quad \dot\gamma^{-1}u_\gamma = \dot\gamma^{-\frac12}u\circ\gamma, \tag{8.18}$$

where

$$\widetilde V = V\big(\gamma(t)\big) + \Big(\frac{h}{\dot\gamma}\Big)^2\Big[\frac14\Big(\frac{\ddot\gamma}{\dot\gamma}\Big)^2 - \frac12\partial_t\Big(\frac{\ddot\gamma}{\dot\gamma}\Big)\Big] = V\circ\gamma + \mathcal{O}(h^2). \tag{8.19}$$

Proposition 8.2. — *If γ is a Stokes curve or an anti-Stokes curve, we have*

$$\Im(\dot\gamma^2 V\circ\gamma) = 0.$$

More precisely, $\dot\gamma^2 V\circ\gamma$ is < 0 in the first case and > 0 in the second case.

Proof. — Stokes and anti-Stokes curves are characterized by the property that $\Im\dot\gamma\phi' = 0$ and $\Re\dot\gamma\phi' = 0$ respectively, where ϕ solves the eiconal equation (8.7). For both types of curves, we have $\Im(\dot\gamma\phi')^2 = 0$ which means that $\Im(\dot\gamma^2 V\circ\gamma) = 0$. On a Stokes curve we have $(\dot\gamma\phi')^2 > 0$, so $\dot\gamma^2 V\circ\gamma < 0$ and on an anti-Stokes curve we have $(\dot\gamma\phi')^2 < 0$, so $\dot\gamma^2 V\circ\gamma > 0$. □

Now complete γ_0 into a smooth family of curves γ_s, $s\in \mathrm{neigh}(0,\mathbb{R})$, so that $x = \gamma_s(t)$ defines local coordinates s, t and the smooth function

$$f(s,t) = \Im\big[(\partial_t\gamma_s)^2 V\big(\gamma_s(t)\big)\big]$$

vanishes for $s = 0$. Assuming, as we may, that $\gamma_0(0) = 0$, $\gamma_0(t) = \gamma_0^\pm(t)$, for $\pm t > 0$, we get for $s = 0$:

$$(\partial_s f)(0,0) = \Im\big(\dot\gamma_0^2 V'(0)\partial_s\gamma_s(0)\big).$$

This is $\neq 0$ since $V'(0)\neq 0$ and $\partial_s\gamma_s(0)_{s=0}$ is not colinear with $\dot\gamma_0$. It follows that $\pm f(s,t)\asymp s$ and we may assume that the plus sign is valid;

$$\Im\big[\big(\partial_t\gamma_s(t)\big)^2 V\big(\gamma_s(t)\big)\big]\asymp s,\quad (s,t)\in \mathrm{neigh}(0). \tag{8.20}$$

Now let $u=u_0$ be a solution of (8.5) as in (8.14) which is exponentially decaying in the Stokes sector Σ_0 containing the anti-Stokes line γ_0^+.

Proposition 8.3. — *The zeros of u_0 are within a distance $\mathcal{O}(h^2)$ from γ_0^- and away from a disc $D(0,h^{\frac{2}{3}}/C)$ if $C>0$ is large enough.*

Proof. — We first prove that the zeros are within a distance $\mathcal{O}(h^2)$ from γ_0. From the WKB structure we already know that they have to be inside a small neighborhood of $\{0\}\cup\gamma_0^-$. Let x_0 be a zero of u and let $s=s_0$ be determined by the property that x_0 belongs to γ_{s_0}, so that $x_0 \overline{=\gamma_{s_0}(t_0)}$ for $-1/\mathcal{O}(1)\leq t_0\leq o(1)$. Take $\gamma=\gamma_{s_0}$ in (8.18): Multiplying by $\overline{\dot\gamma^{-\frac12}u\circ\gamma}$, we get

$$\int_{t_0}^1 \big[\big((-h\partial_t)^2+\dot\gamma^2\widetilde V\big)\dot\gamma^{-\frac12}u\circ\gamma\big]\,\overline{\dot\gamma^{-\frac12}u\circ\gamma}\,\mathrm{d}t=0.$$

Here $u\circ\gamma$ is exponentially decaying for $t\geq 1/\mathcal{O}(1)$ and vanishes at t_0 so we can integrate by parts and get

$$\int_{t_0}^1 \big[\big|h\partial_t(\dot\gamma^{-\frac12}u\circ\gamma)\big|^2+\dot\gamma^2\widetilde V|\dot\gamma^{-\frac12}u\circ\gamma|^2\big]\mathrm{d}t=\mathcal{O}(\mathrm{e}^{-\frac{1}{Ch}}). \tag{8.21}$$

Now $\Im\dot\gamma^2\widetilde V=\Im(\dot\gamma^2V\circ\gamma)+\mathcal{O}(h^2)$ and $\Im(\dot\gamma^2V\circ\gamma)\asymp s_0$, so taking the imaginary part of (8.21), we get

$$(|s_0|-\mathcal{O}(h^2))\int_{t_0}^1|\dot\gamma^{-\frac12}u\circ\gamma|^2\mathrm{d}t\leq\mathcal{O}(\mathrm{e}^{-\frac{1}{Ch}}).$$

Consequently, $s_0=\mathcal{O}(h^2)$ so the zero is at a distance $\leq\mathcal{O}(h^2)$ from γ_0.

It remains to prove that the zeros stay away from $D(0,h^{\frac23}/C)$ and belong to a h^2-neighborhood of γ_0^-. Let $x_0=\gamma_{s_0}(t_0)$ be a zero so that $s_0=\mathcal{O}(h^2)$. Then, with $\gamma=\gamma_{s_0}$ we have $\Re\dot\gamma^2V\asymp t$. Let $v=\dot\gamma^{-\frac12}u\circ\gamma$ and take the real part of (8.21):

$$\int_{t_0}^1\big(|h\partial_t v|^2+\Re(\dot\gamma^2\widetilde V)|v|^2\big)\mathrm{d}t=\mathcal{O}(\mathrm{e}^{-\frac{1}{Ch}}). \tag{8.22}$$

Now,

$$\Re(\dot\gamma^2\widetilde V)\geq\frac{t-t_0}{C}-C\big(|t_0|+h^2\big)$$

and we get

$$\int_{t_0}^{1}\Big(|h\partial_t v|^2+\frac{t-t_0}{C}|v|^2\Big)\,\mathrm{d}t\le \mathcal{O}(\mathrm{e}^{-\frac{1}{Ch}})+C\big(|t_0|+h^2\big)\|v\|^2,$$

where the norm is the one in $L^2([t_0,1])$. Here, we can drop the first term to the right since $\|v\|$ is bounded from below by a power of h. On the other hand, we know (either by using well-known facts about the Dirichlet problem for the Airy operator or by more direct arguments) that the left hand side is bounded from below by $C^{-1}h^{\frac{2}{3}}\|v\|^2$ (using also that $v(1)$ is exponentially small). Hence, $h^{\frac{2}{3}}C\le C\big(|t_0|+h^2\big)$, leading to

$$|t_0|\ge \frac{h^{\frac{2}{3}}}{\widetilde{C}}.$$

Now a second look at (8.22) shows that we cannot have $t_0\ge h^{\frac{2}{3}}/\widetilde{C}$, and the proof is complete. □

Remark 8.4. — By pushing the argument slightly further we see that every zero of u_0 in any fixed disc $D(0,Ch^{\frac{2}{3}})$ is of the form

$$-h^{\frac{2}{3}}V'(0)^{-\frac{1}{3}}\zeta_j+\mathcal{O}(h^{\frac{4}{3}}), \tag{8.23}$$

for some j, where $0<\zeta_1<\zeta_2<\cdots$ are the zeros of $\mathrm{Ai}(-t)$.

In fact, let x_1 be such a zero and consider the equation (8.18) along the curve $\gamma=\gamma_s$ that contains x_1. Assume that the parametrization is chosen with $\gamma(0)=x_1$ and such that γ is oriented in the direction of Σ_0 for increasing t. Choose a similar parametrization of γ_0 so that $\gamma(t)-\gamma_0(t)=\mathcal{O}(h^2)$. Pulling $\dot\gamma^{-\frac{1}{2}}u\circ\gamma$ to γ_0 by means of $\gamma\circ\gamma_0^{-1}$, we get a quasi-mode $\widetilde{u}(t)$ satisfying

$$\big(-(h\partial_t)^2+\dot\gamma_0^2V\big(\gamma_0(t)\big)\big)\widetilde{u}(t)=\mathcal{O}(h^2)\|\widetilde{u}\|\text{ in }L^2\big([0,1/C_0]\big), \tag{8.24}$$

which is exponentially decaying for $t\gg h^{\frac{2}{3}}$ and satisfies the Dirichlet condition $\widetilde{u}(0)=0$. This means that the self-adjoint Dirichlet realization on $[0,1/C_0]$ of the operator to the left in (8.24) has an eigenvalue $=\mathcal{O}(h^2)$. Now it is a routine exercise in self-adjoint semi-classical analysis to see that the eigenvalues of this operator in any interval $]-\infty,Ch^{\frac{2}{3}}]$ are of the form

$$U(0)+h^{\frac{2}{3}}U'(0)^{\frac{2}{3}}\zeta_j+\mathcal{O}(h^{\frac{4}{3}}), \tag{8.25}$$

where $U(t)=\dot\gamma_0^2V(\gamma_0(t))$ is the potential in (8.24). Thus for some j,

$$\dot\gamma_0(0)^2V\big(\gamma_0(0)\big)+h^{\frac{2}{3}}\big(\dot\gamma_0(0)^3V'\big(\gamma_0(0)\big)\big)^{\frac{2}{3}}\zeta_j=\mathcal{O}(h^{\frac{4}{3}}),$$

which simplifies to

$$V(x_1) + h^{\frac{2}{3}} V'(0)^{\frac{2}{3}} \zeta_j + \mathcal{O}(h^{\frac{4}{3}}) = 0,$$

leading to (8.23).

Remark 8.4 allows us to control the exterior Dirichlet problem for $\Im z \geq -c_0 h^{\frac{2}{3}}$ for c_0 as in (3.1).

8.3. The exterior ODE

We are concerned with the operator

$$P = -(h\partial_x)^2 - xQ(x) + ha(x)h\partial_x, \tag{8.26}$$

where Q, a are holomorphic on $\mathrm{neigh}(0, \mathbb{C})$ and $Q > 0$ on the real domain.

Let γ_δ be the contour $x = \gamma_\delta(s)$, $0 \leq s \leq s_0$, $0 < s_0 \ll 1$,

$$\begin{cases} \gamma_\delta(s) = s & \text{for } 0 \leq s \leq \delta, \\ \gamma_\delta(s) = \delta + \mathrm{e}^{\frac{1}{3}i\pi}(s - \delta) & \text{for } \delta \leq s \leq s_0, \end{cases} \tag{8.27}$$

and let $b = \gamma_\delta(s_0)$ be the second end point. Here $\delta \geq 0$ is a small parameter that eventually will take the values 0 and Ch.

Consider the Dirichlet problem

$$(P - z)u = v \text{ on } \gamma_\delta, \quad u(0) = 0, \ u(b) = 0, \tag{8.28}$$

where

$$z = \lambda + h^{\frac{2}{3}} w, \quad \lambda \in \mathbb{R}, \quad |w| \leq \frac{1}{\mathcal{O}(1)}. \tag{8.29}$$

We start by discussing the case $\delta = 0$ and later we indicate the additional arguments in order to treat the case $\delta > 0$. When $\delta = 0$, the operator reduces to the rotated Airy operator with a perturbation,

$$\mathrm{e}^{-\frac{2}{3}\pi i}\big(- (h\partial_s)^2 + sQ(\mathrm{e}^{\frac{1}{3}\pi i} s)\big) + \mathrm{e}^{-\frac{1}{3}\pi i} ha(\mathrm{e}^{\frac{1}{3}\pi i} s) h\partial_s, \tag{8.30}$$

which as in [**15**], [**29**], [**30**], [**31**] can be treated by ressorting to the spectral theory for the Dirichlet problem for the Airy operator. When $\delta > 0$ this appeared as more difficult and in order to cover that case also we chose to use the complex WKB method. The last term $ha(x)h\partial_x$ will have no real importance and can be eliminated by writing

$$\begin{aligned} P &= -\big(h\partial_x - \tfrac{1}{2} hya(x)\big)^2 - xQ(x) + \mathcal{O}(h^2) \\ &= \mathrm{e}^{\frac{1}{2}A}\big[- (h\partial_x)^2 - xQ(x) + \mathcal{O}(h^2)\big]\, \mathrm{e}^{-\frac{1}{2}A}, \end{aligned}$$

where $A = \mathcal{O}(1)$ is a primitive of a. Since the perturbation $\mathcal{O}(h^2)$ can be absorbed in the estimates below, we will assume from now on that $a = 0$. We will also concentrate on the most interesting case when $|\lambda| \leq 1/C$ and indicate later how to treat the easier cases when λ is positive and bounded from above as well as the case when λ is negative and arbitrarily large.

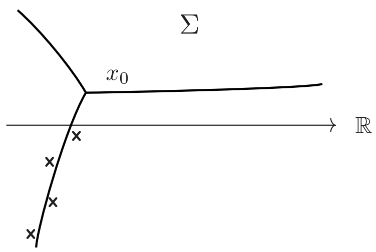
Assuming that $|\lambda| \leq 1/C$, we see that the equation (8.28) has a turning point $x_0(z)$, given by

$$x_0 Q(x_0) + z = 0. \tag{8.31}$$

If $x_1 \in \mathbb{R}$ is the real turning point, given by $x_1 Q(x_1) + \lambda = 0$, then

$$x_0 = x_1 - \frac{1}{\partial V(x_1)} h^{\frac{2}{3}} w + \mathcal{O}(h^{\frac{4}{3}}), \quad \text{where } V(x) = xQ(x). \tag{8.32}$$

We have the following picture



where we draw the three Stokes lines through x_0, the Stokes sector Σ, and notice that the zeros of the corresponding subdominant solution are very close to the Stokes line γ_0^- opposite to Σ and separated from the turning point by a distance $\geq h^{\frac{2}{3}}/C$. A direct calculation from (8.31), (8.23) shows that the imaginary parts of these zeros are $\leq -h^{\frac{2}{3}}/\mathcal{O}(1)$ when $|\lambda| \ll 1$ and $\Im w \geq -Q(0)^{\frac{2}{3}} \zeta_1 \cos \frac{1}{6}\pi + 1/\mathcal{O}(1)$.

From Proposition 8.1, we see that the equation $(P - z)u = 0$ has a solution which is subdominant in Σ, of the form

$$\mathrm{e}^{-\phi(x;h)/h} \tag{8.33}$$

in $(\mathrm{neigh}(x_0, \mathbb{C}) \setminus V_0^-) \cup D(x_0, h^{\frac{2}{3}}/C)$ where V_0^- is a any small "conic" neighborhood of γ_0^- as in Proposition 8.1, such that

$$\phi'(x; h) = \phi_0'(x) + \frac{\mathcal{O}(h)}{x - x_0} \tag{8.34}$$

and ϕ_0 solves the eiconal equation, $(\phi_0')^2 = xQ(x) + z$. (Compared to Proposition 8.1, we have found it convenient to drop the prefactor "i".) Notice that the first term in the right hand side of (8.34) dominates when $|x - x_0| \gg h^{\frac{2}{3}}$.

Moreover, in any set of the form $D(x_0, h^{\frac{3}{2}}/C) \cup (D(x_0, Ch^{\frac{3}{2}}) \setminus V_0^-)$, we have

$$\phi' = \mathcal{O}(h^{\frac{1}{3}}). \tag{8.35}$$

In fact, writing $x - x_0 = h^{\frac{2}{3}}y$ leads to the equation $-(\partial_y^2 + W(y))u = 0$ in a fixed h-independent domain where W is holomorphic and bounded. Rewriting this as a first order system, we see that $|u(y)| + |\partial_y u(y)|$ is of constant order of magnitude, say $\asymp 1$ and the equation tells us that $\partial_y^2 u = \mathcal{O}(1)$. We also know that u is non-vanishing and after shrinking the domain by a fixed rate arbitrarily close to 1, we conclude that $|u(y)| \geq 1/\mathcal{O}(1)$. Indeed, if $|u(y_0)| = \epsilon \ll 1$, then $|u'(y_0)| \asymp 1$ and from the Taylor expansion, $u(y) = u(y_0) + u'(y_0)(y - y_0) + \mathcal{O}((y - y_0)^2)$, we see that u must have a zero in the disc $D(y_0, r)$ if $\epsilon \ll r \ll 1$. Thus $|u(y)| \asymp 1$, $u'(y) = \mathcal{O}(1)$ and hence $\partial_y \ln u = \mathcal{O}(1)$. Hence $h^{\frac{2}{3}}\partial_x \ln u = \mathcal{O}(1)$ and $\partial_x \phi = h\partial_x \ln u = \mathcal{O}(h^{\frac{1}{3}})$ as claimed.

As in Section 8.1 we factor $P - z$ as

$$P - z = (\phi' - h\partial_x)(\phi' + h\partial_x) \tag{8.36}$$

and we shall use this to find a solution u of the equation $(P - z)u = v$. First invert $\phi' - h\partial_x$ by integration from b to get

$$(\phi' + h\partial_x)u = -\frac{1}{h}\int_b^x \mathrm{e}^{(\phi(x)-\phi(y))/h} v(y)dy =: Kv(x). \tag{8.37}$$

In order to estimate the $\mathcal{L}(L^2)$-norm of this integral operator and of similar ones, we collect some useful properties.

Lemma 8.5. — *Assume that* $0 \leq \delta \leq Ch$ *and orient* γ_δ *from* 0 *to* b. *Write* $y \prec x$ *for* $y, x \in \gamma_\delta$ *if* y *precedes* x. *For* $x, y, w \in \gamma_\delta$ *with* $0 \prec y \prec w \prec x \prec b$ *we have with a new constant* $C > 0$*:*

$$\frac{1}{C}\int_y^x \left|\phi'(z)\right| \cdot |\mathrm{d}z| - Ch \leq \Re\phi(x) - \Re\phi(y) \leq \int_y^x \left|\phi'(z)\right| \cdot |\mathrm{d}z|, \tag{8.38}$$

$$\frac{1}{C}\left|\phi'(w)\right| \cdot |x - y| - Ch \leq \int_y^x \left|\phi'(z)\right| \cdot |\mathrm{d}z| \leq C\big(\left|\phi'(x)\right| \cdot |x - y| + h\big), \tag{8.39}$$

$$\frac{1}{C_\epsilon}\mathrm{e}^{-\frac{\epsilon}{h}\int_y^x |\phi'(z)|\cdot|\mathrm{d}z|} \leq \frac{h^{\frac{1}{3}} + |\phi'(x)|}{h^{\frac{1}{3}} + |\phi'(y)|} \leq C_\epsilon \mathrm{e}^{\frac{\epsilon}{h}\int_y^x |\phi'(z)|\cdot|\mathrm{d}z|}, \tag{8.40}$$

for every $\epsilon > 0$. *Here* $C_\epsilon > 0$ *is independent of* h.

Proof. — The second inequality in (8.38) is obvious. By additivity it suffices to show the first inequality in each of the following three cases (where the second case may be void):

1) x, y belong to the horizontal segment $[0, \delta]$;
2) x, y belong to $\gamma_\delta \cap D(x_0, Ch^{\frac{2}{3}})$;
3) x, y are both beyond the cases 1) and 2).

In case 1) both $\int_y^x |\phi'(z)| \cdot |\mathrm{d}z|$ and $\Re(\phi(x) - \phi(y))$ are $\mathcal{O}(h)$ since $\delta = \mathcal{O}(h)$. In the second case this remains true since $|x - y| = \mathcal{O}(h^{\frac{2}{3}})$ and $\phi'(z) = \mathcal{O}(h^{\frac{1}{3}})$ for $y \prec z \prec x$. In the third case the first inequality in (8.38) follows from the fact that γ_δ is here transversal to the Stokes lines and more precisely that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Re\phi\big(\gamma_\delta(t)\big) \asymp \big|\phi'(\gamma_\delta(t))\big|, \quad \text{for } y \prec \gamma_\delta(t) \prec x.$$

Now consider (8.39). If x is as in case 1) or 2) then $\int_y^x |\phi'(z)| \cdot |\mathrm{d}z|$ and $|\phi'(w)| \cdot |x - y|$ are $\mathcal{O}(h)$. If x is as in case 3), then $|\phi'(x)| \geq \frac{1}{C}|\phi'(w)|$ for $w \prec x$ and we get the desired inequalities.

We finally show (8.40). Let I denote the modulus of the logarithmic derivative of $h^{\frac{1}{3}} + |\phi'(x)|$ along γ_δ. Then

$$\mathrm{I} \leq \frac{|\phi''|}{h^{\frac{1}{3}} + |\phi'(x)|}$$

which is $\mathcal{O}(h^{-\frac{2}{3}})$ on $\gamma_\delta \cap D(x_0, Ch^{\frac{2}{3}})$ for every $C > 0$, and on $\gamma_\delta \setminus D(x_0, Ch^{\frac{2}{3}})$:

$$\mathrm{I} = \mathcal{O}(1)\frac{|x - x_0|^{-\frac{1}{2}}}{h^{\frac{1}{3}} + |x - x_0|^{\frac{1}{2}}} = \frac{\mathcal{O}(1)}{|x - x_0|}\cdot$$

Summing up the estimates in both regions, we have

$$\mathrm{I} = \frac{\mathcal{O}(1)}{h^{\frac{2}{3}} + |x - x_0|}\cdot$$

The modulus II of the logarithmic derivative with respect to x of $\mathrm{e}^{\int_y^x |\phi'(z)| \cdot |\mathrm{d}z|/h}$ is bounded by $|\phi'(x)|/h$ which is $\mathcal{O}(h^{-\frac{2}{3}})$ in the first region and $\asymp |x - x_0|^{\frac{1}{2}}/h$ in the second region, provided that C is large enough.

It follows that $\mathrm{I} \leq \epsilon \mathrm{II}$, except in the intersection of γ_δ with the disc $|x - x_0| \leq (h/\epsilon)^{\frac{2}{3}}$. The integrals of both I and II over this exceptional region are $\mathcal{O}_\epsilon(1)$ and (8.40) follows. □

Lemma 8.6. — *The following $\mathcal{L}(L^2)$-norms are $\mathcal{O}(1)$:*

$$(h^{\frac{1}{3}} + |\phi'|) \circ K \quad \textit{and} \quad (h^{\frac{1}{3}} + |\phi'|)^2 \circ K \circ (h^{\frac{1}{3}} + |\phi'|)^{-1}.$$

Proof. — We first notice that we can replace $|\phi'(w)|$ to the left in (8.39) by $|\phi'(w)| + h^{\frac{1}{3}}$.

By Schur's lemma, the $\mathcal{L}(L^2)$-norm of $(h^{\frac{1}{3}} + |\phi'|) \circ K$ is bounded by the geometric mean of the following two quantities:

$$\mathrm{I} = \frac{1}{h} \sup_{x\in\gamma_\delta} \int_b^x \big(h^{\frac{1}{3}} + \big|\phi'(x)\big|\big)\, \mathrm{e}^{\frac{1}{h}\Re(\phi(x)-\phi(y))} |\mathrm{d}y|,$$
$$\mathrm{II} = \frac{1}{h} \sup_{y\in\gamma_\delta} \int_0^y \big(h^{\frac{1}{3}} + \big|\phi'(x)\big|\big)\, \mathrm{e}^{\frac{1}{h}\Re(\phi(x)-\phi(y))} |\mathrm{d}x|.$$

Combining (8.38) and (8.39) with $|\phi'(w)|$ replaced by $h^{\frac{1}{3}} + |\phi'(w)|$, we see that for $x \prec y$,

$$\big(h^{\frac{1}{3}} + \big|\phi'(x)\big|\big)\, \mathrm{e}^{\frac{1}{h}\Re(\phi(x)-\phi(y))} \leq \big(h^{\frac{1}{3}} + \big|\phi'(x)\big|\big)\, \mathrm{e}^{C - \frac{1}{Ch}(h^{\frac{1}{3}} + |\phi'(x)|)|x-y|},$$

implying that $\mathrm{I} = \mathcal{O}(1)$.

In order to estimate II, we also use (8.40) to get

$$\begin{aligned}
&\frac{1}{h}\big(h^{\frac{1}{3}} + \big|\phi'(x)\big|\big)\, \mathrm{e}^{\frac{1}{h}\Re(\phi(x)-\phi(y))} \\
&\quad\leq \frac{1}{h}\big(h^{\frac{1}{3}} + \big|\phi'(x)\big|\big)\, \mathrm{e}^{-\frac{1}{Ch}\int_y^x |\phi'(z)|\cdot|\mathrm{d}z|} \\
&\qquad\leq \frac{\widehat{C}}{h}\big(h^{\frac{1}{3}} + \big|\phi'(y)\big|\big)\, \mathrm{e}^{-\frac{1}{2Ch}\int_y^x |\phi'(z)|\cdot|\mathrm{d}z|} \\
&\qquad\quad\leq \frac{1}{h}(h^{\frac{1}{3}} + |\phi'(y)|)\, \mathrm{e}^{\widetilde{C} - \frac{1}{\widetilde{C}h}(h^{\frac{1}{3}} + |\phi'(y)|)|x-y|},
\end{aligned}$$

and it follows that II is $\mathcal{O}(1)$. Thus the $\mathcal{L}(L^2)$-norm of $(h^{\frac{1}{3}} + |\phi'|) \circ K$ is $\mathcal{O}(1)$ as claimed.

The estimate of the norm of $(h^{\frac{1}{3}} + |\phi'|)^2 \circ K \circ (h^{\frac{1}{3}} + |\phi'|)^{-1}$ is just a slight variation of the above arguments, using (8.40) from the start. $\square$

From the definition of K in (8.37) we get

$$-h\partial_x K v = v - \phi' \circ K v, \tag{8.41}$$

and we conclude that

$$h\partial_x \circ K, \;\; \big(h^{\frac{1}{3}} + \big|\phi'\big|\big) h\partial_x \circ K \circ \big(h^{\frac{1}{3}} + \big|\phi'\big|\big)^{-1} \;\text{ are } \mathcal{O}(1) \text{ in } \mathcal{L}(L^2). \tag{8.42}$$

Now, recall that we can get u from $(\phi'+h\partial_x)u =: w$ by integration outwards from $x=0$:

$$u(x)=\frac{1}{h}\int_0^x \mathrm{e}^{-(\phi(x)-\phi(y))/h}w(y)\mathrm{d}y =: Lw. \tag{8.43}$$

The same estimates apply to L and for the solution $u=LKv$ of the equation $(P-z)u=v$, we get

$$\big\|\big(h^{\frac{1}{3}}+|\phi'|\big)^2u\big\|+\big\|\big(h^{\frac{1}{3}}+|\phi'|\big)h\partial_x u\big\|\leq \mathcal{O}(1)\|v\|. \tag{8.44}$$

Recalling that

$$(P-z)=(\phi'-h\partial)(\phi'+h\partial)=(\phi')^2-h\phi''-(h\partial)^2,$$

and that $\phi''=\mathcal{O}(h^{-\frac{1}{3}})$, we also get $\|(h\partial)^2u\|\leq\mathcal{O}(1)\|v\|$ and thus for $u=LKv$:

$$|||u|||:=\big\|\big(h^{\frac{1}{3}}+|\phi'|\big)^2u\big\|+\big\|(h^{\frac{1}{3}}+|\phi'|)h\partial_x u\big\|+\|(h\partial_x)^2u\|\leq\mathcal{O}(1)\|v\|. \tag{8.45}$$

By construction, $u(0)=0$, but the Dirichlet condition at $x=b$ is not necessarily fulfilled. Now, for instance by using a different factorization

$$(P-z)=(\widetilde{\phi}'+h\partial)(\widetilde{\phi}'-h\partial)$$

and some easy iterations, we see that the problem

$$(P-z)e_b=0,\quad e_b(0)=0,\quad e_b(b)=1 \tag{8.46}$$

has a solution on γ_δ which decays exponentially away from b and satisfies $|||e_b|||=\mathcal{O}(h^{\frac{1}{2}})$.

Moreover, we have $u(b)=\mathcal{O}(h^{-\frac{1}{2}})\|v\|$. In fact, (8.45) shows that $\|u\|_{H^2_h}\leq\mathcal{O}(1)\|v\|$, if we take the H^2_h norm over $\{x\in\gamma_\delta;\, a\prec x\prec b\}$, where $a\in\gamma_\delta$ is close to b, and as in (6.6), we have $|u(b)|\leq\mathcal{O}(h^{-\frac{1}{2}})\|u\|_{H^2_h}$. Thus the function $\widetilde{u}=u-u(b)e_b$ solves $(P-z)\widetilde{u}=v$, $\widetilde{u}(0)=\widetilde{u}(b)=0$ and (8.45) remains valid with u replaced by $\widetilde{u}$. Since our Dirichlet problem is Fredholm of index zero, we also know that $\widetilde{u}$ is the unique solution. Dropping the tildes we get:

Proposition 8.7. — *Consider the problem* (8.28) *for z as in* (8.29) *with $\lambda=1/\mathcal{O}(1)$ and let u be the unique solution constructed above. Then,*

$$\big\|\big(h^{\frac{1}{3}}+|\phi'|\big)^2u\big\|+\big\|(h\partial_x)^2u\big\|+\big\|\big(h^{\frac{1}{3}}+|\phi'|\big)h\partial_x u\big\|\leq\mathcal{O}(1)\|v\|, \tag{8.47}$$

where the L^2 norms are taken over γ_δ.

We make a few remarks about extensions and variants. The first is that we can replace ϕ in (8.47) with ϕ_0, the solution of the eiconal equation,

$(\phi_0')^2 = xQ(x) + z$. Indeed, when $|x - x_0| \leq \mathcal{O}(h^{\frac{2}{3}})$ we have $\phi', \phi_0' = \mathcal{O}(h^{\frac{1}{3}})$ and when $|x - x_0| \geq Ch^{\frac{2}{3}}$, then $|\phi'| \asymp |\phi_0'|$.

The second observation is that along γ_δ, if we let x_1 denote the real turning point (given by $x_1 Q(x_1) + \lambda = 0$, $x_1 \asymp -\lambda$, then

$$h^{\frac{1}{3}} + |\phi_0'| \asymp h^{\frac{1}{3}} + |x - x_0|^{\frac{1}{2}} \asymp \big(|x - x_0| + h^{\frac{2}{3}}\big)^{\frac{1}{2}}$$
$$\asymp \big(|x - x_1| + h^{\frac{2}{3}}\big)^{\frac{1}{2}} \asymp \big(s + |\lambda| + h^{\frac{2}{3}}\big)^{\frac{1}{2}},$$

where we write $x = \gamma_\delta(s)$. Thus (8.47) can be written

$$\big\|\big(h^{\frac{2}{3}} + |\lambda| + s\big)u\big\| + \big\|(h\partial_x)^2 u\big\| + \big\|\big(h^{\frac{2}{3}} + |\lambda| + s\big)^{\frac{1}{2}} h\partial_x u\big\| \leq \mathcal{O}(1)\|v\|. \tag{8.48}$$

CHAPTER 9

PARAMETRIX FOR THE EXTERIOR DIRICHLET PROBLEM

Choose geodesic coordinates (x', x_n) with x' being local coordinates on $\partial\mathcal{O}$, so that the exterior of $\mathcal{O}$ is locally given by $x_n > 0$ and $P = -h^2\Delta$ in $\mathbb{R}^n \setminus \mathcal{O}$ becomes (locally near a boundary point):

$$P = (hD_{x_n})^2 + R(x', hD_{x'}) - x_n Q(x, hD_{x'}) + ha(x)hD_{x_n}. \tag{9.1}$$

(Cf. (5.26), (5.27), (5.28).) Here R is an elliptic second order differential operator with principal symbol $r(x', \xi') = |\xi'|^2$. Similarly, Q is elliptic in the x' variables with principal symbol $q(x, \xi') \asymp |\xi'|^2$. For $z = \lambda + h^{\frac{2}{3}} w$ with $\lambda \in \mathbb{R}$, $\lambda \sim 1$, $|w| \leq 1/\mathcal{O}(1)$, we consider

$$\begin{aligned} P(x', \xi') - z &= P(x', x_n, \xi', hD_{x_n}) - z \\ &= (hD_{x_n})^2 + R(x', \xi') - x_n Q(x, \xi') + ha(x)hD_{x_n} - z \end{aligned} \tag{9.2}$$

as an ODO-valued symbol. We let x_n vary in γ_δ, $0 \leq \delta \leq Ch$.

We investigate three different regions in $T^*\partial\mathcal{O}$.

1) (x', ξ') belongs to a small neighborhood of the glancing hypersurface $\mathcal{G}$: $r(x', \xi') = \lambda$. Then the estimates in Section 8.3 apply with λ there replaced by $\lambda - r(x', \xi')$ and from (8.48) we get

$$\begin{aligned} &\left\| \left(h^{\frac{2}{3}} + \left|\lambda - r(x', \xi')\right| + s\right)u \right\| + \left\| (h\partial_{x_n})^2 u \right\| \\ &\quad + \left\| \left(h^{\frac{2}{3}} + \left|\lambda - r(x', \xi')\right| + s\right)^{\frac{1}{2}} h\partial_{x_n} u \right\| \leq \mathcal{O}(1)\|v\|, \end{aligned} \tag{9.3}$$

when $(P(x', \xi') - z)u = v$ along γ_δ, $u(0) = u(b) = 0$.

2) (x', ξ') belongs to the hyperbolic region $r(x', \xi') \leq \lambda - 1/\mathcal{O}(1)$. Then the turning point x_0 is away from 0 and hence also from γ_δ and the estimates of Section 8.3 still apply and give (9.3), where we notice that

$h^{\frac{2}{3}}+|\lambda-r(x',\xi')|+s\asymp 1$:

$$\|u\|+\|h\partial_{x_n}u\|+\big\|(h\partial_{x_n})^2u\big\|\le\mathcal{O}(1)\|v\|. \tag{9.4}$$

Notice that q may be very small in this region but the estimates now work without any reference to a turning point.

3) (x',ξ') belongs to the elliptic region $r(x',\xi')\ge\lambda+1/\mathcal{O}(1)$. When in addition $r(x',\xi')\le\mathcal{O}(1)$ we get (9.4) again. When $r(x',\xi')\gg 1$ we multiply with $|\xi'|^{-2}$ and get

$$|\xi'|^{-2}(P(x',\xi')-z)=(\widetilde{h}D_{x_n})^2+\widetilde{R}-x_n\widetilde{Q}+\widetilde{h}a(x_n)\widetilde{h}D_{x_n}-\widetilde{z}=\widetilde{P}-\widetilde{z},$$

where $\widetilde{R}=|\xi'|^{-2}R(x',\xi')\asymp 1$, $\widetilde{Q}=|\xi'|^{-2}Q\asymp 1$, $\widetilde{h}=h/|\xi'|\ll 1$, $\widetilde{z}=z/|\xi'|^2$, $|\widetilde{z}|\ll 1$. For the rescaled problem the turning point is well off to the right and γ_δ intersects the Stokes lines transversally. We still get (9.4), now for $(\widetilde{P}-\widetilde{z})u=v$ and h replaced by $\widetilde{h}$ and after scaling back, we get

$$\langle\xi'\rangle^2\|u\|+\langle\xi'\rangle\|h\partial_{x_n}u\|+\big\|(h\partial_{x_n})^2u\big\|\le\mathcal{O}(1)\|v\| \tag{9.5}$$

for solutions of (8.28).

For a fixed $\delta\in\{0,Ch\}$, let $\mathcal{B}(x',\xi')$ be the space of functions on γ_δ vanishing at both end points and equipped with the norm given be the left hand side of (9.3), (9.4), (9.5) respectively when (x',ξ') is as in the three cases.

Then $P(x',\xi')-z=\mathcal{O}(1):\mathcal{B}(x',\xi')\to L^2(\gamma_\delta)$ and has an inverse $E(x',\xi')$ which is $\mathcal{O}(1):L^2(\gamma_\delta)\to\mathcal{B}(x',\xi')$.

Outside a fixed neighborhood of the glancing hypersurface, we have the nice symbol properties

$$\partial_{x'}^\alpha\partial_{\xi'}^\beta P=\mathcal{O}_{\alpha,\beta}\big(\langle\xi'\rangle^{-|\beta|}\big):\mathcal{B}(x',\xi')\longrightarrow L^2(\gamma_\delta). \tag{9.6}$$

Near the glancing hypersurface we have a poblem when derivatives fall on R and we get the weaker estimate

$$\partial_{x'}^\alpha\partial_{\xi'}^\beta P=\mathcal{O}_{\alpha,\beta}(1)\big(h^{\frac{2}{3}}+\big|\lambda-r(x',\xi')\big|\big)^{-(|\alpha|+|\beta|)}. \tag{9.7}$$

This is the reason why traditionally (as in [**31**], [**33**] and other works cited there) one uses some form of second microlocalization. If (x_0,ξ_0) is a point on the glancing hypersurface, we conjugate $P(x,hD)$ with a microlocally defined elliptic Fourier integral operator acting in the tangential variables and get a new operator of the form (9.1) where now R, Q are tangential classical h-pseudodifferential operators and a is replaced by $a(x,hD_{x'};h)$, a classical pseudodifferential operator of order 0 in h, and where

$$R(x',\xi')=\xi_1. \tag{9.8}$$

(See Sections 4 and 5 in [**31**] and [**33**] respectively.) Then the problem appears only when we differentiate with respect to ξ_1:

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}P = \mathcal{O}_{\alpha,\beta}(1)\big(h^{\frac{2}{3}} + \big|\lambda - r(x',\xi')\big|\big)^{-\beta_1}. \tag{9.9}$$

Differentiating the identity $(P-z)E = 1$, we get with $\partial^{\alpha} = \partial^{\alpha}_{x',\xi'}$:

$$(P-z)\partial^{\alpha}E = \sum_{\substack{\alpha'+\alpha''=\alpha\\ \alpha'\neq 0}} c_{\alpha',\alpha''}(\partial^{\alpha'}P)(\partial^{\alpha''}E),$$

and after applying E to the right and using that $E(P-z) = 1$,

$$\partial^{\alpha}E = \sum_{\substack{\alpha'+\alpha''=\alpha\\ \alpha'\neq 0}} c_{\alpha',\alpha''}E(\partial^{\alpha'}P)(\partial^{\alpha''}E).$$

By induction we then get

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}E = \mathcal{O}_{\alpha,\beta}\big(\langle\xi'\rangle^{-|\beta|}\big) : L^2(\gamma_\delta) \longrightarrow \mathcal{B}(x',\xi'), \tag{9.10}$$

outside any fixed neighborhood of the glancing hypersurface $\mathcal{G}$. Near any fixed point of $\mathcal{G}$, we get

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}E = \mathcal{O}_{\alpha,\beta}(1)\big(h^{\frac{2}{3}} + \big|\lambda - r(x',\xi')\big|\big)^{-\beta_1}, \tag{9.11}$$

after conjugation with an elliptic tangential Fourier integral operator, that reduces R to ξ_1.

We now turn to the n-dimensional situation and recall the definition of the singular contour Γ_f in (5.12) and its exterior part $\Gamma_{\mathrm{ext},f}$, where f satisfies (5.31). We take $\theta = \frac{1}{3}\pi$ there and put $\Gamma_0 = \Gamma_f$. For $\delta > 0$, let $\mathcal{O}_{-\delta} = \mathcal{O} + B(0,\delta)$. Then $\mathrm{dist}(x,\mathcal{O}_{-\delta}) = \max(d(x)-\delta, 0)$. Let f_δ be as in (5.31) with $d(x)$ replaced by $\mathrm{dist}(\,.\,,\mathcal{O}_{-\delta})$, still with $\theta = \frac{1}{3}\pi$. Put $\Gamma_\delta = \Gamma_{f_\delta}$. In this section we only work on the exterior parts $\Gamma_{\mathrm{ext},\delta}$ and for simplicity we drop the subscript "ext". Using geodesic coordinates we have

$$\Gamma_{\delta,b} := \big\{x;\, x' \in \partial\mathcal{O},\ x_n \in \gamma_\delta\big\} \subset \Gamma_\delta. \tag{9.12}$$

(Later on we will also include $\mathcal{O}$ into the contour Γ_δ and the Γ_δ above will then be renamed $\Gamma_{\delta,\mathrm{ext}}$.)

Let $\mathcal{B}_b$ be the space of functions $u = u(x',x_n)$ on $\Gamma_{\delta,b}$ with $u(x',0) = u(x',b) = 0$ for which the following norm is finite:

$$\|u\|_{\mathcal{B}} = h^{\frac{2}{3}}\|u\| + \big\|\big(R(x',hD_{x'}) - \lambda\big)u\big\| + \|su\| + \big\|(h\partial_{x_n})^2u\big\|. \tag{9.13}$$

Continuing to treat P as a pseudodifferential operator on $\partial\mathcal{O}$ with operator valued symbol, we obtain a right parametrix of $P-z$ in the following way (cf. [**31**], [**33**]):

Let $\chi_1, \dots, \chi_N \in C_0^\infty(T^*\partial\mathcal{O})$ have their supports in small neighborhoods of the points $\rho_1, \dots, \rho_N \in \mathcal{G}$ that we assume are "evenly distributed" on $\mathcal{G}$ with N sufficiently large and so that $\sum_1^N \chi_j = 1$ near $\mathcal{G}$. Put $\chi_0 = 1 - \sum_1^N \chi_j$. Define corresponding tangential pseudodifferential operators $\chi_j(x', hD_{x'})$ on $\partial\mathcal{O}$ in the standard way, so that $\sum_1^N \chi_j(x', hD_{x'}) = 1$ microlocally near $\mathcal{G}$. With suitable choices of the above quantities, there exist semi-classical elliptic Fourier integral operators of order 0, defined microlocally near ρ_j, such that $R(x', hD_{x'}) = U_j hD_{x_1} U_j^{-1}$ microlocally near $\operatorname{supp}\chi_j$ where U_j^{-1} denotes a microlocal inverse of U_j. Then our parametrix of $P - z$ is an operator $E = \mathcal{O}(1) : L^2(\Gamma_{\delta,b}) \to \mathcal{B}_b$ of the form

$$E = E_0\chi_0(x', hD_{x'}) + \sum_1^N U_j E_j(x', hD_{x'}) U_j^{-1} \chi_j(x', hD_{x'}). \tag{9.14}$$

Here the symbol $E_0(x', \xi')$ belongs to the space $S^0(T^*\partial\mathcal{O}; \mathcal{L}(L^2, \mathcal{B}_b))$ of symbols that satisfy (9.10) and has an asymptotic expansion,

$$E_0 \sim E_{0,0} + hE_{0,1} + h^2 E_{0,2} + \cdots , \tag{9.15}$$

with $E_{0,k} \in S^{-k}$, the space of symbols F satisfying

$$\partial_{x'}^\alpha \partial_{\xi'}^\beta F = \mathcal{O}_{\alpha,\beta}\big(\langle\xi'\rangle^{-k-|\beta|}\big) : L^2(\gamma_\delta) \longrightarrow \mathcal{B}(x', \xi').$$

Moreover, $E_{0,0} = (P(x', \xi') - z)^{-1}$.

For $j = 1, \dots, N$, E_j has the property (9.11) with $r = \xi_1$ and we have an asymptotic expansion

$$E_j \sim E_{j,0} + h^{\frac{1}{3}} E_{j,1} + \cdots , \tag{9.16}$$

with $E_{j,k}$ satisfying (9.11) and with $E_{j,0} = (P(x', \xi') - z)^{-1}$ where it is understood that $P(x', \xi')$ is now simplified with the conjugation by U_j so that $R(x', hD_{x'})$ has become hD_{x_1}. The main property of E is that

$$\big(P(x, hD) - z\big)E = 1 + \mathcal{O}(h^\infty) \ \text{ in } \mathcal{L}(L^2, L^2). \tag{9.17}$$

We can also construct a left parametrix $\widetilde{E}$ with an expression similar to (9.14) but with the cutoff operators to the left, and by a standard argument we see that $\widetilde{E} = E + \mathcal{O}(h^\infty)$ in $\mathcal{L}(L^2, \mathcal{B}_b)$.

Summing up the discussion so far, we have:

Proposition 9.1. — *We can construct an operator* $E = \mathcal{O}(1) : L^2(\Gamma_{\delta,b}) \to \mathcal{B}_b$ *as above, so that*

$$\begin{cases} \big(P(x,hD) - z\big)E = 1 + \mathcal{O}(h^\infty) \text{ in } \mathcal{L}(L^2, L^2), \\ E\big(P(x,hD) - z\big) = 1 + \mathcal{O}(h^\infty) \text{ in } \mathcal{L}(\mathcal{B}_b, \mathcal{B}_b). \end{cases} \tag{9.18}$$

We now consider $P = -h^2\Delta$ on all of Γ_δ and notice that $P - z$ is semiclassically elliptic away from any fixed neighborhood of $\partial\mathcal{O}$, so we have a pseudodifferential parametrix $Q(x, hD; h)$ in that region with symbol $Q(x, \xi; h)$ satisfying $\partial_x^\alpha \partial_\xi^\beta Q = \mathcal{O}(\langle\xi\rangle^{-2-|\beta|})$ such that if $\chi \in C^\infty(\Gamma_\delta)$ is a standard cutoff to a small neighborhood of $\partial\mathcal{O}$, then

$$(P - z)Q(1 - \chi) = (1 - \chi) + K_1,$$
$$(1 - \chi)Q(P - z) = (1 - \chi) + K_2,$$

where K_1, K_2 are negligible operators $\mathcal{O}(h^\infty) : H_h^{-s} \to H_h^s$ for every $s \geq 0$. Further, we may arrange so that the distribution kernel $K_Q(x, y)$ of Q vanishes when $|x - y| > \epsilon$, for any fixed given $\epsilon > 0$.

Assuming that $\operatorname{supp}\chi \subset \Gamma_{\delta,b}$, we choose $\epsilon > 0$ small enough and put

$$F = \chi E \chi + Q(1 - \chi) - Q[P, \chi]E\chi. \tag{9.19}$$

Then, $F = \mathcal{O}(1) : L^2(\Gamma_\delta) \to \mathcal{B}(\Gamma_\delta)$ and

$$(P - z)F = 1 + K_3,$$

where $K_3 = \mathcal{O}(h^\infty) : L^2 \to L^2$. Here $\mathcal{B}(\Gamma_\delta)$ denotes the space of distributions u such that $\chi u \in \mathcal{B}(\Gamma_{\delta,b})$, $(1 - \chi)u \in H_h^2(\Gamma_\delta)$. The construction of a left parametrix is similar, and by a standard argument we see that F is also a left parametrix. Summing up, we get:

Proposition 9.2. — *The operator* F *in* (9.19) *is* $\mathcal{O}(1) : L^2(\Gamma_\delta) \to \mathcal{B}(\Gamma_\delta)$ *and satisfies*

$$(P - z)F = 1 + K_3, \ F(P - z) = 1 + K_4, \tag{9.20}$$

where $K_3 = \mathcal{O}(h^\infty) : L^2(\Gamma_\delta) \to L^2(\Gamma_\delta)$, $K_4 = \mathcal{O}(h^\infty) : \mathcal{B}(\Gamma_\delta) \to \mathcal{B}(\Gamma_\delta)$.

CHAPTER 10

EXTERIOR POISSON OPERATOR AND DN MAP

We need some more estimates in the one dimensional case. Recall that if $u \in C_0^\infty([0,\infty[)$, then

$$\big|u(0)\big|^2 \leq 2\|u\| \cdot \|\partial u\|. \tag{10.1}$$

If $u \in C^\infty([0,\infty[)$, let $\chi \in C_0^\infty([0,\infty[)$, $\chi(0)=0$ and put $\chi_L(x)=\chi(x/L)$. Applying (10.1) to $\chi_L u$ gives

$$\big|u(0)\big|^2 \leq C\Big(\frac{1}{L}\|u\|^2_{[0,L]} + \|u\|_{[0,L]}\| \cdot \partial u\|_{[0,L]}\Big). \tag{10.2}$$

If $\Lambda > 0$ is a continuous function on $[0,\infty[$ of increasing order of magnitude ($\Lambda(x) \geq C^{-1}\Lambda(y)$ when $x \geq y$) we get

$$\big|u(0)\big|^2 \leq C\Big(\frac{1}{L\Lambda(0)^2}\|\Lambda u\|^2_{[0,L]} + \frac{1}{h\Lambda(0)}\|\Lambda u\|_{[0,L]} \cdot \|h\partial u\|_{[0,L]}\Big).$$

Choose L so that $L\Lambda(0)^2 = h\Lambda(0)$, $L = h/\Lambda(0)$. Then,

$$\begin{cases} \big|u(0)\big|^2 \leq \dfrac{C}{h\Lambda(0)}\big(\|\Lambda u\|^2_{[0,h/\Lambda(0)]} + \|h\partial u\|^2_{[0,h/\Lambda(0)]}\big), \\ \sqrt{h\Lambda(0)}\,\big|u(0)\big| \leq C\big(\|\Lambda u\|_{[0,h/\Lambda(0)]} + \|h\partial u\|_{[0,h/\Lambda(0)]}\big). \end{cases} \tag{10.3}$$

Recall that for (x',ξ') near a point on the glancing hypersurface, $r=\lambda$,

$$\|u\|_{\mathcal{B}(x',\xi')} = \|\Lambda^2 u\| + \|\Lambda h\partial_{x_n} u\| + \big\|(h\partial_{x_n})^2 u\big\|, \tag{10.4}$$

where $\Lambda^2 = (h^{\frac{2}{3}} + |r-\lambda| + s)$, $r = r(x',\xi')$, $x_n = \gamma_\delta(s)$, $0 \prec x \prec b$. Since Λ is increasing, we can apply (10.3) and estimate $|u(0)|$ with the first two terms in the $\mathcal{B}$-norm and $|h\partial_{x_n}u(0)|$ using the last two terms:

$$h^{\frac{1}{2}}\Lambda(0)^{\frac{3}{2}}\big|u(0)\big| \leq C\|u\|_{\mathcal{B}}, \tag{10.5}$$

$$h^{\frac{1}{2}}\Lambda(0)^{\frac{1}{2}}\big|h\partial_{x_n}u(0)\big| \leq C\|u\|_{\mathcal{B}}, \tag{10.6}$$

or more explicitly,

$$h^{\frac{1}{2}}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{\frac{3}{4}}\big|u(0)\big|\leq C\|u\|_{\mathcal{B}}, \tag{10.7}$$

$$h^{\frac{1}{2}}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{\frac{1}{4}}\big|h\partial_{x_n}u(0)\big|\leq C\|u\|_{\mathcal{B}}. \tag{10.8}$$

We next estimate the $\mathcal{B}(x',\xi')$-norm of the null-solution in (8.33),

$$u=e_{x',\xi'}=\mathrm{e}^{-\frac{1}{h}\phi(x_n;h)},\quad \phi(x_n;h)=\phi_{x',\xi'}(x_n;h),\quad \phi(0)=0,$$

of $(P(x',\xi')-z)u=0$ along γ_δ. We know that

$$\big(h^{\frac{1}{3}}+|\phi'|\big)^2\asymp h^{\frac{2}{3}}+|r-\lambda|+s,\quad \big(x_n=\gamma_\delta(s)\big),$$

and that

$$\Re\partial_s\phi\asymp|\phi'|\geq\frac{1}{C}\big(h^{\frac{2}{3}}+|r-\lambda|+s\big)^{\frac{1}{2}}$$

when $s+|r-\lambda|\gg h^{\frac{2}{3}}$. Thus with $b=\gamma_\delta(s_0)$,

$$\|e_{x',\xi'}\|^2=\int_0^{s_0}\mathrm{e}^{-\frac{2}{h}\Re\phi(x_n(s))}\mathrm{d}s\leq\int_0^{\infty}\mathrm{e}^{-\frac{1}{Ch}(h^{\frac{2}{3}}+|r-\lambda|)^{\frac{1}{2}}s}\mathrm{d}s,$$

which leads to

$$\|e_{x',\xi'}\|\leq\frac{\mathcal{O}(1)h^{\frac{1}{2}}}{(h^{\frac{2}{3}}+|r-\lambda|)^{\frac{1}{4}}}.$$

We will also use that the same estimate holds for $\|e_{x',\xi'}^{\frac{1}{2}}\|$.

Next look at

$$\big\|(h^{\frac{2}{3}}+|r-\lambda|+s)e_{x',\xi'}\big\|=\big(h^{\frac{2}{3}}+|r-\lambda|\big)\left\|\frac{h^{\frac{2}{3}}+|r-\lambda|+s}{h^{\frac{2}{3}}+|r-\lambda|}e_{x',\xi'}\right\|.$$

From Lemma 8.5 we see that

$$\frac{h^{\frac{2}{3}}+|r-\lambda|+s}{h^{\frac{2}{3}}+|r-\lambda|}\mathrm{e}_{x',\xi'}^{\frac{1}{2}}$$

is bounded, so

$$\left\|\frac{h^{\frac{2}{3}}+|r-\lambda|+s}{h^{\frac{2}{3}}+|r-\lambda|}e_{x',\xi'}\right\|\leq\mathcal{O}(1)\|\mathrm{e}_{x',\xi'}^{\frac{1}{2}}\|\leq\frac{\mathcal{O}(1)h^{\frac{1}{2}}}{(h^{\frac{2}{3}}+|r-\lambda|)^{\frac{1}{4}}}.$$

Thus,

$$\big\|(h^{\frac{2}{3}}+|r-\lambda|+s)e_{x',\xi'}\big\|\leq\mathcal{O}(1)h^{\frac{1}{2}}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{\frac{3}{4}}.$$

The other terms in the $\mathcal{B}$ norm of u satisfy the same estimates and we get

$$\|e_{x',\xi'}\|_{\mathcal{B}}\leq\mathcal{O}(1)h^{\frac{1}{2}}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{\frac{3}{4}}. \tag{10.9}$$

Since $e_{x',\xi'}(0)=1$, we see that this is the reverse inequality to (10.7) up to a bounded factor, so

$$\|e_{x',\xi'}\|_{\mathcal{B}} \asymp h^{\frac{1}{2}}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{\frac{3}{4}}. \tag{10.10}$$

REMARK 10.1. — Using that $e_{x',\xi'}(b)=\mathcal{O}(\mathrm{e}^{-\frac{1}{Ch}})$, we can add an exponentially small reflected term as in (8.46) to get a null solution which vanishes at b and after dividing with a factor $1+\mathcal{O}(\mathrm{e}^{-\frac{1}{Ch}})$ we get a new function $e_{x',\xi'}$ satisfying $(P_{x',\xi'}-z)e_{x',\xi'}=0$, $e_{x',\xi'}(0)=1$, $e_{x',\xi'}(b)=0$ as well as the estimate (10.10).

Recall that $P(x',\xi')-z:\mathcal{B}(x',\xi')\to L^2$ has a uniformly bounded inverse $E(x',\xi')$ and that we have the estimates (9.9), (9.11). Differentiate the equation $(P(x',\xi')-z)e_{x',\xi'}=0$ and notice that $\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}e_{x',\xi'}(0)=\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}e_{x',\xi'}(b)=0$ when $|\alpha|+|\beta|\neq 0$, so that $\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}e_{x',\xi'}\in\mathcal{B}$. We get

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}e_{x',\xi'}=\sum_{\substack{\alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta\\ |\alpha''|+|\beta''|<|\alpha|+|\beta|}} c_{\alpha',\alpha'',\beta',\beta''}E(\partial_{x'}^{\alpha'}\partial_{\xi'}^{\beta'}P)(\partial_{x''}^{\alpha''}\partial_{\xi''}^{\beta''}e_{x',\xi'}). \tag{10.11}$$

By induction, we see that

$$\|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}e_{x',\xi'}\|_{\mathcal{B}}=\mathcal{O}(1)h^{\frac{1}{2}}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{\frac{3}{4}-\beta_1}. \tag{10.12}$$

As a first approximation to the Poisson operator on $\Gamma_{\delta,b}$, we take

$$K^0w=\mathrm{Op}_h(e_{x',\xi'}) \tag{10.13}$$

where Op_h denotes the classical h-quantization in $\mathbb{R}^{n-1}$ also in the case of vector and operator valued symbols, so that our K^0 is microlocally defined in $T^*(\partial\mathcal{O})$ and maps functions of x' to functions of x. (Here it is tacitly assumed that we have reduced R to hD_{x_1} as in (9.11).) Then

$$\gamma K^0=1, \tag{10.14}$$

$$(P-z)K^0=\mathrm{Op}_h(f_{x',\xi'}), \tag{10.15}$$

where

$$f_{x',\xi'}\sim\sum_{\alpha\neq 0}\frac{h^{|\alpha|}}{\alpha!}\partial_{\xi'}^{\alpha}P(x',\xi')D_{x'}^{\alpha}e_{x',\xi'} \tag{10.16}$$

and we have used that $(P(x',\xi')-z)e_{x',\xi'}=0$. From (9.9), (10.12), we see that

$$\|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}f_{x',\xi'}\|_{L^2}=\mathcal{O}(1)h^{\frac{3}{2}}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{-\frac{1}{4}-\beta_1}.$$

We get the microlocal Poisson operator to all orders in h by putting

$$\widetilde{K} = K^0 - E \circ (P-z)K^0.$$

Here

$$E(P-z)K^0 w = \mathrm{Op}_h(\widetilde{r}),$$

where

$$\|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}\widetilde{r}\|_{\mathcal{B}_{x',\xi'}} = \mathcal{O}(1)h^{\frac{3}{2}}(h^{\frac{2}{3}} + |r-\lambda|)^{-\frac{1}{4}-\beta_1}. \tag{10.17}$$

This bound is "better" than (10.12) by a factor

$$h(h^{\frac{2}{3}} + |r-\lambda|)^{-1} \leq h^{\frac{1}{3}},$$

thus we get

$$\widetilde{K}w = \mathrm{Op}_h(e_{x',\xi'} + \widetilde{r}_{x',\xi'}), \tag{10.18}$$

solving

$$\gamma\widetilde{K} = 1, \quad (P-z)\widetilde{K} = \mathcal{O}(h^{\infty}) : \, L^2 \longrightarrow \mathcal{B}. \tag{10.19}$$

As in Proposition 9.2 it is now routine to show that the exact exterior Poisson operator is microlocally given by (10.18) near any fixed point of the glancing hypersurface $\mathcal{G}$.

Away from $\mathcal{G}$ the construction of a Poisson operator on $\Gamma_{\delta,b}$ and on Γ_{δ} is more routine and we omit the details. Using a truncation as in the preceding chapter, we can carry over the construction from $\Gamma_{\delta,b}$ to Γ_{δ}. The preceding chapter gives an approximate Green operator for the exterior problem while the present chapter does the same for the Poisson operator. By simple Neumann series we can replace approximate solution operators by the exact ones and get the following result that summarizes the constructions of this and the preceding sections where we start to use the notation $\Gamma_{\delta}^{\mathrm{ext}}$ to emphasize that $\mathcal{O}$ is not part of this contour.

Proposition 10.2. — *The exterior Dirichlet problem*

$$(P-z)u = v, \;\; \gamma u = w, \quad \text{on } \Gamma_{\delta}^{\mathrm{ext}}, \tag{10.20}$$

where γ is the operator of restriction to the boundary, has a unique solution $u \in H_h^2(\Gamma_{\delta}^{\mathrm{ext}})$ for every $(v,w) \in L^2(\Gamma_{\delta}^{\mathrm{ext}}) \times H_h^{\frac{3}{2}}(\partial\mathcal{O})$, of the form

$$u = G_{\mathrm{ext}}v + K_{\mathrm{ext}}w. \tag{10.21}$$

If $\chi \in C^\infty(\Gamma_\delta^{\mathrm{ext}})$ has its support away from a fixed distance to $\partial\mathcal{O}$ and is equal to one near infinity (and satisfies uniform estimates with all its derivatives when $h \to 0$), then

$$\chi G_{\mathrm{ext}},\ G_{\mathrm{ext}}\chi = \mathcal{O}(1) : L^2 \longrightarrow H_h^2, \tag{10.22}$$

$$\chi K_{\mathrm{ext}} = \mathcal{O}(h^\infty) : H_h^{\frac{3}{2}}(\partial\mathcal{O}) \longrightarrow H_h^2. \tag{10.23}$$

If we choose local geodesic coordinates x', x_n near a boundary point, then near that point G_{ext} is a pseudodifferential operator with operator valued symbol,

$$G_{\mathrm{ext}} = E(x', hD_{x'}; h), \tag{10.24}$$

where E fulfills (9.10), (9.11) (and for the latter estimate it is assumed that P has been conjugated by a tangential Fourier integral operator in order to straighten out $R - \lambda$).

In the same coordinates

$$\chi K_{\mathrm{ext}} = K(x', hD_{x'}; h), \tag{10.25}$$

where

$$\left\| \partial_{x'}^\alpha \partial_{\xi'}^\beta K(x', \xi'; h) \right\|_{\mathcal{B}_{x',\xi'}} = \mathcal{O}(1) h^{\frac{1}{2}} \left(h^{\frac{2}{3}} + |r - \lambda| \right)^{\frac{3}{4} - \beta_1} \tag{10.26}$$

near $\mathcal{G}$ (after straightening of $R - \lambda$), while away from $\mathcal{G}$:

$$\left\| \partial_{x'}^\alpha \partial_{\xi'}^\beta K(x', \xi'; h) \right\|_{\mathcal{B}_{x',\xi'}} = \mathcal{O}(1) h^{\frac{1}{2}} \langle \xi' \rangle^{-\frac{3}{2} - |\beta|}. \tag{10.27}$$

By construction, $G_{\mathrm{ext}} = \mathcal{O}(1) : L^2 \to \mathcal{B}$ near $\partial\mathcal{O}$ and (cf. (10.4)) we get the first part of

Corollary 10.3. — *We have*

$$G_{\mathrm{ext}} = \mathcal{O}(h^{-\frac{2}{3}}) : L^2 \longrightarrow H_h^2, \tag{10.28}$$

$$K_{\mathrm{ext}} = \mathcal{O}(h^{-\frac{1}{6}}) : H_h^{\frac{3}{2}}(\partial\mathcal{O}) \longrightarrow H_h^2. \tag{10.29}$$

For the second part, we combine (10.4) and (10.27).

Finally, we consider the exterior Dirichlet to Neumann (DN) map

$$\mathcal{N}_{\mathrm{ext}} = hD_\nu K_{\mathrm{ext}}, \tag{10.30}$$

where ν denotes the exterior unit normal. From (10.25), (10.26), (10.8), we see that this is a pseudodifferential operator with symbol

$$\gamma h D_{x_n}\big(K(x', \xi'; h)\big) =: n_{\mathrm{ext}}(x', \xi'; h)$$

satisfying

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta} n_{\mathrm{ext}}\big(x', \xi'; h) = \mathcal{O}(\langle \xi' \rangle^{1-|\beta|}\big) \tag{10.31}$$

away from $\mathcal{G}$ and

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta} n_{\mathrm{ext}}(x', \xi'; h) = \mathcal{O}(1)\big(h^{\frac{2}{3}} + |r - \lambda|\big)^{\frac{1}{2}-\beta_1}, \tag{10.32}$$

near $\mathcal{G}$ after the usual straightening. In particular, we have

Corollary 10.4. — *For every $s \in \mathbb{R}$ we have that*

$$\mathcal{N}_{\mathrm{ext}} = \mathcal{O}(1) : H_h^{s+1} \longrightarrow H_h^{s}.$$

CHAPTER 11

THE INTERIOR DN MAP

We work here inside $\mathcal{O}$ and assume that

$$P = -h^2\Delta + V(x), \tag{11.1}$$

where we will first assume only that $V \in L^\infty(\mathcal{O};\mathbb{R})$ and soon make stronger assumptions. The results will be applied to V_0 in (2.3), but for simplicity we drop the subscript 0 in this chapter.

We study the interior Poisson operator $K_{\text{in}}(z) = H^{\frac{3}{2}}(\partial\mathcal{O}) \to H^2(\mathcal{O})$ associated to $P - z$ and the interior DN-map

$$\mathcal{N}_{\text{in}} = \gamma h D_\nu K_{\text{in}} : H^{\frac{3}{2}}(\partial\mathcal{O}) \longrightarrow H^{\frac{1}{2}}(\partial\mathcal{O}) \tag{11.2}$$

under the assumption that,

$$\Re z = \lambda \asymp 1, \quad \frac{h^{\frac{2}{3}}}{\mathcal{O}(1)} \leq |\Im z| \leq \mathcal{O}(1)h^{\frac{2}{3}}. \tag{11.3}$$

Using the right inverse of γ in (6.7), we can write

$$K_{\text{in}} = \gamma^{-1} - (P_{\text{in}} - z)^{-1}\gamma^{-1}$$

and see that

$$\big\|K_{\text{in}}(z)\big\|_{\mathcal{L}(H^{\frac{3}{2}},H^2)} = \mathcal{O}(1)(h^{\frac{1}{2}} + h^{-\frac{2}{3}+\frac{1}{2}}) = \mathcal{O}(1)h^{-\frac{1}{6}} \tag{11.4}$$

where P_{in} is the Dirichlet realization of P. Consequently,

$$\|\mathcal{N}_{\text{in}}(z)\|_{\mathcal{L}(H^{\frac{3}{2}},H^{\frac{1}{2}})} \leq \mathcal{O}(h^{-\frac{1}{2}})\big\|K_{\text{in}}(z)\big\|_{\mathcal{L}(H^{\frac{3}{2}},H^2)} = \mathcal{O}(h^{-\frac{2}{3}}). \tag{11.5}$$

We now assume that

$$V \in C^\infty(\overline{\mathcal{O}};\mathbb{R}), \quad \gamma V = 0, \quad \gamma\partial_\nu V \leq 0, \tag{11.6}$$

where the last two assumptions can be somewhat weakened. Using parametrix constructions, we shall improve the estimate (11.5) to:

PROPOSITION 11.1. — *Under the assumption* (11.3), *we have*

$$\big\|\mathcal{N}_{\text{in}}(z)\big\|_{\mathcal{L}(H^{\frac{3}{2}},H^{\frac{1}{2}})} = \mathcal{O}(1). \tag{11.7}$$

Proof. — We make parametrix constructions in different regions of $T^*\partial\mathcal{O}$ and start with the hyperbolic region

$$\mathcal{H} = \big\{(x',\xi') \in T^*\partial\mathcal{O};\, r(x',\xi') < \lambda\big\},$$

where we write the operator in geodesic coordinates (with $\mathcal{O}$ given by $x_n \leq 0$) as in (9.1). Near a point $(x_0',\xi_0') \in \mathcal{H}$ we construct a microlocal approximation to the Poisson operator of the form

$$\widetilde{K}_{\text{in}}(z)w(x) = \frac{1}{(2\pi h)^{n-1}} \iint \mathrm{e}^{\frac{i}{h}(\phi(x,\eta')-y'\eta')} a(x,\eta';h)w(y')\mathrm{d}y'\mathrm{d}\eta'. \tag{11.8}$$

We write P as in (9.1):

$$\begin{cases} P = (hD_{x_n})^2 + R(x,hD_{x'}) + ha(x)hD_{x_n}, \\ R(x,hD_{x'}) = R(x',hD_{x'}) - x_nQ(x,hD_{x'}), \end{cases} \tag{11.9}$$

where we recall that V is incorporated in P and hence in the term $-x_nQ$ and the condition (11.6) together with the strict convexity of $\mathcal{O}$ assures that $q > 0$ for $\xi' \neq 0$. Recall that a can be eliminated and assume for simplicity that $a = 0$. As before p denotes the semi-classical principal symbol of P.

Now consider the eiconal equation

$$p(x,\phi') - z = 0 \quad \text{for } x \in \text{neigh}(x_0',0)\cap\mathcal{O}, \quad \phi(x',0,\eta') = x'\eta'. \tag{11.10}$$

With $r(x,\xi') = r(x',\xi') - x_nq(x,\xi')$ it becomes

$$\partial_{x_n}\phi = \pm\big(\lambda + h^{\frac{2}{3}}w - r(x,\phi'_{x'})\big)^{\frac{1}{2}}, \quad \mp\Im w > 0.$$

Using the principal branch of the square root we choose the sign as indicated. If ϕ_0 is the real solution of the corresponding eiconal problem when $w = 0$, we can solve (11.10) to all orders in h by the asymptotic expansion,

$$\phi(x,\eta') = \phi_0(x,\eta') + h^{\frac{2}{3}}\phi_1(x,\eta') + h^{\frac{4}{3}}\phi_2(x,\eta';h),$$

where $\phi_1,\phi_2,\ldots. = \mathcal{O}(x_n)$,

$$\partial_{x_n}\phi_1 = \pm\frac{1}{2}\big(\lambda - r(x,\partial_{x'}\phi_0)\big)^{-\frac{1}{2}}w,$$

so that

$$\Im\phi \asymp h^{\frac{2}{3}}\Im\phi_1 \asymp |x_n\Im w|h^{\frac{2}{3}}. \tag{11.11}$$

By solving the transport equations in the usual way, we get the amplitude a as a symbol of order 0 and if $\chi \in C_0^\infty(\mathcal{H})$ has its support in a small neighborhood of (x_0', ξ_0') we get a Fourier integral operator $\widetilde{K}_{\text{in}}(z) : C^\infty(\partial\mathcal{O}) \to C^\infty(\overline{\mathcal{O}})$ solving

$$(P - z)\widetilde{K}_{\text{in}}(z) = \mathcal{O}(h^\infty) : \mathcal{D}'(\partial\mathcal{O}) \longrightarrow C^\infty(\overline{\mathcal{O}}), \tag{11.12}$$

$$\gamma\widetilde{K}_{\text{in}}(z) = \chi(x', hD_{x'}). \tag{11.13}$$

Here (11.11) is important, since it assures that the distribution kernel $\widetilde{K}_{\text{in}}(x, y', z)$ of $\widetilde{K}_{\text{in}}(z)$ is $\mathcal{O}(h^\infty)$ with all its derivatives when $\operatorname{dist}(x, \partial\mathcal{O}) \geq h^{\frac{1}{3}-\delta}$ for any fixed $\delta > 0$. (Another standard fact, implicitly used here, is that the distribution kernel is $\mathcal{O}(h^\infty)$ with all its derivatives as soon as (x', y') is outside any fixed neighborhood of the diagonal.)

From (11.11) we get additional damping, leading to

$$\widetilde{K} = \mathcal{O}(h^{\frac{1}{6}}) : H_h^{\frac{3}{2}} \longrightarrow H_h^2. \tag{11.14}$$

It also follows that

$$\gamma hD_\nu \widetilde{K}_{\text{in}}(z) = \widetilde{\chi}(x', hD_{x'}; h) \tag{11.15}$$

where $\widetilde{\chi}(x', \xi'; h)$ is a classical symbol of order 0 in h and of order $-\infty$ in ξ' which is $\mathcal{O}(h^\infty)$ with all its derivatives outside any fixed neighborhood of the support of χ.

A similar even more standard construction works in the elliptic region

$$\mathcal{E} = \big\{(x', \xi') \in T^*\partial\mathcal{O};\ r(x', \xi') > \lambda\big\}.$$

We get an operator $\widehat{K} = \mathcal{O}(h^{\frac{1}{2}}) : H_h^{\frac{3}{2}} \to H_h^2$ such that

$$(P - z)\widehat{K} = \mathcal{O}(h^\infty), \tag{11.16}$$

$$\gamma\widehat{K} = 1 - \chi(x', hD_{x'}), \tag{11.17}$$

$$\gamma hD_\nu \widehat{K} = n_\chi(x', hD_{x'}; h), \tag{11.18}$$

where $\chi \in C_0^\infty(T^*\partial\mathcal{O})$ is any function equal to one in a neighborhood of $\mathcal{G} \cup \mathcal{H}$. $\widetilde{\chi}$ has the same properties as χ and $n_\chi \in S^1(T^*\partial\Omega)$ is equal to $\mathcal{O}(h^\infty)$ with all its derivatives away from $\operatorname{supp}(1 - \chi)$.

We next turn to the more difficult study near the glancing hypersurface

$$\mathcal{G} = \big\{(x', \xi') \in T^*\partial\mathcal{O};\ r(x', \xi') = \lambda\big\},$$

and we shall start by pushing the construction in $\mathcal{H}$ closer to $\mathcal{G}$ and almost up to a distance $\gg h^{\frac{2}{3}}$ from that set. We write the operator in geodesic coordinates as in (9.1). Let $\rho_0 = (x_0', \xi_0') \in \mathcal{G}$ and assume, after conjugation with an elliptic tangential Fourier integral operator that microlocally,

$$R(x', hD_{x'}) - \lambda = hD_{x_1},\ (x_0', \xi_0') = (0,0). \tag{11.19}$$

Let $\eta' \in \mathbb{R}^{n-1}$ satisfy

$$(\eta_2, \dots, \eta_{n-1}) = \frac{1}{\mathcal{O}(1)}, \quad \eta_1 = -\epsilon, \quad h^{\frac{2}{3}} \ll \epsilon \ll 1.$$

We shall construct an asymptotic solution to the problem

$$(P-z)u = 0, \quad u(x', 0) = a(x')\,\mathrm{e}^{\frac{i}{h}x'\eta'}, \tag{11.20}$$

or equivalently with $u = \mathrm{e}^{ix'\eta'/h}\widetilde{u}$,

$$\mathrm{e}^{-\frac{i}{h}x'\eta'}(P-z)\,\mathrm{e}^{\frac{i}{h}x'\eta'}\widetilde{u} = 0, \quad \widetilde{u}(x',0) = a(x'). \tag{11.21}$$

The conjugated operator to the left can be written

$$(hD_{x_n})^2 + hD_{x_1} - x_n Q(x, \eta' + hD_{x'}) - (\epsilon + h^{\frac{2}{3}} w). \tag{11.22}$$

From looking at the eiconal equation $p(x, \phi') - z = 0$ with boundary condition $\phi'_{x'}(x', 0) = \eta'$, it is natural to make the dilation in x_n,

$$x_n = \epsilon \widetilde{x}_n, \quad x' = \widetilde{x}'. \tag{11.23}$$

Then $hD_{x_n} = \frac{h}{\epsilon} D_{\widetilde{x}_n}$, $hD_{x'} = hD_{\widetilde{x}'}$ and a direct calculation shows that

$$\mathrm{e}^{-\frac{i}{h}x'\eta'}(P-z)\,\mathrm{e}^{\frac{i}{h}x'\eta'} = \epsilon\big(\widetilde{P} - (1 + \widetilde{h}^{\frac{2}{3}} w)\big), \tag{11.24}$$

where $\widetilde{h} = h\epsilon^{-\frac{3}{2}} \ll 1$ and

$$\widetilde{P} = (\widetilde{h} D_{\widetilde{x}_n})^2 + \epsilon^{\frac{1}{2}} \widetilde{h} D_{\widetilde{x}_1} - \widetilde{x}_n Q(\widetilde{x}', \epsilon\widetilde{x}_n, \eta' + \epsilon^{\frac{3}{2}} \widetilde{h} D_{\widetilde{x}'}). \tag{11.25}$$

Thus after dilation, we are in a "uniformly hyperbolic" situation and we get a solution

$$\widetilde{u} = b(\widetilde{x}; \widetilde{h})\,\mathrm{e}^{\frac{i}{\widetilde{h}}\widetilde{\phi}(\widetilde{x})}, \quad \widetilde{x} = \Big(x', \frac{x_n}{\epsilon}\Big),$$

of the problem

$$\big(\widetilde{P} - (1 + \widetilde{h}^{\frac{2}{3}} w)\big)\widetilde{u} = \mathcal{O}(\widetilde{h}^{\infty}), \quad \widetilde{u}(\widetilde{x}', 0) = a(\widetilde{x}'), \tag{11.26}$$

defined in a region

$$|\widetilde{x}'| \le \mathcal{O}(1), \quad 0 \le -\widetilde{x}_n < \frac{1}{\mathcal{O}(1)},$$

where b is a classical symbol of order 0 and $\widetilde{\phi}(\widetilde{x})$ is uniformly bounded with all its derivatives in the same region. $\widetilde{\phi}$ is here the solution of the eiconal equation,

$$\widetilde{p}(\widetilde{x}, \widetilde{\phi}'_{\widetilde{x}}) - (1 + \widetilde{h}^{\frac{2}{3}} w) = 0, \quad \widetilde{\phi}_{|\widetilde{x}_n = 0} = 0, \tag{11.27}$$

which satisfies

$$\Im \widetilde{\phi} \asymp |\widetilde{x}_n| \widetilde{h}^{\frac{2}{3}}. \tag{11.28}$$

Thus,

$$|\widetilde{u}| = \mathcal{O}(1)\, \mathrm{e}^{-|\widetilde{x}_n|/(C\widetilde{h}^{\frac{1}{3}})},$$

which is $\mathcal{O}(\widetilde{h}^{\infty})$ in any region $-\widetilde{x}_n \geq \widetilde{h}^{\frac{1}{3}-\delta}$ for any fixed $\delta > 0$.

In the original coordinates, we get the asymptotic solution of (11.20)

$$u(x; \eta'; h) = b\Big(\frac{x_n}{\epsilon}, x', \eta'; \widetilde{h}\Big)\, \mathrm{e}^{\frac{i}{h}(x'\eta' + \epsilon^{\frac{3}{2}} \widetilde{\phi}(x_n/\epsilon, x', \eta'))}. \tag{11.29}$$

These solutions can be superposed to build a microlocal Poisson operator, if we take $a = 1$, and we get $\check{K} = \mathcal{O}(\widetilde{h}^{1/6}) : H_h^{\frac{3}{2}} \to H_h^2$, where we use the modified norm

$$\sum_{|\alpha| \leq 2} \big\| (hD_{x'})^{\alpha'} (\widetilde{h} D_{\widetilde{x}_n})^{\alpha_n} v \big\|$$

on H_h^2 with $L^2(\mathrm{d}x' \mathrm{d}\widetilde{x}_n)$ as the underlying L^2-norm. This gives in the original coordinates,

$$\sum_{|\alpha| \leq 2} \big\| (hD_{x'})^{\alpha'} (h\epsilon^{-\frac{1}{2}} D_{x_n})^{\alpha_n} \check{K} u \big\|_{L^2(\mathrm{d}x)} \leq \mathcal{O}(1) h^{\frac{1}{6}} \epsilon^{\frac{1}{4}} \|u\|_{H_h^{3/2}}. \tag{11.30}$$

In particular, with the ordinary H^2 norm,

$$\check{K} = \mathcal{O}(1) h^{\frac{1}{6}} \epsilon^{\frac{1}{4}} : H_h^{\frac{3}{2}} \longrightarrow H_h^2. \tag{11.31}$$

We get the approximation to the DN map:

$$\mathcal{N}_{\mathrm{in}}^{\mathrm{approx}} = \mathrm{Op}_h\Big(\epsilon^{\frac{1}{2}} \partial_{\widetilde{x}_n} \widetilde{\phi}(x', 0, \xi') + \frac{h}{i\epsilon} (\partial_{\widetilde{x}_n} b)(x', 0, \xi'; \widetilde{h}) \Big). \tag{11.32}$$

Here we must recall that $\epsilon = -\xi_1$, so the symbol of $\mathcal{N}_{\mathrm{in}}^{\mathrm{approx}}$ is singular in that variable but good enough for our 2-microlocal calculus, in view of the fact that $\epsilon \gg h^{\frac{2}{3}}$ and it is a uniformly bounded operator: $H_h^{\frac{3}{2}} \to H_h^{\frac{1}{2}}$.

It remains to study the region

$$-h^{\frac{2}{3}-\delta} \leq r(x', \xi') - \lambda \leq \widetilde{\delta}, \tag{11.33}$$

where $\delta, \widetilde{\delta} > 0$ are small and independent of h. Again, we reduce R to the form (11.19) and restrict ξ' to a set

$$(\xi_2, \dots, \xi_{n-1}) = \frac{1}{\mathcal{O}(1)}, \quad -h^{\frac{2}{3}-\delta} \le \xi_1 \le \widetilde{\delta}.$$

We consider (cf. (11.22))

$$P(x, \xi', hD_{x_n}) - z = (hD_{x_n})^2 + \xi_1 - x_n Q(x, \xi') - h^{\frac{2}{3}} w, \tag{11.34}$$

and we follow the approach for the exterior problem started in Section 8.3, with two not very essential differences:

- ▷ x_n remains real and we study the Dirichlet problem on an interval $[-b, 0]$ for $0 < b \ll 1$ independent of h;
- ▷ there will be a slight degeneration when $\xi_1 \ll -h^{\frac{2}{3}}$.

We review the one-dimensional analysis with x', ξ' as parameters, writing x instead for x_n and $Q(x)$ instead of $Q(x', x_n, \xi')$. *We first assume that Q is analytic.* Let x_0 be the complex turning point, given by

$$x_0 Q(x_0) = \xi_1 - h^{\frac{2}{3}} w,$$

and we let $x_1 \asymp \xi_1$ be the corresponding real turning point given by

$$x_1 Q(x_1) = \xi_1.$$

Then

$$x_0 = x_1 - \frac{h^{\frac{2}{3}} w}{V'(x_1)} + \mathcal{O}(h^{\frac{4}{3}}), \quad \text{where } V(x) = xQ(x).$$

As in the exterior case we take a null solution of the form $u = \mathrm{e}^{-\phi(x;h)/h}$ which is subdominant in the direction of negative x and increasing in order of magnitude when x increases. More precisely, for $x - x_1 \ll -h^{\frac{2}{3}}$ we have

$$-\partial_x(\Re \phi) \asymp |\partial_x \phi| \asymp |x - x_1|^{\frac{1}{2}} \tag{11.35}$$

and for $|x - x_1| \le \mathcal{O}(h^{\frac{2}{3}})$ we have $\partial_x \phi = \mathcal{O}(h^{\frac{1}{3}})$.

For $x - x_1 \gg h^{\frac{2}{3}}$ (as well as for $x - x_1 \ll -h^{\frac{2}{3}}$) we have (8.34), where

$$-\phi_0' = \left(\xi_1 - xQ(x) - h^{\frac{2}{3}} w\right)^{\frac{1}{2}},$$

and we choose the principal branch of the square root with a cut along $\overline{\mathbb{R}}_-$, which has positive real part. Then for $x - x_1 \gg h^{\frac{2}{3}}$ we get when $\pm\Im w > 0$:

$$\begin{aligned} -\phi_0' &= \mp i(xQ(x) - \xi_1 + h^{\frac{2}{3}} w)^{\frac{1}{2}} \\ &= \mp i(xQ(x) - \xi_1)^{\frac{1}{2}} \Big(1 + \frac{h^{\frac{2}{3}} w}{xQ(x) - \xi_1}\Big)^{\frac{1}{2}} \\ &= \mp i(xQ(x) - \xi_1)^{\frac{1}{2}} \mp \frac{ih^{\frac{2}{3}} w}{2(xQ(x) - \xi_1)^{\frac{1}{2}}} + \frac{\mathcal{O}(h^{\frac{4}{3}})}{(xQ(x) - \xi_1)^{\frac{3}{2}}}. \end{aligned}$$

It follows that

$$-\Re\phi_0' \asymp \frac{h^{\frac{2}{3}}}{|x - x_1|^{\frac{1}{2}}} \quad \text{when } x - x_1 \gg h^{\frac{2}{3}}. \tag{11.36}$$

This quantity dominates over the remainder $\mathcal{O}(h)|x - x_0|^{-1}$ in (8.34) when $|x - x_0| \gg h^{\frac{2}{3}}$,

$$\frac{h^{\frac{2}{3}}}{|x - x_0|^{\frac{1}{2}}} \gg \frac{h}{|x - x_0|}$$

and hence

$$-\Re\phi' \asymp \frac{h^{\frac{2}{3}}}{|x - x_1|^{\frac{1}{2}}} \quad \text{when } x - x_1 \gg h^{\frac{2}{3}}. \tag{11.37}$$

This is slightly worse than (11.35) and if that estimate had been valid also for $x - x_1 \gg h^{\frac{2}{3}}$, then we would get exactly the same estimates as in the case of the exterior problem.

It is natural to ask how much worse (11.37) is than (11.35). Recall that we work on an interval $[-b, 0]$ and that $x_1 \asymp \xi_1 \geq -h^{\frac{2}{3}-\delta}$, so $x - x_1 \leq -x_1 \leq h^{\frac{2}{3}-\delta}$. Thus we get

$$\frac{\mathrm{RHS}(11.35)}{\mathrm{RHS}(11.37)} = \frac{|x - x_1|}{h^{\frac{2}{3}}} \leq h^{-\delta}. \tag{11.38}$$

For $-b \leq y \leq w \leq x \leq 0$ we have

$$\frac{1}{C} h^{\delta} \int_y^x \big|\phi'(t)\big| \mathrm{d}t - Ch \leq -\Re\phi(x) + \Re\phi(y) \leq \int_y^x \big|\phi'(t)\big| \mathrm{d}t, \tag{11.39}$$

$$\frac{1}{C}|\phi'(w)| \cdot |x - y| - Ch \leq \int_y^x \big|\phi'(t)\big| \mathrm{d}t \leq C\big(|\phi'(\widetilde{z}(x, y))| \cdot |x - y| + h\big), \tag{11.40}$$

where $\widetilde{z}$ is the point in $\{x, y\}$ maximizing $|\widetilde{z} - x_1|$.

The majoration (8.40) remains valid and we even have

$$\frac{1}{C_\epsilon}\mathrm{e}^{-\frac{\epsilon}{h}(-\Re\phi(x)+\Re\phi(y))} \leq \frac{h^{\frac{1}{3}}+|\phi'(x)|}{h^{\frac{1}{3}}+|\phi'(y)|} \leq C_\epsilon \mathrm{e}^{\frac{\epsilon}{h}(-\Re\phi(x)+\Re\phi(y))}, \tag{11.41}$$

as can be seen by comparing the logarithmic derivative of $h^{\frac{1}{3}}+|\phi'(x)|$ with $-\Re\phi'/h$ in the region $x-x_1 \gg h^{\frac{2}{3}}$, where $\phi''(x)=\mathcal{O}(|x-x_0|^{-\frac{1}{2}})$ and (11.36) holds.

The factor h^δ in (11.39) gives slight losses in the estimates of Subsection 8.3 and we get

LEMMA 11.2. — *If* $(P(x',\xi')-z)u=0$ *on* $[-b,0]$, $u(0)=u(-b)=0$, *then*

$$\left\|(h^{\frac{1}{3}}+|\phi'|)^2 u\right\| + \left\|(h\partial_x)^2 u\right\| + \left\|\big(h^{\frac{1}{3}}+|\phi'|\big)h\partial_x u\right\| \leq \mathcal{O}(h^{-2\delta})\|v\|, \tag{11.42}$$

when $\xi_1 \geq -h^{\frac{2}{3}-\delta}$.

Proof. — We solve the Dirichlet problem on $[-b,0]$ as in Section 8.3 and start with applying the natural modification of the operator K:

$$Kv(x) = -\frac{1}{h}\int_{-b}^{x} \mathrm{e}^{\phi(x)-\phi(y))/h} v(y)dy \tag{11.43}$$

and Lemma 8.6 deteriorates slightly to

LEMMA 11.3. — *The* $\mathcal{L}(L^2)$*-norms of*

$$\big(h^{\frac{1}{3}}+|\phi'|\big)\circ K, \quad \big(h^{\frac{1}{3}}+|\phi'|\big)^2\circ K\circ\big(h^{\frac{1}{3}}+|\phi'|\big)^{-1}, \quad K\circ\big(h^{\frac{1}{3}}+|\phi'|\big)$$

are $\mathcal{O}(1)h^{-\delta}$.

Proof. — We use Schur's lemma as in the proof of Lemma 8.6. Thus for instance, the L^2-norm of $(h^{\frac{1}{3}}+|\phi'|)\circ K$ is bounded by the geometric mean of

$$\mathrm{I} = \frac{1}{h}\sup_{-b\leq x\leq 0}\int_{-b}^{x}\big(h^{\frac{1}{3}}+|\phi'(x)|\big)\mathrm{e}^{\frac{1}{h}(\Re(\phi(x)-\phi(y))}\mathrm{d}y,$$

$$\mathrm{II} = \frac{1}{h}\sup_{-b\leq y\leq 0}\int_{y}^{0}\big(h^{\frac{1}{3}}+|\phi'(x)|\big)\mathrm{e}^{\frac{1}{h}(\Re(\phi(x)-\phi(y))}\mathrm{d}x.$$

Here, by (11.39), (11.40),

$$\mathrm{e}^{\frac{1}{h}\Re(\phi(x)-\phi(y))} \leq C\,\mathrm{e}^{-\frac{1}{Ch^{1-\delta}}\int_y^x|\phi'(t)|\mathrm{d}t} \leq \widetilde{C}\,\mathrm{e}^{-\frac{1}{\widetilde{C}h^{1-\delta}}(h^{\frac{1}{3}}+|\phi'(x)|)|x-y|}, \tag{11.44}$$

and we get $\mathrm{I}=\mathcal{O}(h^{-\delta})$.

To get the same estimate for II we also use (11.41). The other L^2-norms are estimated similarly. □

The proof of Lemma 11.2 can now be finished as in Section 8.3. □

We next eliminate the analyticity assumption in Lemma 11.2. Let x_1 be the real turning point determined by $x_1Q(x_1)=\xi_1$, so that $x_1 \leq \mathcal{O}(1)h^{\frac{2}{3}-\delta}$. Let $x_2 = x_1 - h^{\frac{2}{3}-\delta}$. For a large but fixed N, put

$$\widetilde{Q}(x) = \begin{cases} Q(x) & \text{if } x \leq x_2, \\ \displaystyle\sum_0^{N-1} \frac{1}{\alpha!} Q^{(\alpha)}(x_2)(x-x_2)^\alpha & \text{if } x \geq x_2. \end{cases}$$

Since $\widetilde{Q}$ is holomorphic in a $h^{\frac{2}{3}-\delta}$-neighborhood of x_1, we see that if $\widetilde{P}$ is the corresponding operator then we have a null solution $\mathrm{e}^{-\widetilde{\phi}/h}$ of $P-z$ with the same properties as $\mathrm{e}^{-\phi/h}$ in the analytic case above and such that Lemma 11.2 applies. Now $\widetilde{Q}-Q = \mathcal{O}(1)h^{(\frac{2}{3}-\delta)N}$ and if we choose N large enough, it follows that $P-z$ has a null solution $\mathrm{e}^{-\phi/h}$, where

$$\widetilde{Q}-Q, \phi-\widetilde{\phi}, \phi'-\widetilde{\phi}', \phi''-\widetilde{\phi}'' = \mathcal{O}(h).$$

Another perturbation argument shows that Lemma 11.2 holds for $P-z$.

Let $x_{n,1}(x',\xi')$ be the real turning point determined by

$$-x_{n,1}Q(x',x_{n,1},\xi')+\xi_1 = 0$$

where we recall that $\xi_1 = r(x',\xi')-\lambda$. In analogy with (9.3), we can reformulate (11.42) as

$$\begin{aligned} (11.45)\quad & \big\|\big(h^{\frac{2}{3}}+|x_n-x_{n,1}|\big)u\big\| + \big\|(h\partial_{x_n})^2u\big\| \\ & +\big\|\big(h^{\frac{2}{3}}+|x_n-x_{n,1}|\big)^{\frac{1}{2}}(h\partial_{x_n})u\big\| \leq \mathcal{O}(h^{-2\delta})\big\|(P(x',\xi')-z)u\big\| \end{aligned}$$

for smooth functions u on $[-b,0]$, vanishing at the end points. Notice here that

$$\big(h^{\frac{1}{3}}+|\phi'|\big)^2 \asymp h^{\frac{2}{3}}+|x-x_{n,1}|.$$

Define the $\mathcal{B}(x',\xi')$ norm to be the left hand side in (11.45) and let $\mathcal{B}$ be the space of functions on $[-b,0]$ with finite $\mathcal{B}$-norm that vanish at the end points. Then we still have the symbol property (9.9) for $P(x',\xi'):\mathcal{B}(x',\xi')\to L^2$ and we get (9.11) for $E=(P(x',\xi')-z)^{-1}$ with a slight loss:

$$(11.46)\quad \partial_{x'}^\alpha\partial_{\xi'}^\beta E = \mathcal{O}_{\alpha,\beta}\big(h^{-2\delta(1+|\alpha|+|\beta|)}\big)\big(h^{\frac{2}{3}}+|\lambda-r(x',\xi')|\big)^{-\beta_1}, \quad L^2 \longrightarrow \mathcal{B}.$$

Due to the non-monotonicity of $\Lambda = (h^{\frac{2}{3}}+|\lambda-r(x,\xi')|)^{\frac{1}{2}}$ as a function of x_n between $x_{n,1}$ and 0 when $x_{n,1}<0$, we get (10.7), (10.8) with loss:

$$(11.47)\qquad h^{\frac{1}{2}}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{\frac{3}{4}}\big|u(0)\big| \leq Ch^{-3\delta/4}\|u\|_{\mathcal{B}},$$

$$(11.48)\qquad h^{\frac{1}{2}}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{\frac{1}{4}}\big|h\partial_{x_n}u(0)\big| \leq Ch^{-\delta/4}\|u\|_{\mathcal{B}}.$$

Normalize ϕ by imposing the condition $\phi(0)=0$ and let $e_{x',\xi'}=\mathrm{e}^{-\frac{1}{h}\phi}$ be the null solution of $P(x',\xi')-z$ so that $e_{x',\xi'}(0)=1$ and $e_{x',\xi'}(-b)$ is exponentially small. Using (11.41), (11.44), we get (10.9) with a δ loss:

$$\|e_{x',\xi'}\|_{\mathcal{B}}\leq \mathcal{O}(1)h^{\frac{1}{2}(1-\delta)}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{\frac{3}{4}}. \tag{11.49}$$

Adding an exponentially small reflected null solution to $e_{x',\xi'}$ and renormalizing, we get a new null solution, that we denote by $e_{x',\xi'}$ instead of the earlier one, which satisfies the boundary conditions $e_{x',\xi'}(0)=1$, $e_{x',\xi'}(-b)=0$ and which also satisfies (11.49). Then we get the weakened version of (10.12):

$$\|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}e_{x',\xi'}\|_{\mathcal{B}}=\mathcal{O}(1)h^{\frac{1}{2}(1-\delta)-2\delta(|\alpha|+|\beta|)}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{\frac{3}{4}-\beta_1}. \tag{11.50}$$

As a first approximation to the microlocal interior Poisson operator on $\{x;\, -b\leq x_n\leq 0,\, |x'|\leq \mathcal{O}(1)\}$ we take (cf. (10.13))

$$K^0w=\mathrm{Op}_h(e_{x',\xi'}). \tag{11.51}$$

Then $\gamma K^0=1$, $(P-z)K^0=\mathrm{Op}_h(f_{x',\xi'})$, where,

$$f_{x',\xi'}=\sum_{\alpha\neq 0}\frac{h^{|\alpha|}}{\alpha!}\partial_{\xi'}^{\alpha}P(x',\xi')D_{x'}^{\alpha}e_{x',\xi'},$$

and by (9.9), (11.50),

$$\|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}f_{x',\xi'}\|_{L^2}=\mathcal{O}(1)h^{\frac{3}{2}-\frac{5}{2}\delta-2\delta(|\alpha|+|\beta|)}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{-\frac{1}{4}-\beta_1}.$$

Using E as a first approximation, we can construct an operator-valued symbol $\widetilde{E}(x',\xi';h)$ such that $\widetilde{E}(x',hD_{x'};h)$ inverts $P(x',hD_{x'})-z$ to all orders in h. We get a microlocal Poisson operator to all orders in h by putting

$$\widetilde{K}=K^0-\widetilde{E}\circ(P-z)K^0=K^0+\mathrm{Op}_h(\widetilde{r}),$$

and $\widetilde{r}$ fulfills the slightly deteriorated version of (10.17):

$$\|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}\widetilde{r}\|_{\mathcal{B}}=\mathcal{O}(1)h^{\frac{3}{2}-\frac{5}{2}\delta-2\delta(1+|\alpha|+|\beta|)}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{-\frac{1}{4}-\beta_1}.$$

Now $\widetilde{K}$ can be written as in (10.18) and we have (10.19). The symbol $e_{x',\xi'}+\widetilde{r}_{x',\xi'}$ there satisfies

$$\big\|\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}(e_{x',\xi'}+\widetilde{r}_{x',\xi'})\big\|_{\mathcal{B}}=\mathcal{O}(1)h^{\frac{1}{2}-\frac{1}{2}\delta-2\delta(|\alpha|+|\beta|)}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{\frac{3}{4}-\beta_1},$$

when $\delta>0$ is small enough. From this estimate and the similar ones in the other regions we get

$$\widetilde{K}=\mathcal{O}(h^{\frac{1}{6}}):H_h^{\frac{3}{2}}\longrightarrow H_h^2, \tag{11.52}$$

and this also holds for the exact Poisson operator $K_{\mathrm{in}}=K_{\mathrm{in}}^V$.

The corresponding DN-map is a pseudodifferential operator with symbol

$$n(x',\xi';h)=\gamma hD_{x_n}(e+\widetilde{r}),$$

and combing the above estimate with (11.48), we get the estimate

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}n=\mathcal{O}(1)h^{-\frac{3}{4}\delta-2\delta(|\alpha|+|\beta|)}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{\frac{1}{2}-\beta_1}. \tag{11.53}$$

This is a bounded symbol in the region where $h^{-\frac{3}{4}\delta}|r-\lambda|^{\frac{1}{2}}=\mathcal{O}(1)$, *i.e.* where $|r-\lambda|=\mathcal{O}(1)h^{\frac{3}{2}\delta}$ and to get an better conclusion, we take a closer look.

First, we see that

$$\gamma hD_{x_n}e_{x',\xi'}=i\partial_{x_n}\phi(0)=\mathcal{O}(1)\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{\frac{1}{2}}$$

is bounded. Secondly, from the above estimate on the $\mathcal{B}$ norm of $\widetilde{r}$ and (11.48), we conclude that

$$\gamma hD_{x_n}\widetilde{r}=\mathcal{O}(1)h^{1-(\frac{5}{2}+\frac{1}{4})\delta}\big(h^{\frac{2}{3}}+|r-\lambda|\big)^{-\frac{1}{2}}$$

which is also bounded. Thus we have an improvement of (11.53) when $\alpha=\beta=0$, and we conclude that n is in a sufficiently good symbol class to conclude that its quantization is L^2 bounded.

Patching together the different microlocal Poisson operators, we get an approximation mod $\mathcal{O}(h^{\infty})$ in $\mathcal{L}(H_h^{\frac{3}{2}},H_h^2)$ of K_{in} and also the conclusion of Proposition 11.1 from the boundedness of the corresponding microlocal DN-maps. □

Let V be as in Proposition 11.1 and let K_{in}^V and $\mathcal{N}_{\mathrm{in}}^V$ denote the corresponding Poisson and Dirichlet to Neumann operators. Let $W\in L^{\infty}(\Omega;\mathbb{R})$. Then

$$K_{\mathrm{in}}^{V+W}=K_{\mathrm{in}}^V-(P_{\mathrm{in}}^{V+W}-z)^{-1}WK_{\mathrm{in}}^V=:K_{\mathrm{in}}^V+A,$$

where in view of (11.52):

$$\|A\|_{\mathcal{L}(H_h^{\frac{3}{2}},H_h^2)}\le\mathcal{O}(1)h^{-\frac{2}{3}+\frac{1}{6}}\|W\|_{L^\infty}=\mathcal{O}(1)h^{-\frac{1}{2}}\|W\|_{L^\infty}.$$

Thus $\mathcal{N}_{\mathrm{in}}^{V+W}=\mathcal{N}_{\mathrm{in}}^V+B$, $B=\gamma hD_\nu A$, and we get

$$\|B\|_{\mathcal{L}(H_h^{\frac{3}{2}},H_h^2)}=\mathcal{O}(1)h^{-\frac{1}{2}-\frac{1}{2}}\|W\|_{L^\infty}=\mathcal{O}(1)h^{-1}\|W\|_{L^\infty}.$$

This implies:

Proposition 11.4. — *The conclusion of Proposition* 11.1 *remains valid if we replace V in there with $V+W$, where $W\in L^{\infty}(\Omega;\mathbb{R})$ satisfies*

$$\|W\|_{L^\infty}\le\mathcal{O}(h). \tag{11.54}$$

When $W = \delta\Theta q_\omega$ is as in Theorem 2.2, we have (11.54), provided α is large enough. See Remark 15.1.

For a greater generality of our results it is of interest to have a the following variant of the last proposition, where the perturbation W can be independent of h. We start with some simple exponentially weighted estimates. Let $\phi \in C^\infty(\overline{\mathcal{O}};\mathbb{R})$ and consider

$$P^{V,\epsilon} = \mathrm{e}^{\frac{\epsilon\phi}{h}} P^V \mathrm{e}^{-\frac{\epsilon\phi}{h}} = P^V + F,$$

where

$$F = i\epsilon(\phi' \cdot hD_x + hD \cdot \phi') - \epsilon^2(\phi')^2 = \mathcal{O}(\epsilon) : H_h^1 \longrightarrow H_h^0.$$

Since $(P_{\mathrm{in}}^V - z)^{-1} = \mathcal{O}(h^{-\frac{2}{3}}) : H_h^0 \to H_h^2$ when $\frac{1}{2} < \Re z < 2$, $|\Im z| \asymp h^{\frac{2}{3}}$, we get the same conclusion for $(P_{\mathrm{in}}^{V,\epsilon} - z)^{-1} = \mathrm{e}^{\epsilon\phi/h}(P_{\mathrm{in}}^V - z)^{-1}\mathrm{e}^{-\epsilon\phi/h}$, provided that $\epsilon \ll h^{\frac{2}{3}}$.

Let $\phi_{|\partial\mathcal{O}} = 0$. Then $K^{V,\epsilon} = \mathrm{e}^{\epsilon\phi/h}K^V$ is the Poisson operator for $P^{V,\epsilon} - z$. We can also get $K^{V,\epsilon}$ by a perturbative argument, writing

$$\begin{aligned} K^{V,\epsilon} &= K^V - (P_{\mathrm{in}}^{V,\epsilon} - z)^{-1} F K^V \\ &= K^V + \mathcal{O}(h^{-\frac{2}{3}}\epsilon h^{\frac{1}{6}}) = \mathcal{O}(h^{\frac{1}{6}}) : H_h^{\frac{3}{2}} \longrightarrow H_h^2. \end{aligned}$$

Thus $\mathrm{e}^{\epsilon\phi/h}K^V(z) = \mathcal{O}(h^{1/6}) : H_h^{\frac{3}{2}} \to H_h^2$. Now assume that

$$W(x) = \mathcal{O}\big(\operatorname{dist}(x,\partial\mathcal{O})^{N_0}\big),$$

for some $N_0 > 0$, to be determined. Then $WK^V = W\mathrm{e}^{-\epsilon\phi/h}\mathrm{e}^{\epsilon\phi/h}K^V$ and taking $\phi \asymp \operatorname{dist}(\cdot,\partial\mathcal{O})$, $\epsilon \geq h^{\frac{2}{3}}/\mathcal{O}(1)$, we see that

$$W\mathrm{e}^{-\epsilon\phi/h} = \mathcal{O}\big(\operatorname{dist}^{N_0}\mathrm{e}^{-\operatorname{dist}/(Ch^{\frac{1}{3}})}\big) = \mathcal{O}(h^{\frac{1}{3}N_0}).$$

Then as in the discussion prior to Proposition 11.4, we have $K_{\mathrm{in}}^{V+W} = K_{\mathrm{in}}^V + A$, where

$$A = (P_{\mathrm{in}}^{V+W} - z)^{-1} W K_{\mathrm{in}}^V = \mathcal{O}(1) h^{-\frac{2}{3}+\frac{N_0}{3}+\frac{1}{6}} : H_h^{\frac{3}{2}} \longrightarrow H_h^2.$$

The choice $N_0 = 3$ gives $A = \mathcal{O}(h^{\frac{1}{2}}) : H_h^{\frac{3}{2}} \to H_h^2$ and we get the following variant and extension of Proposition 11.4:

Proposition 11.5. — *The conclusion of Proposition* 11.1 *remains valid if we replace V there with $V + W$, where $W \in L^\infty(\Omega;\mathbb{R})$ satisfies*

$$W(x) = \mathcal{O}\big(\operatorname{dist}(x,\partial\mathcal{O})^3\big). \tag{11.55}$$

More generally, we can take $W = W_1 + W_2$, where W_1 and W_2 fulfill (11.54) *and* (11.55) *respectively.*

CHAPTER 12

SOME DETERMINANTS

Let V_0 is as in (11.6) and

$$V = V_0 + W, \tag{12.1}$$

where the real-valued term W is $\mathcal{O}(1)$ in L^∞. We let

$$P = -h^2\Delta + V =: P^V, \quad P_0 = -h^2\Delta + V_0. \tag{12.2}$$

Recall the definitions of P_{out}, $\mathcal{P}_{\mathrm{out}}$, $\mathcal{P}_{\mathrm{in}}$, P_{in} in Chapter 7, with the potential V as above.

Our first task is to define the determinants of the factors in (7.19). In the following, H^s denotes H^s_h if nothing else is indicated.

Proposition 12.1. — *The three factors in* (7.19) *are meromorphic families of Fredholm operators in the region* $\frac{1}{2} < \Re z < \frac{3}{2}$, $\Im z > -h^{\frac{2}{3}} c_0$, *where* c_0 *is as in* (3.1). *More precisely,*

$$\mathcal{P}_{\mathrm{in}}(z) : H^2(\mathcal{O}) \longrightarrow H^0(\mathcal{O}) \times H^{\frac{3}{2}}(\partial\mathcal{O}),$$

$$\mathcal{P}_{\mathrm{out}}(z) : H^2(\mathcal{O}) \longrightarrow H^0(\mathcal{O}) \times H^{\frac{1}{2}}(\partial\mathcal{O})$$

are holomorphic Fredholm families, while

$$\begin{pmatrix} 1 & 0 \\ h^{\frac{1}{2}} B G_{\mathrm{in}} & \mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{out}} \end{pmatrix} : H^0(\mathcal{O}) \times H^{\frac{3}{2}}(\partial\mathcal{O}) \longrightarrow H^0(\mathcal{O}) \times H^{\frac{1}{2}}(\partial\mathcal{O})$$

is a meromorphic Fredholm family.

Proof. — This is clear for $\mathcal{P}_{\mathrm{in}}$, $\mathcal{P}_{\mathrm{out}}$, and the factorization (7.19) then implies that the remaining factor is a meromorphic Fredholm family. □

From (7.19) and the last proposition, we get

$$\det \mathcal{P}_{\text{out}}(z) = \det(\mathcal{N}_{\text{in}} - \mathcal{N}_{\text{ext}}) \det \mathcal{P}_{\text{in}}(z). \tag{12.3}$$

The next result will permit us to do some analysis.

PROPOSITION 12.2. — *The determinants of the factors in* (7.19) *can also be defined as in Section* 4.4.

Proof. — We have

$$\partial_z \mathcal{P}_{\text{in}}(z) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \partial_z^2 \mathcal{P}_{\text{in}}(z) = 0. \tag{12.4}$$

Thus the C_p-norm of $\partial_z \mathcal{P}_{\text{in}}(z) : H^2 \to H^0 \times H^{\frac{3}{2}}$ can be bounded by that of the inclusion map $\iota : H^2(\mathcal{O}) \to H^0(\mathcal{O})$. Here we can consider $\mathcal{O}$ as a bounded subset with smooth boundary of a torus T and choose a uniformly bounded Seeley extension $\sigma : H^2(\mathcal{O}) \to H^2(T)$ so that $\iota = \rho \iota_T \sigma$, where $\iota_T : H^2(T) \to H^0(T)$ is the inclusion map and $\rho : H^0(T) \to H^0(\mathcal{O})$ is the restriction map. ρ and σ being uniformly bounded, it suffices to study the Schatten class norm of ι_T. Here $H^2(T) = (1 - h^2\Delta)^{-1}(H^0(T))$ so the problem is that of the C_p-norm of $(1 - h^2\Delta)^{-1} : H^0(T) \to H^0(T)$.

By Weyl's law we get for $p > \frac{1}{2}n$,

$$\begin{aligned} \|(1 - h^2\Delta)^{-1}\|_{C_p}^p &= \int_0^\infty (1 + h^2\lambda)^{-p} d\mathcal{O}(\lambda^{\frac{1}{2}n}) \\ = \mathcal{O}(h^2) \int_0^\infty \frac{\lambda^{\frac{1}{2}n}}{(1 + h^2\lambda)^{p+1}} \mathrm{d}\lambda &= \mathcal{O}(h^{-n}) \int_0^\infty \frac{t^{\frac{1}{2}n}}{(1+t)^{p+1}} \mathrm{d}t \end{aligned}$$

and then $\|\iota_T\|_{C_p}^p = \mathcal{O}(h^{-n})$, so

$$\big\|\partial_z \mathcal{P}_{\text{in}}\big\|_{C_p} = \mathcal{O}(h^{-\frac{n}{p}}), \quad p > \tfrac{1}{2}n. \tag{12.5}$$

This implies that $\mathcal{P}_{\text{in}}(z)$ satisfies (4.30) for any $p > \frac{1}{2}n$, so its determinant can be defined as in Section 4.4.

In order to treat the other two operators, we need to collect some more information about $\mathcal{N}_{\text{ext}}$.

LEMMA 12.3. — *For z as in Proposition* 12.1*, we have for all $s \in \mathbb{R}$, $k \in \mathbb{N}$:*

$$\partial_z^k \mathcal{N}_{\text{ext}}(z) = \mathcal{O}\big((\Im z + c_0 h^{\frac{2}{3}})^{-k}\big) : H^s \longrightarrow H^{s-1+2k}. \tag{12.6}$$

Proof. — Microlocally near the glancing hypersurface and in the hyperbolic region, this follows from Corollary 10.4 and the Cauchy inequalities. The extra regularization comes from the elliptic region and here $K_{\rm ext}(z)$ is the Poisson operator of an elliptic boundary value problem and satisfies

$$\partial_z^k K_{\rm ext}(z) = C_k(P_{\rm ext}-z)^{-k}K_{\rm ext}(z). \qquad \Box$$

Applying the lemma to $B = B(z)$ in (7.6), we get

$$\partial_z^k B(z) = \mathcal{O}(1)h^{-\frac{1}{2}}(\Im z + c_0 h^{\frac{2}{3}})^{-k} : H^2(\mathcal{O}) \longrightarrow H^{\frac{1}{2}+2k}(\partial\mathcal{O}). \tag{12.7}$$

The C_p-norm of the inclusion map $H^{\frac{1}{2}+2k} \to H^{\frac{1}{2}}$ is bounded by a constant times the C_p-norm of $(1-h^2\Delta_{\partial\mathcal{O}})^{-k}$ which by Weyl asymptotics is finite and $\mathcal{O}(h^{(1-n)/p})$ when $p \geq 1$ and $p > (n-1)/(2k)$. Thus for each such p,

$$\partial_z^k B \in C_p(H^2, H^{\frac{1}{2}}), \quad \|\partial_z^k B\|_{C_p} = \mathcal{O}(h^{-\frac{1}{2}+\frac{1-n}{p}}\big(\Im z + c_0 h^{\frac{2}{3}})^{-k}\big).$$

It then follows as in the proof of (12.5) that when $p \geq 1$ and $p > n/(2k)$,

$$\begin{aligned}\partial_z^k \mathcal{P}_{\rm out}(z) &\in C_p(H^2, H^0 \times H^{\frac{1}{2}}),\\ \big\|\partial_z^k \mathcal{P}_{\rm out}(z)\big\|_{C_p} &= \mathcal{O}\big(h^{-\max(\frac{n}{p},\frac{1}{2}+\frac{n-1}{p}+\frac{2}{3}k)}\big).\end{aligned} \tag{12.8}$$

Thus we have verified (4.30) with $p = \frac{1}{2}(n+\epsilon)$ and $\det \mathcal{P}_{\rm out}(z)$ can indeed be defined as in Section 4.4.

In that chapter we have seen that if $P(z)$ fulfills (4.30), then so does $P(z)^{-1}$ on the open subset of bijectivity. We also saw that if $P_1(z) \in \mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ and $P_2(z) \in \mathcal{L}(\mathcal{H}_2,\mathcal{H}_3)$ satisfy (4.30), then so does $P_1(z)P_2(z)$. Having checked that $\mathcal{P}_{\rm in}(z)$ and $\mathcal{P}_{\rm out}(z)$ satisfy (4.30), we conclude from (7.19) that $\left(\begin{smallmatrix}1 & 0\\ h^{\frac{1}{2}}BG_{\rm in} & \mathcal{N}_{\rm in}-\mathcal{N}_{\rm ext}\end{smallmatrix}\right)$ also satisfies (4.30) and the proposition follows from Section 4.4. $\Box$

CHAPTER 13

UPPER BOUNDS ON THE BASIC DETERMINANT

The first task will be to get an upper bound on $\ln|\det \mathcal{P}_{\text{out}}|$ in the whole region

$$|\Im z| < c_0 h^{\frac{2}{3}}, \quad \tfrac{1}{2} < \Re z < 2 \tag{13.1}$$

by some negative power of h.

Using the addendum at the end of Section 4.4, we shall derive a rough upper bound on $\ln|\det \mathcal{P}_{\text{out}}(z)|$. Let

$$\widetilde{P} = P + i1_{\mathcal{O}}, \quad \widetilde{\mathcal{P}}_{\text{out}}(z) = \begin{pmatrix} \widetilde{P} - z \\ B(z) \end{pmatrix}.$$

Assume first that $W = 0$ so that $V = V_0$ is smooth. Thanks to the perturbation $i1_{\mathcal{O}}$,

$$\widetilde{\mathcal{P}}_{\text{in}} := \begin{pmatrix} \widetilde{P} - z \\ h^{\frac{1}{2}}\gamma \end{pmatrix} : H^{s+2}(\mathcal{O}) \longrightarrow H^s(\mathcal{O}) \times H^{s+\frac{3}{2}}(\partial\mathcal{O}) \tag{13.2}$$

is bijective with an inverse $\widetilde{E}_{\text{in}}(z) = \big(\widetilde{G}_{\text{in}}(z) \;\; h^{-\frac{1}{2}}\widetilde{K}_{\text{in}}(z)\big)$, where

$$\widetilde{G}_{\text{in}} = \mathcal{O}_s(1) : H^s \longrightarrow H^{s+2}, \quad \widetilde{K}_{\text{in}} = \mathcal{O}_s(h^{\frac{1}{2}}) : H^{s+\frac{3}{2}} \longrightarrow H^s,$$

for $0 < h \le h(s)$, $0 \le s < \infty$. This is the inverse of an elliptic boundary value problem and we see that $\widetilde{\mathcal{N}}_{\text{in}}$, defined as in (7.17), is a nice h-pseudodifferential operator on $\partial\mathcal{O}$ of order 0 in h and of order 1 in ξ', with leading symbol $-i(i + (\xi')^2 - z)^{\frac{1}{2}}$, where we use the principal branch of the square root with a cut along the negative real axis. This symbol takes its values in the interior of the fourth quadrant. Then in analogy with (7.19), we have

$$\widetilde{\mathcal{P}}_{\text{out}}(z) = \begin{pmatrix} 1 & 0 \\ h^{\frac{1}{2}} B\widetilde{G}_{\text{in}} & \widetilde{\mathcal{N}}_{\text{in}} - \mathcal{N}_{\text{ext}} \end{pmatrix} \widetilde{\mathcal{P}}_{\text{in}}(z), \tag{13.3}$$

where B was given in (7.6).

We have already investigated $\mathcal{N}_{\rm ext}$ and found that it is an h-pseudodifferential operator whose symbol is nice away from $\mathcal{G}$ where it becomes exotic but small. Away from that set it is of order $(0,1)$ in (h,ξ') with leading part $i((\xi')^2-z)^{\frac{1}{2}}$. When $\Im z\geq 0$ its values are confined to the first quadrant.

From this it follows that $\widetilde{\mathcal{N}}_{\rm in}-\mathcal{N}_{\rm ext}$ is an elliptic h-pseudodifferential operator of order (0,1) whose symbol has a small exotic part near $\mathcal{G}$. Consequently, for every $s\in\mathbb{R}$;

$$\widetilde{\mathcal{N}}_{\rm in}-\mathcal{N}_{\rm ext}:\ H^{s+\frac{3}{2}}\longrightarrow H^{s+\frac{1}{2}} \tag{13.4}$$

is bijective with a uniformly bounded inverse for $0<h\leq h(s)\ll 1$.

It now follows from (13.3) and from the fact that

$$B=\mathcal{O}_s(h^{-\frac{1}{2}}):H^{s+2}\longrightarrow H^{s+\frac{3}{2}}$$

for every $s\geq 0$, that

$$\begin{aligned}\widetilde{\mathcal{P}}_{\rm out}^{-1}&=\widetilde{\mathcal{P}}_{\rm in}(z)^{-1}\begin{pmatrix}1&0\\-(\widetilde{\mathcal{N}}_{\rm in}-\mathcal{N}_{\rm ext})^{-1}h^{\frac{1}{2}}B\widetilde{G}_{\rm in}&(\widetilde{\mathcal{N}}_{\rm in}-\mathcal{N}_{\rm ext})^{-1}\end{pmatrix}\\&=\big(\widetilde{G}_{\rm in}-\widetilde{K}_{\rm in}(\widetilde{\mathcal{N}}_{\rm in}-\mathcal{N}_{\rm ext})^{-1}B\widetilde{G}_{\rm in}\quad h^{-\frac{1}{2}}\widetilde{K}_{\rm in}(\widetilde{\mathcal{N}}_{\rm in}-\mathcal{N}_{\rm ext})^{-1}\big).\end{aligned}\tag{13.5}$$

We conclude that for every $s\in[0,+\infty[$,

$$\begin{cases}\widetilde{\mathcal{P}}_{\rm out}(z)=H^{s+2}\to H^s\times H^{s+\frac{1}{2}}\ \text{has an inverse,}\\ \widetilde{\mathcal{E}}_{\rm out}(z)=\big(\widetilde{G}_{\rm out}\quad h^{-\frac{1}{2}}\widetilde{K}_{\rm out}\big)\ \text{with}\ \widetilde{G}_{\rm out}=\mathcal{O}_s(1):H^s\to H^{s+2},\\ \widetilde{K}_{\rm out}=\mathcal{O}_s(h^{\frac{1}{2}}):H^{s+\frac{1}{2}}\to H^{s+2},\ \text{for}\ 0<h\leq h(s).\end{cases}\tag{13.6}$$

Now drop the assumption that $W=0$ and take again $V=V_0+W$ where we assume that $\|W\|_{L^\infty}\leq 1/C$ with C large enough. Then from (13.6) (where we had $V=V_0$) and a simple perturbation argument we see that

$$\text{(13.6) remains valid for } s=0. \tag{13.7}$$

Write

$$\mathcal{P}_{\rm out}(z)=\big(1+\mathcal{K}(z)\big)\widetilde{\mathcal{P}}_{\rm out}(z), \tag{13.8}$$

where

$$\mathcal{K}(z)=\begin{pmatrix}P-\widetilde{P}\\0\end{pmatrix}\widetilde{\mathcal{E}}_{\rm out}(z).$$

Now $\widetilde{\mathcal{P}}_{\rm out}(z)$ satisfies (12.8) when $p\geq 1$ and $p>n/(2k)$ and hence also (4.30) with p there equal to $\frac{1}{2}(n+\epsilon)$. Moreover, as in the case of $\mathcal{P}_{\rm out}$, the corresponding Schatten class norm of $\partial_z^k\widetilde{\mathcal{P}}_{\rm out}$ is bounded by some negative power of h. Using the bounds on the norm $\widetilde{\mathcal{E}}_{\rm out}$, we see that this operator has the same property. Consequently we have the same properties for $\mathcal{K}(z)$ and Proposition 4.12 applies and shows that $\det(1+\mathcal{K}(z))$ can be defined as in Section 4.4 and satisfies the upper bound

$$\ln\left|\det\left(1+\mathcal{K}(z)\right)\right| \le \mathcal{O}(h^{-N}) \tag{13.9}$$

for some $N \ge 0$. Similarly, $\det\widetilde{\mathcal{P}}_{\mathrm{out}}(z)$ is well-defined and can be realized so that

$$\left|\ln|\det\widetilde{\mathcal{P}}_{\mathrm{out}}|\right| \le \mathcal{O}(h^{-N}). \tag{13.10}$$

Combining this with (13.8), we get

Proposition 13.1. — *There exists $N_0 > 0$ such that*

$$\ln\left|\det\mathcal{P}_{\mathrm{out}}(z)\right| \le \mathcal{O}(1)h^{-N_0}. \tag{13.11}$$

We next start a more precise study of $\det\mathcal{P}_{\mathrm{out}}$ in the region

$$\tfrac{1}{2} < \Re z < 2, \quad ch^{\frac{2}{3}} < |\Im z| < c_0 h^{\frac{2}{3}}, \tag{13.12}$$

where $c > 0$ can be chosen arbitrarily small. For that we shall use Proposition 12.2 and study the two factors to the right in (12.3).

We start with $\det(\mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}})$ and the aim is to write this function as a product of two factors, one being holomorphic and non-vanishing in the whole rectangle $]\frac{1}{2}, 2[+ i\,] - h^{\frac{2}{3}}c_0, h^{\frac{2}{3}}c_0[$, the other being of the form $\det(1+T(z))$, where T is a meromorphic family of trace class operators on $\partial\mathcal{O}$ with poles at $\sigma(P_{\mathrm{in}})$ and whose trace class norm is $\mathcal{O}(h^{1-n})$ when $|\Im z| > h^{\frac{2}{3}}c$. Let

$$P = P^V = -h^2\Delta + V, \quad P_0 = P^{V_0} = -h^2\Delta + V_0, \quad V = V_0 + W$$

with V_0 as before, $W = \mathcal{O}(h)$ in L^∞ and we shall have to strengthen the assumptions on W. In geodesic coordinates,

$$P = (hD_{x_n})^2 + R(x, hD_{x'}), \quad P_0 = (hD_{x_n})^2 + R_0(x, hD_{x'}). \tag{13.13}$$

Let $S : C^\infty(\overline{\mathcal{O}}) \to C^\infty(\overline{\mathcal{O}})$ be of the form $S = S(x, hD_{x'})$ near $\partial\mathcal{O}$ in geodesic coordinates, where $S \ge 0$ has compact support in ξ'. In the interior of $\mathcal{O}$ we arrange by cutting and pasting so that S is a pseudodifferential operator in all the variables of order 0 in h and with symbol of compact support in ξ. Put

$$\widetilde{P}_0 = P_0 + S, \quad \widetilde{P} = P + S. \tag{13.14}$$

Let $\chi = \chi(x', \xi') \in C_0^\infty(T^*\partial\mathcal{O})$ be equal to 1 near $\mathcal{H} \cup \mathcal{G}$. Let $\mathcal{N} = \mathcal{N}_{\mathrm{in}}$ be the Dirichlet to Neumann map associated to $P - z$ (and we will write $P = P_{\mathrm{in}}$

when we wish to emphasize that we take the Dirichlet realization). We start with the trivial decomposition

$$\mathcal{N} = \mathcal{N}\chi(x', hD_{x'}) + \mathcal{N}\big(1 - \chi(x', hD_{x'})\big). \tag{13.15}$$

By Proposition 11.4 the first term to the right is of trace class $C_1(H^{\frac{3}{2}}, H^{\frac{1}{2}})$ and the corresponding trace class norm is $\mathcal{O}(h^{1-n})$ when $|\Im z| \geq h^{\frac{2}{3}}c$.

Now S can be chosen so that

$$\begin{pmatrix} \widetilde{P}_0 - z \\ h^{\frac{1}{2}}\gamma \end{pmatrix} : H^2 \longrightarrow H^0 \times H^{\frac{3}{2}}$$

is bijective with a uniformly bounded inverse $\big(\widetilde{G}_0 \;\; h^{-\frac{1}{2}}\widetilde{K}_0\big)$. Since $\|W\|_{L^\infty} = \mathcal{O}(h) \ll 1$, we have the same fact for

$$\begin{pmatrix} \widetilde{P} - z \\ h^{\frac{1}{2}}\gamma \end{pmatrix} : H^2 \to H^0 \times H^{\frac{3}{2}}$$

and we let $\big(\widetilde{G} \;\; h^{-\frac{1}{2}}\widetilde{K}\big)$ be the inverse.

$K = K_{\text{in}}$ satisfies

$$K(1 - \chi) = \widetilde{K}(1 - \chi) + (P_{\text{in}} - z)^{-1}S\widetilde{K}(1 - \chi). \tag{13.16}$$

Hence

$$\begin{cases} \mathcal{N}(1 - \chi) = \mathrm{I} + \mathrm{II}, \;\; \mathrm{I} = \widetilde{\mathcal{N}}(1 - \chi), \\ \mathrm{II} = \gamma hD_\nu(P - z)^{-1}S\widetilde{K}(1 - \chi). \end{cases} \tag{13.17}$$

Here $\widetilde{K} = \widetilde{K}_0 - (\widetilde{P} - z)^{-1}W\widetilde{K}_0 = \widetilde{K}_0 + \mathcal{O}(h^{\frac{1}{2}})\|W\|_{L^\infty} : H^{\frac{3}{2}} \to H^2$, so

$$\widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}_0 + \mathcal{O}(1)\|W\|_{L^\infty} : H^{\frac{3}{2}} \longrightarrow H^{\frac{1}{2}}. \tag{13.18}$$

Now, as we saw earlier in a slightly different situation, $\widetilde{\mathcal{N}}_0$ is a nice h-pseudodifferential operator of order (0,1) in $(h, \xi'))$ with leading symbol $-i(s(x', \xi') + (\xi')^2 - z)^{\frac{1}{2}}$ and as in (13.4) $\widetilde{\mathcal{N}}_0 - \mathcal{N}_{\text{ext}} = H^{s+\frac{3}{2}} \to H^{s+\frac{1}{2}}$ is bijective with a uniformly bounded inverse for $0 < h < h(s) \ll 1$. From (13.18) we get the same conclusion for $\widetilde{\mathcal{N}} - \mathcal{N}_{\text{ext}} : H^{\frac{3}{2}} \to H^{\frac{1}{2}}$.

We shall next estimate the norm of $S\widetilde{K}(1 - \chi) : H^{\frac{3}{2}} \to H^0$ and for that we try to "commute" $1 - \chi$ and K and exploit that $S(1 - \chi) = \mathcal{O}(h^\infty)$. From $\gamma[\widetilde{K}, \chi] = 0$, $(\widetilde{P} - z)[\widetilde{K}, \chi] = -[\widetilde{P}, \chi]\widetilde{K}$, we get

$$[\widetilde{K}, \chi] = -(\widetilde{P}_{\text{in}} - z)^{-1}[\widetilde{P}, \chi]\widetilde{K}. \tag{13.19}$$

Moreover,

$$S\widetilde{K}(1 - \chi) = S(1 - \chi)\widetilde{K} - S[\widetilde{K}, \chi], \tag{13.20}$$

where the first term to the right is $\mathcal{O}(h^\infty) : H^{\frac{3}{2}} \to H^0$ and we shall see that $[\widetilde{K}, \chi] = \mathcal{O}(h^{\frac{3}{2}}) : H^{\frac{3}{2}} \to H^0$, provided that $\nabla W = \mathcal{O}(1)$ in L^∞: Assume

$$\partial^\alpha W = \mathcal{O}(1) \text{ in } L^\infty, \quad \text{for } |\alpha| \leq 1, \tag{13.21}$$

in addition to the previous assumption that $\|W\| = \mathcal{O}(h)$. As in the remark after Proposition 11.4, this will hold for $W = \delta\Theta q_\omega$ as in Theorem 2.2.

LEMMA 13.2. — *Under the assumption* (13.21), *we have*

$$[\widetilde{K}, \chi] = \mathcal{O}(h^{\frac{3}{2}}) : H^{\frac{3}{2}} \longrightarrow H^2. \tag{13.22}$$

Proof. — If $Q \in C_0^\infty(\mathbb{R}^{2n})$ we have the following representation of the h-pseudodifferential operator $Q(x, hD_x)$ in the classical quantization, obtained in [**1**]:

$$\begin{aligned} Q(x,hD) = \Big(-\frac{1}{\pi}\Big)^{2n} \int \cdots \int & (z_1 - x_1)^{-1} \cdots (z_n - x_n)^{-1} (\zeta_1 - hD_{x_1})^{-1} \\ & (\zeta_n - hD_{x_n})^{-1} \partial_{\bar{z}_1} \cdots \partial_{\bar{z}_n} \partial_{\bar{\zeta}_1} \cdots \partial_{\bar{\zeta}_n} \\ & \widetilde{Q}(z_1, \dots, z_n, \zeta_1, \dots, \zeta_n) L(\mathrm{d}z) L(\mathrm{d}\zeta), \end{aligned} \tag{13.23}$$

where $\widetilde{Q} \in C_0^\infty$ is an almost holomorphic extension satisfying

$$\partial_{(\bar{z},\bar{\zeta})} \widetilde{Q} = \mathcal{O}\big((|\Im z_1| \cdots |\Im z_n| \cdot |\Im \zeta_1| \cdots |\Im \zeta_n|)^\infty \big).$$

From this representation we recover the well-known fact that $Q = \mathcal{O}(1) : L^2 \to L^2$ and for $[Q, W]$ we get a similar formula with $2n$ terms, obtained by replacing one of $(z_j - x_j)^{-1}$ or $(\zeta_j - hD_{x_j})^{-1}$ by $(z_j - x_j)^{-1}[x_j, W](z_j - x_j)^{-1}$ or $(\zeta_j - hD_{x_j})^{-1}[hD_{x_j}, W](\zeta_j - hD_{x_j})^{-1}$ respectively. Then from the boundedness of W and ∇W we see that

$$[Q(x, hD_x), W] = \mathcal{O}(h) : L^2 \longrightarrow L^2. \tag{13.24}$$

The lemma now follows from (13.24) and (13.19). □

Returning to (13.20), we see that

$$S\widetilde{K}(1 - \chi) = \mathcal{O}(h^{\frac{3}{2}}) : H^{\frac{3}{2}} \to H^0. \tag{13.25}$$

We use this in the expression for II in (13.17) together with the telescopic formula

$$(P - z)^{-1} = (\widetilde{P} - z)^{-1} \sum_0^{N-1} (S(\widetilde{P} - z)^{-1})^k + (P - z)^{-1} (S(\widetilde{P} - z)^{-1})^N \tag{13.26}$$

to see that

$$\mathrm{II}(z) = \mathrm{III}(z) + \mathrm{IV}(z), \tag{13.27}$$

where

$$\mathrm{III}(z) = \gamma h D_\nu (\widetilde{P} - z)^{-1} \sum_0^{N-1} (S(\widetilde{P} - z)^{-1})^k S\widetilde{K}(1-\chi) \tag{13.28}$$

is holomorphic and $\mathcal{O}(h) : H^{\frac{3}{2}} \to H^{\frac{1}{2}}$ in the whole rectangle $]\frac{1}{2}, 2[+ i] - h^{\frac{2}{3}}c_0, h^{\frac{2}{3}}c_0[$ and

$$\mathrm{IV}(z) = \gamma h D_\nu (P - z)^{-1} (S(\widetilde{P} - z)^{-1})^N S\widetilde{K}(1-\chi). \tag{13.29}$$

Let N be the smallest integer with

$$N > \tfrac{1}{2}(n-1) \tag{13.30}$$

and assume that

$$\partial^\alpha W = \mathcal{O}(1) \text{ in } L^\infty \text{ for } |\alpha| \leq 2N. \tag{13.31}$$

Again this will hold for $W = \delta\Theta q_\omega$ as in Theorem 2.2 if $\alpha(\,.\,)$ there is large enough. Then $\mathrm{IV}(z)$ is locally uniformly bounded $H^{\frac{3}{2}} \to H^{2(N+1)-\frac{3}{2}} = H^{2N+\frac{1}{2}}$ away from $\sigma(P_{\mathrm{in}})$ and when $|\Im z| \geq h^{\frac{2}{3}}c$ the norm is uniformly

$$\leq \mathcal{O}(h^{\frac{3}{2}-\frac{1}{2}-\frac{2}{3}}) = \mathcal{O}(h^{\frac{1}{3}}).$$

Since $2N > n-1$, we see that $\mathrm{IV}(z) \in C_1(H^{\frac{3}{2}}, H^{\frac{1}{2}})$ and that when $|\Im z| \geq h^{\frac{2}{3}}c$ the corresponding trace class norm is $\leq \mathcal{O}(h^{\frac{1}{3}+1-n})$. Summing up the discussion so far, we have:

Proposition 13.3. — *$\mathcal{N} = \mathcal{N}_{\mathrm{in}}$ can be decomposed as*

$$\mathcal{N} = \widetilde{\mathcal{N}} + \mathrm{III} + (\mathcal{N} - \widetilde{\mathcal{N}})\chi + \mathrm{IV}, \tag{13.32}$$

where $\widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}_0 + \mathcal{O}(1)\|W\|_{L^\infty} = \mathcal{O}(1) : H^{\frac{3}{2}} \to H^{\frac{1}{2}}$ and $\mathrm{III} = \mathcal{O}(h) : H^{\frac{3}{2}} \to H^{\frac{1}{2}}$ are holomorphic in the whole rectangle $]\frac{1}{2}, 2[+ i] - h^{\frac{2}{3}}c_0, h^{\frac{2}{3}}c_0[$, while $(\mathcal{N} - \widetilde{\mathcal{N}})\chi$ and $\mathrm{IV}(z)$ are holomorphic away from $\sigma(P_{\mathrm{in}})$ with values in $C_1(H^{\frac{3}{2}}, H^{\frac{1}{2}})$ and

$$\|(\mathcal{N} - \widetilde{\mathcal{N}})\chi\|_{C_1} + \|\mathrm{IV}\|_{C_1} = \mathcal{O}(h^{1-n}), \ |\Im z| \geq h^{\frac{2}{3}}c. \tag{13.33}$$

Now write

$$\mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}} = \widehat{A}(z) + (\mathcal{N} - \widetilde{\mathcal{N}})\chi + \mathrm{IV}, \tag{13.34}$$

where

$$\widehat{A}(z) := \widetilde{\mathcal{N}} + \mathrm{III} - \mathcal{N}_{\mathrm{ext}} : H^{\frac{3}{2}} \longrightarrow H^{\frac{1}{2}}, \tag{13.35}$$

is holomorphic, uniformly bounded and uniformly invertible in the whole rectangle, and factorize,

$$\mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}} = \widehat{A}(z)\widehat{B}(z), \tag{13.36}$$

$$\widehat{B}(z) = 1 + \widehat{A}(z)^{-1}\left((\mathcal{N} - \widetilde{\mathcal{N}})\chi + \mathrm{IV}\right) =: 1 + \widehat{C}(z), \tag{13.37}$$

where $\widehat{C}(z)$ belongs to $C_1(H^{\frac{3}{2}}, H^{\frac{3}{2}})$ away from $\sigma(P_{\mathrm{in}})$ and the corresponding trace class norm is $\mathcal{O}(h^{1-n})$ when $|\Im z| \geq h^{\frac{2}{3}}c$.

We conclude that

$$\ln\left|\det \widehat{B}(z)\right| \leq \mathcal{O}(h^{1-n}), \quad \text{when } |\Im z| \geq h^{\frac{2}{3}}c. \tag{13.38}$$

$\widehat{A}(z)$ in (13.35) is holomorphic in the whole rectangle. It follows from Lemma 12.3 and the discussion after (12.7) that the C_p-norm of $\partial_z^k \mathcal{N}_{\mathrm{ext}} : H^{\frac{3}{2}} \to H^{\frac{1}{2}}$ is bounded by a negative power of h when p is ≥ 1 and $> (n-1)/(2k)$.

As in the proof of that lemma, we write

$$\partial_z^k \widetilde{\mathcal{N}}(z) = C_k \gamma h D_\nu (\widetilde{P}_{\mathrm{in}} - z)^{-k} \widetilde{K}_{\mathrm{in}}$$

and using (13.31) we see that $\partial_z^k \widetilde{\mathcal{N}}(z) = \mathcal{O}(1) : H^{\frac{3}{2}} \to H^{\frac{1}{2}+2k}$ for $2k \leq 2N+2$ and hence the C_p-norm of $\partial_z^k \widetilde{\mathcal{N}} : H^{\frac{3}{2}} \to H^{\frac{1}{2}}$ is bounded by some negative power of h when p is ≥ 1 and $> (n-1)/(2k)$, for $k \leq N+1$. For $k = N+1$ we have $k > \frac{1}{2}(n-1)$, so $n/(2k) < 1$. From (13.28) we get the same estimates for $\partial_z^k \mathrm{III}$. Thus the C_p-norm of $\partial_z^k \widehat{A}(z) : H^{\frac{3}{2}} \to H^{\frac{1}{2}}$ is bounded by some negative power of h when p is ≥ 1 and $> (n-1)/(2k)$, $k \leq N+1$.

In conclusion, $\det \widehat{A}(z)$ and its inverse $\det \widehat{A}(z)^{-1}$ can be defined in the whole rectangle as in Section 4.4, such that for some N_0,

$$\ln\left|\det \widehat{A}(z)\right| = \mathcal{O}(h^{-N_0}).$$

The desired factorization of $\det(\mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}})$ is now

$$\det(\mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}}) = \det \widehat{A}(z) \det \widehat{B}(z), \tag{13.39}$$

where $\det \widehat{A}(z)$ and its inverse are holomorphic in the whole rectangle and bounded from above by $C\exp(Ch^{-N_0})$ for some $C, N_0 > 0$.

Before continuing, we sum up and compare the two main results so far. Proposition 4.12, applied to $1 + \mathcal{K}(z)$ in (13.8), gives

$$1 + \mathcal{K}(z) = A(z)B(z), \tag{13.40}$$

where in the rectangle (13.1),

$$\ln\left|\det A(z)\right| = \mathcal{O}(h^{-N}), \tag{13.41}$$

$$\ln\left|\det B(z)\right| \leq \mathcal{O}(h^{-N}). \tag{13.42}$$

More precisely, $B(z) = 1 + R_N(\mathcal{K})\mathcal{K}^N =: 1 + C(z)$, where $C(z)$ is holomorphic with values in the trace class operators and

$$\big\|C(z)\big\|_{C_1} \leq \mathcal{O}(h^{-N}). \tag{13.43}$$

Here, the exponent N may take a new value at each appearance. Further (see (13.8))

$$\det \mathcal{P}_{\mathrm{out}} = \det \widetilde{\mathcal{P}}_{\mathrm{out}} \det A(z) \det B(z), \tag{13.44}$$

where $\det \widetilde{\mathcal{P}}_{\mathrm{out}}$ can be defined as in Section 4.4 such that

$$\big|\ln|\det \widetilde{\mathcal{P}}_{\mathrm{out}}|\big| = \mathcal{O}(h^{-N}). \tag{13.45}$$

On the other hand we have (7.19), (12.3):

$$\det \mathcal{P}_{\mathrm{out}}(z) = \det\big(\mathcal{P}_{\mathrm{in}}(z)\big) \det(\mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}}), \tag{13.46}$$

where

$$\det(\mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}}) = \det \widehat{A}(z) \det \widehat{B}(z), \quad \widehat{B}(z) = 1 + \widehat{C}(z). \tag{13.47}$$

Here, $\det \widehat{A}(z)$ is holomorphic and

$$\ln\big|\det \widehat{A}(z)\big| = \mathcal{O}(h^{-N}) \tag{13.48}$$

in the whole rectangle, while $\widehat{C}(z)$ is meromorphic with values in $C_1(H^{\frac{3}{2}}, H^{\frac{3}{2}})$ with the poles at the (real) eigenvalues of P_{in}. Moreover, for $|\Im z| \geq h^{\frac{2}{3}}c$ we have $\|\widehat{C}(z)\|_{C_1} \leq \mathcal{O}(h^{1-n})$, so in that region

$$\ln\big|\det\big(1 + \widehat{C}(z)\big)\big| \leq \mathcal{O}(h^{1-n}). \tag{13.49}$$

We shall now compare the expressions (13.44) and (13.46).

In (13.44) the first two factors to the left are well defined up to factors of the form $\exp p(z)$ where p is a polynomial of degree $\leq N$ and as we have seen, we can choose realizations satisfying (13.44), (13.41). As for $\det B(z)$, defined as a determinant of a trace class perturbation of 1 (which is a special case of the definition in Section 4.4), we only have the upper bound (13.42).

In (13.46), $\det \mathcal{P}_{\mathrm{in}}(z) = \det(P_{\mathrm{in}} - z)$ can be defined as in Section 4.4 up to a factor $\exp p(z)$ as before, in such a way that $\ln|\det \mathcal{P}_{\mathrm{in}}| \leq \mathcal{O}(h^{-N})$ and when $|\Im z| \geq h^{\frac{2}{3}}/\widetilde{C}$, we even have $\ln|\det \mathcal{P}_{\mathrm{in}}(z)| = \mathcal{O}(h^{-N})$. This factor will be further studied below. Similarly, we have (13.47), (13.48) and again we define $\det \widehat{B}$ as the determinant of a trace class perturbation of the identity.

When writing the identity

$$\det \mathcal{P}_{\mathrm{out}}(z) = \det \widetilde{\mathcal{P}}_{\mathrm{out}} \det A(z) \det B(z) = \det \mathcal{P}_{\mathrm{in}} \det \widehat{A}(z) \det \widehat{B}(z), \tag{13.50}$$

it is not *a priori* clear that we can choose $\det \widetilde{P}_{\mathrm{out}}$, $\det A(z)$, $\det \widehat{A}(z)$, $\det \mathcal{P}_{\mathrm{in}}$ all satifying the above bounds simultaneously, since we have made definite choices of $\det B(z)$ and $\det \widehat{B}(z)$. However, if we restrict the attention to the region $|\Im z| \geq h^{\frac{2}{3}}c$ we know that $B(z)^{-1}$ and $\widehat{B}(z)^{-1}$ are bounded in operator norm by some negative power of h, and this additional information implies that $B(z)^{-1} = 1 + D(z)$, $\widehat{B}(z)^{-1} = 1 + \widehat{D}(z)$, where $D(z)$ and $\widehat{D}(z)$ are bounded in trace class norm by negative powers of h, so in that region we also get

$$\ln \big|\det B(z)\big|,\, \ln \big|\det \widehat{B}(z)\big| = \mathcal{O}(h^{-N}).$$

Then if we choose the other factors with moduli that have polynomially bounded logarithms, we can modify one of them by a factor $\exp p(z)$, where $p(z)$ is a polynomial of degree $\leq N$ with real part $= \mathcal{O}(h^{-N})$ and achieve (13.50) in such a way that

- ▷ $\ln|x| = \mathcal{O}(h^{-N})$ when $x = \det A$, $\det \widehat{A}$, $\det \widetilde{\mathcal{P}}_{\mathrm{out}}$ in the whole rectangle;
- ▷ $\ln|x| = \mathcal{O}(h^{-N})$ for $|\ln z| \geq h^{\frac{2}{3}}c$, when $x = \det B(z)$, $\det \widehat{B}(z)$, $\det \mathcal{P}_{\mathrm{in}}$;
- ▷ $\ln|x| \leq \mathcal{O}(h^{-N})$ in the whole rectangle, when $x = \det B(z)$, $\det \mathcal{P}_{\mathrm{in}}$.

Moreover, as we have seen,

$$\ln \big|\det \widehat{B}(z)\big| \leq \mathcal{O}(h^{1-n}), \quad \text{when } |\Im z| \geq h^{\frac{2}{3}}c. \tag{13.51}$$

The aim is to study the zeros of $\det \mathcal{P}_{\mathrm{out}}(z)$ in the rectangle (13.1), using the upper bound (13.11) and the more precise upper bound for $|\Im z| \geq h^{\frac{2}{3}}c$ resulting from the last expression in (13.50) together with (13.51) and the fact that $\ln|\det \widehat{A}| = \mathcal{O}(h^{-N})$. After division with $\det \widehat{A}(z)$ we can concentrate on the function

$$f(z) = \det \mathcal{P}_{\mathrm{in}} \det \widehat{B}(z), \tag{13.52}$$

for which

$$\ln \big|f(z)\big| \leq \mathcal{O}(h^{-N}). \tag{13.53}$$

Next, look at $\det \mathcal{P}_{\mathrm{in}}(z)$. Let $\widetilde{K} = \mathcal{O}(h^{\frac{1}{2}}) : H^s \to H^{s+\frac{1}{2}}$, $s \in \mathbb{R}$ be a right inverse of γ. Then,

$$\big(1 \;\; \widetilde{K}\big) : \mathcal{D}(P_{\mathrm{in}}) \times H^{\frac{3}{2}} \longrightarrow H^2$$

is a bijection with a bounded inverse and

$$\mathcal{P}_{\mathrm{in}}(z)\big(1 \;\; h^{-\frac{1}{2}}\widetilde{K}\big) = \begin{pmatrix} P_{\mathrm{in}} - z & h^{-\frac{1}{2}}(P-z)\widetilde{K} \\ 0 & 1 \end{pmatrix},$$

so

$$\det \mathcal{P}_{\mathrm{in}}(z) \det \big(1 \;\; h^{-\frac{1}{2}}\widetilde{K}\big) = \det(P_{\mathrm{in}} - z)$$

and since $\widetilde{K}$ is independent of z, we can take $\det(1\ h^{-\frac{1}{2}}\widetilde{K})$ to be an arbitrary non-vanishing constant, say 1 and get

$$\det \mathcal{P}_{\mathrm{in}}(z) = \det(P_{\mathrm{in}} - z). \tag{13.54}$$

The method in Section 4.4 shows that

$$\partial_z^N \ln\det(P_{\mathrm{in}} - z) = -(N-1)!\,\mathrm{tr}(P_{\mathrm{in}} - z)^{-N}, \tag{13.55}$$

for $N > \frac{1}{2}n$, so that $(P_{\mathrm{in}} - z)^{-N}$ is of trace class.

Let $\chi \in C_0^\infty(]\frac{1}{4}, 4[; [0,1])$ be equal to 1 in a neighborhood of $[\frac{1}{3}, 3]$. If

$$N(\lambda) = \#\big(\sigma(P_{\mathrm{in}}) \cap]-\infty, \lambda]\big),$$

we get

$$\begin{aligned}\partial_z^N \ln\det(P_{\mathrm{in}} - z) &= -(N-1)!\int (\lambda - z)^{-N}\mathrm{d}N(\lambda)\\ &= -(N-1)!\int (\lambda - z)^{-N}\chi(\lambda)\mathrm{d}N(\lambda)\\ &\qquad -(N-1)!\int (\lambda - z)^{-N}\big(1-\chi(\lambda)\big)\mathrm{d}N(\lambda).\end{aligned} \tag{13.56}$$

Thus,

$$\ln\det(P_{\mathrm{in}} - z) = \mathrm{I}(z) + \mathrm{II}(z), \tag{13.57}$$

where

$$-\partial_z^N \mathrm{I}(z) = (N-1)!\int (\lambda - z)^{-N}\chi(\lambda)\mathrm{d}N(\lambda) \tag{13.58}$$

$$-\partial_z^N \mathrm{II}(z) = (N-1)!\int (\lambda - z)^{-N}\big(1-\chi(\lambda)\big)\mathrm{d}N(\lambda). \tag{13.59}$$

Up to a polynomial, we have for $\Im z \neq 0$:

$$\mathrm{I}(z) = \int \ln(\lambda - z)\chi(\lambda)\mathrm{d}N(\lambda), \tag{13.60}$$

where we use the standard branch of ln with a cut along $]-\infty, 0[$. In particular,

$$\Re\mathrm{I}(z) = \int \ln|\lambda - z|\chi(\lambda)\mathrm{d}N(\lambda). \tag{13.61}$$

In order to estimate $\mathrm{II}(z)$, we shall use the rough estimate

$$N(\lambda) = \mathcal{O}(h^{-n}\lambda^{\frac{1}{2}n}), \tag{13.62}$$

which is valid uniformly for $0 < h \ll 1$, $\lambda \geq 1$. It follows from (13.62) and an integration by parts in (13.59), that

$$\partial_z^N \mathrm{II}(z) = \mathcal{O}(h^{-n}) \tag{13.63}$$

in the domain (13.1). By integration, we see that we can choose $\mathrm{II}(z)$ holomorphic in this domain such that

$$\mathrm{II}(z) = \mathcal{O}(h^{-n}). \tag{13.64}$$

This will allow us to replace $\det \mathcal{P}_{\mathrm{in}}$ by $\exp \mathrm{I}(z)$ in the definition of $f(z)$ in (13.52), without affecting the validity of (13.53).

Before that we will discuss some harmonic majorants of $\Re \mathrm{I}(z)$. Recall that if $\Omega \Subset \mathbb{C}$ has piecewise smooth boundary and if $G = G_\Omega$, $K = K_\Omega$ are the corresponding Dirichlet and Poisson kernels for the Dirichlet problem for the Laplacien, then by Green's formula, we have

$$K(x,y) = \partial_{\nu_y} G(x,y),$$

where ν is the exterior unit normal. This still holds when $\Omega = \Omega_r$ is the infinite strip $\{x \in \mathbb{C};\ |\Im x| < r\}$ and we consider the solutions to the Dirichlet problem that are bounded when the data are bounded. In the case $\Omega = \Omega_1$ we have (see for instance [**26**]) that $G(x,y)$ is of class $C^\infty(\overline{\Omega} \times \overline{\Omega})$ away from the diagonal and there exists $C_0 > 0$ such that for every $r > 0$ and all $\alpha, \beta \in \mathbb{N}$, there exists a constant $C = C_{\alpha,\beta,r}$ such that

$$\left|\nabla_x^\alpha \nabla_y^\beta G(x,y)\right| \le C \exp -\frac{1}{C_0} |\Re x - \Re y|, \quad \text{when } |x-y| > r > 0. \tag{13.65}$$

Moreover,

$$G_{r\Omega}(x,y) = G_\Omega\Big(\frac{x}{r}, \frac{y}{r}\Big), \quad K_{r\Omega}(x,y) = \frac{1}{r} K_\Omega\Big(\frac{x}{r}, \frac{y}{r}\Big). \tag{13.66}$$

Consider first the subharmonic function $\ln|x|$ on Ω_r and its smallest harmonic majorant there, given by

$$\Delta h_r = 0, \quad h_{r|\partial\Omega_r} = \ln|x|.$$

Then, $\psi_r := h_r - \ln|x| \ge 0$ is equal to $-2\pi G_{\Omega_r}(x,0)$ and we are interested in

$$f_r := -\partial_\nu \psi_r = 2\pi \partial_\nu G_{\Omega_r}(x,0) = 2\pi K_{\Omega_r}(0,x) = \frac{2\pi}{r} K_{\Omega_1}\Big(0, \frac{x}{r}\Big) = \frac{1}{r} f_1\Big(\frac{x}{r}\Big),$$

which is a non-negative function defined on the boundary and satisfies

$$\partial_x^\alpha f_r = \mathcal{O}_\alpha(1) r^{-1-|\alpha|} \mathrm{e}^{-\frac{1}{C_0 r}|\Re x|}. \tag{13.67}$$

Also,

$$\int_{\partial\Omega_r} f_r |\mathrm{d}x| = 2\pi, \quad f_r(\bar{x}) = f_r(x). \tag{13.68}$$

The smallest harmonic majorant in Ω_r of

$$\phi_{\mathrm{in}} := \Re \mathrm{I}(x) = \sum \chi(\lambda_j) \ln|z - \lambda_j| \tag{13.69}$$

is

$$h_{r,\mathrm{in}}(x) = \sum \chi(\lambda_j) h_r(x-\lambda_j). \tag{13.70}$$

The function

$$\Phi_r = \begin{cases} \phi_{\mathrm{in}} \text{ outside } \Omega_r \\ h_{\mathrm{in}} \text{ in } \Omega_r \end{cases} \tag{13.71}$$

is subharmonic, $\Delta\Phi_r$ is supported in $\partial\Omega_r$ and equal to

$$\sum \chi(\lambda_j)\big(f_r(x-\lambda_j)\delta(\Im x - r) + f_r(x-\lambda_j)\delta(\Im x + r)\big). \tag{13.72}$$

If $\frac{1}{2} \leq a < b \leq 2$, we get with

$$g_r(t) = \frac{1}{2\pi}\big(f_r(t+ir) + f_r(t-ir)\big) =: \frac{1}{r} g_1\Big(\frac{t}{r}\Big) \geq 0, \tag{13.73}$$

that

$$\int_{a\leq \Re x \leq b} \Delta\Phi_r(x) L(\mathrm{d}x) = 2\pi \int_a^b g_r * (\chi \mathrm{d}N)(t)\mathrm{d}t. \tag{13.74}$$

Notice that $g_r(t) = (1/r)g_1(t/r)$ is an approximation of δ and we will use (13.74) with $r = h^{\frac{2}{3}}c$.

Returning to (13.52), (13.53), we see that the zeros of f in the rectangle (13.1) will not change if we replace $\det \mathcal{P}_{\mathrm{in}}$ in (13.52) by $\exp \mathrm{I}(z)$, so we now redefine f to be

$$f(z) = \mathrm{e}^{\mathrm{I}(z)} \det \widehat{B}(z), \tag{13.75}$$

and notice that (13.53) still holds because of (13.64). Moreover,

$$\ln \big|f(z)\big| = \phi_{\mathrm{in}}(z) + \ln \big|\det \widehat{B}(z)\big|, \tag{13.76}$$

and (13.53) tells us that

$$\ln \big|f(z)\big| \leq \mathcal{O}(h^{-N}) \tag{13.77}$$

in the whole rectangle, while (13.51) shows that

$$\ln \big|f(z)\big| \leq \phi_{\mathrm{in}}(z) + \mathcal{O}(h^{1-n}), \tag{13.78}$$

in the part of the rectangle where $|\Im z| \geq h^{\frac{2}{3}}c$.

Clearly, the whole discussion so far remains valid if we enlarge the rectangle (13.1) by replacing $\frac{1}{2}$ by a slightly smaller constant and the bound 2 by a slightly larger constant. We can find α, β with $\frac{1}{2} - \alpha \asymp 1/\mathcal{O}(1)$, $\beta - 2 \asymp 1/\mathcal{O}(1)$ such that $\phi_{\mathrm{in}} \geq -\mathcal{O}(h^{-N})$ for $\Re z = \alpha, \beta$, and (13.53) tells us that

$$\ln \big|f(z)\big| \leq h_r(z) + \mathcal{O}(h^{-N}), \tag{13.79}$$

on the same vertical segments, while (13.78) tells us that

$$\ln\big|f(z)\big| \le h_r(z) + \mathcal{O}(h^{1-n}) \tag{13.80}$$

on the horizontal parts of the boundary of $[\alpha,\beta]+ir[-1,1]$. By the maximum principle, we get in the latter rectangle

$$\ln\big|f(z)\big| \le \widetilde{h}(z) + \mathcal{O}(h^{1-n}),$$

where $\widetilde{h}$ is the harmonic function on $[\alpha,\beta]+ir[-1,1]$ which is equal to a constant$=\mathcal{O}(h^{-N})$ on the vertical parts of the boundary and equal to $h_r(z)$ on the horizontal parts. Using that r is of the order of $h^{\frac{2}{3}}$ together with simple estimates on the Poisson kernel in thin rectangles (see [**26**], Section 2), we see that

$$\widetilde{h}(z) \le \mathcal{O}(1)h^{-N}\exp\Big(-\frac{1}{\mathcal{O}(1)r}\Big) + h_r(z) \le h_r(z) + \mathcal{O}(h^{1-n})$$

on $[\frac{1}{2},2]+ir[-1,1]$ and we get the estimate

$$\ln|f(z)| \le h_r(z) + \mathcal{O}(h^{1-n})$$

on the latter rectangle, leading to

$$\ln\big|f(z)\big| \le \Phi_r(z) + \mathcal{O}(h^{1-n}) \ \text{ in the rectangle (13.1).} \tag{13.81}$$

This estimate together with (13.74) form the main conclusion of this chapter.

CHAPTER 14

SOME ESTIMATES FOR P_{out}

In this and the next two chapters we shall construct a suitable perturbation W as in Theorem 2.2 such that we get a lower bound for $f(z)$ in (13.52) that matches (13.81). Here z is any given point in the set (13.12) and the perturbation will depend on that point. As we shall see, this amounts to getting a good bound on the smallest singular value on $\widehat{B}$ (cf. (13.47)) or equivalently on that of $\mathcal{P}_{\mathrm{out}}$, or of $\mathcal{N}_{\mathrm{in}}(z) - \mathcal{N}_{\mathrm{out}}(z)$.

For $\mu > 0$, let $E(\mu) \subset L^2(\mathcal{O})$ be the spectral subspace associated to all eigenvalues $< \mu^2$ of $P_{\mathrm{out}}(z)^* P_{\mathrm{out}}(z)$. We shall show that if μ is small enough (to be specified below) and $u \in E(\mu)$ is normalized, then $\|u\|_{L^2(\mathcal{O}_h \setminus \mathcal{O}_{2h})}$ cannot be too small. When $c \geq 0$, we define

$$\mathcal{O}_c = \big\{ x \in \mathcal{O};\ \mathrm{dist}(x, \partial\mathcal{O}) > c \big\}.$$

If $u \in E(\mu)$, we have $u = \sum_1^N u_j e_j$, where $e_1, \dots, e_N$ is an orthonormal basis of eigenfunctions in $E(\mu)$, $P_{\mathrm{out}}(z)^* P_{\mathrm{out}}(z) e_j = t_j^2 e_j$, $0 \leq t_j < \mu$, and

$$\big\| P_{\mathrm{out}}(z) u \big\|^2 = \big(P_{\mathrm{out}}(z)^* P_{\mathrm{out}}(z) u \,|\, u \big) = \sum_1^N |u_j|^2 t_j^2 \leq \mu^2 \sum |u_j|^2 = \mu^2 \|u\|^2,$$

where all norms are in L^2 if nothing else is specified. Thus, if $u \in E(\mu)$, and $\|u\| = 1$,

$$P_{\mathrm{out}}(z) u = v, \quad \|v\| < \mu. \tag{14.1}$$

By standard elliptic estimates, combined with the dilation $x = hy$, $hD_{x_j} = D_{y_j}$, we have for every fixed θ with $0 < \theta \ll 1$,

$$\begin{aligned} \|u\|_{H_h^2(\mathcal{O}_{(1+\theta)h} \setminus \mathcal{O}_{2h/(1+\theta)})} &\leq C_\theta \big(\|v\| + \|u\|_{L^2(\mathcal{O}_h \setminus \mathcal{O}_{2h})} \big) \\ &\leq C_\theta \big(\mu + \|u\|_{L^2(\mathcal{O}_h \setminus \mathcal{O}_{2h})} \big). \end{aligned} \tag{14.2}$$

Let $\chi \in C_0^\infty(\mathcal{O}_{(1+\theta)h}; [0,1])$ be equal to 1 on $\mathcal{O}_{3h/2}$ and satisfy $\partial^\alpha \chi = \mathcal{O}(h^{-|\alpha|})$, $\alpha \in \mathbb{N}^n$. Let $\Gamma = \Gamma_f$ be a Lipschitz contour as in and around (5.31) with $\theta = \frac{1}{3}\pi$. Let P_{ext} be the Dirichlet realization of P on $\Gamma \setminus \mathcal{O}_{2h}$. Then

$$(P_{\text{ext}} - z)(1-\chi)u = (1-\chi)v - [P,\chi]u, \tag{14.3}$$

where we let u also denote the outgoing extension of u which is well-defined since $u \in \mathcal{D}(P_{\text{out}}(z))$ and where v also denotes the 0 extension. Similarly,

$$(P_{\text{in}} - z)\chi u = \chi v + [P,\chi]u. \tag{14.4}$$

If V vanishes outside $\mathcal{O}_{2h}$, we know from Chapter 9 (with $\mathcal{O}$ there replaced by $\mathcal{O}_{2h}$) that $\|(P_{\text{ext}}-z)^{-1}\|_{\mathcal{L}(L^2,L^2)} = \mathcal{O}(h^{-\frac{2}{3}})$. More generally, we shall assume that

$$\|V\|_{L^\infty(\mathcal{O}\setminus\mathcal{O}_{2h})} \ll h^{\frac{2}{3}}, \tag{14.5}$$

and we notice that this holds for $V = V_0 + \delta\Theta q_\omega$ in Theorem 2.2 if α is large enough. Then by a simple perturbation argument, the preceding estimate on the exterior resolvent remains valid and we get from (14.2), (14.3),

$$h^{\frac{2}{3}} \big\|(1-\chi)u\big\|_{L^2(\mathcal{O})} \leq \mathcal{O}(1)\big(\mu + \|u\|_{L^2(\mathcal{O}_h\setminus\mathcal{O}_{2h})}\big). \tag{14.6}$$

Similarly, by using that $\|(P_{\text{in}} - z)^{-1}\|_{\mathcal{L}(L^2,L^2)} = \mathcal{O}(h^{-\frac{2}{3}})$, we get

$$h^{\frac{2}{3}} \|\chi u\|_{L^2(\mathcal{O})} \leq \mathcal{O}(1)\big(\mu + \|u\|_{L^2(\mathcal{O}_h\setminus\mathcal{O}_{2h})}\big). \tag{14.7}$$

Combining the two estimates and recalling that $\|u\| = 1$, we get

$$h^{\frac{2}{3}} \leq \mathcal{O}(1)\big(\mu + \|u\|_{L^2(\mathcal{O}_h\setminus\mathcal{O}_{2h})}\big), \tag{14.8}$$

and if $\mu \ll h^{\frac{2}{3}}$, for all $u \in E(\mu)$ with $\|u\|_{L^2(\mathcal{O})} = 1$,

$$\|u\|_{L^2(\mathcal{O}_h\setminus\mathcal{O}_{2h})} \geq \frac{h^{\frac{2}{3}}}{\mathcal{O}(1)}. \tag{14.9}$$

Next we make a remark about the H^s regularity of of elements in $E(\mu)$. Assume that for some fixed $s > \frac{1}{2}n$, we have $V = V_1 + V_2$

$$\|V_1\|_{H_1^s} + h^{-\frac{1}{2}n}\|V_2\|_{H_h^s} \leq \mathcal{O}(1). \tag{14.10}$$

When $V = V_0 + W = V_0 + \delta\Theta q_\omega$ is a potential as in Theorem 2.2, we take $V_1 = V_0$, $V_2 = W$ and get (14.10), provided $\alpha(n, v_0, s, \epsilon, \theta, M, \widetilde{M})$ in (2.9) is large enough (cf. Remark 15.1). So far we have systematically used the semi-classical Sobolev spaces $H^s = H_h^s$ but in (14.10) we also use the standard Sobolev space $H^s = H_1^s$ (with $h = 1$). Following standard conventions, we let

$$H_\bullet^\sigma(\mathcal{O}) = H_\bullet^\sigma(\mathbb{R}^n)_{|\mathcal{O}}, \quad H_\bullet^\sigma(\overline{\mathcal{O}}) = \{u \in H_\bullet^\sigma(\mathbb{R}^n);\, \operatorname{supp} u \subset \overline{\mathcal{O}}\}.$$

If $u = \sum_1^N u_j e_j \in E(\mu)$, we have $(P_{\mathrm{out}}^* P_{\mathrm{out}})^k u = \sum_1^N t_j^{2k} u_j e_j$, so

$$\big\|(P_{\mathrm{out}}^* P_{\mathrm{out}})^k u\big\| \le \mu^{2k}\|u\|, \quad k \in \mathbb{N}. \tag{14.11}$$

We will assume that $\mu = \mathcal{O}(1)$ and limit the attention to k in a bounded interval, so the right hand side of (14.11) will be $\mathcal{O}(\|u\|)$. We study *a priori* estimates in the interior. Let $\Omega_2 \subset \Omega_1 \subset \mathcal{O}$ be open with $\mathrm{dist}(\Omega_2, \complement\Omega_1) \ge h/C$. If $P_{\mathrm{out}} u = v,\ \ u, v \in H_h^\sigma(\Omega_1)$, $0 \le \sigma \le s$, we can write

$$-h^2 \Delta u = v + (z - V)u =: w,$$

where

$$\|w\|_{H_h^\sigma(\Omega_1)} \le \mathcal{O}(1)\big(\|v\|_{H_h^\sigma(\Omega_1)} + \|u\|_{H_h^\sigma(\Omega_1)}\big)$$

and standard *a priori* estimates for $-\Delta$ (after the dilation $x = hy$) give

$$\|u\|_{H_h^{\sigma+2}(\Omega_2)} \le \mathcal{O}(1)\big(\|v\|_{H_h^\sigma(\Omega_1)} + \|u\|_{H_h^\sigma(\Omega_1)}\big). \tag{14.12}$$

If $s < \sigma < s + 2$, we only get

$$\|u\|_{H_h^{s+2}(\Omega_2)} \le \mathcal{O}(1)\big(\|v\|_{H_h^\sigma(\Omega_1)} + \|u\|_{H_h^\sigma(\Omega_1)}\big). \tag{14.13}$$

The same *a priori* estimate holds for P_{out}^*.

We shall now use these estimates to study elements of $E(\mu)$ and first assume for simplicity that (14.10) holds for all $s > 0$. From the fact that

$$(P_{\mathrm{out}}^* P_{\mathrm{out}})^k u = \mathcal{O}_k(1)\|u\|$$

in $H^0(\mathcal{O})$ for all $k \in \mathbb{N}$ we first infer by integration by parts, that

$$P_{\mathrm{out}}(P_{\mathrm{out}}^* P_{\mathrm{out}})^{k-1} u = \mathcal{O}(1)$$

in $H^0(\mathcal{O})$. Using the *a priori* estimate for P_{out}^*, we get

$$\begin{aligned}&\big\|P_{\mathrm{out}}(P_{\mathrm{out}}^* P_{\mathrm{out}})^{k-1} u\big\|_{H^2(\mathcal{O}_{h/C})}\\ &\quad \le \mathcal{O}(1)\big(\|(P_{\mathrm{out}}^* P_{\mathrm{out}})^k u\|_{H^0(\mathcal{O})} + \|P_{\mathrm{out}}(P_{\mathrm{out}}^* P_{\mathrm{out}})^{k-1} u\|_{H^0(\mathcal{O})}\big) \le \mathcal{O}(1),\end{aligned}$$

and using the one for P_{out}, we get

$$\begin{aligned}&\big\|(P_{\mathrm{out}}^* P_{\mathrm{out}})^{k-1} u\big\|_{H^2(\mathcal{O}_{h/C})}\\ &\quad \le \mathcal{O}(1)\big(\|P_{\mathrm{out}}(P_{\mathrm{out}}^* P_{\mathrm{out}})^k u\|_{H^0(\mathcal{O})} + \|(P_{\mathrm{out}}^* P_{\mathrm{out}})^{k-1} u\|_{H^0(\mathcal{O})}\big) \le \mathcal{O}(1).\end{aligned}$$

Thus for all $k \in \mathbb{N}$,

$$\|(P_{\mathrm{out}}^* P_{\mathrm{out}})^k u\|_{H^2(\mathcal{O}_{h/C})} + \|P_{\mathrm{out}}(P_{\mathrm{out}}^* P_{\mathrm{out}})^k u\|_{H^2(\mathcal{O}_{h/C})} \le \mathcal{O}(1).$$

Here we use again the *a priori* estimates for P_{out}^* and P_{out} and get that for every $k \in \mathbb{N}$,

$$\big\|(P_{\mathrm{out}}^* P_{\mathrm{out}})^k u\big\|_{H^4(\mathcal{O}_{2h/C})} + \big\|P_{\mathrm{out}}(P_{\mathrm{out}}^* P_{\mathrm{out}})^k u\big\|_{H^4(\mathcal{O}_{2h/C})} \le \mathcal{O}(1).$$

Iterating this argument, we get for every $j \in \mathbb{N}$ that for every $k \in \mathbb{N}$,

$$\left\|(P_{\mathrm{out}}^* P_{\mathrm{out}})^k u\right\|_{H^{2j}(\mathcal{O}_{2jh/C})} + \left\|P_{\mathrm{out}}(P_{\mathrm{out}}^* P_{\mathrm{out}})^k u\right\|_{H^{2j}(\mathcal{O}_{2jh/C})} \leq \mathcal{O}(1).$$

Now if we make the assumption (14.10) for a fixed $s > \frac{1}{2}n$, we see that the above iteration works as long as $2j \leq s+2$, then if this last j is strictly less than $\frac{1}{2}(s+2)$, we can make one more iteration and reach the degree of regularity $s+2$. Hence the final conclusion is that if $\mu = \mathcal{O}(1)$ and we assume (14.10) for a fixed $s > \frac{1}{2}n$, then for every $C > 0$, we have

$$\left\|(P_{\mathrm{out}}^* P_{\mathrm{out}})^k u\right\|_{H^{s+2}(\mathcal{O}_{h/C})} + \left\|P_{\mathrm{out}}(P_{\mathrm{out}}^* P_{\mathrm{out}})^k u\right\|_{H^{s+2}(\mathcal{O}_{h/C})} \leq \mathcal{O}(1). \tag{14.14}$$

We end this chapter with some estimates relating the small singular values of $P_{\mathrm{out}}(z)$ to those of $\mathcal{P}_{\mathrm{out}}$ and when z belongs to the set (13.12), to those of $\mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{out}}$ and of $\widehat{B}(z) = 1 + \widehat{C}(z)$ in (13.36) and (13.37).

Recall that $\mathcal{P}_{\mathrm{out}}(z)$ is bijective precisely when $P_{\mathrm{out}}(z)$ is, and when so is the case it easy to check that

$$\mathcal{P}_{\mathrm{out}}(z)^{-1} = \big(P_{\mathrm{out}}(z)^{-1} \quad (1 - P_{\mathrm{out}}(z)^{-1}(P-z))h^{-\frac{1}{2}}\widehat{K}\big), \tag{14.15}$$

where we recall that $\widehat{K} = \mathcal{O}(h^{\frac{1}{2}}) : H^{\frac{1}{2}} \to H^2$ is a right inverse of B.

Recall that when $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator between two Hilbert spaces, then the singular values $s_1(A) \geq s_2(A) \geq \cdots$ are defined by the fact that $s_j(A)^2$ is the decreasing sequence formed first by all discrete eigenvalues of A^*A above the essential spectrum and then (when $\mathcal{H}_1$ is infinite dimensional only) by an infinite repetition of $\sup \sigma_{\mathrm{ess}}(A^*A)$. It is well known and easy to see that the non vanishing singular values of A and of A^* are the same.

We have the Ky Fan inequalities

$$\begin{cases} s_{n+k-1}(A+B) \leq s_n(A) + s_k(B), \\ s_{n+k-1}(BA) \leq s_n(A)s_k(B), \end{cases} \tag{14.16}$$

in the cases when $B : \mathcal{H}_1 \to \mathcal{H}_2$ and $\mathcal{H}_2 \to \mathcal{H}_3$ respectively.

Applying this to (14.15), we get

$$s_j\big(\mathcal{P}_{\mathrm{out}}(z)^{-1}\big) \geq s_j\big(P_{\mathrm{out}}(z)^{-1}\big). \tag{14.17}$$

If $\Pi_1 : H^0 \times H^{\frac{1}{2}} \to H^0$, $\Pi_2 : H^0 \times H^{\frac{1}{2}} \to H^{\frac{1}{2}}$ are the natural projections (of norm 1), we can rewrite (14.15) as

$$\begin{aligned} \mathcal{P}_{\mathrm{out}}(z)^{-1} &= P_{\mathrm{out}}(z)^{-1}\Pi_1 + (1 - P_{\mathrm{out}}(z)^{-1}(P-z))h^{-\frac{1}{2}}\widehat{K}\Pi_2 \\ &= P_{\mathrm{out}}(z)^{-1}(\Pi_1 - (P-z)h^{-\frac{1}{2}}\widehat{K}\Pi_2) + h^{-\frac{1}{2}}\widehat{K}\Pi_2, \end{aligned}$$

which leads to

$$s_j\big(\mathcal{P}_{\rm out}(z)^{-1}\big)\le \mathcal{O}(1)\big(1+s_j\big(P_{\rm out}(z)^{-1}\big)\big). \tag{14.18}$$

We now restrict z to (13.12) and consider (7.19) which can be written

$$\mathcal{P}_{\rm out}(z)^{-1}=\mathcal{P}_{\rm in}(z)^{-1}\begin{pmatrix}1 & 0\\ 0 & (\mathcal{N}_{\rm in}-\mathcal{N}_{\rm ext})^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ -h^{\frac12}BG_{\rm in} & 1\end{pmatrix} \tag{14.19}$$

and also

$$\begin{pmatrix}1 & 0\\ 0 & (\mathcal{N}_{\rm in}-\mathcal{N}_{\rm ext})^{-1}\end{pmatrix}=\mathcal{P}_{\rm in}(z)\mathcal{P}_{\rm out}(z)^{-1}\begin{pmatrix}1 & 0\\ h^{\frac12}BG_{\rm in} & 1\end{pmatrix}. \tag{14.20}$$

Here the operator norms of $\mathcal{P}_{\rm in}^{-1}$ and $h^{\frac12}BG_{\rm in}$ are $\mathcal{O}(h^{-\frac23})$. From (14.19) we get

$$s_j\big(\mathcal{P}_{\rm out}(z)^{-1}\big)\le \mathcal{O}(h^{-\frac43})\big(1+s_j\big((\mathcal{N}_{\rm in}-\mathcal{N}_{\rm ext})^{-1}\big)\big), \tag{14.21}$$

while (14.20) leads to

$$s_j((\mathcal{N}_{\rm in}-\mathcal{N}_{\rm ext})^{-1})\le \mathcal{O}(h^{-\frac23})s_j\big(\mathcal{P}_{\rm out}(z)^{-1}\big). \tag{14.22}$$

Finally, from (13.36), (13.37) and the uniform boundedness of $\widehat{A}(z)$ and its inverse, we get

$$s_j((\mathcal{N}_{\rm in}-\mathcal{N}_{\rm ext})^{-1})\asymp s_j(\widehat{B}(z)^{-1})=s_j\big(\big(1+\widehat{C}(z)\big)^{-1}\big). \tag{14.23}$$

When $A:\mathcal{H}_1\to\mathcal{H}_2$ is a Fredholm operator of index 0, we let $t_1^2\le t_2^2\le\cdots$ with $t_j\ge 0$ describe the lower part of the spectrum of A^*A in analogy with s_j^2. Again $t_j(A)=t_j(A^*)$ and when A is bijective we have $t_j(A)=1/s_j(A^{-1})$.

Let N be the number of singular values $0\le t_1\le\cdots\le t_N$ of $1+\widehat{C}(z)$ that are $\le\frac12$. If $e_1,\dots,e_N$ is a corresponding orthonormal family of eigenfunctions of $(1+\widehat{C}(z))^*(1+\widehat{C}(z))$, then $\|(1+\widehat{C}(z))u\|\le\frac12\|u\|$ and consequently $\|\widehat{C}(z)u\|\ge\frac12\|u\|$, for all $u\in\mathbb{C}e_1\oplus...\oplus\mathbb{C}e_N$. By the mini-max characterization of singular values, we get $s_N(\widehat{C}(z))\ge\frac12$ and using that the trace class norm of $\widehat{C}(z)$ is $\mathcal{O}(h^{1-n})$, we conclude that $N=\mathcal{O}(h^{1-n})$. Combining this with (14.23), (14.21), (14.17), we see that there exists a constant $C>0$ such that

$$t_j\big(P_{\rm out}(z)\big)\ge h^{\frac43}/C,\quad \text{for } j\ge Ch^{1-n}. \tag{14.24}$$

CHAPTER 15

PERTURBATION MATRICES AND THEIR SINGULAR VALUES

We shall use a general estimate from [**25**]. Let $e_1, \dots, e_N \in C^0(\Omega) \cap L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open. Let $\mathcal{E}_\Omega = ((e_j \,|\, e_k)_{L^2(\Omega)})_{1\le j,k\le N}$ be the corresponding Gramian and let $0 \le \epsilon_1 \le \cdots \le \epsilon_N$ be its eigenvalues. Then (see [**25**], Prop. 5.5), there exists $a_1, \dots, a_N \in \Omega$ such that the singular values $s_1 \ge \cdots \ge s_N \ge 0$ of the $N \times N$ matrix $M = M_{\delta_a}$, given by

$$M_{j,k} = \sum_{\nu=1}^{N} e_j(a_\nu) e_k(a_\nu) = \int \delta_a(x) e_j(x) e_k(x),$$

satisfy the estimates,

$$s_1 \ge \frac{(E_1 \cdots E_N)^{\frac{1}{N}}}{\mathrm{vol}\,(\Omega)} \quad \text{and} \quad s_k \ge s_1 \Big(\prod_1^N \Big(\frac{E_j}{s_1 \mathrm{vol}\,(\Omega)} \Big) \Big)^{\frac{1}{N-k+1}}.$$

Here $E_j = \epsilon_1 + \cdots + \epsilon_{N+1-j}$, and we write $\delta_a = \sum \delta(\cdot - a_\nu)$.

Let $\widehat{e}_1, \dots, \widehat{e}_N$ be an orthonormal basis in $E(\mu)$, $\mu \ll h^{\frac{2}{3}}$, and choose $\Omega = \mathcal{O}_h \setminus \mathcal{O}_{2h}$, $e_j = \widehat{e}_{j\,|\Omega}$. Define $\mathcal{E}_\Omega$ as above and let $a_1, \dots, a_N \in \Omega$ be a corresponding set of points. The eigenvalues ϵ_j and the singular values $s_j = s_j(M_{\delta_a})$ remain unchanged if we replace $\widehat{e}_1, \dots, \widehat{e}_j$ by another orthonormal basis in $E(\mu)$.

Applying (14.9) to $u = \sum u_j \widehat{e}_j$, when $\vec{u} := (u_1, \dots, u_N)^{\mathrm{t}}$ is normalized in ℓ^2, we see that $\mathcal{E}_\Omega(\vec{u} \,|\, \vec{u}) \ge h^{\frac{4}{3}}/\mathcal{O}(1)$, so $E_j \ge (N-j+1)h^{\frac{4}{3}}/\mathcal{O}(1)$. Thus, for a suitable choice of $a_1, \dots, a_N \in \Omega$, we get after a simple calculation:

$$s_1 \ge \frac{(N!)^{\frac{1}{N}}}{h\mathcal{O}(1)} h^{\frac{4}{3}}, \tag{15.1}$$

$$s_k \ge s_1^{-\frac{k-1}{N-k+1}} h^{\frac{1}{3}\frac{N}{N-k+1}} (N!)^{\frac{1}{N-k+1}} C^{-\frac{N}{N-k+1}}. \tag{15.2}$$

We will also need an upper bound on $s_1 = s_1(M_{\delta_a})$. Let $s > \frac{1}{2}n$ and adopt the assumption (14.10). If $\vec{u} = (u_1, \dots, u_N)^{\mathrm{t}}$, $\vec{v} = (v_1, \dots, v_N)^{\mathrm{t}}$ are normalized, (14.14) with $k = 0$ implies that $\|u\|_{H_h^s(\mathcal{O}_{h/C})}$, $\|v\|_{H_h^s(\mathcal{O}_{h/C})}$ are $\mathcal{O}(1)$ when $u = \sum u_j \widehat{e}_j$, $v = \sum v_j \widehat{e}_j$ and also from Proposition 6.1 that $uv = \mathcal{O}(h^{-\frac{1}{2}n})$ in $H_h^s(\mathcal{O})$. Furthermore, we know from [**25**] that $\|\delta_a\|_{H_h^{-s}(\overline{\mathcal{O}}_{h/C})} = \mathcal{O}(Nh^{-\frac{1}{2}n})$. Hence,

$$\langle M_{\delta_a} u, v\rangle = \int \delta_a uv \mathrm{d}x = \mathcal{O}(1)\|\delta_a\|_{H_h^{-s}(\overline{\mathcal{O}}_{h/C})} \|uv\|_{H_h^s(\mathcal{O}_{h/C})} \leq \mathcal{O}(1)Nh^{-n},$$

and varying u, v we conclude that

$$s_1(M_{\delta_a}) = \|M_{\delta_a}\| \leq \mathcal{O}(1)Nh^{-n}. \tag{15.3}$$

Using this in (15.2) gives

$$s_k(M_{\delta_a}) \geq C^{-\frac{N+k-1}{N-k+1}} \mathrm{e}^{-\frac{N}{N-k+1}} Nh^{\frac{\frac{1}{3}N+n(k-1)}{N-k+1}}. \tag{15.4}$$

If we restrict k to the range $1 \leq k \leq \theta N$ for some $0 < \theta < 1$, we get

$$s_k(M_{\delta_a}) \geq C^{-\frac{1+\theta}{1-\theta}} \mathrm{e}^{-\frac{1}{1-\theta}} Nh^{\frac{\frac{1}{3}+n\theta}{1-\theta}}. \tag{15.5}$$

Recall the form of the perturbed operator in (2.5), (2.6), (2.7), where Θ in $C^\infty(\overline{\mathcal{O}})$ is also described. Clearly, $\Theta \asymp \widetilde{\Theta}(h) := h^{v_0}$ in $\mathcal{O}_h \setminus \mathcal{O}_{2h}$. The potential δ_a/Θ satisfies

$$\|\Theta^{-1}\delta_a\|_{H_h^{-s}(\overline{\mathcal{O}})} \leq \mathcal{O}(1)\frac{N}{\widetilde{\Theta}(h)h^{\frac{1}{2}n}}\cdot \tag{15.6}$$

As in [**25**], (6.15)–(6.18), we get the decomposition

$$\Theta^{-1}\delta_a = q + r, \quad q = \sum_{\mu_k \leq L} \alpha_k \epsilon_k, \tag{15.7}$$

where

$$\|q\|_{H_h^{-s}(\mathcal{O})} \leq \frac{CN}{\widetilde{\Theta}(h)h^{\frac{1}{2}n}}, \tag{15.8}$$

$$\|r\|_{H_h^{-s}(\mathcal{O})} \leq \mathcal{O}(1)L^{-(s-\frac{1}{2}n-\epsilon)}\frac{N}{\widetilde{\Theta}(h)h^{\frac{1}{2}n}}, \tag{15.9}$$

$$\|\alpha\|_{\ell^2} \leq C\frac{L^{\frac{1}{2}n+\epsilon}N}{\widetilde{\Theta}(h)h^{\frac{1}{2}n}}\cdot \tag{15.10}$$

We also denote by Θ the zero extension of Θ to all of $\mathbb{R}^n$. Under the assumption (2.6), we have for $|\alpha| = v_0 + 1$,

$$D^\alpha \Theta = f_\alpha + g_\alpha, \tag{15.11}$$

where $f_\alpha \in C^\infty(\overline{\mathcal{O}}) 1_{\mathcal{O}}$ and g_α is a smooth boundary layer ($\in C^\infty(\partial\mathcal{O}) \otimes \delta(\omega(x))$ where $\omega \in C^\infty(\mathbb{R}^n; \mathbb{R})$, $\omega^{-1}(0) = \partial\mathcal{O}$, $d\omega \neq 0$ on $\partial\mathcal{O}$). Using the strict convexity and stationary phase, we see that $\widehat{g}_\alpha(\xi) = \mathcal{O}(\langle\xi\rangle^{-\frac{1}{2}(n-1)})$ and by integration by parts, it follows that

$$\widehat{\Theta}(\xi) = \mathcal{O}(1)\langle\xi\rangle^{-v_0-1-\frac{1}{2}(n-1)}.$$

Here the hat indicates the ordinary (h-independent) Fourier transform. In the following, we shall assume that

$$\frac{1}{2}n < s < v_0 + \frac{1}{2}, \tag{15.12}$$

and then

$$\Theta \in H_1^s(\overline{\mathcal{O}}). \tag{15.13}$$

From [**27**], we recall that if $s > \frac{1}{2}n$, $u \in H^s(\mathbb{R}^n)$, $v \in H^\sigma(\mathbb{R}^n)$ for some $\sigma \in [-s, s]$, then $uv \in H^\sigma(\mathbb{R}^n)$ and we have

$$\|uv\|_{H_h^\sigma} \leq \mathcal{O}(1)\|u\|_{H_1^s} \cdot \|v\|_{H_h^\sigma}.$$

From (15.7)–(15.9), we now deduce that

$$\delta_a = \Theta q + \widetilde{r}, \quad \widetilde{r} = \Theta r, \tag{15.14}$$

where

$$\|\widetilde{r}\|_{H_h^{-s}(\overline{\mathcal{O}})} \leq \mathcal{O}(1) L^{-(s-\frac{1}{2}n-\epsilon)} \frac{N}{\widetilde{\Theta}(h) h^{\frac{1}{2}n}}, \tag{15.15}$$

$$\|\Theta q\|_{H_h^{-s}(\overline{\mathcal{O}})} \leq \frac{CN}{\widetilde{\Theta}(h) h^{\frac{1}{2}n}}. \tag{15.16}$$

We also need to control the $H_h^s(\mathcal{O})$-norm of Θq. Recall from [**25**], [**27**] that

$$\|q\|^2_{H_h^s(\mathcal{O})} \leq \mathcal{O}(1) \sum_{\mu_k \leq L} |\alpha_k|^2 \langle\mu_k\rangle^{2s} \leq \mathcal{O}(1) L^{2s} \|\alpha\|^2_{\ell^2},$$

so

$$\|\Theta q\|_{H_h^s(\mathcal{O})} \leq \mathcal{O}(1)\|q\|_{H_h^s(\mathcal{O})} \leq \mathcal{O}(1) L^{\frac{1}{2}n+s+\epsilon} \frac{N}{\widetilde{\Theta}(h) h^{\frac{1}{2}n}}, \tag{15.17}$$

and in particular,

$$\|\Theta q\|_{L^\infty(\mathcal{O})} \leq \mathcal{O}(h^{-\frac{1}{2}n})\|\Theta q\|_{H_h^s(\mathcal{O})} \leq \mathcal{O}(1) L^{\frac{1}{2}n+s+\epsilon} \frac{N}{\widetilde{\Theta}(h) h^n}. \tag{15.18}$$

From (15.15) we deduce (as above for M_{δ_a}) that

$$\|M_{\widetilde{r}}\| \leq \mathcal{O}(1)\|\widetilde{r}\|_{H_h^{-s}(\overline{\mathcal{O}})} h^{-\frac{1}{2}n} \leq \mathcal{O}(1) L^{-(s-\frac{1}{2}n-\epsilon)} \frac{N}{\widetilde{\Theta}(h)h^n}, \tag{15.19}$$

and returning to the decomposition (15.14) and the lower bound (15.5), we get for $1 \leq k \leq \theta N$, $0 < \theta < 1$:

$$s_k(M_{\Theta q}) \geq C^{-\frac{1+\theta}{1-\theta}} \mathrm{e}^{-\frac{1}{1-\theta}} N h^{\frac{\frac{1}{3}+n\theta}{1-\theta}} - \mathcal{O}(1) \frac{N}{L^{s-\frac{1}{2}n-\epsilon}\widetilde{\Theta}(h)h^n}. \tag{15.20}$$

The lower bounds on L will imply that the first term to the right dominates over the second.

Remark 15.1. — For a general perturbation $W = \delta\Theta q_\omega$ as in Theorem 2.2, the discussion above shows that

$$\|W\|_{H_h^{\widetilde{s}}(\mathbb{R}^n)} \leq \mathcal{O}(\delta) L^{\widetilde{s}} \|\alpha\|_{\ell^2} \leq \mathcal{O}(\delta) L^{\widetilde{s}} R, \tag{15.21}$$

provided that $\frac{1}{2}n < \widetilde{s} < v_0 + \frac{1}{2}$.

CHAPTER 16

END OF THE CONSTRUCTION

To start with we choose z in the full rectangle (13.1) and later on we will restrict the attention to $ch^{\frac{2}{3}} < |\Im z| < c_0 h^{\frac{2}{3}}$. We recall that $\mathcal{P}_{\mathrm{out}}(z)$ is an elliptic boundary value problem in the semi-classical sense in the region $|\xi'| \gg 1$. It follows that

$$\|u\|_{H^2} \leq \mathcal{O}(1)(\|(P-z)u\| + \|u\|) \tag{16.1}$$

for $u \in \mathcal{D}(P_{\mathrm{out}}(z))$. From this estimate we see that the small singular values $t_1(P_{\mathrm{out}}(z)) \leq t_2(P_{\mathrm{out}}(z)) \leq \cdots$ are of the same order of magnitude as the small singular values $\widetilde{t}_j$ in the L^2-sense defined as the square roots of the small eigenvalues of $P_{\mathrm{out}}(z)^* P_{\mathrm{out}}(z)$ where $P_{\mathrm{out}}(z)^*$ is the adjoint of $P_{\mathrm{out}}(z)$ as a closed densely defined operator: $L^2(\mathcal{O}) \to L^2(\mathcal{O})$. This follows from (16.1) and the mini-max characterizations of t_j and of $\widetilde{t}_j$. In this section it will be convenient to work with the $\widetilde{t}_j$ and we shall drop the tildes in order to simplify the notation.

Recall that $\widetilde{\Theta}(h) = h^{v_0}$. Let $\tau_0 \in]0, h^{\frac{4}{3}}/\mathcal{O}(1)]$ and let N be determined by

$$0 \leq t_1(P_{\mathrm{out}}) \leq \cdots \leq t_N(P_{\mathrm{out}}) < \tau_0 \leq t_{N+1}(P_{\mathrm{out}}), \tag{16.2}$$

so that $N \leq \mathcal{O}(h^{1-n})$ in view of (14.24). The basic iteration step (cf. Prop. 7.2 in [**25**]) is

Proposition 16.1. — *Let $0 < \theta < \frac{1}{2}$ be the parameter in* (2.8)*, let $\widetilde{\theta} \in]0, \theta[$ and $\kappa > 0$. If N is sufficiently large, depending on θ, $\widetilde{\theta}$ only, there exists an admissible potential q as in* (2.7) *with $L = L_{\min}$ and $R = R_{\min}$* (*as introduced in and after* (2.8))*, such that if*

$$P_\delta = P - \delta \Theta q, \quad \delta = C^{-1} h^{\alpha} \tau_0, \tag{16.3}$$

$C \gg 1$, $\alpha \geq \alpha(n, v_0, s, \epsilon, \theta, \widetilde{\theta}, \kappa)$ large enough, then

$$t_\nu(P_{\delta,\mathrm{out}}) \geq t_\nu(P_{\mathrm{out}}) - \mathcal{O}(1)\delta N h^{-(\frac{1}{2}n+s+\epsilon)M_{\min}-v_0-n}, \quad \nu \geq N+1, \tag{16.4}$$

$$t_\nu(P_{\delta,\mathrm{out}}) \geq \tau_0 h^{N_2}, \quad \big[(1-\widetilde{\theta})N\big]+1 \leq \nu \leq N. \tag{16.5}$$

Here we put $N_2 = \alpha + (\frac{1}{3}+2n\theta)/(1-2\theta)+\kappa$ and let $[a] = \max(\mathbb{Z}\cap]-\infty, a])$ denote the integer part of a.

When $N = \mathcal{O}(1)$ we have the same result, provided that we replace (16.5) *by the estimate $t_N(P_{\delta,\mathrm{out}}) \geq \tau_0 h^{N_2}$.*

Proof. — The estimate (16.4) follows from the mini-max characterization of singular values, which gives

$$t_\nu(P_{\delta,\mathrm{out}}) \geq t_\nu(P_{\mathrm{out}}) - \delta\|\Theta q\|_{L^\infty}, \tag{16.6}$$

to which we can apply (15.18).

Let $e_1, \dots, e_N \in L^2(\mathcal{O})$ be an orthonormal family of eigenfunctions of $P_{\mathrm{out}}^* P_{\mathrm{out}}$, corresponding to the eigenvalues $t_1^2, \dots, t_N^2$. Using the symmetry of P_{out}, established in Proposition 7.4 we see as in [**25**] that a corresponding family of eigenfunctions of $P_{\mathrm{out}}P_{\mathrm{out}}^*$ is given by

$$f_j = \Gamma e_j,$$

where Γ denotes the antilinear operator of complex conjugation. The f_j form an orthonormal family corresponding to

$$\sigma(P_{\mathrm{out}}P_{\mathrm{out}}^*) \cap [0, \tau_0^2[= \{t_1^2, \dots, t_N^2\}.$$

Let $E_N = \bigoplus_1^N \mathbb{C}e_j$, $F_N = \bigoplus_1^N \mathbb{C}f_j$. Then $P_{\mathrm{out}} : E_N \to F_N$ and $P_{\mathrm{out}}^* : F_N \to E_N$ have the same singular values $t_1, \dots, t_N$. Define $R_+ : L^2(\mathcal{O}) \to \mathbb{C}^N$, $R_- : \mathbb{C}^N \to L^2(\mathcal{O})$, by

$$R_+u(j) = (u\,|\,e_j), \quad R_-u_- = \sum_1^N u_-(j)f_j.$$

Then

$$\mathcal{P} = \begin{pmatrix} P_{\mathrm{out}} & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{D}(P_{\mathrm{out}}) \times \mathbb{C}^N \to L^2 \times \mathbb{C}^N \tag{16.7}$$

has the bounded inverse

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}, \tag{16.8}$$

where

$$\|E\| \leq \frac{1}{t_{N+1}} \leq \frac{1}{\tau_0}, \quad E_+v_+ = \sum_1^N v_+(j)e_j, \quad E_-v(j) = (v|f_j), \tag{16.9}$$

and E_{-+} has the singular values $t_j(E_{-+}) = t_j(P_{\rm out})$ or equivalently, $s_j(E_{-+}) = t_{N+1-j}(P_{\rm out})$.

When N is large, we consider two cases:

▷ Case 1. — $s_j(E_{-+}) \geq \tau_0 h^{N_2}$ for $1 \leq j \leq N - [(1-\widetilde{\theta})N]$. We get the proposition with $q = 0$, $P_\delta = P$.

▷ Case 2. — $s_j(E_{-+}) < \tau_0 h^{N_2}$ for some $j \leq N - [(1-\widetilde{\theta})N]$. Put $P_\delta = P + \delta\Theta q$ with q as in Chapter 15. From (16.3) we deduce that

$$\delta \frac{CN}{\widetilde{\Theta}(h)h^n} L^{\frac{1}{2}n+s+\epsilon} \leq \frac{\tau_0}{2}, \tag{16.10}$$

and then by (15.18) that $\delta\|\Theta q\|_{L^\infty} \leq \tau_0/2$. We can therefore replace $P_{\rm out}$ by $P_{\delta,\rm out}$ in (16.7) and still get a bijective operator

$$\mathcal{P}_\delta = \begin{pmatrix} P_{\delta,\rm out} & R_- \\ R_+ & 0 \end{pmatrix}$$

with the inverse

$$\mathcal{E}_\delta = \begin{pmatrix} E^\delta & E_+^\delta \\ E_-^\delta & E_{-+}^\delta \end{pmatrix}.$$

As in [**25**], we have

$$\begin{cases} E_{-+}^\delta = E_{-+} + \delta E_-\Theta q E_+ + \delta^2 E_-\Theta q E \Theta q E_+ + \cdots, \\ E^\delta = E + \sum_1^\infty \delta^k E(\Theta q E)^k, \\ E_+^\delta = E_+ + \sum_1^\infty \delta^k (E\Theta q)^k E_+, \\ E_-^\delta = E_- + \sum_1^\infty \delta^k E_-(\Theta q E)^k. \end{cases} \tag{16.11}$$

Here $\|E_\pm\| \leq 1$, $\|E\| \leq 1/\tau_0$ and in view of (16.10), we have $\delta\|\Theta q\|_{L^\infty} \leq \frac{1}{2}\tau_0$, leading to:

(16.12)

$$\begin{cases} E^\delta = E + \mathcal{O}\Big(\frac{1}{\tau_0}\frac{\delta\|\Theta q\|_{L^\infty}}{\tau_0}\Big), \quad E_+^\delta = E_+ + \mathcal{O}\Big(\frac{\delta\|\Theta q\|_{L^\infty}}{\tau_0}\Big), \\ E_-^\delta = E_- + \mathcal{O}\Big(\frac{\delta\|\Theta q\|_{L^\infty}}{\tau_0}\Big), \quad E_{-+}^\delta = E_{-+} + \delta E_-\Theta q E_+ + \mathcal{O}\Big(\frac{(\delta\|\Theta q\|_{L^\infty})^2}{\tau_0}\Big). \end{cases}$$

The leading perturbation in E_{-+}^{δ} is $\delta M = \delta E_{-}\Theta q E_{+}$, where $M = M_{\Theta q}$: $\mathbb{C}^N \to \mathbb{C}^N$ has the matrix

$$M_{j,k} = (\Theta q e_k \,|\, f_j) = \int \Theta q e_k e_j \, \mathrm{d}x. \tag{16.13}$$

From the Ky Fan inequalities, we get

$$\delta s_{k+\ell-1}(M_{\Theta q}) \leq s_k(E_{-+}^{\delta}) + s_\ell(E_{-+}) + \mathcal{O}\Big(\frac{(\delta \|\Theta q\|_{L^\infty})^2}{\tau_0}\Big),$$

which we write

$$s_k(E_{-+}^{\delta}) \geq \delta s_{k+\ell-1}(M_{\Theta q}) - s_\ell(E_{-+}) - \mathcal{O}\Big(\frac{(\delta \|\Theta q\|_{L^\infty})^2}{\tau_0}\Big). \tag{16.14}$$

Let $\ell = N - [(1-\widetilde{\theta})N]$ so that $s_\ell(E_{-+}) < \tau_0 h^{N_2}$ and let $k \leq N - [(1-\widetilde{\theta})N]$ so that

$$k + \ell - 1 \leq 2\big(N - [(1-\widetilde{\theta})N]\big) - 1 \leq 2\theta N,$$

for N large enough. Here, $2\theta < 1$, so we can apply (15.20) with θ there replaced by 2θ and get a q as in the proposition such that

$$s_{k+\ell-1}(M_{\Theta q}) \geq \frac{N}{C(\theta)} h^{\frac{\frac{1}{3}+2n\theta}{1-2\theta}} - \mathcal{O}(1) \frac{N}{L^{s-\frac{1}{2}n-\epsilon}\widetilde{\Theta}(h) h^n}. \tag{16.15}$$

Then (16.14) gives

$$\begin{aligned} s_k(E_{-+}^{\delta}) \geq \delta N \Bigg(\frac{h^{\frac{\frac{1}{3}+2n\theta}{1-2\theta}}}{C(\theta)} - \frac{\mathcal{O}(1)}{L^{s-\frac{1}{2}n-\epsilon}\widetilde{\Theta}(h) h^n} \Bigg) \\ - \tau_0 h^{N_2} - \mathcal{O}\Big(\frac{(\delta \|\Theta q\|_{L^\infty})^2}{\tau_0}\Big). \end{aligned} \tag{16.16}$$

Here we notice that with our choice of $L = L_{\min}$ large enough, we have

$$\frac{\mathcal{O}(1)}{L^{s-\frac{1}{2}n-\epsilon}\widetilde{\Theta}(h) h^n} \leq \frac{h^{\frac{\frac{1}{3}+2n\theta}{1-2\theta}}}{2C(\theta)}.$$

Thus for $k \leq N - [(1-\widetilde{\theta})N]$:

$$s_k(E_{-+}^{\delta}) \geq \frac{\delta N}{2C(\theta)} h^{\frac{\frac{1}{3}+2n\theta}{1-2\theta}} - \tau_0 h^{N_2} - \mathcal{O}\Big(\frac{(\delta \|\Theta q\|_{L^\infty})^2}{\tau_0}\Big),$$

and using (15.18):

$$\begin{aligned}(16.17)\ s_k(E^\delta_{-+}) &\geq \delta N\Big(\frac{1}{2C(\theta)}h^{\frac{\frac{1}{3}+2n\theta}{1-2\theta}} - \mathcal{O}(1)\frac{\delta}{N\tau_0}\|\Theta q\|^2_{L^\infty}\Big) - \tau_0 h^{N_2}\\ &\geq \delta N\Big(\frac{1}{2C(\theta)}h^{\frac{\frac{1}{3}+2n\theta}{1-2\theta}} - \frac{\mathcal{O}(1)\delta N}{\tau_0}h^{-2(\frac{1}{2}n+s+\epsilon)M-2v_0-2n}\Big) - \tau_0 h^{N_2}\\ &\geq \delta N\Big(\frac{1}{2C(\theta)}h^{\frac{\frac{1}{3}+2n\theta}{1-2\theta}} - \frac{\mathcal{O}(1)\delta}{\tau_0}h^{1-3n-2v_0-2(\frac{1}{2}n+s+\epsilon)M}\Big) - \tau_0 h^{N_2}\\ &\geq \frac{\delta N}{4C(\theta)}h^{\frac{\frac{1}{3}+2n\theta}{1-2\theta}} - \tau_0 h^{N_2},\end{aligned}$$

where the last estimate follows from the choice of δ in (16.3) and we recall that α is large enough.

Here by the choice of N_2 the last term is subdominant when $h > 0$ is small enough and we get

$$(16.18)\qquad s_k(E^\delta_{-+}) \geq \tau_0 h^{N_2},\quad \text{for } 1 \leq k \leq N - \big[(1-\widetilde{\theta})N\big].$$

After an arbitrarily small abstract perturbation of $P_{\delta,\mathrm{out}}$, we may assume that this operator is bijective, and we can then write the standard identity

$$P^{-1}_{\delta,\mathrm{out}} = E^\delta - E^\delta_+(E^\delta_{-+})^{-1}E^\delta_-$$

and apply the Ky Fan inequalities to get for $1 + [(1-\widetilde{\theta})N] \leq \nu \leq N$:

$$s_\nu(P^{-1}_{\delta,\mathrm{out}}) \leq s_1(E^\delta) + \|E^\delta_+\| \cdot \|E^\delta_-\| s_\nu\big((E^\delta_{-+})^{-1}\big) \leq \mathcal{O}(1)\frac{1}{h^{N_2}\tau_0},$$

since $s_\nu((E^\delta_{-+})^{-1}) = 1/s_{N+1-\nu}(E^\delta_{-+})$ and $1 \leq N+1-\nu \leq N - [(1-\widetilde{\theta})N]$, or in other terms,

$$t_\nu(P_{\delta,\mathrm{out}}) \geq \frac{\tau_0 h^{N_2}}{\mathcal{O}(1)}.$$

This is (16.5) apart from the factor $1/\mathcal{O}(1)$, which can be eliminated by increasing N_2 slightly.

When $N = \mathcal{O}(1)$ we consider the cases $s_1(E_{-+}) \geq \tau_0 h^{N_2}$ and $s_1(E_{-+}) < \tau_0 h^{N_2}$. In the first case we take the perturbation 0 as before. In the second case, we repeat the proof above with $k = \ell = 1$ and reach first (16.18) with $k = 1$ and finally (16.5) with $\nu = N$. □

REMARK 16.2. — 1) In the proof we have seen that $\delta\|\Theta q\|_{L^\infty} \leq \frac{1}{2}\tau_0$ and (16.6) shows that

$$t_\nu(P_{\delta,\mathrm{out}}) \geq t_\nu(P_{\mathrm{out}}) - \frac{\tau_0}{2} \geq \frac{\tau_0}{2},\ \nu \geq N+1.$$

2) From (16.10), (15.17), we get

$$\|\delta\Theta q\|_{H_h^s} \leq \mathcal{O}(1)\tau_0 h^{\frac{1}{2}n}.$$

3) Let $\widetilde{s} > \frac{1}{2}n + 2N$, where N is the smallest integer in $]\frac{1}{2}(n-1), +\infty[$. If we choose α in (2.9) sufficiently large, then

$$\|\delta\Theta q\|_{H_h^{\widetilde{s}}} \leq \mathcal{O}(h^{\frac{1}{2}n}).$$

We see that the perturbed operator P_δ satisfies the general assumptions of our discussion, including (11.54), (13.31), (14.10) for $W = \delta\Theta q$.

The last remark shows that we can apply Proposition 16.1 to $P_{\delta,\mathrm{out}}$ with τ_0 replaced by $\tau_0 h^{N_2}$ and N replaced by an $N_{\mathrm{new}} \leq [(1-\widetilde{\theta})N]$. The procedure can be iterated at most $\mathcal{O}(1)\ln\frac{1}{h}$ times until we get a perturbation $P_{\mathrm{final},\delta,\mathrm{out}}$ with $t_1(P_{\mathrm{final},\delta,\mathrm{out}}) \geq \tau_0 h^{\mathcal{O}(1)\ln\frac{1}{h}}$. Thus in the end we get:

Proposition 16.3. — *Let $0 < \theta < \frac{1}{2}$ be the parameter in* (2.8) *and let τ_0 in $]0, h^{\frac{4}{3}}]$. Then there exists an admissible potential q as in (2.7) with $L = L_{\min}$ and $R = R_{\min}$ (as introduced in and after* (2.8)) *such that if*

$$P_\delta = P + \delta\Theta q, \quad \delta = C^{-1}h^\alpha\tau_0, \tag{16.19}$$

$C \gg 1$, $\alpha \geq \alpha(n, v_0, s, \epsilon, \theta)$ large enough, then

$$t_1(P_{\delta,\mathrm{out}}) \geq \tau_0 h^{\mathcal{O}(1)\ln\frac{1}{h}}. \tag{16.20}$$

From (14.22) we get for the special perturbation above

$$s_1\big((\mathcal{N}_{\mathrm{in}} - \mathcal{N}_{\mathrm{ext}})^{-1}\big) \leq \frac{\mathcal{O}(1)}{h^{\frac{2}{3}}t_1(\mathcal{P}_{\mathrm{out}})} \leq \frac{\mathcal{O}(1)}{\tau_0 h^{\mathcal{O}(1)\ln\frac{1}{h}}}, \tag{16.21}$$

and (14.23) then gives

$$s_1\big((1+\widehat{C}(z))^{-1}\big) \leq \frac{\mathcal{O}(1)}{\tau_0 h^{\mathcal{O}(1)\ln\frac{1}{h}}}. \tag{16.22}$$

Recall from Proposition 13.3 and (13.36)–(13.37) that

$$\widehat{C}(z) = \mathcal{O}(1) : H^{\frac{3}{2}} \longrightarrow H^{\frac{3}{2}}, \quad |\Im z| \geq h^{\frac{2}{3}}c, \tag{16.23}$$

in addition to the fact that the trace class norm of the same operator is $\mathcal{O}(h^{1-n})$. We now work with $H^{\frac{3}{2}}(\partial\mathcal{O})$ as the underlying Hilbert space and let $\widehat{C}^*$ denote the adjoint of $\widehat{C}$. Consider,

$$\big|\det(1+\widehat{C})\big|^2 = \det(1+\widehat{C}^*)(1+\widehat{C}) = \det(1+D), \tag{16.24}$$

where $D = \widehat{C} + \widehat{C}^* + \widehat{C}^*\widehat{C}$ is self-adjoint, $\mathcal{O}(1)$ in operator norm and $\mathcal{O}(h^{1-n})$ in trace class norm. Let $\lambda_1, \lambda_2, \dots$ denote the non-vanishing eigenvalues of D, so that

$$1 + \lambda_j \geq \frac{\tau_0^2}{\mathcal{O}(1)} h^{2\mathcal{O}(1) \ln \frac{1}{n}} \tag{16.25}$$

by (16.22) (which is a bound on the norm of $(1 + \widehat{C})^{-1}$). We also know that $\sum |\lambda_j| = \mathcal{O}(h^{1-n})$, so there are at most $\mathcal{O}(h^{1-n})$ values j for which $|\lambda_j| \geq \frac{1}{2}$. Thus we get from (16.24):

$$\begin{aligned} \left|\det(1 + \widehat{C})\right|^2 = \prod (1 + \lambda_j) &= \prod_{j;\, |\lambda_j| \geq \frac{1}{2}} (1 + \lambda_j) \prod_{j;\, |\lambda_j| < \frac{1}{2}} (1 + \lambda_j) \\ &\geq \Big(\frac{\tau_0^2}{\mathcal{O}(1)} h^{2\mathcal{O}(1) \ln \frac{1}{h}} \Big)^{\mathcal{O}(h^{1-n})} \prod_{j;\, |\lambda_j| \leq \frac{1}{2}} \mathrm{e}^{-\mathcal{O}(1)|\lambda_j|}. \end{aligned}$$

Since $\sum |\lambda_j| = \mathcal{O}(h^{1-n})$, we get

$$\ln \left|\det(1 + \widehat{C})\right| \geq -\mathcal{O}(h^{1-n}) \Big(\Big(\ln \frac{1}{h} \Big)^2 + \ln \frac{1}{\tau_0} \Big). \tag{16.26}$$

Now return to the function $f(z)$ that was (re)defined in (13.75). From (13.76), (16.26) and (13.78) we get for our special perturbation $V = V_0 + W$ (where W depends on z with $ch^{\frac{2}{3}} \leq |\Im z| \leq c_0 h^{\frac{2}{3}}$):

$$\phi_{\mathrm{in}}(z) - \mathcal{O}(h^{1-n}) \Big(\Big(\ln \frac{1}{h} \Big)^2 + \ln \frac{1}{\tau_0}) \leq \ln \left|f(z)\right| \leq \phi_{\mathrm{in}}(z) + \mathcal{O}(h^{1-n}). \tag{16.27}$$

Here the upper bound is valid for all perturbations V of V_0 in our class independently of z with $|\Im z| \asymp h^{\frac{2}{3}}/C$, while the lower bound is valid for our special z-dependent perturbation.

ϕ_{in} (cf. (13.69)) is defined in terms of the interior Dirichlet problem for the perturbed potential $V_0 + W$ where W also depends on z, and we would like to replace this function by one which is independent of the perturbation W. To emphasize the presence of the perturbation we write

$$\phi_{\mathrm{in}}^{\delta}(z) = \sum \chi(\lambda_j^{\delta}) \ln |z - \lambda_j^{\delta}|$$

for the function in (16.27), and

$$\phi_{\mathrm{in}}^{0}(z) = \sum \chi(\lambda_j^{0}) \ln |z - \lambda_j^{0}|$$

for the corresponding function, associated to the unperturbed operator P_0^{in}.

From the mini-max principle, we get

$$|\lambda_j^{\delta} - \lambda_j^{0}| \leq \|W\|_{\infty}.$$

For $|\Im z| \geq r$, $0 < r \leq 1$, we see that

$$\Big|\frac{\partial}{\partial \lambda}\big(\chi(\lambda)\ln|z-\lambda|\big)\Big| \leq \mathcal{O}(\frac{1}{r}),$$

so

$$\big|\chi(\lambda_j^\delta)\ln|z-\lambda_j^\delta| - \chi(\lambda_j^0)\ln|z-\lambda_j^0|\big| \leq \mathcal{O}(1)\frac{\|W\|_\infty}{r}.$$

The number of eigenvalues of P_{in}^δ and of P_{in}^0 in $\operatorname{supp}\chi$ is $\mathcal{O}(h^{-n})$ and it follows that

$$|\phi_{\mathrm{in}}^\delta(z) - \phi_{\mathrm{in}}^0(z)| \leq \mathcal{O}(1)\frac{\|W\|_\infty}{rh^n}.$$

Here we take $r \asymp h^{\frac{2}{3}}$ as in (16.27). From the second part of Remark 16.2 we know that $W = \delta\Theta q$ satisfies

$$\|W\|_\infty \leq \mathcal{O}(1)h^{-\frac{1}{2}n}\|W\|_{H_h^s} \leq \mathcal{O}(1)\tau_0$$

and thus

$$\big|\phi_{\mathrm{in}}^\delta(z) - \phi_{\mathrm{in}}^0(z)\big| \leq \mathcal{O}(1)\tau_0 h^{-\frac{2}{3}-n}.$$

In Proposition 16.3 we have assumed that $0 < \tau_0 \leq h^{\frac{4}{3}}$. We now strengthen that assumption to

$$\tau_0 \in]0, h^{\frac{5}{3}}]. \tag{16.28}$$

Then,

$$\big|\phi_{\mathrm{in}}^\delta(z) - \phi_{\mathrm{in}}^0(z)\big| \leq \mathcal{O}(1)h^{1-n} \tag{16.29}$$

and we obtain

Proposition 16.4. — *In* (16.27) *we can replace* $\phi_{\mathrm{in}} = \phi_{\mathrm{in}}^\delta$ *by the function* ϕ_{in}^0*, defined for the unperturbed operator* P_{in}^0 *as in* (13.69).

CHAPTER 17

END OF THE PROOF OF THEOREM 2.2 AND PROOF OF PROPOSITION 2.4

Let ϕ_{in}^0 be defined in (13.69) with respect to the unperturbated operator P_{in}^0. With $r = \frac{1}{4}h^{\frac{2}{3}}c$, let $h^0 = h_r^0$ be the harmonic majorant in Ω_r and define $\Phi_r^0 = \Phi^0$ as in (13.71). Recall that f is defined in (13.75) (for the perturbed operator P_δ). Since $\phi_{\mathrm{in}}^\delta - \phi_{\mathrm{in}}^0 = \mathcal{O}(h^{1-n})$ by (16.29), we have the same estimate for $h_r - h_r^0$ and hence for $\Phi_r - \Phi_r^0$. Then by (13.81) we conclude that

$$\ln\left|f(z)\right| \leq \Phi_r^0(z) + \mathcal{O}(h^{1-n}) \text{ in the rectangle (13.1).} \tag{17.1}$$

For each z as in (13.12) we have constructed a perturbation $W = \delta\Theta q$ as in and after (2.8) with $L = L_{\min}$, $R = R_{\min}$ such that (cf. Proposition 16.4)

$$\Phi_r^0 - \mathcal{O}(h^{1-n})\Big(\Big(\ln\frac{1}{h}\Big)^2 + \ln\frac{1}{\tau_0}\Big) \leq \ln\left|f(z)\right|. \tag{17.2}$$

Let

$$\epsilon_0(h) = Ch\Big(\Big(\ln\frac{1}{h}\Big)^2 + \ln\frac{1}{\tau_0}\Big) \tag{17.3}$$

so that

$$\ln\left|f(z)\right| \leq \Phi_r^0(z) + h^{-n}\epsilon_0(h) \tag{17.4}$$

for all z in the rectangle (13.1) and so that for every z as in (13.12), there is a perturbation as in (17.2) such that

$$\ln\left|f(z)\right| \geq \Phi_r^0 - h^{-n}\epsilon_0(h). \tag{17.5}$$

If we fix such a value of z and work in the α-variables, we are in the same situation as in Section 8 in [**25**] and we can apply Proposition 8.2 and Remark 8.3 of that paper to obtain

Proposition 17.1. — *Let $\epsilon > 0$ be small enough so that $\epsilon \exp(\mathcal{O}(\epsilon_0)h^{-n}) \leq 1$. For each z as in* (13.12)*, we have*

$$P\big(|f(z)| \leq \mathrm{e}^{\Phi_r^0}\epsilon\big) \leq \mathcal{O}(1)\frac{\epsilon_0(h)}{h^{n+N_6}} \exp\Big(\frac{h^n}{\mathcal{O}(1)\epsilon_0(h)} \ln \epsilon\Big). \tag{17.6}$$

Here $N_6 = \max(N_3, N_5)$, where $N_3 = n(M+1)$, $N_5 = N_4 + \widetilde{M}$ (cf. (2.11)*).*

If we write $\epsilon = \mathrm{e}^{-\widetilde{\epsilon}/h^n}$, then the condition on ϵ is fulfilled when

$$\widetilde{\epsilon} \geq \mathrm{Const.}\, \epsilon_0 \tag{17.7}$$

and (17.6) becomes

$$P\big(|f(z)| \leq \mathrm{e}^{\Phi_r^0(z) - \widetilde{\epsilon}/h^n}\big) \leq \mathcal{O}(1)\frac{\epsilon_0(h)}{h^{n+N_6}} \exp\Big(-\frac{\widetilde{\epsilon}}{\mathcal{O}(1)\epsilon_0(h)}\Big). \tag{17.8}$$

Let $\frac{1}{2} \leq a < b \leq 2$ and put $\Gamma = [a,b] + ih^{\frac{2}{3}}c[-1,1]$, $r = \frac{1}{4}h^{\frac{2}{3}}c$. We shall apply Theorem 1.2 in [**26**] to the function $u = f$, with h there replaced by h^n and with $\phi = h^n\Phi_r$. Let

$$\rho(t) = \max\big(4ch^{\frac{2}{3}} - \tfrac{1}{2}(t-a), \tfrac{1}{2}h^{\frac{2}{3}}c, 4ch^{\frac{2}{3}} - \tfrac{1}{2}(b-t)\big), \quad a \leq t \leq b,$$

and define the function $\widetilde{r} : \partial\Gamma \to]0, \infty[$ by

$$\widetilde{r}(z) = \rho(\Re t).$$

Then $\widetilde{r}$ has Lipschitz modulus $\leq \frac{1}{2}$ and this will be our function "r" in [**26**]. Choose points $z_1^0, \dots, z_N^0 \in \partial\Gamma$ as in the introduction of [**26**]. This can be done in a such a way that $|\Im z_j^0| = h^{\frac{2}{3}}c$ for all j. Moreover, we see that $N \asymp h^{-\frac{2}{3}}$ and further $\Delta\Phi_r = 0$ in $D(z_j^0, r(z_j^0))$ except for at most $\mathcal{O}(1)$ values of j . Let $\widetilde{z}_j \in D(z_j^0, r(z_j^0)/(2C_1))$ be as in Theorem 1.2 in [**26**], where we recall that these points depend on $\Phi_r, \Gamma, \widetilde{r}$ but not on the function f. Moreover we notice that C_1 can be chosen arbitrarily large. Then according to (17.8) we have

$$\big|f(\widetilde{z}_j)\big| \geq \mathrm{e}^{\Phi_r(\widetilde{z}_j) - \widetilde{\epsilon}/h^n}, \quad j = 1, 2, \dots, N \tag{17.9}$$

with probability

$$\geq 1 - \mathcal{O}(1)\frac{N\epsilon_0(h)}{h^{n+N_6}}\,\mathrm{e}^{-\frac{\widetilde{\epsilon}}{\mathcal{O}(1)\epsilon_0(h)}} = 1 - \mathcal{O}(1)\frac{\epsilon_0(h)}{h^{n+N_6+\frac{2}{3}}}\,\mathrm{e}^{-\frac{\widetilde{\epsilon}}{\mathcal{O}(1)\epsilon_0(h)}}. \tag{17.10}$$

Here we recall that (17.7) holds and that $|f| \leq \mathrm{e}^{\Phi_r + \widetilde{\epsilon}/h^n}$ in a neighborhood of Γ. Theorem 1.2 in [**26**] then shows that with $\sigma(P_\delta)$ denoting the set of resonances of P_δ,

$$\text{(17.11)}\Big|\#\big(\sigma(P_\delta)\cap\big([a,b]+ih^{\frac{2}{3}}c[-1,0]\big)\big)-\frac{1}{2\pi}\int_{[a,b]+ih^{\frac{2}{3}}c[-1,1]}\Delta\Phi_r^0L(\mathrm{d}z)\Big|$$
$$\leq C_2\Big(\sum_{w=a,b}\int_{[w-Ch^{\frac{2}{3}},w+Ch^{\frac{2}{3}}]+ih^{\frac{2}{3}}c[-1,1]}\Delta\Phi_r^0L(\mathrm{d}z)+h^{-n}\sum_1^N\widetilde{\epsilon}\Big),$$

with a probability as in (17.10). Here we assume for simplicity that $c\ll c_0$, otherwise we have to slightly modify the choice of ρ, r, z_j^0 above.

Now recall (13.74) where $g_r(t)=r^{-1}g_1(t/r)$, $0\leq g_1\in\mathcal{S}(\mathbb{R})$, $\int g_1\mathrm{d}t=1$. With N_0 denoting the eigenvalue counting function for P_{in}^0, we get with probability as in (17.10),

$$\text{(17.12)}\quad \#\Big|\big(\sigma(P_\delta)\cap\big([a,b]+ih^{\frac{2}{3}}c[-1,0]\big)\big)-\int_a^b g_r*(\chi\mathrm{d}N_0)(t)\mathrm{d}t\Big|$$
$$\leq C_2\Big(\sum_{w=a,b}\int_{w-Ch^{\frac{2}{3}}}^{w+Ch^{\frac{2}{3}}}g_r*(\chi\mathrm{d}N_0)(t)\mathrm{d}t+\mathcal{O}(h^{-\frac{2}{3}-n}\widetilde{\epsilon})\Big).$$

This is a slightly stronger version of the main result (2.16) as we shall see next. Consider

$$J:=\int_a^b g_r*(\chi\mathrm{d}N_0)(t)\mathrm{d}t=\int_a^b\int_{\mathbb{R}}g_r(t-s)\chi(s)\mathrm{d}N_0(s)\mathrm{d}t,$$

where we recall that $r=\frac{1}{4}h^{\frac{2}{3}}c$. We split the integral into $\mathrm{I}+\mathrm{II}$, where I is obtained by retricting the s integration to the interval $[a-\rho,b+\rho]$ and II is obtained from integration in s over $\mathbb{R}\setminus[a-\rho,b+\rho]$. Here we take $\rho=h^{-\delta+\frac{2}{3}}$, where $\delta>0$ can be arbitrarily small but independent of h.

Carrying out first the t integration, we see that

$$\mathrm{I}\leq\int_{[a-\rho,b+\rho]}\chi(s)\mathrm{d}N_0(s)=N_0(b+\rho)-N_0(a-\rho).$$

As for II, we have uniformly for $t\in[a,b]$ that

$$\int_{\mathbb{R}\setminus[a-\rho,b+\rho]}g_r(t-s)\chi(s)\mathrm{d}N(s)\leq\int_{|t-s|\geq\rho}\frac{1}{r}g_1\Big(\frac{t-s}{r}\Big)\chi(s)\mathrm{d}N(s)=\mathcal{O}(h^\infty),$$

since $\rho/r\geq\frac{1}{4}h^{-\delta}c$ so that $g_1((t-s)/r)/r=\mathcal{O}(h^\infty)$ and $\int\chi(s)\mathrm{d}N(s)=\mathcal{O}(h^{-n})$. Thus,

$$J\leq N_0(b+\rho)-N_0(a-\rho)+\mathcal{O}(h^\infty).$$

To get a corresponding lower bound, assume $b-a \geq 2\rho$ (in order to exclude a trivial case), and write

$$J \geq \int_a^b \int_{a+\rho}^{b-\rho} g_r(t-s)\chi(s)\mathrm{d}N_0(s)\mathrm{d}t.$$

For $a+\rho \leq s \leq b-\rho$, we have $1 \geq \int_a^b g_r(t-s)\mathrm{d}t \geq 1-\mathcal{O}(h^\infty)$, so

$$\begin{aligned} J &\geq \int_{a+\rho}^{b-\rho} \big(1-\mathcal{O}(h^\infty)\big)\mathrm{d}N_0(s) \\ &\geq \big(1-\mathcal{O}(h^\infty)\big)\big(N_0(b-\rho)-N_0(a+\rho)\big) \\ &\geq N_0(b-\rho)-N_0(a+\rho)-\mathcal{O}(h^\infty). \end{aligned}$$

In conclusion, for $r = \frac{1}{4}h^{\frac{2}{3}}c$ and$\rho = h^{-\delta+\frac{2}{3}}$, we get from (17.12),

$$\begin{aligned} &N_0(b-\rho)-N_0(a+\rho)-\mathcal{O}(h^\infty) \\ &\quad\leq \int_a^b g_r*(\chi\mathrm{d}N_0)(t)\mathrm{d}t \leq N_0(b+\rho)-N_0(a-\rho)+\mathcal{O}(h^\infty). \end{aligned} \tag{17.13}$$

Applying this to (17.12), we get with a probability as in (17.10)

$$\begin{aligned} &\Big|\#\big(\sigma(P_\delta)\cap\big([a,b]+ih^{\frac{2}{3}}c[-1,0]\big)\big)-\big(N_0(b)-N_0(a)\big)\Big| \\ &\quad\leq \mathcal{O}(1)\Big(\sum_{w=a,b}\big(N_0(w+\rho)-N_0(w-\rho)\big)+h^{-\frac{2}{3}-n}\widetilde{\epsilon}\Big). \end{aligned} \tag{17.14}$$

This concludes the proof of Theorem 2.2.

Proof of Proposition 2.4. — Let V_0 be as in Theorem 2.2 and let W_0 satisfy the assumptions of the proposition. Our unperturbed operator is now

$$P_0 = -h^2\Delta + V_0 + W_0 = P^{V_0+W_0}. \tag{17.15}$$

rather than the right hand side of (2.3) that we now denote by P_0^0. The proof will consist in checking the proof of Theorem 2.2 with this new operator P_0.

Nothing changes until Chapter 11. Here Proposition 11.5 can be used instead of Proposition 11.4 to see that the conclusion of Proposition 11.1 is valid for (the new) unperturbed operator P_0 as well as for the perturbed operator P^V in (12.2), where now $V = V_0 + W_0 + W$ and as before $W = \mathcal{O}(h)$ in L^∞.

The discussion in Chapter 12 remains valid.

In Chapter 13 the first change appears after (13.6), where we now take $V = V_0 + W_0 + W$ with $\|W\|_{L^\infty} = \mathcal{O}(1)$. Then we still have (13.7) provided that we modify the definition of $\widetilde{P}$ prior to (13.2) by taking $\widetilde{P} = P + Ci1_{\mathcal{O}}$ with C large enough. We obtain Proposition 13.1 as before.

In the subsequent discussion, P_0 is the same operator but with the new notation $P_0^0 = P^{V_0}$, while $P = P^V$ with $V = V_0 + W_0 + W$ with the initial assumption that $W = \mathcal{O}(h)$ in L^∞. After (13.15) we just have to invoke Proposition 11.5 instead of Proposition 11.4.

In the expression for $\widetilde{K}$ after (13.17) we have to replace W by $W_0 + W$ and as in the proof of Proposition 11.5, we have $(\widetilde{P} - z)^{-1} W_0 \widetilde{K}_0 = \mathcal{O}(h^2) : H^{\frac{3}{2}} \to H^2$. Thus instead of (13.18) we get

$$\widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}_0 + \mathcal{O}(1)\|W\|_{L^\infty} + \mathcal{O}(h^2) : H^{\frac{3}{2}} \longrightarrow H^{\frac{1}{2}}. \tag{17.16}$$

Lemma 13.2 remains valid since W_0 also satisfies (13.21). Since W_0 satisfies (13.31), the following discussion goes through without any changes until Proposition 13.3, where we just have to add a term $\mathcal{O}(h^2)$ to the estimate of $\widetilde{\mathcal{N}} - \widetilde{\mathcal{N}}_0$ after (13.32). The remainder of Chapter 13 goes through without any changes.

After that, there are no changes. P_{in}^0 in Proposition 16.4 is the Dirichlet realization of (the new) $P_0 = P^{V_0 + W_0}$. □

APPENDIX A

WKB ESTIMATES ON AN INTERVAL

We follow [**11**], [**36**]. See also [**2**]. Let $V \in C^2([a,b])$, $-\infty < a < b < +\infty$ and assume that $V(x) \neq 0$ for all $x \in [a,b]$. Choose a branch of $\ln V(x)$ and put $V(x)^\theta = \exp(\theta \ln V(x))$. Put

$$\begin{aligned} y_\pm(x) &= V(x)^{-\frac{1}{4}} \, \mathrm{e}^{\pm\phi(x)/h} = \mathrm{e}^{\psi_\pm(x)/h}, \\ \psi_\pm &= \pm\phi - \frac{1}{4} h \ln V(x), \quad \phi'(x) = V(x)^{\frac{1}{2}}. \end{aligned}$$

Then

$$\begin{gathered} \mathrm{e}^{-\psi_\pm/h} \circ \big(V(x) - (h\partial)^2\big) \circ \mathrm{e}^{\psi_\pm/h} = -(h\partial)^2 - 2\psi'_\pm \circ h\partial + h^2 r, \\ r = \frac{1}{4} \cdot \frac{V''}{V} - \frac{5}{16}\Big(\frac{V'}{V}\Big)^2, \end{gathered}$$

so

$$\big(V - (h\partial)^2\big) y_\pm = h^2 r y_\pm.$$

The equation $(V - (h\partial)^2) y = 0$ can be written

$$\Big(h\partial - \begin{pmatrix} 0 & 1 \\ V & 0 \end{pmatrix}\Big)\begin{pmatrix} y \\ h\partial y \end{pmatrix} = 0. \tag{A.1}$$

Put

$$e_\pm = \begin{pmatrix} 1 \\ h\partial y_\pm / y_\pm \end{pmatrix} = \begin{pmatrix} 1 \\ \partial\psi_\pm \end{pmatrix}.$$

From the identity

$$\Big(h\partial - \begin{pmatrix} 0 & 1 \\ V & 0 \end{pmatrix}\Big)\begin{pmatrix} y_\pm \\ h\partial y_\pm \end{pmatrix} + h^2 r y_\pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

we get

$$\Big(h\partial + \psi'_\pm - \begin{pmatrix} 0 & 1 \\ V & 0 \end{pmatrix}\Big) e_\pm + h^2 r \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0. \tag{A.2}$$

If $u_\pm$ is a scalar C^1-function, we get

$$\Big(h\partial - \begin{pmatrix} 0 & 1 \\ V & 0 \end{pmatrix}\Big)u_\pm e_\pm = h\partial(u_\pm)e_\pm - u_\pm\psi'_\pm e_\pm - u_\pm h^2 r\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{A.3}$$

Here,

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}V^{-\frac{1}{2}}(e_+ - e_-)$$

and with the substitution

$$\begin{pmatrix} y \\ h\partial y \end{pmatrix} = u_+e_+ + u_-e_- \iff \begin{cases} y = u_+ + u_-, \\ h\partial y = u_+\partial\psi_+ + u_-\partial\psi_-, \end{cases} \tag{A.4}$$

we find after some calculation that (A.1) is equivalent to

$$\Big(h\partial - \begin{pmatrix} \psi'_+ & 0 \\ 0 & \psi'_- \end{pmatrix} - h^2 r\frac{1}{2}V^{-\frac{1}{2}}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\Big)\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = 0. \tag{A.5}$$

Here,

$$\frac{r}{V^{\frac{1}{2}}} = \frac{1}{4}\frac{V''}{V^{\frac{3}{2}}} - \frac{5}{16}\frac{(V')^2}{V^{\frac{5}{2}}}. \tag{A.6}$$

Let $E(x,y)$ be the forward fundamental solution of the differential operator in (A.5), i.e. the one which vanishes for $x < y$. Then for $a \le y \le x \le b$:

$$\big\|E(x,y)\big\| \le \frac{1}{h}\exp\frac{1}{h}\int_y^x \big(\max(\Re\psi'_+, \Re\psi'_-)(t) + Ch^2|rV^{-\frac{1}{2}}|(t)\big)\,\mathrm{d}t. \tag{A.7}$$

Assume from now on that

$$\Re V(x)^{\frac{1}{2}} \ge 0, \quad x \in [a,b]. \tag{A.8}$$

Then (A.7) simplifies to

$$\big\|E(x,y)\big\| \le \frac{1}{h}\,\mathrm{e}^{\frac{1}{h}(\Re\psi_+(x) - \Re\psi_+(y))}\,\mathrm{e}^{Ch\int_y^x |rV^{-\frac{1}{2}}|(t)\,\mathrm{d}t}. \tag{A.9}$$

Let us consider the situation of a simple turning point:

$$\begin{cases} |V(x)| \asymp |x - z_0|, \ \ V', V'' = \mathcal{O}(1), \\ |x - z_0| \ge h^{\frac{2}{3}}/C \ \text{ for } x \in [a,b], \end{cases} \tag{A.10}$$

where $z_0 \in \mathbb{C}$. Then from (A.6) we have $\int_y^x |r/V^{\frac{1}{2}}|\mathrm{d}z = \mathcal{O}(1/h)$ and the last exponential in (A.9) is $\mathcal{O}(1)$. We get

$$\big\|E(x,y)\big\| \le \mathcal{O}\Big(\frac{1}{h}\Big)\,\mathrm{e}^{\frac{1}{h}(\Re\psi_+(x) - \Re\psi_+(y))}, \quad a \le y \le x \le b. \tag{A.11}$$

Apply the operator in (A.5) to

$$u^0 = \begin{pmatrix} u_+^0 \\ u_-^0 \end{pmatrix} = \begin{pmatrix} y_+ \\ 0 \end{pmatrix}.$$

We get

$$\Big(h\partial - \begin{pmatrix} \psi_+' & 0 \\ 0 & \psi_-' \end{pmatrix} - h^2 \frac{r}{2V^{\frac{1}{2}}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\Big) u^0 = -h^2 \frac{r}{2V^{\frac{1}{2}}} \begin{pmatrix} y_+ \\ -y_+ \end{pmatrix},$$

and we have the solution

$$\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = u^0 + \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$$

of (A.5), where

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} = \int_a^x E(x,y) h^2 \frac{r}{2V^{\frac{1}{2}}}(y) \begin{pmatrix} y_+ \\ -y_+ \end{pmatrix}(y) \mathrm{d}y.$$

Here

$$\frac{r}{V^{\frac{1}{2}}}(y) = \frac{\mathcal{O}(1)}{|y - z_0|^{\frac{5}{2}}}$$

and using (A.11), we get

$$\Big\| \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \Big\| \leq Ch \,\mathrm{e}^{\frac{\psi_+(x)}{h}} \int_a^x \frac{1}{|y - z_0|^{\frac{5}{2}}} \mathrm{d}y \leq \mathcal{O}(1)\,\mathrm{e}^{\psi_+(x)/h}. \tag{A.12}$$

Thus we have the exact solution of (A.5):

$$\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \mathrm{e}^{\psi_+/h} \mathcal{O}(1). \tag{A.13}$$

If we make the substitution (A.4), we see that y is an exact solution of

$$(V - (h\partial)^2) y = 0, \tag{A.14}$$

which satisfies

$$y = \mathcal{O}(1)\,\mathrm{e}^{\psi_+/h}, \tag{A.15}$$

$$h\partial y = \mathcal{O}(1)\,\mathrm{e}^{\psi_+/h}. \tag{A.16}$$

Using this with (A.14), we get similar approximations for the higher derivatives of y.

The inhomogeneous equation

$$\big(V - (h\partial)^2\big) y = z, \tag{A.17}$$

can be transformed into a system

$$\Big(h\partial - \begin{pmatrix} 0 & 1 \\ V & 0 \end{pmatrix}\Big)\begin{pmatrix} y \\ h\partial y \end{pmatrix} = \begin{pmatrix} 0 \\ -z \end{pmatrix}, \tag{A.18}$$

where the right hand side can be written $z_+e_+ + z_-e_-$, $z_+ = -z_- = -z/(2V^{\frac{1}{2}})$. The substitution (A.4) gives

$$\Big(h\partial - \begin{pmatrix} \psi_+' & 0 \\ 0 & \psi_-' \end{pmatrix} - \frac{1}{2}h^2 rV^{-\frac{1}{2}}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\Big)\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = -\frac{z}{2V^{\frac{1}{2}}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{A.19}$$

which has the solution

$$\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = -\int_a^x E(x,y)\frac{z(y)}{2V(y)^{\frac{1}{2}}}\,\mathrm{d}y\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{A.20}$$

Writing

$$E(x,y) = \begin{pmatrix} E_{++} & E_{+-} \\ E_{-+} & E_{--} \end{pmatrix},$$

we get (cf. (A.4))

$$\begin{cases} u_+(x) = \displaystyle\int_a^x \big(-E_{++}(x,y) + E_{+-}(x,y)\big)\frac{z(y)}{2V(y)^{\frac{1}{2}}}\,\mathrm{d}y, \\ u_-(x) = \displaystyle\int_a^x \big(-E_{-+}(x,y) + E_{--}(x,y)\big)\frac{z(y)}{2V(y)^{\frac{1}{2}}}\,\mathrm{d}y. \end{cases} \tag{A.21}$$

Now we add the assumption that $V \in C^\infty([a,b])$. Assume for simplicity that $\Re z_0 = 0$ and assume that $b \leq 0$. It is standard that we have exact solutions to

$$\big(V - (h\partial)^2\big)\big(a(x;h)\,\mathrm{e}^{\psi(x)/h}\big) = 0, \quad \psi = \psi_+ \tag{A.22}$$

for which a has a complete asymptotic expansion in $C^\infty([a,c])$ of the form

$$a \sim \sum_{j=0}^{\infty} a_j(x)h^j, \tag{A.23}$$

where c is any fixed number in $]a, b - 1/\mathcal{O}(1)[$.

By solving the usual sequence of transport equations, we have a unique continuation of the a_j to the full interval $[a,b]$ so that $\mathrm{e}^{\psi/h}\sum_0^\infty a_j h^j$ is a formal asymptotic solution of (A.22) and as we have seen in Section 8.2, we have

$$\partial^\alpha a_j(x) = \mathcal{O}\big(|x - z_0|^{-\frac{3j}{2}-\alpha}\big). \tag{A.24}$$

The power $|x - z_0|^{-\frac{1}{4}}$ in Section 8.2 corresponds to the factor $V(x)^{-\frac{1}{4}}$ which is no longer counted in a but in the exponential factor $\mathrm{e}^{\psi/h} = V^{-\frac{1}{4}}\,\mathrm{e}^{\phi/h}$.

On the other hand $a\,\mathrm{e}^{\psi/h}$ has a unique extension to the full interval $[a,b]$ as a solution of (A.21) that we can still write on the same form and we shall show that the asymptotic expansion (A.23) still holds in sup norm and with the natural remainder estimates. Write $a=\sum_0^N a_j h^j + r_N = a^N + r_N$, so that

$$(V-(h\partial)^2)(r_N\,\mathrm{e}^{\psi/h}) = \big((h\partial)^2 - V\big)(a^N\,\mathrm{e}^{\psi/h}).$$

We know that $r_N = \mathcal{O}(h^{N+1})$ with all its derivatives on $[a,c]$.

Let $\chi\in C^\infty([a,b];[0,1])$ vanish near a and be equal to one in a neighborhood of $[c,b]$. Write

$$\begin{aligned}\big(V-(h\partial)^2\big)&(\chi r_N\,\mathrm{e}^{\psi/h})\\ &= \big((h\partial)^2-V\big)(a^N\,\mathrm{e}^{\psi/h}) + \big((h\partial)^2-V\big)\big((1-\chi)r_N\,\mathrm{e}^{\psi/h}\big).\end{aligned}\tag{A.25}$$

Here $((h\partial)^2-V)((1-\chi)r_N\,\mathrm{e}^{\psi/h}) = b_N\,\mathrm{e}^{\psi/h}$, where $b_N=\mathcal{O}(h^{N+2})$ with all its derivatives. On the other hand, using that $\mathrm{e}^{\psi/h}\sum_0^\infty a_j h^j$ is a formal asymptotic solution, we get

$$\mathrm{e}^{-\psi/h}\big((h\partial)^2-V\big)(a^N\,\mathrm{e}^{\psi/h}) = h^{N+2}c_N,$$

where $\partial^\alpha c_N = \mathcal{O}(|x-z_0|^{-\frac{3}{2}N-2-\alpha})$, so

$$(V-(h\partial)^2)(\chi r_N\,\mathrm{e}^{\psi/h}) = h^{N+2}d_N\,\mathrm{e}^{\psi/h},$$

where $\partial^\alpha d_N = \mathcal{O}(|x-z_0|^{-\frac{3}{2}N-2-\alpha})$.

We conclude that

$$\chi r_N = \mathcal{O}\left(\frac{1}{h}\right)\int_a^x \frac{h^{N+2}}{|y-z_0|^{\frac{3}{2}N+2+\frac{1}{2}}}\,\mathrm{d}y = \mathcal{O}(1)\frac{h^{N+1}}{|x-z_0|^{\frac{3}{2}(N+1)}}.\tag{A.26}$$

Thus r_N satisfies the same estimate.

In principle we could also show that $\partial^\alpha r_N = \mathcal{O}(1)h^{N+1}/|x-z_0|^{\frac{3}{2}(N+1)+\alpha}$, but content ourselves with the observation that this is the case in the situation of Section 8.2, since the holomorphy then allows us to use the Cauchy inequalities.

BIBLIOGRAPHY

[1] Andersson (M.) & Sjöstrand (J.) – *Functional calculus for non-commuting operators with real spectra via an iterated Cauchy formula*, J. Funct. An., t. **210** (2004), pp. 341–375.

[2] Berry (M. V.) & Mount (K. E.) – *Semiclassical approximations in wave mechanics*, Rep. Prog. Phys., t. **35** (1972), pp. 315–397.

[3] Bony (J.-F.), Bruneau (V.) & Raikov (G.) – *Counting function of characteristic values and magnetic resonances*, `http://arxiv.org/abs/1109.3985`.

[4] Bordeaux Montrieux (W.) & Sjöstrand (J.) – *Almost sure Weyl asymptotics for non-self-adjoint elliptic operators on compact manifolds*, Ann. Fac. Sci. Toulouse, t. **19** (2010), pp. 567–587.

[5] Carron (G.) – *Déterminant relatif et la fonction Xi*, Amer. J. Math., t. **124** (2002), pp. 307–352.

[6] Christiansen (T. J.) – *Schrödinger operators and the distribution of resonances in sectors*, Anal. PDE, t. **5** (2012), pp. 961–982.

[7] Davies (E. B.), Exner (P.) & Lipovský (J.) – *Non-Weyl asymptotics for quantum graphs with general coupling conditions*, J. Phys. A 43, t. **47** (2010), 474013, 16 pp.

[8] Davies (E. B.) & Pushnitski (A.) – *Non-Weyl resonance asymptotics for quantum graphs*, Anal. PDE, t. **4** (2011), pp. 729–756.

[9] Dinh (Tien-Cuong) & Vu (Duc-Viet) – *Asymptotic number of scattering resonances for generic Schrödinger operators*, `http://arxiv.org/abs/1207.4273`.

[10] Exner (P.) & Lipovský (J.) – *Non-Weyl resonance asymptotics for quantum graphs in a magnetic field*, Phys. Lett. A, t. **375** (2011), pp. 805–807.

[11] Fédoriouk (M. V.) – *Méthodes asymptotiques pour les équations différentielles ordinaires linéaires*, Éditions Mir, 1987.

[12] Froese (R.) – *Asymptotic distribution of resonances in one dimension*, J. Differential Equations, t. **137** (1997), pp. 251–272.

[13] Gohberg (I.) & Leiterer (J.) – *Holomorphic operator functions of one variable and applications*, Operator Theory: Advances and Applications, vol. 192, Birkhäuser Verlag, 2009.

[14] Gohberg (I. C.) & Krein (M. G.) – *Introduction to the theory of linear non-selfadjoint operators*, Translations of mathematical monographs, vol. 18, Amer. Math. Soc., 1969.

[15] Hargé (T.) & Lebeau (G.) – *Diffraction par un convexe*, Inv. Math., t. **118** (1994), pp. 161–196.

[16] Helffer (B.) & Sjöstrand (J.) – *Résonances en limite semi-classique*, Mém. Soc. Math. France (N.S.), t. **24-25** (1986).

[17] Ivrii (V.) – *Sharp spectral asymptotics for operators with irregular coefficients. II. Domains with boundaries and degenerations*, Comm. Partial Differential Equations, t. **28** (2003), pp. 103–128.

[18] Melin (A.) & Sjöstrand (J.) – *Bohr-Sommerfeld quantization condition for non-selfadjoint operators in dimension 2*, Astérique, t. **284** (2003), pp. 181–244.

[19] Nakamura (S.), Stefanov (P.) & Zworski (M.) – *Resonance expansions of propagators in the presence of potential barriers*, J. Funct. Anal., t. **205** (2003), pp. 180–205.

[20] Regge (T.) – *Analytic properties of the scattering matrix*, Il Nuovo Cimento, t. **8** (1958), pp. 671–679.

[21] Simon (B.) – *Resonances in one dimension and Fredholm determinants*, J. Funct. Anal., t. **178** (2000), pp. 396–420.

[22] ———, *The definition of molecular resonance curves by the method of exterior complex scaling*, Physics Lett. 71A, t. **2,3** (30 April 1979), pp. 211–214.

[23] Sjöstrand (J.) – *Lectures on resonances*, http://math.u-bourgogne.fr/IMB/sjostrand/Coursgbg.pdf.

[24] ———, *Resonances for bottles and trace formulae*, Math. Nachr., t. **221** (2001), pp. 95–149.

[25] ———, *Eigenvalue distribution for non-self-adjoint operators with small multiplicative random perturbations*, Ann. Fac. Sci. Toulouse, t. **18** (2009), pp. 739–795, http://arxiv.org/abs/0802.3584.

[26] ———, *Counting zeros of holomorphic functions of exponential growth*, J. pseudodifferential operators and applications, t. **1** (2010), pp. 75–100, http://arxiv.org/abs/0910.0346.

[27] ———, *Eigenvalue distribution for non-self-adjoint operators on compact manifolds with small multiplicative random perturbations*, Ann. Fac. Sci. Toulouse, t. **19** (2010), pp. 277–301.

[28] Sjöstrand (J.) & Zworski (M.) – *Complex scaling and the distribution of scattering poles*, J. Amer. Math. Soc., t. **4** (1991), pp. 729–769.

[29] ———, *Estimates on the number of scattering poles near the real axis for strictly convex obstacles*, Ann. Inst. Fourier, t. **43** (1993), pp. 769–790.

[30] ———, *The complex scaling method for scattering by strictly convex obstacles*, Ark. Mat., t. **33** (1995), pp. 135–172.

[31] ———, *Asymptotic distribution of resonances for convex obstacles*, Acta Math., t. **183** (2000), pp. 191–253.

[32] ———, *Elementary linear algebra for advanced spectral problems*, Ann. Inst. Fourier, t. **57** (2007), pp. 2095–2141.

[33] ———, *Fractal upper bounds on the density of semiclassical resonances*, Duke Math J., t. **137** (2007), pp. 381–459.

[34] Stefanov (P.) – *Sharp upper bounds on the number of the scattering poles*, J. Funct. Anal., t. **231** (2006), pp. 111–142.

[35] Vodev (G.) – *Sharp bounds on the number of scattering poles in even-dimensional spaces*, Duke Math. J., t. **74** (1994), pp. 1–17.

[36] Voros (A.) – *Spectre de l'équation de Schrödinger et méthode BKW*, Publications Mathématiques d'Orsay, Université de Paris-Sud (1982), 75 pp., http://mathdoc.emath.fr/PMO/PDF/V_VOROS-167.pdf.

[37] ZIELIŃSKI (L.) – *Semiclassical distribution of eigenvalues for elliptic operators with Hölder continuous coefficients. I. Non-critical case*, Colloq. Math., t. **99** (2004), pp. 157–174.

[38] ZWORSKI (M.) – *Distribution of poles for scattering on the real line*, J. Funct. Anal., t. **73** (1987), pp. 277–296.

[39] ———, *Sharp polynomial bounds on the number of scattering poles*, Duke Math. J., t. **59** (1989), pp. 311–323.

[40] ———, *Sharp polynomial bounds on the number of scattering poles of radial potentials*, J. Funct. Anal., t. **82** (1989), pp. 370–403.