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DIAGONAL OPERATORS ON SPACES OF MEASURABLE FUNCTIONS

by

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1. Introduction.

We denote by L the set of equivalence classes of real-valued measurable functions on a fixed measure space (X , Σ , μ) . L is an algebra with unit and a vector lattice with respect to almost everywhere pointwise operations. The space of essentially bounded real-valued functions $L^{\infty} = L^{\infty}(\mu)$ is a normed subalgebra of L and L is a module over L^{∞} with respect to almost everywhere pointwise multiplication. A subspace M of L is a solid sublattice of L if and only if M is an L^{∞}-submodule of L [4] . We will can an L^{∞}-submodule M of L a <u>locally convex</u> L^{∞}-<u>module</u> if M is a locally convex vector space whose topology is given by a family of seminorms p satisfying

$$p(af) \leqslant \| a\|_{\infty} p(f) \qquad a \in L^{\infty} , \ f \in M .$$

Such a seminorm is called a <u>scalar</u> L^{∞}-<u>seminorm</u> [7]. Since a scalar L^{∞}-seminorm defined on a solid sublattice of L is a lattice seminorm and vice versa, M is a locally convex L^{∞}-module if and only if it is a locally convex vector lattice and solid in L [4]. The Banach spaces $L^{p}(\mu)$, $1 \leqslant p \leqslant \infty$, and Köthe spaces equipped with Köthe topologies [12] are examples of locally convex L^{∞}-modules.

A linear operator T mapping a subspace M of L into another subspace of L will be called <u>diagonal</u> if there is a locally measurable real-valued function g on X such that $Tf = gf$ for every f in M . A linear operator T mapping an L^{∞}-submodule M into another L^{∞}-submodule of L will be called L^{∞}-<u>linear</u> if $T(af) = a\, T(f)$ for every a in L^{∞} and f in M .

From now on M and N will denote locally convex L^{∞}-modules (or equivalently, locally convex solid sublattices of L). Further N is assumed to be order complete. If A is a subset of L , then A^{+} denotes the set of positive elements of A .

We present our results without proofs ; a full account will appear elsewhere. Finally, we wish to express our gratitude to the Scientific and Technical Research Council of Turkey for their support.

2. L^∞-linear operators.

Let $\mathcal{C}$ be the set of positive continuous linear operators from M into N. Then $\mathcal{L}(M,N) = \mathcal{C} - \mathcal{C}$ is a solid sublattice of the space $L^b(M,N)$ of order bounded linear operators from M into N. By $\mathcal{H}_\infty(M,N)$ we denote the space of continuous L^∞-linear operators from M into N.

LEMMA. - $\mathcal{H}_\infty(M,N)$ is a sublattice of $\mathcal{L}(M,N)$.

A locally convex L^∞-module M is said to have the dominated convergence property if for every sequence (f_n) in L with $|f_n| \leqslant g$ for some g in M and $\lim f_n(x) = f(x)$ on X, we have $\lim f_n = f$ in M.

PROPOSITION 1. - Let Λ be a Köthe space, T a Köthe topology on Λ and $\Lambda^\times$ the α-dual of Λ. Consider the following conditions :

a) T is compatible with the duality $(\Lambda, \Lambda^\times)$.

b) If $f_n \in \Lambda$ and $f_n(x) \downarrow 0$ on X then $\lim f_n = 0$ in $\Lambda(T)$

c) $\Lambda(T)$ has the dominated convergence property.

d) If p is one of the scalar L^∞-seminorms defining the topology T on Λ and $f \in \Lambda$, then for every $\varepsilon>0$ there exists $\delta>0$ such that $\mu(E)<\delta$ implies $p(\chi_E f) < \varepsilon$.

We have (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d).

We will also consider the following condition.

(A) for every $f \in M^+$ there is an increasing sequence (s_n) of positive simple functions of bounded support such that $s_n(x) \uparrow f_n(x)$ on X and $\lim s_n = f$ in M.

The Banach spaces $L^p(\mu)$, $1 \leqslant p < \infty$, satisfy this condition.

PROPOSITION 2. - If a Köthe space $\Lambda(T)$ has the dominated convergence property, it satisfies (A).

From now on we assume $L^1(\mu)' = L^\infty(\mu)$.

A diagonal operator is certainly L^∞-linear. Under certain assumptions the converse is also true.

PROPOSITION 3. - a) Let M satisfy condition (A). If for every set of finite measure B, the characteristic function $\chi_B \in M$, then every element of $\mathcal{H}_\infty(M, N)$ is a diagonal operator.

b) If M is a Köthe space which has the dominated convergence property, then every element of $\mathcal{H}_\infty(M, N)$ is a diagonal operator.

Remark : The hypothesis of the proposition is satisfied by $L^p(\mu)$, $1\leqslant p<\infty$. On the other hand, if $T : L^\infty \to N$ is L^∞-linear, since $T(f) = T(1)f$ for every $f\in L^\infty$, it is also diagonal.

The set of idempotents in L^∞ is denoted by I_∞ and non-negative finite linear combinations of elements of I_∞ are dense in $(L^\infty)^+$. If $\chi\in I_\infty$, then $\chi' = 1-\chi\in I_\infty$ also.

PROPOSITION 4. - There is a projection P of $\mathcal{L}(M,N)$ onto $\mathcal{H}_\infty(M,N)$ with $0\leqslant P\leqslant I$.

The projection is constructed in successive steps. First, for $T\in\mathcal{C}$ and $f\in M^+$ we define an element of N by

$$P(T)(f) = \bigwedge_{I_\infty} \{\chi\, T(\chi f) + \chi' T(\chi' f)\} .$$

We prove that $P(T)$ is additive on M^+ and then extend it to a positive linear operator on M . In the next step P is proved to be additive on $\mathcal{C}$ and then extended to $\mathcal{L}(M,N)$.

Remark 1. If we define an L^∞-module structure on $\mathcal{L}(M,N)$ by letting $(a.T)(f)=T(af)$ for f in M and a in L^∞ , then P is also L^∞-linear.

Remark 2. If we take μ to be the counting measure on the set of positive integers, a Köthe space becomes a solid sequence space [5] . Certain operators on sequence spaces can be represented by infinite-matrices [8 ; p. 20] . If (t_{ij}) is the matrix which represents the operators T , then $P(T)$ is the operator represented by the diagonal of the matrix (t_{ij}) .

Let M and N be Banach sublattices of L , and $\mathcal{N}(M,N)$ the space of nuclear operators from M into N with the nuclear norm $r(\ .\ )$. Every nuclear operator can be written as the difference of two positive nuclear operators. If $u_i\in M'$ and $g_i\in N$, $i=1,\dots,n$, by $\sum^n u_i\otimes g_i$ we denote the nuclear operator which sends each $f\in M$ to $\sum^n u_i(f)\, g_i$. We consider the following conditions on a Banach L^∞- module Q .

(B) Given $f\in Q$ and $\varepsilon>0$, there is $\delta>0$ such that $\mu(E)<\delta$ implies $\|f\chi_E\|<\varepsilon$.

(C) The support of each $f\in Q$ is σ-finite.

(D) Q has the dominated convergence property.

By $\mathcal{N}_\infty(M,N)$ we will denote the space of nuclear L^∞-linear operators from M into N with the nuclear norm.

PROPOSITION 5. - Let M and N the Banach L^{∞}- modules. If M satisfies (B), N satisfies (C) and (D) and further for every finite family of atoms $\{\chi_1, \dots, \chi_n\}$, $u \in M'$ and $g \in N$ we have

$$(*) \qquad r(\sum^{n} \chi_k u \otimes \chi_k g) \leqslant \|(\sum^{n} \chi_k) u\| \quad \|(\sum^{n} \chi_k) g\|$$

then the projection P maps $\mathcal{N}(M,N)$ onto $\mathcal{N}_{\infty}(M,N)$ such that $r(P(T)) \leqslant r(T)$ for each $T \in \mathcal{N}(M,N)$.

Remark : If M' has property (B) instead of M, M has property (C) instead of N or if M' has property (D) instead of N, the result still holds.

3. Diagonal and nuclear diagonal operators on L^p-spaces.

Let M and N be two normed L^{∞}- modules and $M \otimes N$ the complete projective tensor product as defined by Grothendieck [3]. Let K be the smallest closed subspace of $M \otimes N$ containing all elements of the form $(af \otimes g) - (f \otimes ag)$ for every $a \in L^{\infty}$, $f \in M$ and $g \in N$. The quotient space $M \otimes N/K$ with the quotient norm is called the normed L^{∞}-tensor product of M and N, and denoted by $M \otimes_{\infty} N$. If $f \otimes_{\infty} g$ denotes $f \otimes g$ mod K for each $f \in M$, $g \in N$, then for $u \in M \otimes_{\infty} N$ the norm is given by [4 and 9]

$$\gamma_{\infty}(u) = \inf\{\sum^{\infty} \|f_i\| \ \|g_i\| : u = \sum^{\infty} f_i \otimes_{\infty} g_i \ , \ f_i \in M \ , \ g_i \in N\} .$$

With a measure space (X, Σ, μ) we associate for every real number $s > o$ a weighted counting measure space as follows : ψ is the set of equivalence classes of atoms of μ together with the equivalence class of sets of μ-measure zero. We let $\mu_{\alpha} = \mu(A)$ for any $A \in \alpha$, where $\alpha \in \psi$. For any subset S of ψ we define

$$\tilde{\mu}^s(S) = \sum_{\alpha \in S} \mu_{\alpha}^s \quad .$$

PROPOSITION 6. - (Harte). Let $1/p + 1/q = 1/r \leqslant 1$ where $1 \leqslant p$, $q \leqslant \infty$. Then $L^p(\mu) \otimes_{\infty} L^q(\mu)$ is isometrically L^{∞}-isomorphic with $L^r(\mu)$.

In the result complementary to this we have to use the weighted counting measure constructed above.

PROPOSITION 7. - Let $s = 1/p + 1/q > 1$ where $1 \leqslant p$, $q \leqslant \infty$. Then $L^p(\mu) \otimes_{\infty} L^q(\mu)$ is isometrically L^{∞}-isomorphic with $L^1(\tilde{\mu}^s)$.

This result can be found in [6]. Next we give characterizations of diagonal operators between L^p-spaces as another L^p-space. Again we have two cases, the first due to Harte [4] and the second to Orhon [6].

PROPOSITION 8. - Let $1/q - 1/p = 1/r$ where $1 \leqslant p$, $q \leqslant \infty$. Then $\mathcal{H}_\infty(L^p(\mu), L^q(\mu))$ is isometrically L^∞-isomorphic with $L^r(\mu)$.

In the result complementary to this we again need the weighted counting measure.

PROPOSITION 9. - Let $1 \leqslant p < q \leqslant \infty$. Then $\mathcal{H}_\infty(L^p(\mu) , L^q(\mu))$ is isometrically L^∞-isomorphic with $L^\infty(\tilde{\mu})$.

Remark : Diagonal operators between l^p-spaces were characterized by A. Tong [11]. G. Crofts [1] has considered diagonal operators between sequence spaces.

Using the projection constructed in proposition 4 and its properties discussed in proposition 5, we can define a continuous linear operator from $L^{p'}(\mu) \otimes_\infty L^q(\mu)$ onto the space $\mathcal{N}_\infty(L^p , L^q)$ of diagonal nuclear operators. This enables us to characterize $\mathcal{N}_\infty(L^p , L^q)$ by using propositions 6 and 7 .

PROPOSITION 10. - $\mathcal{N}_\infty(L^p(\mu) , L^q(\mu))$ is isometrically isomorphic with

(i) $L^1(\tilde{\mu}^{1/r})$, if $1 \leqslant q < p < \infty$ and $1/r = 1/q - 1/p$.

(ii) $l^1(\psi_0)$, if $1 \leqslant p = q < \infty$ where ψ_0 denotes the set of equivalence classes of atoms of μ .

(iii) $L^s(\tilde{\mu}^{1-s})$, if $1 \leqslant p < \infty$ and $s = pq/pq - q + p$.

(iv) $L^{p'}(\tilde{\mu}^{1-p'})$, if $1 < p < \infty$ and $q = \infty$.

Remark : In proposition 10 the cases $\mathcal{N}_\infty(L^\infty, L^p)$, $1 \leqslant p \leqslant \infty$ and $\mathcal{N}_\infty(L^1 , L^\infty)$ are not covered. In the case $\mathcal{N}_\infty(L^1 , L^\infty)$ our method breaks down, since in this case the projection p (cap.)does not take nuclear operators to nuclear diagonal operators. Nuclear diagonal operators on l^p-spaces were characterized by A. Tong [11].

BIBLIOGRAPHIE

[1] CROFTS (G.). - Concerning perfect Fréchet spaces and diagonal transformations. Math. Ann. 182, 67-76, (1969).

[2] DIEUDONNÉ (J.). - Sur les espaces de Köthe. J. Analyse Math. 1, 81-115, (1951).

[3] GROTHENDIECK (A.). - Produits tensoriels topologiques et espaces nucléaires. Mém. Amer. Math. Soc., N° 16 (1955).

[4] HARTE (R. E.). - Modules over a Banach algebra. Ph. D. thesis. Cambridge, 1964.

[5] KOTHE (G.). - Topologische lineare Räume I. Springer-Verlag, 1960.

[6] ORHON (M.). - Vector semi-norms on modules. Ph. D. thesis, University of Wales, 1969.

[7] ORHON (M.). - On the Hahn-Banach theorem for modules over C(S). J. London Math. Soc. (2), 1, 363-368 (1969).

[8] PERESSINI (A. L.). - Ordered topological vector spaces. Harper and Row. 1967.

[9] RIEFFEL (M. A.). - Induced Banach representations of Banach algebras and locally compact groups. J. Func. Anal. 1, 443-491, (1967).

[10] SCHAEFER (H. H.). - Topological vector spaces. Macmillan 1966.

[11] TONG (A.). - Diagonal nuclear operators on l_p-spaces. Trans. Amer. Math. Soc., 143, 237-247, (1969).

[12] WELLAND (R.). - On Köthe spaces. Trans. Amer. Math. Soc., 112, 267-277, (1964).

[13] ZAANEN (A. C.). - Integration. North-Holland 1967.

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