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Wavelet transform of Beurling–Björck type ultradistributions

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Abstract – Wavelet transform of a distribution in $\mathcal{M}'_\omega$ involving wavelet of infraexponential decay (subexponential decay) is studied. An inversion formula is obtained which is valid in the weak topology of $\mathcal{D}'$. A discussion on extension of the results to ultradistribution space of compact support is also given.

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1. Introduction

Wavelet analysis has been used for intrinsic characterizations of important function and distribution spaces ([10], [11]). Recently, the wavelet transform has been extended to distributions, and inversion formulae have been established in distribution setting by Pathak [13, 14], Pathak *et al* [16, 17, 18] and Pandey [12] using duality arguments.

Wavelets of subexponential decay whose Fourier transform have compact support i.e. band limited wavelets, were investigated by Dziubański and Hernández [7]. Pathak and Singh [17] extended the work of Dziubański and Hernández and studied wavelets with more general decay (infraexponential decay) whose Fourier transforms have compact support. The aim of the present paper is to develop the

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theory of wavelet transform involving these wavelets using ultradistribution theory of Beurling [3] and Björck [4].

Now, we recall definitions and properties of the desired test function and ultradistribution spaces from [4], [6], [9], and [19]. Let $\mathfrak{M}$ be the set of all real-valued functions ω on $\mathbb{R}$ which can be represented as $\omega(x) = \sigma(|x|)$, where $\sigma(t)$ is an increasing continuous concave function on $[0, \infty)$ satisfying the following conditions [9, p. 14]:

$$(1) \qquad (\alpha) \quad 0 = \omega(0) \le \omega(\xi + \eta) \le \omega(\xi) + \omega(\eta), \quad \text{for all } \xi, \eta \in \mathbb{R};$$

$$(2) \qquad (\beta) \quad \int_{\mathbb{R}} \frac{\sigma(\xi)}{(1 + |\xi|)^2} \, \mathrm{d}\xi < \infty;$$

there exists real number p and positive real number q such that

$$(3) \qquad (\gamma) \quad \sigma(\xi) \ge p + q \log(1 + t), \quad t \ge 0.$$

Let $\omega \in \mathfrak{M}$. We denote by $\mathcal{M}_\omega$ the set of all functions $\psi(t) \in C^\infty(\mathbb{R})$ which satisfy

$$(4) \qquad P_{k,\lambda}(\psi) = \sup_{t \in \mathbb{R}} \{ e^{\lambda \omega(t)} |D^k \psi(t)| \} < \infty$$

for all non-negative integers k and all non-negative real λ. The topology on $\mathcal{M}_\omega$ is defined by the semi-norms $\{P_{k,\lambda}\}$. It can be readily seen that $\mathcal{M}_\omega$ is a vector space. A sequence $\{\psi_\nu\}_{\nu=1}^\infty$ is a Cauchy sequence in $\mathcal{M}_\omega$ if for each non-negative integer m and k, $P_{k,\lambda}(\psi_\mu - \psi_\nu) \to 0$ as $\mu, \nu \to \infty$ independently of each other. The space $\mathcal{M}_\omega$ is a sequentially complete space and therefore it is a complete countably multinormed space and so a Fréchet space. The dual of $\mathcal{M}_\omega$ is denoted by $\mathcal{M}_\omega{}'$; it is a distribution space [6, p. 170]. The Schwartz space $\mathcal{D}(\mathbb{R})$ consisting of C^∞–functions of compact support is a subspace of $\mathcal{M}_\omega(\mathbb{R})$ and $\mathcal{M}'_\omega \subset \mathcal{D}'$.

Suppose that the Fourier transform of $\psi \in \mathcal{M}_\omega$, defined by

$$\hat{\psi}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} \psi(x) \, \mathrm{d}x,$$

satisfies

$$(5) \qquad \pi_{k,\lambda}(\psi) = \sup_{\xi \in \mathbb{R}} \{ e^{\lambda \omega(\xi)} |D^k \hat{\psi}(\xi)| \} < \infty, \quad \lambda \ge 0, \ k \in \mathbb{N}_0.$$

Then the space of all functions $\psi \in L^1(\mathbb{R})$ such that $\psi, \hat{\psi} \in C^\infty(\mathbb{R})$ and (4) and (5) hold, is denoted by $\mathcal{S}_\omega$ the topology of $\mathcal{S}_\omega$ is defined by the seminorms $P_{k,\lambda}$ and $\pi_{k,\lambda}$, see [4, p. 377].

Let K be a compact subset of $\mathbb{R}$. The space $\mathcal{D}_\omega$ (K) is the set of all ψ in $L^1(\mathbb{R})$ such that ψ has support in K and

$$\|\psi\|_\lambda := \int_{\mathbb{R}} |\widehat{\psi}(\xi)| e^{\lambda\omega(\xi)}\, \mathrm{d}\xi < \infty, \quad \text{for all } \lambda > 0. \tag{6}$$

Let $\{K_n\}$ be a sequence of compact set in $\mathbb{R}$ such that $\bigcup_{n=1}^{\infty} K_n = \mathbb{R}$ and K_n is contained in the interval of K_{n+1} for all n. Then $\mathcal{D}_\omega(\mathbb{R}) = \lim\operatorname{ind} \mathcal{D}_\omega(K_n)$. Since $\mathcal{D}_\omega \subset \mathcal{S}_\omega$ and the topology of $\mathcal{D}_\omega$ is stronger than that induced on $\mathcal{D}_\omega$ by $\mathcal{S}_\omega$, it follows that the restriction of any $f \in \mathcal{S}'_\omega$ to $\mathcal{D}_\omega$ is in $\mathcal{D}'_\omega$. The elements of $\mathcal{D}'_\omega$ are called ultradistributions [6].

Now we recall from [5] some definitions and results related to wavelet transform needed in the present investigation.

Let $\psi \in L^2(\mathbb{R})$. Define

$$\psi_{b,a}(t) := \frac{1}{\sqrt{|a|}} \psi\Big(\frac{t-b}{a}\Big), \quad t \in \mathbb{R},\ b \in \mathbb{R},\ a \in \mathbb{R}_0 = \mathbb{R}\backslash\{0\}. \tag{7}$$

The wavelet transform $W(b,a)$ of $f \in L^2(\mathbb{R})$ with respect to the wavelet $\psi_{b,a}(t) \in L^2(\mathbb{R})$ is defined by

$$W(b,a) := \int_{\mathbb{R}} f(t)\overline{\psi_{b,a}(t)}\, \mathrm{d}t \tag{8}$$

and the corresponding wavelet inversion formula is given by

$$\frac{1}{C_\psi} \int_{\mathbb{R}} \int_{\mathbb{R}_0} \frac{1}{\sqrt{|a|}} W(b,a) \psi\Big(\frac{x-b}{a}\Big) \frac{\mathrm{d}b\, \mathrm{d}a}{a^2} = f(x), \tag{9}$$

where

$$C_\psi = \int_{\mathbb{R}} \frac{|\widehat{\psi}(w)|^2}{|w|}\, \mathrm{d}w < \infty \quad ([5, \text{p. } 9]).$$

In the present work we shall investigate properties of the wavelet $\psi_{b,a}(t) \in \mathcal{M}_\omega(\mathbb{R})$. Wavelet transform of $f \in \mathcal{M}'_\omega$ will be studied and the inversion formula (9) will be extended to distribution space $\mathcal{M}'_\omega$. It has been noted by Constantinescu *et al* [6, p. 169] that the Schwartz space $\mathcal{S}(\mathbb{R})$ and Gelfand-Shilov space $S^{\alpha,A}$ are special cases of the space $\mathcal{M}_\omega$. Therefore, some of the results obtained in this paper are more general than those derived in [13] and [16].

2. Wavelet transform on $\mathcal{M}'_\omega$

In this section, certain basic properties of the wavelets in $\mathcal{M}_\omega$ and wavelet transform of $f \in \mathcal{M}'_\omega$ are obtained.

LEMMA 2.1. *If $\psi \in \mathcal{M}_\omega$, then $\psi(\frac{t-b}{a}) \in \mathcal{M}_\omega$ for arbitrary but fixed $b, a \in \mathbb{R}$, $a \neq 0$.*

PROOF. Let a and b be fixed real numbers. Then for $k = 0, 1, 2, \dots,$

$$\sup_{-\infty<t<\infty}\left|e^{\lambda\omega(t)}D^k\psi\Big(\frac{t-b}{a}\Big)\right|$$
$$= \sup_{-\infty<t<\infty}\left|e^{\lambda\omega(\frac{t-b}{a})}\psi^{(k)}\Big(\frac{t-b}{a}\Big)\Big(\frac{1}{a^k}\Big)\right|\left|\frac{e^{\lambda\omega(t)}}{e^{\lambda\omega(\frac{t-b}{a})}}\right|.$$

Then by Property (α) we get

$$\sup_{-\infty<t<\infty}\left|e^{\lambda\omega(t)}D^k\psi\Big(\frac{t-b}{a}\Big)\right| \le \frac{1}{|a|^k}P_{k,\lambda}(\psi)\sup_{-\infty<t<\infty}\left|\frac{e^{\lambda\omega(\frac{t-b}{a}+\frac{b}{a})}}{e^{\lambda\omega(\frac{t-b}{a})}}\right| \quad \text{(by (4))}$$
$$\le P_{k,\lambda}(\psi)\Big(\frac{1}{|a|^k}\Big)e^{\lambda\omega(\frac{b}{a})} < \infty \quad \text{(by (1))},$$

for all fixed real numbers b and $a \neq 0$. □

In what follows we shall assume that $\psi \in \mathcal{M}_\omega(\mathbb{R})$ is the basic function generating the wavelet $\psi_{b,a}(t)$ given by (7). Since function $\psi(\frac{t-b}{a})$ belongs to $\mathcal{M}_\omega$ for fixed b and $a \neq 0$ as a function of t under conditions of Lemma 2.1, for $f \in \mathcal{M}'_\omega$ the wavelet transform $W(b,a)$ of f is defined by

$$W(b,a) = \frac{1}{\sqrt{|a|}}\Big\langle f(t),\ \overline{\psi\Big(\frac{t-b}{a}\Big)}\Big\rangle = \langle f(t),\ \overline{\psi_{b,a}(t)}\rangle, \quad a \neq 0,\ a,b \in \mathbb{R}. \tag{10}$$

THEOREM 2.2. *For real b and $a \neq 0$ let $W(b,a)$ be defined by (10), then under conditions of Lemma 2.1, there exists $m \in \mathbb{N}_0$ such that*

$$|W(b,a)| \le C(m,\psi)(|a|^{-m-1/2}\exp(m\omega(b/a))), \quad \textit{for some } m \in \mathbb{N}_0.$$

PROOF. To very $f \in \mathcal{M}'_\omega$ there exists a non-negative integer m and a constant $C > 0$ such that, for all $\psi \in \mathcal{M}_\omega$,

$$|W(b,a)| = \left|\frac{1}{\sqrt{|a|}}\Big\langle f(t),\ \overline{\psi\Big(\frac{t-b}{a}\Big)}\Big\rangle\right|$$
$$\le \frac{C}{\sqrt{|a|}}\max_{0\le k\le m}\sup_{b,t\in\mathbb{R}}\left|e^{m\omega(\frac{t}{a})}D_t^k\overline{\psi\Big(\frac{t-b}{a}\Big)}\right|$$
$$= \frac{C}{\sqrt{|a|}}\max_{0\le k\le m}\sup_{b,t\in\mathbb{R}}\left|e^{m\omega(\frac{t-b}{a})}\overline{\psi^{(k)}\Big(\frac{t-b}{a}\Big)}\Big(\frac{1}{a^k}\Big)\right|\left|\frac{e^{m\omega(\frac{t}{a})}}{e^{m\omega(\frac{t-b}{a})}}\right|$$
$$= \frac{C}{\sqrt{|a|}}\left[\sup_{b,t\in\mathbb{R}}\left|\frac{e^{m\omega(\frac{t}{a})}}{e^{m\omega(\frac{t-b}{a})}}\right|\right]\Big(\frac{1}{|a|^m}\Big)\max_{0\le k\le m}P_{k,m}(\psi)$$
$$\le C\max_{0\le k\le m}P_{k,m}(\psi)\frac{e^{m\omega(\frac{b}{a})}}{|a|^{m+1/2}}$$

by using property (α), as in the Lemma 2.1. This gives the required result. □

3. Inversion of the wavelet transform on $\mathcal{M}'_\omega$

In order to study properties of the wavelet transform of $f \in \mathcal{M}'_\omega$ we obtain an appropriate structure formula for $f \in \mathcal{M}'_\omega$.

Assume that $f \in \mathcal{M}'_\omega$ then as in the proof of the above theorem there exists a non-negative integer m and constant $C > 0$ such that for all $\phi \in \mathcal{M}_\omega$,

$$|\langle f, \phi\rangle| \leq C \max_{0\leq k\leq m} \sup_{t\in\mathbb{R}} |e^{m\omega(t)} D_t^k \phi(t)|. \tag{11}$$

Then following [8, p. 112] we have two cases.

Case I. For $t > 0$,

$$\begin{aligned} e^{m\omega(t)}|D_t^k\phi(t)| &= e^{m\omega(t)} \left| \int_t^\infty \frac{d}{dz}(D_z^k\phi(z))\,\mathrm{d}z \right| \\ &\leq \max_{0\leq k\leq m} \int_t^\infty e^{m\omega(z)} \left| \frac{d}{dz}(D_z^k\phi(z))\,\mathrm{d}z \right| \\ &\leq \max_{0\leq k\leq m+1} \int_{-\infty}^\infty |e^{m\omega(z)} D_z^k\phi(z)|\,\mathrm{d}z \\ &\leq \max_{0\leq k\leq m+1} \int_{-\infty}^\infty |e^{-\omega(z)} e^{(m+1)\omega(z)} D_z^k\phi(z)|\,\mathrm{d}z \\ &\leq \max_{0\leq k\leq m+1} \|e^{-\omega(z)}\|_2 \; \|e^{(m+1)\omega(z)} D_z^k\phi(z)\|_2. \end{aligned}$$

Case II. For $t < 0$,

$$\begin{aligned} e^{m\omega(t)}|D_t^k\phi(t)| &= e^{m\omega(t)} \left| \int_{-\infty}^t \frac{d}{dz}(D_z^k\phi(z))\,\mathrm{d}z \right| \\ &\leq \max_{0\leq k\leq m} \int_{-\infty}^t e^{m\omega(z)} \left| \frac{d}{dz}(D_z^k\phi(z))\,\mathrm{d}z \right| \\ &\leq \max_{0\leq k\leq m+1} \int_{-\infty}^\infty |e^{m\omega(z)} D_z^k\phi(z)|\,\mathrm{d}z \\ &\leq \max_{0\leq k\leq m+1} \int_{-\infty}^\infty |e^{-\omega(z)} e^{(m+1)\omega(z)} D_z^k\phi(z)|\,\mathrm{d}z \\ &\leq \max_{0\leq k\leq m+1} \|e^{-\omega(z)}\|_2 \; \|e^{(m+1)\omega(z)} D_z^k\phi(z)\|_2. \end{aligned}$$

Therefore, there exists $C > 0$ such that

$$|\langle f, \phi\rangle| \leq C \max_{0\leq k\leq m+1} \|e^{(m+1)\omega(t)} D_t^k\phi(t)\|_2, \quad \phi \in \mathcal{M}_\omega. \tag{12}$$

Now, we show that the above L^2-norm exists finitely. Indeed, for some $\mu > 0$, we have

$$\begin{aligned}
&\max_{0\le k\le m+1}\left(\int_{-\infty}^{\infty}|e^{(m+1)\omega(t)}D_t^k\phi(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\\
&=\max_{0\le k\le m+1}\left(\int_{-\infty}^{\infty}|e^{-\mu\omega(t)}e^{(m+1+\mu)\omega(t)}D_t^k\phi(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\\
&\le\max_{0\le k\le m+1}P_{k,m+1+\mu}(\phi)\left(\int_{-\infty}^{\infty}e^{-2\mu[p+q\log(1+t)]}\,\mathrm{d}t\right)^{\frac{1}{2}}\qquad\text{(by (3))}
\end{aligned}$$

we can choose μ large so that the last integral is finite. Thus

$$\max_{0\le k\le m+1}\|e^{(m+1)\omega(t)}D_t^k\phi(t)\|_2<\infty.$$

Now, applying Hahn-Banach theorem and Riesz representation theorem to (12) we get g_k belonging to the space $L^2(\mathbb{R})$ such that

$$\begin{aligned}
|\langle f,\phi\rangle| &= \sum_{k=0}^{m+1}\langle g_k(t), e^{(m+1)\omega(t)}D_t^k\phi(t)\rangle\\
&= \sum_{k=0}^{m+1}\langle D_t^k(e^{(m+1)\omega(t)}g_k(t)),\,\phi(t)\rangle.
\end{aligned}$$

Therefore desired structure formula is

$$f=\sum_{k=0}^{m+1}D_t^k(e^{(m+1)\omega(t)}g_k(t)), \tag{13}$$

where $g_k\in L^2(\mathbb{R})$ and $k=0,1,2,3,\ldots$.

THEOREM 3.1. *Let $f\in\mathcal{M}'_\omega$, $\psi\in\mathcal{M}_\omega$ and $W(b,a)$ be defined by (10). Then*

$$D_b^rW(b,a)=\frac{\partial^rW}{\partial b^r}=\Big\langle f(t),\,\frac{\partial^r}{\partial b^r}\frac{1}{\sqrt{|a|}}\psi\Big(\frac{t-b}{a}\Big)\Big\rangle,$$

and

$$D_a^rW(b,a)=\frac{\partial^rW}{\partial a^r}=\Big\langle f(t),\,\frac{\partial^r}{\partial a^r}\frac{1}{\sqrt{|a|}}\psi\Big(\frac{t-b}{a}\Big)\Big\rangle.$$

Proof. Using the structure formula for f as given in (13) and following [12] we have

$$\begin{aligned} W(b,a) &= \langle f(t),\ \psi_{b,a}(t)\rangle \\ &= \Big\langle \sum_{k=0}^{m+1} D_t^k(e^{(m+1)\omega(t)} g_k(t)),\ \psi_{b,a}(t)\Big\rangle \\ &= \Bigg\langle g_k(t),\ \sum_{k=0}^{m+1}(e^{(m+1)\omega(t)})(-1)^k D_t^k\Bigg(\frac{\psi\left(\frac{t-b}{a}\right)}{\sqrt{|a|}}\Bigg)\Bigg\rangle. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{\partial^r W}{\partial b^r}(b,a) &= \sum_{k=0}^{m+1}\int_{-\infty}^{\infty} g_k(t)e^{(m+1)\omega(t)}\frac{\partial^r}{\partial b^r}(-1)^k(D_t^k\overline{\psi_{b,a}(t)})\,\mathrm{d}t \\ &= \sum_{k=0}^{m+1}\int_{-\infty}^{\infty} g_k(t)e^{(m+1)\omega(t)}(-1)^{k+r} D_t^{k+r}\overline{\psi_{b,a}(t)}\,\mathrm{d}t \\ &= \sum_{k=0}^{m+1}\int_{-\infty}^{\infty} g_k(t)e^{(m+1)\omega(t)}(-1)^k D_t^k\frac{\partial^r}{\partial b^r}\overline{\psi_{b,a}(t)}\,\mathrm{d}t \\ &= \sum_{k=0}^{m+1}\Big\langle D_t^k(e^{(m+1)\omega(t)}g_k(t)),\ \frac{\partial^r}{\partial b^r}\psi_{b,a}(t)\Big\rangle \\ &= \Big\langle f(t),\ \frac{\partial^r}{\partial b^r}\psi_{b,a}(t)\Big\rangle \quad \text{(by structure formula (13))}. \end{aligned}$$

Similarly result for differentiation with respect to a can be proved. □

In order to derive inversion formula for the wavelet transform of $f \in \mathcal{M}'_\omega$ we define function $g_\nu(t)$ as follows [1]:

$$g_\nu(t) = \begin{cases} g(t) & \text{if} -\nu \le t \le \nu, \\ 0 & \text{elsewhere.} \end{cases}$$

Also define $f_\nu \in \mathcal{M}'_\omega$ by

$$\langle f_\nu,\ \phi\rangle = \sum_{k=0}^{m+1}\langle g_\nu(t),\ (e^{(m+1)\omega(t)} D_t^k\phi(t))\rangle, \quad \phi \in \mathcal{D}_\omega, \tag{14}$$

then $g_\nu \to g$ in $L^2(\mathbb{R})$ as $\nu \to \infty$ therefore, $\langle f_\nu, \phi\rangle \to \langle f, \phi\rangle$ as $\nu \to \infty$.

THEOREM 3.2. *Assume that the wavelet transform $W(b,a)$ of $f \in \mathcal{M}'_\omega$ with respect to $\psi \in \mathcal{M}_\omega$ is defined by (10). Then*

$$\lim_{\substack{N\to\infty \\ R\to\infty}} \left\langle \frac{1}{C_\psi} \int_{-R}^{R} \int_{-N}^{N} W(b,a)\psi_{b,a}(x) \frac{\mathrm{d}b\,\mathrm{d}a}{a^2},\ \phi(x) \right\rangle = \langle f, \phi \rangle, \tag{15}$$

for each $\phi \in \mathcal{D}_\omega$ and $b, a \in \mathbb{R}$, $a \neq 0$ where $\psi_{b,a}(x)$ is given by (7).

PROOF. Using the structure formula for f_ν as given in (14) we have

$$\begin{aligned} (W f_\nu)(b,a) &= \langle f_\nu(t),\ \overline{\psi_{b,a}(t)} \rangle \\ &= \int_{-\infty}^{\infty} g_\nu(t) \sum_{k=0}^{m+1} e^{(m+1)\omega(t)} D_t^k \overline{\psi_{b,a}(t)}\, \mathrm{d}t. \end{aligned} \tag{16}$$

We wish to derive the inversion formula

$$\frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W f_\nu)(b,a)\psi_{b,a}(x) \frac{dbda}{a^2} = f_\nu,$$

interpreting convergence in the weak topology of $\mathcal{D}'$, i.e.

$$J \equiv \left\langle \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (W f_\nu)(b,a)\psi_{b,a}(x) \frac{\mathrm{d}b\,\mathrm{d}a}{a^2},\ \phi(x) \right\rangle = \langle f_\nu, \phi \rangle,$$

for all $\phi \in \mathcal{D}_\omega$. From (16) we have

$$\begin{aligned} J &= \left\langle \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_\nu(t) \sum_{k=0}^{m+1} e^{(m+1)\omega(t)} D_t^k \overline{\psi_{b,a}(t)} \psi_{b,a}(x) \frac{\mathrm{d}t\,\mathrm{d}b\,\mathrm{d}a}{a^2},\ \phi(x) \right\rangle \\ &= \left\langle \frac{1}{C_\psi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} g_\nu(t) \right.\right.\right. \\ &\qquad \left.\left.\left. \sum_{k=0}^{m+1} e^{(m+1)\omega(t)} (-1)^k D_b^k \overline{\psi_{b,a}(t)} \right\} \psi_{b,a}(x)\, \mathrm{d}t \right] \frac{\mathrm{d}b\,\mathrm{d}a}{a^2},\ \phi(x) \right\rangle \end{aligned}$$

as $D_t\psi_{b,a}(t) = -D_b\psi_{b,a}(t)$. Therefore, by integration by parts with respect to b we have

$$\begin{aligned} J &= \left\langle \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_\nu(t) \sum_{k=0}^{m+1} e^{(m+1)\omega(t)} \overline{\psi_{b,a}(t)} D_b^k \psi_{b,a}(x) \frac{\mathrm{d}t\,\mathrm{d}b\,\mathrm{d}a}{a^2},\ \phi(x) \right\rangle \\ &= \left\langle \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_\nu(t) \right. \\ &\qquad \left. \sum_{k=0}^{m+1} e^{(m+1)\omega(t)} \overline{\psi_{b,a}(t)} (-1)^k D_x^k \psi_{b,a}(x) \frac{\mathrm{d}t\,\mathrm{d}b\,\mathrm{d}a}{a^2},\ \phi(x) \right\rangle. \end{aligned}$$

Hence, by distributional differentiation,

(17)

$$J = \left\langle \frac{1}{C_\psi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_\nu(t) \sum_{k=0}^{m+1} e^{(m+1)\omega(t)} \overline{\psi_{b,a}(t)} \psi_{b,a}(x) \frac{\mathrm{d}t\,\mathrm{d}b\,\mathrm{d}a}{a^2},\ D_x^k \phi(x) \right\rangle.$$

The integrand

$$D_x^k \phi(x) \psi_{b,a}(x) \overline{\psi_{b,a}(t)} g_\nu(t) \frac{e^{(m+1)\omega(t)}}{a^2}$$

is absolutely integrable with respect to x and t in the x, t-plane and so Fubini's theorem is applicable with respect to integration by x and t. Therefore (17) yields

$$\begin{aligned}
J &= \frac{1}{C_\psi} \sum_{k=0}^{m+1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \\
&\qquad D_x^k \phi(x) \psi_{b,a}(x) \overline{\psi_{b,a}(t)} g_\nu(t) e^{(m+1)\omega(t)} \frac{\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}b\,\mathrm{d}a}{a^2} \\
&= \frac{1}{C_\psi} \sum_{k=0}^{m+1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \\
&\qquad \left[\overline{\overline{W_\psi\{D_x^k \phi(x)\}}(b,a) \psi_{b,a}(t) \frac{\mathrm{d}b\,\mathrm{d}a}{a^2}} \right] g_\nu(t) e^{(m+1)\omega(t)}\,\mathrm{d}t \\
&= \sum_{k=0}^{m+1} \int_{-\infty}^{\infty} \overline{D_t^k \phi(t)} g_\nu(t) e^{(m+1)\omega(t)}\,\mathrm{d}t \qquad \text{(by inversion formula (9))} \\
&= \sum_{k=0}^{m+1} \langle g_\nu(t),\ (-1)^k e^{(m+1)\omega(t)} D_t^k \phi(t) \rangle \\
&= \left\langle \sum_{k=0}^{m+1} D_t^k (e^{(m+1)\omega(t)} g_\nu(t)),\ \phi(t) \right\rangle \\
&= \langle f_\nu, \phi \rangle \qquad \text{(by structure formula (14))} \\
&\longrightarrow \langle f, \phi \rangle \quad \text{as } \nu \to \infty.
\end{aligned}$$

This completes the proof of the theorem. □

4. Wavelet transform on $\mathcal{S}'_\omega$

In this section we assume that wavelets are of infraexponential decay so that their Fourier transforms are of compact support. To deal with such wavelets we suppose that $\psi \in \mathcal{S}_\omega(\mathbb{R})$, then $\psi_{b,a} \in \mathcal{S}_\omega$ for fixed $a, b \in \mathbb{R}$, $a \neq 0$. Now, we extend the wavelet transform in Fourier space defined by

$$W(b,a) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ib\omega}\hat{f}(\omega)\overline{\hat{\psi}(a\omega)}d\omega. \tag{18}$$

Assume that $\hat{f}(\omega) \in \mathcal{D}'_\omega(\mathbb{R})$ and $\overline{\hat{\psi}(\omega)} \in \mathcal{S}_\omega(\mathbb{R})$ is of compact support, then $\overline{\hat{\psi}(a\omega)}\hat{f}(\omega) \in \mathcal{E}'_\omega(\mathbb{R})$ [2, pp. 121–127]. Now, we define generalized wavelet transform of $f \in \mathcal{Z}'_\omega(\mathbb{R})$ [2, pp. 127] as generalized Fourier transform of $\hat{f}(\cdot)\overline{\hat{\psi}(a\cdot)}$:

$$\begin{aligned} W(b,a) &= \frac{1}{2\pi}\langle \hat{f}(\omega), \overline{\hat{\psi}(a\omega)}e^{ib\omega}\rangle \\ &= \frac{1}{2\pi}\langle \hat{f}(\omega)\overline{\hat{\psi}(a\omega)}, e^{ib\omega}\rangle \\ &= \frac{1}{2\pi}\langle \frac{1}{a}\hat{f}(u/a)\overline{\hat{\psi}(u)}, e^{i\zeta u/a}\rangle \\ &= \frac{1}{2\pi a}\langle g_a(u), e^{i\zeta u/a}\rangle. \end{aligned} \tag{19}$$

where $g_a(u) = \hat{f}(u/a)\overline{\hat{\psi}(u)},\ a \neq 0$.

Assume that $\text{supp}\hat{\psi}(u) = [-\beta, \beta],\ \alpha > 0$. Then $\text{supp} g_a(u) = [-\beta, \beta],\ \beta > 0$.

Theorem 4.1. *If $\overline{\hat{\psi}(u)}\hat{f}(u/a) \in \mathcal{E}'_\omega(\mathbb{R})$, its wavelet transform is a C^∞ function in $\mathbb{R}$ given by*

$$W(b,a) = \frac{1}{2\pi a}\langle g_\alpha(u), e^{ibu/a}\rangle. \tag{20}$$

Moreover, $W(b,a)$ can be extended to the complex space $\mathbb{C}$ as an entire analytic function given by

$$W(\zeta,a) = \frac{1}{2\pi a}\langle g_\alpha(u), e^{i\zeta u/a}\rangle. \tag{21}$$

Proof. In (20), g_a is a distribution with compact support while $e^{ibu/a}$ is a C^∞-function of u. Thus, the right-hand side of (20) is well defined. Also, by [2, Theorem 4.6, p. 124] it follows that the right-hand side of (20) is a C^∞-function of $b \in \mathbb{R}$ and $W(b,a)$ can be extended to the complex plane as an entire analytic function given by (21).

Further proof is very similar to that given in [2, p. 124] in the case of Fourier transform of distributions. □

Remark 4.2. Relation (21) can be used to study Paley–Wiener–Schwartz theorem for wavelet transform of ultradistribution of compact support.

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