Rend. Sem. Mat. Univ. Padova, Vol. 138 (2017), 283–288
DOI 10.4171/RSMUP/138-15

Centralizers of finite subgroups in Hall's universal group

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Abstract – The structure of the centralizers of elements and finite abelian subgroups in Hall's universal group is studied by B. Hartley by using the property of existential closed structure of Hall's universal group in the class of locally finite groups. The structure of the centralizers of arbitrary finite subgroups were an open question for a long time. Here by using basic group theory and the construction of P. Hall we give a complete description of the structure of centralizers of arbitrary finite subgroups in Hall's universal group. Namely we prove the following. Let U be the Hall's universal group and F be a finite subgroup of U. Then the centralizer $C_U(F)$ is isomorphic to an extention of $Z(F)$ by U.

Mathematics Subject Classification (2010). 20E32, 20F50.

Keywords. Infinite symmetric groups, universal groups, centralizers of subgroups.

1. Introduction

A locally finite group U satisfying:

(i) every finite group can be embedded into U,

(ii) any two isomorphic finite subgroups of U are conjugate in U

is called a *universal group.*

Philip Hall proved the existence and uniqueness of universal groups in the countable case in (1959), see [3]. This group is referred to as Hall's universal

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group. Hall constructed his group as a union of a tower of finite symmetric groups;

$$U_1 \leq U_2 \leq \cdots$$

where U_1 is a symmetric group of order greater than 2 and if U_n is given, then U_{n+1} is the symmetric group on the set U_n and the group U_n is embedded into U_{n+1} by right regular representation.

Hence

$$U = \varinjlim U_i$$

Hall's universal group U satisfies the following properties some of which are quite unusual; for the proofs, see [3] and [7, Chapter 6].

Proposition 1.1. *Let U be Hall's universal group.*

(a) *Let C_m denote the set of all elements of order $m > 1$ of U. Then C_m is a single class of conjugate elements and $U = C_m C_m$. In particular U is simple.*

The automorphism α of the group G is called locally inner if for every finite set F of elements of G, there is an element $g = g_F$ of G such that $f^\alpha = f^g$ for every element $f \in F$.

(b) *If G is any locally finite universal group, then every automorphism of G is locally inner.*

(c) *The cardinality of the automorphism group of Hall's universal group U is* $|\operatorname{Aut}(U)| = 2^{\aleph_0}$.

(d) *Every countably infinite locally finite group can be embedded into U.*

Hall's universal group can be written as a direct limit of alternating groups. Then the question of whether Hall's universal group can be written as a direct limit of other families of finite simple groups is answered by F. Leinen in [8]. He proved that Hall's universal group can be constructed as a direct limit of simple linear groups $\{\mathrm{PSL}(n_i, \mathbf{F}_q)\}$, $\{\mathrm{PSU}(n_i, \mathbf{F}_q)\}$, $\{\mathrm{PSp}(2n_i, \mathbf{F}_q)\}$, $\{\mathrm{P}\Omega^+(2n_i, \mathbf{F}_q)\}$, $\{\mathrm{P}\Omega(2n_i + 1, \mathbf{F}_q)\}$, $\{\mathrm{P}\Omega^-(2n_i + 2, \mathbf{F}_q)\}$.

2. Main result

About the centralizers of elements (subgroups) in Hall's universal group, the following results were announced by Hartley in [4, Proposition 1.8].

PROPOSITION 2.1. (a) *If F is a finite subgroup of U with trivial center, then $C_U(F)$ is isomorphic to U.*

(b) *If A is a finite abelian subgroup of U, then $C_U(A)/A$ is an infinite simple group.*

For the structure of the centralizers of subgroups in permutation groups one can see [1, Chapter 4] and [9, Chapter 6]. For the centralizers of subgroups in algebraically closed groups see; [5] and [6, Chapter 2].

Is it possible to find the structure of centralizers of finite subgroups in U by using basic group theory?

The answer is positive, but first, we recall some of the facts on centralizers of subgroups in finite symmetric groups. Since U is a direct limit of finite symmetric groups, we may use the centralizers of subgroups in symmetric groups to find the centralizers of finite subgroups in Hall's universal group. In the following some of the results are well known but for the reader's convenience we give the proof here.

Both the right and left regular representations of G are subgroups of the group $\mathrm{Sym}(G)$ and they commute with each other elementwise in $\mathrm{Sym}(G)$.

Indeed as $(g_1^{-1}x)g_2 = g_1^{-1}(xg_2)$, we have $xl(g_1)r(g_2) = xr(g_2)l(g_1)$ for any $x \in G$. So

$$l(g_1)r(g_2) = r(g_2)l(g_1).$$

It follows that $l(G) \leq C_{\mathrm{Sym}(G)}((r(G))$ and $r(G) \leq C_{\mathrm{Sym}(G)}((l(G))$.

LEMMA 2.2 ([2, p. 86]). *The centralizer of the right regular representation $r(G)$ in* $\mathrm{Sym}(G)$ *is the left regular representation $l(G)$. Similarly $C_{\mathrm{Sym}(G)}(l(G)) = r(G)$.*

PROOF. Let π be a permutation in $\mathrm{Sym}(G)$ belonging to the centralizer of $r(G)$. Let $(1)\pi = g^{-1}$. We show that $\pi = l(g)$. Let $\pi l(g)^{-1} = \pi^*$. Then π^* belongs to the centralizer of $r(G)$ and fixes the identity i.e. $(1)\pi^* = 1$. Here $(1)\pi^* r(x) = x$. By $\pi^* r(x) = r(x)\pi^*$ we have $(1)r(x)\pi^* = (x)\pi^* = x$, and so $(x)\pi^* = x$ for any $x \in G$, whence π^* is the identity permutation and so $\pi l(g)^{-1} = \pi^* = id$, we obtain $\pi = l(g)$. Hence the centralizer of $r(G)$ is $l(G)$. Moreover there exists a permutation t of $\mathrm{Sym}(G)$ of order 2 namely the permutation taking every element to its inverse in G satisfying $t^{-1}r(G)t = l(G)$. As for any $x \in G$ we have

$$x.t^{-1}r(g)t = x^{-1}r(g)t = (x^{-1}g)t = (x^{-1}g)^{-1} = g^{-1}x = xl(g) \quad \text{for all } x \in G.$$

So $t^{-1}r(g)t = l(g)$. Observe that t has order two in $\mathrm{Sym}(G)$ and so

$$C_{\mathrm{Sym}(G)}(r(G))^t = C_{\mathrm{Sym}(G)}(l(G)) = r(G). \qquad \square$$

LEMMA 2.3. $l(G) \cap r(G) = l(Z(G)) = r(Z(G)) \cong Z(G)$.

PROOF. Let $x \in l(G) \cap r(G)$. Then $x = l(h) = r(s)$ for some $h, s \in G$. Then $1.l(h) = 1.r(s)$ implies that $h^{-1} = s$. Then for any $g \in G$, we have $g.l(h) = g.r(h^{-1})$ i.e. $h^{-1}g = gh^{-1}$ for all $g \in G$. It follows that $h \in Z(G)$ and we have

$$l(G) \cap r(G) \subseteq r(Z(G)) = l(Z(G))$$

Conversely for any element $z \in Z(G)$ we have $l(z) = r(z^{-1}) \in l(G) \cap r(G)$. Hence

$$l(Z(G)) = r(Z(G)) \leq l(G) \cap r(G).$$

So

$$l(Z(G)) = r(Z(G)) = l(G) \cap r(G) \cong Z(G). \qquad \square$$

We now prove the main theorem.

THEOREM 2.4. *Let U be the Hall's universal group and F be a finite subgroup of U. Then the centralizer $C_U(F)$ is isomorphic to an extention of $Z(F)$ by U.*

PROOF. Let F be a finite subgroup of U. As $U = \bigcup_{i=1}^{\infty} U_i$ is a direct limit of finite symmetric groups U_i where U_i is embedded by right regular representation into $U_{i+1} = \mathrm{Sym}(U_i)$, we may assume that $F \leq U_i$ for some $i \in \mathbb{N}$. As F is also contained in U_{i+1} the orbits of F under right regular representation in U_{i+1} are the left cosets of F in U_i. The action of F on each of its orbits are equivalent to the action of F on itself by right multiplication. So we have $m_{i+1} = \frac{|U_i|}{|F|}$ permutationally equivalent orbits. By Lemma 2.2 the centralizer of the right regular representation $r(F)$ in $\mathrm{Sym}(F)$ is the left regular representation $l(F)$. The structure of centralizers of intransitive subgroups in symmetric groups is well known; see [1, Page 109]. Hence

$$C_{\mathrm{Sym}(U_{i+1})}(r(F)) \cong l(F)^{x_1^{-1}} \times \cdots \times l(F))^{x_{m_{i+1}}^{-1}} \rtimes \mathrm{Sym}(m_{i+1})$$

where x_i's are left coset representatives of F in U_i and the elements of $\mathrm{Sym}(m_{i+1})$ permutes the permutationally isomorphic pairs $l(F)^{x_i^{-1}}$ for $i = 1, 2, \ldots, m_{i+1}$. In fact $C_{\mathrm{Sym}(U_{i+1})}(r(F)) \cong l(F) \wr \mathrm{Sym}(m_{i+1})$ where the wreath product is the permutational wreath product.

As the groups U_i are embedded in U_{i+1} by right regular representation, the centralizers $C_{U_i}(F)$ are also embedded into $C_{U_{i+1}}(F)$ by right regular representation and hence the group $C_U(F) = \bigcup C_{U_i}(F)$ will be the direct limits of the centralizers $C_{U_i}(F)$. This is one of the differences between the diagonal embedding and the regular embedding, since during the diagonal embedding usually the group theoretical properties are preserved but by regular embedding some of these properties are not preserved like cycle structure of a permutation.

But the subgroup $l(F)^{x_1^{-1}}$ and also the copies of it, in the centralizer $C_{U_{i+1}}(F)$ will not contribute new elements when we take their right regular representation in U_{i+1} in the next step, as the centralizer of F in the next step is known and there are no elements coming from the right regular representation. By Lemma 2.3 $l(G) \cap r(G) = l(Z(G)) = r(Z(G))$, the only elements goes from the base group of $C_{U_i}(F)$ to $C_{U_{i+1}}(F)$ are the $r(Z(F))$. The group $Z(F)$ which is contained in each U_i will consists of diagonal elements in the wreath product, but when we take the direct limit of these, as the subgroup $r(F)$ the center $Z(r(F))$ will be diagonally embedded.

In the semidirect product part, the embedding of symmetric groups $\mathrm{Sym}(m_i)$ into $\mathrm{Sym}(m_{i+1})$ are embedded by right regular representations and the sequence of integers m_i is an infinite increasing sequence. Moreover when we take right regular representation of F on U_i and then on U_{i+1} the only thing we have, more cosets than before, modulo identification. But the cosets of F in U_i can be preserved, only new cosets comes. The permutations in U_i which permutes the cosets will permute them as elements of U_{i+1} the only thing that, they are embedded into U_{i+1} by regular embeddings. Since countable locally finite universal group is unique up to isomorphism [7, Theorem 6.4] from the direct limit of the symmetric groups by right regular representation, we obtain an isomorphic copy of U. So $C_U(F)$ is isomorphic to an extention of $Z(F)$ by U.

In particular if $Z(F) = \{1\}$, then $C_U(F) \cong U$. □

Then the results of Hartley in Proposition 2.1 will be a corollary of the above theorem. Moreover we have the following corollary.

Corollary 2.5. *The centralizer $C_U(F)$ of every finite subgroup F of U has U as an epimorphic image.*

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Manoscritto pervenuto in redazione il 27 settembre 2016.