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Multi-Catalan tableaux and the two-species TASEP

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Abstract. The goal of this paper is to provide a combinatorial expression for the steady state probabilities of the two-species ASEP. In this model, there are two species of particles, one *heavy* and one *light*, on a one-dimensional finite lattice with open boundaries. Both particles can swap places with adjacent holes to the right and left at rates 1 and q. Moreover, when the heavy and *light* particles are adjacent to each other, they can swap places as if the *light* particle were a hole. Additionally, the *heavy* particles can hop in and out at the boundary of the lattice. Our main result is a combinatorial interpretation for the stationary distribution at $q = 0$ in terms of certain multi-Catalan tableaux. We provide an explicit determinantal formula for the steady state probabilities and the partition function, as well as some general enumerative results for this case. We also describe a Markov process on these tableaux that projects to the two-species ASEP, and thus directly explains the connection between the two. Finally, we give a conjecture that gives a formula for the stationary distribution to the $q = 1$ case, using certain two-species alternative tableaux.

Mathematics Subject Classification (2010). 05E99.

Keywords. TASEP, multispecies, tableaux.

1. Introduction

The asymmetric simple exclusion process (ASEP) is a well-studied model that describes the dynamics of particles hopping on a finite one-dimensional lattice on n sites with open boundaries, with the rule that there is at most one particle in a site, and at most one particle hops at a time. Figure 1 shows the parameters of this process, with α, β, and q denoting the rates of the hopping particles. Processes of this flavor have been studied in many contexts, in particular for their connections to some very nice combinatorics. For instance, see [4] and the references therein.

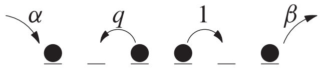

Figure 1. The parameters of the ASEP.

In this work we consider a two-species ASEP, studied by Uchiyama [10] and others (see [7, 1] and the references therein). The two-species ASEP has two species of particles, *heavy* and *light*. Many variations of multi-species exclusion processes have been studied for some of their interesting combinatorics and thermodynamic properties. In our process, both types of particles can swap places with an adjacent hole to the right and left with rates 1 and q, respectively. Furthermore, *heavy* particles can enter from the left and exit on the right of the lattice with respective rates α and β, and they can treat the *light* particles as holes and swap places with them to the right and left, also at rates 1 and q. Since the *light* particles cannot enter or exit, their number stays fixed. In particular, when we fix the number of *light* particles to be zero, we recover the original ASEP.

We denote the *heavy* and *light* particles by $\bullet$ and ${\color{gray}\bullet}$ respectively, and we denote a *hole*, or absence of a particle, by $\circ$. We let r denote the number of ${\color{gray}\bullet}$'s. Then the two-species ASEP of size (n,r) is a Markov chain, whose states are words of length n in the letters $\{\bullet, \circ, {\color{gray}\bullet}\}^n$ with exactly r ${\color{gray}\bullet}$'s.

Figure 2 shows the parameters of the two-species process. More precisely, the transitions in the Markov chain are the following, with X and Y arbitrary words in $\{\bullet, \circ, {\color{gray}\bullet}\}^*$.

$$X{\color{gray}\bullet}\circ Y \underset{q}{\overset{1}{\rightleftharpoons}} X\circ{\color{gray}\bullet}Y \qquad X\bullet\circ Y \underset{q}{\overset{1}{\rightleftharpoons}} X\circ\bullet Y \qquad X\bullet{\color{gray}\bullet}Y \underset{q}{\overset{1}{\rightleftharpoons}} X{\color{gray}\bullet}\bullet Y$$

$$\circ X \xrightarrow{\alpha} \bullet X \qquad X\bullet \xrightarrow{\beta} X\circ$$

where by $X \xrightarrow{u} Y$ we mean that the transition from X to Y has probability $\frac{u}{n+1}$, n being the length of X (and also Y).

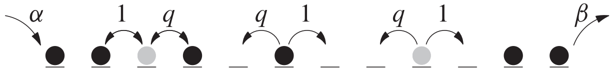

Figure 2. The parameters of the two-species ASEP.

A Matrix Ansatz solution and corresponding matrices in the work of Uchiyama in [10, Theorem 2.9] give exact expressions for the steady state distribution of the two-species ASEP. We denote the steady state probability of a state X by $\mathrm{Prob}(X)$.

Theorem 1.1 (Uchiyama, 2008). *Let $X = X_1 \dots X_n$ with $X_i \in \{\bullet, \textcolor{lightgray}{\bullet}, \circ\}$ represent a state of the two-species ASEP of length n with r **light** particles. Suppose there are matrices D,E, and A and vectors $\langle w|$ and $|v\rangle$ which satisfy the following conditions*:

$$
\begin{aligned}
DE &= D + E + qED,\\
DA &= A + qAD,\\
AE &= A + qEA,\\
\langle w|E &= \frac{1}{\alpha}\langle w|,\\
D|v\rangle &= \frac{1}{\beta}|v\rangle.
\end{aligned}
$$

Then

$$
\mathrm{Prob}(X) = \frac{1}{Z_{n,r}}\langle w|\prod_{i=1}^{n} D\,\mathbb{1}_{(X_i=\bullet)} + A\,\mathbb{1}_{(X_i=\textcolor{lightgray}{\bullet})} + E\,\mathbb{1}_{(X_i=\circ)}\,|v\rangle
$$

where $Z_{n,r}$ is the coefficient of y^r in $\frac{\langle w|(D+yA+E)^n|v\rangle}{\langle w|A^r|v\rangle}$.

This result generalizes a previous Matrix Ansatz solution for the regular ASEP of Derrida et. al. in [6].

In his work, Uchiyama provides matrices (that are neither positive or rational) that satisfy the conditions of Theorem 1.1. From these, the matrix product yields steady state probabilities in the form of polynomials in α, β, and q with positive integer coefficients. Therefore one would hope for a combinatorial interpretation of these probabilities, with results akin to those of Corteel and Williams [5] for the original ASEP. Such results could yield explicit general formulae for both the desired probabilities and the partition function.

The goal of this paper is to provide some combinatorial solutions to the two-species ASEP for some special cases. In Section 2 of this paper, we describe certain tableaux which we call *multi-Catalan tableaux* that give an interpretation for the steady state distributions of the two-species TASEP, which is the ASEP at $q = 0$. In Section 3 we provide some enumerative results for the multi-Catalan tableaux. In Section 4 we describe a Markov process on the multi-Catalan tableaux that projects to the two-species TASEP, and which gives another proof of our main result in Section 2. Finally, in Section 5 we define some more general multi-Catalan tableaux that we believe give an interpretation for the steady state distributions of the two-species ASEP at $q = 1$. Note that our forthcoming paper with X. Viennot [9] will give another combinatorial solution to the two-species ASEP for general q.

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2. Multi-Catalan tableaux

A number of combinatorial objects have been introduced to study the combinatorics of the original ASEP, including permutation tableaux [4], tree-like tableaux [2], staircase tableaux, and the *alternative tableaux* of Viennot [11]. The cardinality of the set of alternative tableaux corresponding to the usual TASEP (i.e. the ASEP at $q = 0$) is the Catalan number (these are sometimes called the *Catalan tableaux*). In this Section, we introduce the multi-Catalan tableaux which generalize the Catalan tableaux, in order to give a combinatorial interpretation for the two-species TASEP.

Definition 2.1. A *multi-Catalan tableau* of *size* n is a filling of a Young diagram of shape $(n, n-1, \dots, 1)$ with the symbols α, β, and x as follows:

(1) every box on the anti-diagonal must contain an α, β, or x;
(2) a box that sees an α to its right and a β below must contain an α or β;
(3) a box that sees an α to its right and an x below must contain a β;
(4) a box that sees an x to its right and a β below must contain an α;
(5) every other box must be empty.

In the definition above, when we refer to the symbol that a box "sees" to its right or below, we mean the first symbol encountered in the same row or column, respectively. For example, in the first tableau of Figure 3, x is the first symbol that the β in the top row sees below it. Finally, note that Rule 5 implies that all boxes in the same row and left of a β must be empty, and also that all boxes in the same column and above an α must be empty.

Definition 2.2. The *weight* $\mathrm{wt}(T)$ of a multi-Catalan tableau T is the product of all the α's and β's it contains.

Definition 2.3. The *type* $\mathrm{type}(T)$ of the tableau T is the word in $\{\bullet, \circ, \textcolor{lightgray}{\bullet}\}^*$ that is read from the anti-diagonal from top to bottom, by assigning a $\bullet$ to α, a $\circ$ to β, and a $\textcolor{lightgray}{\bullet}$ to x.

Definition 2.4. The *weight* of a word X in $\{●, ○, ●\}^*$ is

$$\text{weight}(X) = \sum_T \text{wt}(T),$$

where the sum is over all multi-Catalan tableaux T such that $\text{type}(T) = X$.

Our main result is the following.

Theorem 2.1. *Consider the two-species ASEP of size n. Let X be a state described by a word in $\{●, ○, ●\}^n$ with r ●'s. Let $Z^0_{n,r} = \sum_{X'} \text{weight}(X')$ where the sum is over all words X' of length n with r ●'s. Then the steady state probability of state X is*

$$\text{Prob}(X) = \frac{\text{weight}(X)}{Z^0_{n,r}}.$$

We show as an example all possible multi-Catalan tableaux and their weights of type ● ○ ○ ● ○ in Figure 3. Thus Theorem 2.1 implies that

$$\text{Prob}(● ○ ○ ● ○) = \frac{1}{Z^0_{5,1}}(\alpha^4\beta^4 + \alpha^3\beta^4 + \alpha^2\beta^4).$$

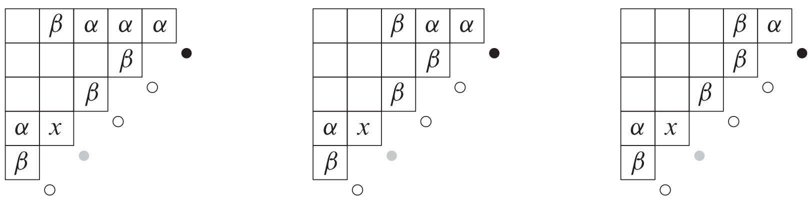

Figure 3. The multi-Catalan tableaux of type ● ○ ○ ● ○ with weights $\alpha^4\beta^4$, $\alpha^3\beta^4$, and $\alpha^2\beta^4$ respectively.

In keeping with common ASEP notation, where the ●, ●, and ○ are called the *type* 2, *type* 1, and *type* 0 particles respectively, we introduce the following notation for the rows, columns, and boxes of the multi-Catalan tableaux.

Definition 2.5. A 2-*row* is a row whose right-most box contains an α and a 1-*row* is one whose right-most box contains an x. A 0-*column* is a column whose bottom-most box contains a β and a 1-*column* is one whose bottom-most box contains an x. Then a $2-0$ *box* is one that lies in a 2-row and a 0-column (and correspondingly for 2-1, 1-0, and 1-1 boxes).

Note that we can ignore the rows with right-most box containing a β or columns with bottom-most box containing an α since they are automatically required to be empty according to Definition 2.1.

To connect back to the two-species TASEP, let the word X in $\{●, ○, ●\}^n$ describe a state. Then we fill a Young diagram of shape $(n, n-1, \ldots, 1)$ as follows: from top to bottom, we fill the anti-diagonal with symbols α, β, and x by reading the word X from left to right, and placing an α for a ●, a β for a ○, and an x for a ●. Then any valid filling of the rest of the diagram according to the rules (2)–(5) of Definition 2.1 will result in a multi-Catalan tableau of type X.

2.1. Condensed multi-Catalan tableaux. We provide a condensed version of the characterization of the multi-Catalan tableaux, which offers a more natural proof of our results. We introduce the following definitions.

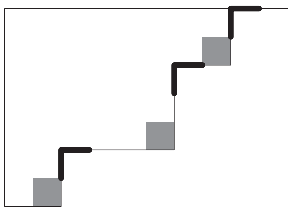

Figure 4. The grey squares mark the corners and the darkened edges mark the inner corners.

Definition 2.6. An *inner corner* of a Young diagram is a consecutive pair of a west edge and a south edge on the boundary of the tableau. A *corner* is simply the box that is both the right-most box of some row and the bottom-most box of some column. Figure 4 shows some examples.

Definition 2.7. A *condensed* multi-Catalan tableau T of *size* (n, k, r) is a Young diagram $Y = Y(T)$ with at least r inner corners, that is justified to the northwest and contained in a rectangle of size $k + r \times n - k$. Y is identified with the lattice path $L = L(T)$ that takes the steps south and west and follows the southeast border of Y. In addition, we have the following:

- each edge of L is labelled with a 0, 1, or 2 such that exactly r inner corners have both edges labeled with 1's, and the remaining west edges have the label 0, and the remaining south edges have the label 2;
- a 0-column is a column with a 0 labeling its bottom-most edge (a 1-column is defined correspondingly);

- a 2-row is a row with a 2 labeling its right-most edge (a 1-row is defined correspondingly);
- a 2-0 box is a box in a 2-row and a 0-column (the 2-1, 1-0, 1-1 boxes are defined correspondingly).

Finally, we fill T with α's and β's according to the following rules.

i. A box in the same row and left of a β must be empty.

ii. A box in the same column and above of an α must be empty.

iii. A 2-0 box that is not forced to be empty must contain an α or a β.

iv. A 2-1 box that is not forced to be empty must contain a β.

v. A 1-0 box that is not forced to be empty must contain an α.

We identify the Young diagram Y with a partition $\lambda = \lambda(T)$, which we also call the *shape* of Y and of T. Specifically, $\lambda = (\lambda_1, \ldots, \lambda_{k+r})$ where λ_i is the number of boxes of Y in row i of the $k + r \times n - k$ rectangle.

Definition 2.8. The *labeling word* of the multi-Catalan tableau is the word in $\{2, 1, 0\}$ that is read from the labels on the lattice path $L(T)$ from top to bottom, but with 1 counted only once for each 1-labeled inner corner of T. The *type* of T is the word in {●, ●, ○} that is associated to the labeling word, by assigning a ● to a 2, a ● to a 1, and a ○ to a 0.

Definition 2.9. The *weight* of the condensed tableau is the weight of the symbols inside it times the weight of the lattice path $L(T)$, which is obtained by giving each 2-edge weight α and each E edge weight β. In particular, for a tableau of size (n, k, r), the weight of $L(T)$ is $\alpha^k \beta^{n-k-r}$.

In Figure 5, we demonstrate by example the conversion from a staircase multi-Catalan tableau to a condensed multi-Catalan tableau. Specifically, we remove the anti-diagonal from the staircase along with the 0-rows and 2-columns, and then glue together all the 2-0, 2-1, 1-0, and 1-1 boxes. Then we label the boundary edges of the tableau. We label a vertical edge with a 2 if it belongs to a 2-row and with a 1 if it belongs to a 1-row. Similarly, we label a horizontal edge with a 0 if it belongs to a 0-column and with a 1 if it belongs to a 1-column. It is easy to check that the types and weights (according to Definitions 2.2 and 2.9) of the two tableaux are equal.

Another way to obtain a condensed tableau from a word X in $\{\bullet, \circ, \bullet\}^*$ is to draw a lattice path $L = L(X)$ with steps south and west, by reading X from left to right and drawing a 2-labeled south edge for a $\bullet$, a 0-labeled west edge for a $\circ$, and a 1-labeled pair of a west edge and a south edge for a $\bullet$. L is then identified with the Young diagram Y whose southeast border it coincides with. (More precisely, Y has shape $\lambda = (\lambda_1, \lambda_2, \ldots)$, where λ_i is the number of $\circ$'s and $\bullet$'s in X following the ith instance of either $\bullet$ or $\bullet$.) Any filling of Y according to rules (i)-(v) of Definition 2.7 results in a tableau of type X.

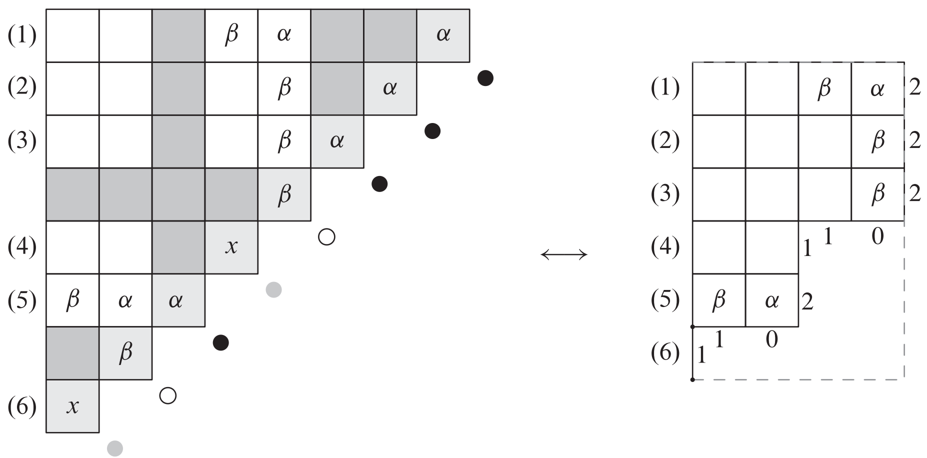

Figure 5. The staircase multi-Catalan tableau and its corresponding condensed multi-Catalan tableau with size $(10, 4, 2)$ and shape $\lambda = (4, 4, 4, 2, 2, 0)$ have type $\bullet\bullet\bullet\circ\bullet\bullet\circ\bullet$ and weight $\alpha^6\beta^6$. The condensed tableau is formed by gluing together the white boxes of the staircase tableau.

Note that if T_s denotes the staircase version of a multi-Catalan tableau of size (n, r, k) and T_c is the corresponding condensed version, and $\mathrm{wt}(T_s)$ is the product of the symbols α, β, and q in the filling of T_s, then $\mathrm{wt}(T_c)$ is $\alpha^k \beta^{n-r-k}$ times the product of the symbols α, β, and q in the filling of T_c. By construction, it is easy to see that $\mathrm{wt}(T_c) = \mathrm{wt}(T_s)$. Since the staircase version of the multi-Catalan tableaux is in simple bijection with the condensed version, we will call them both *multi-Catalan tableaux,* and refer to them interchangeably.

The usual proof of Theorem 2.1 uses the by now standard technique of showing that the weight generating functions of the multi-Catalan tableaux satisfy the same recurrences as the steady state probabilities of the two-species TASEP as given by the Matrix Ansatz of Theorem 1.1. See [5] for an example of such a proof for the original ASEP. Instead of the Matrix Ansatz style proof, we provide a more illuminating proof of Theorem 2.1 in Section 4 by constructing a Markov chain on the multi-Catalan tableaux that projects to the two-species TASEP.

3. Enumeration of multi-Catalan tableaux

Building on some enumerative results in [8] for regular Catalan tableaux that correspond to the usual TASEP, we can deduce some properties of the multi-Catalan tableaux. We include the proofs for these results to give some intuition for the structure of the tableaux.

Theorem 3.1. *The number of multi-Catalan tableaux corresponding to a two-species TASEP of size n and with r* ●*'s is*

$$Z^0_{n,r}(\alpha = \beta = 1) = \frac{2(r+1)}{n+r+2}\binom{2n+1}{n-r}.$$

Proof. We make two observations about the structure of the tableaux. First, any box that lies in a 1-row or column is either empty or automatically determined by the rules (iv)-(v) from Definition 2.7. Second, any box that lies *left* of a 1-column or *above* a 1-row must be empty. In particular, note that any box that lies in a 1-column is either empty if there's already a β to the right in the same row, or is forced to contain a β otherwise. In both of these cases, any box to the left of that 1-column must be empty. Similarly, any box that lies in a 1-row is either empty if there's already an α below in the same column, or must contain an α otherwise. In both of these cases, any box above that 1-row must be empty. Figure 6 shows an example of this structure.

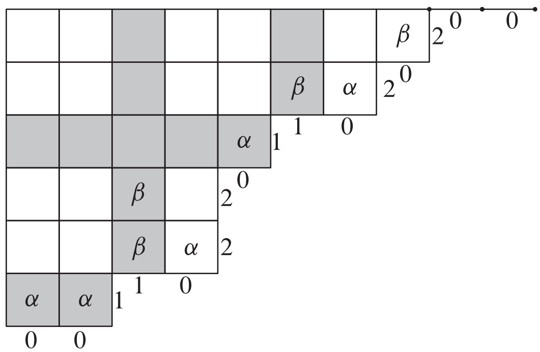

Figure 6. The grey boxes indicate the boxes that belong to a 1-row or 1-column. Observe that any box above a 1-row or left of a 1-column is forced to be empty.

Consequently, the filling of the multi-Catalan tableau can be recreated from the fillings of just the 2-0 boxes that do not lie north or west of any 1-rows or columns. Thus to enumerate these fillings, we can remove all the boxes that lie in the r 1-rows and 1-columns along with all the boxes respectively north and west of these rows and columns. We are left with a disjointed set of $r+1$ smaller tableaux, each of which is a multi-Catalan tableau whose type has zero ●'s. The sum of the sizes of these $r+1$ tableaux is $n-r$.

Multi-Catalan tableaux whose type has zero ●'s are the same as the Catalan tableaux from [8], which are a well-known specialization of the aforementioned alternative tableaux. In particular, the number of such tableaux of size n is the Catalan number $C_{n+1} = \frac{1}{n+1}\binom{2n+2}{n+1}$. We obtain the equation in the theorem as the appropriately chosen coefficient of the convolution of $r+1$ Catalan numbers. □

Theorem 3.2. *Let $r+k+\ell = n$. The number of multi-Catalan tableaux corresponding to a two-species TASEP of size n and with r ●'s and k ●'s is*

$$\frac{r+1}{n+1}\binom{n+1}{k}\binom{n+1}{\ell}.$$

Proof. We refine the proof of Theorem 3.1 by keeping track of the number of 2-rows. More precisely, after removing the 1-rows and 1-columns and the boxes that lie respectively above and west of the 1-rows and 1-columns, we are left with a disjointed list of $r+1$ smaller tableaux, the sum of whose sizes is $n-r$. We let these smaller tableaux (starting from top to bottom) have sizes $n_1,\dots,n_{r+1}$ with $n_1+\cdots+n_{r+1} = n-r$. Furthermore, if we wish to have a total of k 2-rows, we let the smaller tableaux have, respectively, $k_1,\dots,k_{r+1}$ 2-rows with $k_1+\cdots+k_{r+1} = k$.

The number of Catalan tableaux of size n whose type has k ●'s is $\mathcal{N}_{n+1,k+1}$ (such tableaux are in bijection with pairs of nested lattice paths contained in a $k \times n-k$ box). Thus we have that the number of multi-catalan tableaux of size (n,r,k) is

$$\sum_{\substack{k_1\le n_1,\dots,k_{r+1}\le n_{r+1},\\ n_1+\cdots+n_{r+1}=n-r,\\ k_1+\cdots+k_{r+1}=k}} \prod_{i=1}^{r+1} \mathcal{N}_{n_i+1,k_i+1}$$

where $\mathcal{N}_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ is the n,k-Narayana number. The theorem follows. □

The formulae above can also be obtained through a different combinatorial interpretation of the two-species TASEP in [7, Section 4].[1]

[1] In their paper, Duchi and Schaeffer study a different version of a two-species TASEP with four parameters α,β,γ, and ϵ, which does not reduce to the version studied in our paper. However, it turns out that the specialization $\alpha=\beta=\gamma=\epsilon=1$ in their model is equal to the

Moreover, using the structure described above, one can derive the partition function $Z^0_{n,r}$ with some manipulations of sums from the following formula of Derrida [6]:

$$Z^0_{n,0} = (\alpha\beta)^n \sum_{p=1}^{n-r} \frac{p}{2n-p} \binom{2n-p}{n} \frac{\alpha^{-p-1} - \beta^{-p-1}}{\alpha^{-1} - \beta^{-1}}. \tag{3.1}$$

The following formula is also given in [1, Appendix C].

Theorem 3.3. *The weight generating function for the multi-Catalan tableaux of size n and whose type has r $\textcolor{gray}{\bullet}$'s is*

$$Z^0_{n,r} = (\alpha\beta)^{n-r} \sum_{p=1}^{n-r} \frac{2r+p}{2n-p} \binom{2n-p}{n+r} \frac{\alpha^{-p-1} - \beta^{-p-1}}{\alpha^{-1} - \beta^{-1}}.$$

Proof. Direct calculations of $Z^0_{n,r}$ are not particularly illuminating. Instead we prove the formula by induction on n and r. Equation (3.1) gives us for free the case $r = 0$. Now define $Z^{\circ}_{n,r} = \sum_X \mathrm{weight}(\circ X)$ and $Z^{\bullet}_{n,r} = \sum_X \mathrm{weight}(\bullet X)$, where X ranges over all words in $\{\bullet, \circ, \textcolor{gray}{\bullet}\}^{n-1}$ with r $\textcolor{gray}{\bullet}$'s. Similarly, define $Z^{\textcolor{gray}{\bullet}}_{n,r} = \sum_Y \mathrm{weight}(\textcolor{gray}{\bullet} Y)$ where Y ranges over all words in $\{\bullet, \circ, \textcolor{gray}{\bullet}\}^{n-1}$ with $r-1$ $\textcolor{gray}{\bullet}$'s. Then

$$Z^0_{n,r} = Z^{\circ}_{n-1,r} + Z^{\bullet}_{n-1,r} + Z^{\textcolor{gray}{\bullet}}_{n-1,r-1}. \tag{3.2}$$

We obtain the following recursions.

- The tableaux of type $\circ X$ can be built by adding a single boundary 0-edge of total weight β to each tableau of type X, so

$$Z^{\circ}_{n,r}(\alpha, \beta) = \beta Z^0_{n-1,r}(\alpha, \beta).$$

- The tableaux of type $\bullet X$ can be built by adding a 2-row to the top of each tableau of type X. If a tableau of type X has j α-free columns, then there are the following options for that 2-row:
  - there is an α in every free box, or
  - there is a β in one of the free boxes and an α in every free box to the right of the β.

two-species TASEP we describe in our paper when $\alpha = \beta = 1$. Thus in their paper, one can also find the formulae given in Theorems 3.1 and 3.2.

The coefficient of α^{-j} in $\frac{Z^0_{n-1,r}}{(\alpha\beta)^{n-r-1}}$ is the sum of the weights of the tableaux of type X with exactly j α-free rows. Also, note that if $r > 0$, then that 2-row will necessarily have a β from one of the 2-1 boxes in that row. Thus

$$Z^{\bullet}_{n,r}(\alpha,\beta) = (\alpha\beta)^{n-r}\sum_{j=0}^{n}\Big(\sum_{\ell=0}^{j}\alpha^{-\ell}\Big)[\alpha^{-j}]\Big(\frac{Z^0_{n-1,r}(\alpha,\beta)}{(\alpha\beta)^{n-r-1}}\Big).$$

- The tableaux of type ${\color{gray}\bullet}Y$ can be built by adding a 1-row to the top of a tableau of type Y, where Y has $r-1$ ${\color{gray}\bullet}$'s. Every free DA box in that new 1-row must contain an α. Thus

$$Z^{\color{gray}\bullet}_{n,r}(\alpha,\beta) = \alpha^{n-r}Z^0_{n-1,r-1}(1,\beta).$$

Now, we suppose that the formula in Proposition 3.3 holds for all $Z^0_{n',r'}$ with $n' < n$, and all r'. We show that it also holds for $Z^0_{n,r}$. In particular, the formula holds for $Z^{\circ}_{n-1,r}$, $Z^{\bullet}_{n-1,r}$, and $Z^{\color{gray}\bullet}_{n-1,r-1}$. Thus it is sufficient to check that

$$[\alpha^{-s}\beta^{-t}]\frac{1}{(\alpha\beta)^{n-r}}(Z^{\circ}_{n-1,r} + Z^{\bullet}_{n-1,r} + Z^{\color{gray}\bullet}_{n-1,r-1})$$

equals the desired

$$[\alpha^{-s}\beta^{-t}]Z^0_{n,r} = \frac{2r+s+t}{2n-s-t}\binom{2n-s-t}{n+r},$$

where by $[x^k]f(x)$ we denote the coefficient of x^k in $f(x)$.

Indeed, a careful computation confirms that the coefficients of $\alpha^{-s}\beta^{-t}$ of both the expressions above are equal, and so the formula in Proposition 3.3 does hold for $Z^0_{n,r}$. □

For the next theorem, we make some more precise definitions to describe the structure of the multi-Catalan tableaux.

Definition 3.1. We represent a word X in $\{\bullet,\circ,{\color{gray}\bullet}\}^n$ with exactly r ${\color{gray}\bullet}$'s by a list of $r+1$ words in $\{\bullet,\circ\}^*$, where each word of the list is the longest possible continuous sub-word of X that does not contain a ${\color{gray}\bullet}$. We call this *list of* $\bullet$–$\circ$ *sub-words* $(X_1,\dots,X_{r+1})$. We then represent that list by a list of partitions $\Lambda = (\Lambda_1,\dots,\Lambda_{r+1})$, where the partition $\Lambda_i = \lambda(X_i)$ is the shape obtained from applying the definition of the partition λ to the ith $\bullet$–$\circ$ word.

As an example for the above, the tableau in Figure 6 has type ○ ○ ● ○ ● ○ ◍ ○ ● ● ○ ◍ ○ ○, which can be rewritten as a list of three ●–○ words (○○●○●○, ○●●○, ○○). Then the list of partitions is $\Lambda = ((2,1),(1,1),(\emptyset))$.

For our final result in this section, we define the matrix $A_\lambda^{\alpha,\beta} = (A_{ij})_{1\le i,j\le k}$, where λ is some partition $(\lambda_1,\dots,\lambda_k)$, and

$$\begin{aligned} A_{ij} &= \beta^{j-i}\alpha^{\lambda_i-\lambda_{j+1}}\left(\binom{\lambda_{j+1}}{j-i} + \beta\binom{\lambda_{j+1}}{j-i+1}\right) \\ &\quad + \beta^{j-i}\alpha^{\lambda_i-\lambda_j}\sum_{\ell=0}^{\lambda_j-\lambda_{j+1}-1}\left(\alpha^\ell\binom{\lambda_j-\ell-1}{j-i-1} + \beta\binom{\lambda_j-\ell-1}{j-i}\right). \end{aligned}$$

From [8], $\operatorname{weight}(X) = \det A^{\alpha,\beta}_{\lambda(X)}$ for X a word in $\{●,○\}^*$ corresponding to a state of the two-species ASEP with zero ◍'s. Thus we obtain the following exact closed formula for the steady state probabilities of the two-species TASEP.

Theorem 3.4. *Consider the two-species TASEP of size n, and a state X with exactly r ◍'s. Let $\Lambda = (\Lambda_1,\dots,\Lambda_{r+1})$ be the list of partitions that corresponds to X according to Definition 3.1. Let Λ_i have k_i rows and m_i columns. Then*

$$\operatorname{Prob}(X) = \alpha^{n-m_1}\beta^{n-k_{r+1}}\det A_{\Lambda_1}(\alpha,1)\det A_{\Lambda_{r+1}}(1,\beta)\prod_{i=2}^{r}\det A_{\Lambda_i}(1,1)$$

is the unnormalized steady state probability of state X.

4. A Markov chain on the multi-Catalan tableaux that projects to the two-species TASEP

In this section we construct a Markov chain on the multi-Catalan tableaux that provides a proof of Theorem 2.1 and generalizes the construction of Corteel and Williams from [3]. We start by defining projection for Markov chains, from [3, Definition 3.2-0].

Definition 4.1. Let M and N be Markov chains on finite sets X and Y, and let F be a surjective map from X to Y. We say that M *projects* to N if the following properties hold.

- If $x_1, x_2 \in X$ with $\operatorname{Prob}_M(x_1 \to x_2) > 0$, then

$$\operatorname{Prob}_M(x_1 \longrightarrow x_2) = \operatorname{Prob}_N(F(x_1) \longrightarrow F(x_2)).$$

- If y_1 and y_2 are in Y and $\mathrm{Prob}_N(y_1 \to y_2) > 0$, then for each $x_1 \in X$ such that $F(x_1) = y_1$, there is a unique $x_2 \in X$ such that $F(x_2) = y_2$ and $\mathrm{Prob}_M(x_1 \to x_2) > 0$; moreover,

$$\mathrm{Prob}_M(x_1 \longrightarrow x_2) = \mathrm{Prob}_N(y_1 \longrightarrow y_2).$$

This means that if M projects to N via the map F, then the steady state probability that N is in state y is equal to the sum of the steady state probabilities over all the states $x \in \{z \in X | F(z) = y\}$. In our case, N is the two-species TASEP, and M is the Markov chain on the multi-Catalan tableaux which we describe below. Corteel and Williams defined a Markov chain on permutation tableaux (in bijection with alternative tableaux) that projects to the ASEP. In the two-species TASEP, we have an analogous result using similar transitions.

Figure 7. The tableau corner associated to the transition ●●○●○● → ●○●●○●.

Definition 4.2. Define a *corner* of the tableau to be a 2-0, 2-1, or 1-0 box that is both the right-most box of some row and the bottom-most box of some column. Define a *right leg* to be the set of 0-edges of the lattice path $L(T)$ that lie on the north boundary of the $k + r \times n - k$ rectangle. In other words, T has a right leg when type(T) begins with a ○. Analogously, define a *left leg* to be the set of 2-edges of the lattice path $L(T)$ that lie on the west boundary of the $k + r \times n - k$ rectangle. In other words, T has a left leg when type(T) ends with a ●. We call the *transition points* the union of the set of corners along with the left leg and right leg (if those are present).

We describe the process by examining the possible transitions out of some multi-Catalan tableau T which corresponds to the two-species TASEP state X for which type$(T) = X$. Every transition is associated to some chosen transition point (namely, either a chosen corner or a right leg or a left leg). In particular, the right leg corresponds to a transition $\bigcirc X' \to \bullet X'$ for $X = \bigcirc X'$, and the left leg corresponds to a transition $X'\bullet \to X'\bigcirc$ for $X = X'\bullet$. On the other hand, each corner of T corresponds to a transition from X on the TASEP that does not involve particles entering or exiting at the boundary – these transitions occur with rate 1. Specifically, a transition on state X at TASEP location i corresponds to a transition at the corner box in T that has its east edge precisely the ith edge of the lattice path $L(T)$ (from top to bottom). Moreover, a 2-0 corner of T corresponds to a transition ●○ → ○●, a DA corner corresponds to a transition ●● → ●●, and an AE corner corresponds to a transition ●○ → ○● out of state X. (We note here that 1-1 boxes are not included in the set of corners since there is no TASEP transition they correspond to.) Figure 7 shows a transition location in a two-species TASEP word X along with the corresponding corner of a multi-Catalan tableau of type X.

To obtain a transition at a chosen corner, we first strip off the labels on the boundary, then perform certain column or row removal and re-insertion, and finally reapply new labels. We describe the row/column procedure below for the two possible cases for the Greek symbol that corner box could contain.

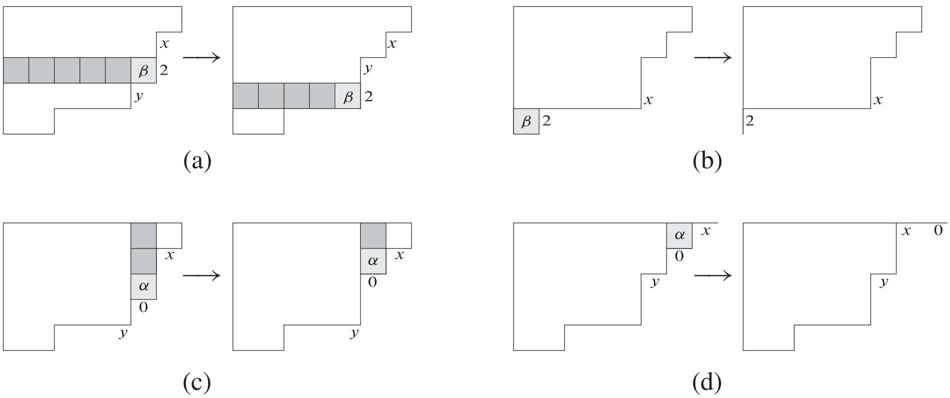

Figure 8. The row removal and re-insertion procedure for (a) and (b) the chosen corner containing a β, and (c) and (d) the chosen corner containing an α. The rows and columns labeled by x and y are preserved.

The corner contains a β. Remove the row containing the corner (which is a horizontal stack of empty boxes with a β on the right), cut off one of the empty boxes, and insert the row (with the β still at the right of it) in the bottom-most location possible so that the resulting shape is still a Young shape. Figure 8 (a) shows an example. If the row originally had a single box, cutting off a box means it becomes an empty row, and so it should be placed at the south end of the shape with the rest of the empty rows. Figure 8 (b) shows an example of this case.

The corner contains an α. Remove the column containing the corner (which is a stack of empty boxes above an α), cut off one of the empty boxes, and insert the column (with the α still at the bottom of it) in the right-most location possible so that the resulting shape is still a Young shape. Figure 8 (c) shows an example. If the column originally had a single box, cutting off a box means it becomes an empty column, and so it should be placed at the east end of the shape with the rest of the empty columns. Figure 8 (d) shows an example of this case.

Now we put the labels back on the edges of the boundary after exchanging the relevant two letters in the labeling word. For example, if the original state was X● ○Y for some words X and Y, and a ● hopped to get the state X○ ●Y, then the labels on the boundary change from $X\ 2\ 0\ Y$ to $X\ 0\ 2\ Y$.

The following lemmas verify that the above actions are well-defined.

Lemma 4.1. *Let X be a word in* {●, ○, ●}*, *and let T be a multi-Catalan tableau with* $\text{type}(T) = X$. *A transition as defined above* ***at a corner that contains a*** $\boldsymbol{\beta}$ *results in a valid multi-Catalan tableau.*

Proof. Let Y be the Young diagram and $\lambda = (\lambda_1, \ldots, \lambda_{r+k})$ be the partition associated to T with r and k the number of ●'s and ●'s respectively in $\text{type}(T)$. Let $L(T)$ be the lattice path associated to T. The edges of $L(T)$ are labeled with ●'s, ○'s, and ●'s according to the labeling word X. Let X' and X'' denote arbitrary words in {●, ○, ●}*.

Suppose the chosen corner of T occurs in row i of length λ_i. For the following, we identify the partition $(\lambda_1, \ldots, \lambda_s)$ with the partition $(\lambda_1, \ldots, \lambda_s, 0, \ldots, 0)$
Let $\lambda_i > 1$. After removing row i of T and reinserting a row of length $\lambda_i - 1$ into the lowest position possible, we obtain a tableau T' of shape $(\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_{r+k})$. In other words, T' has the shape λ with the single box removed in row i.

Since row i contains a β, the its label must be 2. The fact that its right-most box is a corner implies that for any $j > i$, $\lambda_j < \lambda_i$. Thus we have two cases, depending on whether λ_{i+1} is equal to or less than $\lambda_i - 1$. For both of these cases, we check that applying to $L(T')$ the new labeling word corresponding to the two-species TASEP transition from X is consistent with the shape of T'.

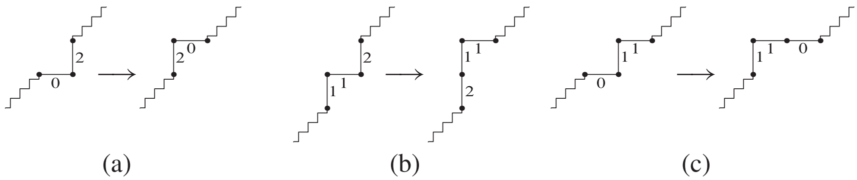

Figure 9. The transition on a multi-Catalan tableau that corresponds to the two-species TASEP transition (a) ● ○ → ○ ●, (b) ● ● → ● ●, and (c) ● ○ → ○ ●.

CASE 1: $\lambda_{i+1} = \lambda_i - 1$. The label of the column containing the chosen corner is either 0 or 1. Since 1 always labels a pair of edges belonging to an inner corner, if the label of the column containing the chosen corner is 0, then the label of row $i + 1$ is necessarily 2, and similarly if the label of that column is 1, then the label of row $i + 1$ is necessarily 1. In the first case, we can write $X = X'● ○X''$, and in the second case we can write $X = X'● ◍X''$. From the above, the shape of T' is simply the shape of T with the right-most box of row i removed. Figure 9 (a) and (b) show that labeling $L(T')$ with the labeling word X' 0 2 X'' and X' 1 2 X'' for the first and second case respectively is consistent with the shape of T'.

CASE 2: $\lambda_{i+1} < \lambda_i - 1$. The label of the column containing the chosen corner *must be 0*, since otherwise that column would have to belong to an inner corner labeled 1, which would require $\lambda_{i+1} = \lambda_i - 1$ as in Case 1. Thus we can write $X = X'● ○X''$. From the above, the shape of T' is simply the shape of T with the right-most box of row i removed. Figure 9 (a) shows that labeling the lattice path $L(T')$ with the labeling word X' 0 2 X'' is consistent with the shape of T'.

For both of these cases, the newly inserted row is labeled by 2, and since it was placed in the south-most location possible for its length, its right-most box is a corner box. Thus inserting this row with a β in its right-most box indeed results in a valid multi-Catalan tableau. Furthermore, neither the weight of the filling or the weight of the boundary of the tableau changed after the transition and so $\mathrm{wt}(T') = \mathrm{wt}(T)$.
Let $\lambda_i = 1$. As before, row i is labelled with a 2. The transition from T to T' is completed by simply removing row i and replacing it with a single 2-labeled south edge on the west boundary of T to make T'. The column of T containing the corner box of row i is labeled by 0 or 1. In the first case, X must necessarily have the form $X'● ○ ●^j$ for some j, and so the type and labeling word of T' become $X'◍ ●^{j+1}$ and X' 0 2^{j+1} respectively. In the second case X must have the form $X'● ◍ ●^j$ for some j, and so the type and labeling word of T' become $X'◍ ●^{j+1}$ and X' 1 2^{j+1} respectively. For both of these special cases, the labeling word of T' is consistent with its shape. Furthermore, the weight of the boundary of T' is the same as for T, but the filling of T' lost one β, so $\mathrm{wt}(T') = \frac{1}{\beta}\,\mathrm{wt}(T)$.

This concludes the proof of the lemma. □

Lemma 4.2. *Let X be a word in $\{●, ○, ◍\}^*$, and let T be a multi-Catalan tableau with* $\mathrm{type}(T) = X$. *A transition as defined above* ***at a corner that contains an α*** *results in a valid multi-Catalan tableau.*

Proof. By the symmetry of the rules for the multi-Catalan tableaux, the proof is exactly the same as the one for Lemma 4.1, except if we take the transpose of the tableau and exchange the roles of α and β. For this case, see Figure 9 (a) and (c) for the transition $L(T) \to L(T')$. It will be useful further on that if the transition from T to T' occurs at a corner that belongs to a column of length $\mu_i > 1$, then as before, $\mathrm{wt}(T') = \mathrm{wt}(T)$. Otherwise, if $\mu_i = 1$, then $\mathrm{wt}(T') = \frac{1}{\alpha}\mathrm{wt}(T)$. In that case, if the transition occurred at a 2-0 corner, type(T) necessarily has the form $\circ^j \bullet \circ X$, and if the transition occurred at a 1-0 corner, type(T) necessarily has the form $\circ^j$ ● $\circ X$ for some j and some word X in $\{\bullet, \circ, ●\}^*$. □

Transitions at the boundary. For an arbitrary two-species TASEP word X in $\{\bullet, \circ, ●\}^*$, we describe the transition that corresponds to the two-species TASEP transition $\circ X \to \bullet X$ from a tableau T of type $\circ X$ to a tableau T' of type $\bullet X$. T must necessarily have at least one empty 0-column on its right, so after stripping off the labels of the tableau, we remove the right-most empty column and instead insert a row with a β in its right-most box, of maximal possible length such that the semi-perimeter stays fixed, but at the lowest position possible for that length. (In the degenerate case, if type$(T) = \circ \bullet^j$, we simply insert a row of length zero to make T' with type $\bullet^{j+1}$.) Finally, we apply the labeling word $\bullet X$ to the edges of $L(T')$. Figure 10 (a) shows an example where $\lambda_1 < n - k - 1$.

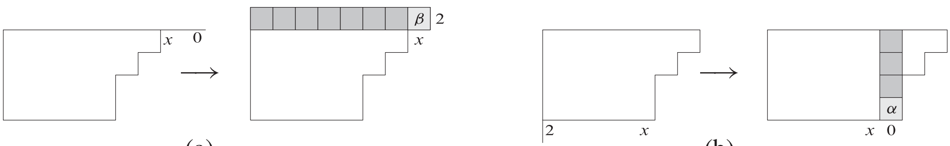

Figure 10. The transitions on multi-Catalan tableau that correspond to the two-species TASEP transitions (a) $\circ X \to \bullet X$ and (b) $X\bullet \to X\circ$.

Let such a transition occur with rate α.

If a new row of nonzero length was inserted into T, then effectively one 0-labeled boundary edge of T was replaced with a 2-labeled boundary edge for T', plus the filling of T' gained one β. Then $\mathrm{wt}(T') = \alpha\,\mathrm{wt}(T)$. Otherwise, in the degenerate case when type$(T) = \circ \bullet^j$, we have type$(T') = \bullet^{j+1}$ and so $\mathrm{wt}(T') = \frac{\alpha}{\beta}\mathrm{wt}(T)$.

The transition $X\bullet \to X\circ$ is symmetric to the above, if one were to take the transpose of the tableau and exchange the roles of α and β. For a transition from T of type $X\bullet$ to T' of type $X\circ$, T must necessarily have at least one empty 2-row at the bottom of it. After stripping off the labels of the tableau, we remove the bottom-most empty row and instead insert a column with an α in its bottom-most box, of maximal possible length such that the semi-perimeter stays fixed, at the right-most position possible for that length. (In the degenerate case, if $\mathrm{type}(T) = \circ^j \bullet$, we simply insert a column of length zero to make T' with type $\circ^{j+1}$.) Finally, we apply the labeling word $X\circ$ to the edges of $L(T')$. Figure 10 (b) shows an example where the number of nonzero rows of T equals $r + k - 1$.

Let such a transition occur with rate β.

Similarly to the above, if the new column added has nonzero length, we obtain that $\mathrm{wt}(T') = \beta\,\mathrm{wt}(T)$. Otherwise, in the degenerate case where $\mathrm{type}(T) = \circ^j \bullet$ and $\mathrm{type}(T') = \circ^{j+1}$, we have $\mathrm{wt}(T') = \frac{\beta}{\alpha}\,\mathrm{wt}(T)$.

The following lemmas prove that these boundary transitions are well-defined.

Lemma 4.3. *Let X be a word in $\{\bullet, \circ, \bullet\}^*$, and let T be a multi-Catalan tableau with* $\mathrm{type}(T) = \circ X$. *A transition on the boundary of T as defined above, corresponding to the two-species TASEP transition $\circ X \to \bullet X$ results in a valid multi-Catalan tableau.*

Proof. Let the shape of T of size (n, r, k) with associated lattice path $L(T)$ be $\lambda = (\lambda_1, \ldots, \lambda_s)$ for some $s > 0$. We assume the non-degenerate case $\lambda_1 > 0$. Suppose $\lambda_1 = \ldots = \lambda_i > \lambda_{i+1}$ for some i. Then the shape of T' must be $\lambda' = (\lambda'_1, \ldots, \lambda'_{s+1})$ where $\lambda'_1 = \ldots = \lambda'_{i+1} = n - k - 1$ and $\lambda'_j = \lambda_{j-1}$ for $j > i + 1$. It is easy to check that labeling $L(T')$ with the labeling word $2X$ is consistent.

It remains to check that placing a β in row $i + 1$ of T' was valid. Since $\lambda'_i = \lambda'_{i+1}$, the vertical edge of row $i + 1$ of T' does not belong to an inner corner, and thus is necessarily labeled by 2. Therefore since the right-most box of row $i + 1$ of T' is a corner box in a 2-row, it can certainly contain a β, and so T' is a valid multi-Catalan tableau with labeling word $2X$. $\square$

Lemma 4.4. *Let X be a word in $\{\bullet, \circ, \bullet\}^*$, and let T be a multi-Catalan tableau with* $\mathrm{type}(T) = X\bullet$. *A transition on the boundary of T as defined above, corresponding to the two-species TASEP transition $X\bullet \to X\circ$ results in a valid multi-Catalan tableau.*

Proof. The proof is equivalent to the proof of Lemma 4.3 above, except that instead we take the transpose of the tableaux and exchange the roles of α and β. $\square$

We carefully summarize the transitions from a multi-Catalan tableau T to the tableau S, depending on the chosen corner at which the transition occurs. We will be referring to these cases further on. First we make the following definitions. Let T have size (n,k,r) and let $\lambda = (\lambda_1,\dots,\lambda_{k+r})$ be the shape of T. Assume that λ has at least one non-zero part.

Definition 4.3. We define λ_R be the indicator that equals 1 if T has a right leg, and 0 otherwise. We define λ_L be the indicator that equals 1 if T has a left leg, and 0 otherwise.

Definition 4.4. We call a *top-most corner* a corner such that the length of the row containing it equals λ_1. We define the indicator δ_β which equals 1 if the top-most corner contains a β, and 0 otherwise. Analogously, we call a *bottom-most corner* a corner such that the length of the row containing it equals the length of the smallest non-zero row of λ. We define the indicator δ_α which equals 1 if the bottom-most corner contains an α, and 0 otherwise. We call a *middle corner* a corner that is neither a top-most corner or a bottom-most corner.

Remark 4.1. Denote by $\pi(T \to S)$ the rate of transition from tableau T to S (where by rate we mean the unnormalized probability). We obtain the following cases for the transitions from T to S.

(1) For a transition at a middle corner, a top-most corner with $\delta_\beta = 1$, or a bottom-most corner with $\delta_\alpha = 1$, we have $\mathrm{wt}(S) = \mathrm{wt}(T)$ and $\pi(T \to S) = 1$.

(2) For a transition at a top-most corner with $\delta_\beta = 0$ such that the length of the column containing it is greater than 1, we have $\mathrm{wt}(S) = \mathrm{wt}(T)$ and $\pi(T \to S) = 1$. Then S will have top-most corner that contains an α.

(3) For a transition at a bottom-most corner with $\delta_\alpha = 0$ such that the length of the row containing it is greater than 1, we have $\mathrm{wt}(S) = \mathrm{wt}(T)$ and $\pi(T \to S) = 1$. Then S will have a bottom-most corner that contains a β.

(4) For a transition at a top-most corner with $\delta_\beta = 0$ such that the length of the column containing it is 1, we have $\mathrm{wt}(S) = \frac{1}{\alpha}\,\mathrm{wt}(T)$ and $\pi(T \to S) = 1$.

(5) For a transition at a bottom-most corner with $\delta_\alpha = 0$ such that the length of the row containing it is 1, we have $\mathrm{wt}(S) = \frac{1}{\beta}\,\mathrm{wt}(T)$ and $\pi(T \to S) = 1$.

(6) For a transition at a right leg, we have $\mathrm{wt}(S) = \alpha\,\mathrm{wt}(T)$ and $\pi(T \to S) = \alpha$. S will not have a right leg, and it will have a top-most corner that contains a β.

(7) For a transition at a left leg, we have $\mathrm{wt}(S) = \beta\,\mathrm{wt}(T)$ and $\pi(T \to S) = \beta$. S will not have a left leg, and it will have a bottom-most corner that contains an α.

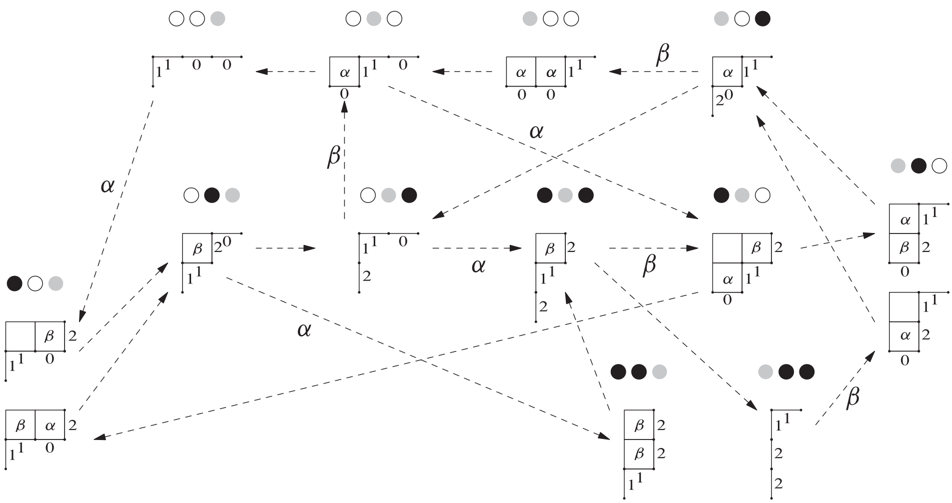

Figure 11. This is the state diagram of a two-species TASEP of size $(3, 1)$. The words in $\{\circ, \text{●}, \bullet\}^3$ represent the corresponding states, and the unlabelled arrows are transitions with rate 1.

Figure 11 shows the transitions on all the states of the two-species TASEP of size 3 with one ●.

Theorem 4.5. *In the Markov chain on multi-Catalan tableaux, the steady state probability of a multi-Catalan tableau T is* $\mathrm{wt}(T)$. *This Markov chain projects to the two-species TASEP.*

Proof. Let T be a multi-Catalan tableau of size (n, k, r). Let

$$\mathcal{S} = \{S : \pi(T \longrightarrow S) > 0\}$$

and

$$\mathcal{T} = \{T' : \pi(T' \longrightarrow T) > 0\}.$$

We show the following.

i. The unique stationary probability of T in this Markov chain is proportional to $\mathrm{wt}(T)$ due to the fact that detailed balance holds:

$$\mathrm{wt}(T)\sum_{S\in\mathcal{S}}\pi(T\to S)=\sum_{T'\in\mathcal{T}'}\mathrm{wt}(T')\pi(T'\to T). \tag{4.1}$$

ii. For every $S\in\mathcal{S}$, we have that $\frac{1}{n+1}\pi(T\to S)$ equals the probability of the transition from the state $\mathrm{type}(T)$ to $\mathrm{type}(S)$ of the two-species TASEP.

iii. For every state X of the two-species TASEP and every state Y for which there is a nonzero probability $\frac{p}{n+1}$ of transition from X, for any tableau T with $\mathrm{type}(T)=X$, there exists a unique tableau S with $\mathrm{type}(S)=Y$, and moreover $\pi(T\to S)=p$.

Condition (i) implies that $\mathrm{wt}(T)$ is proportional to the steady state probability of T. Satisfying condition (ii) for all T and (iii) for all states X of the two-species TASEP implies that $\mathrm{weight}(X)$ is proportional to the steady state probability of X. Thus proving (i)-(iii) is sufficient to show that our Markov chain on the multi-Catalan tableaux indeed projects to the two-species TASEP.

Condition (i). Let $X=\mathrm{type}(T)$. First we treat the transitions going out of T to $S\in\mathcal{S}$. By the construction of our Markov chain on the tableaux, it is clear that there is a transition with probability 1 for every corner, a transition with probability α for a right leg, and a transition with probability β for a left leg. These transitions directly correspond to all the possible transitions out of the two-species TASEP state X. Suppose X has C corners (note that 1-1 boxes are excluded). Thus we obtain

$$\sum_{S\in\mathcal{S}}\pi(T\to S)=C+\alpha\delta_L+\beta\delta_R. \tag{4.2}$$

For the transitions going into T from $T'\in\mathcal{T}$, we observe that any transition from one tableau to another ends with a corner, an edge on the right leg, or an edge on the left leg. This is because for a transition that involves either inserting into the tableau a nonempty column containing an α or a nonempty row containing a β, then the box containing the Greek symbol is the aforementioned corner. Otherwise, for a transition that involves inserting into the tableau an empty column or an empty row, the result is a contribution of an edge to the right leg or an edge to the left leg, respectively. Thus it is sufficient to examine the corners and the right leg and left leg of T to enumerate all the possibilities for $T'\in\mathcal{T}$. We examine the pre-image of the cases for the possible transitions going out of T to obtain the following cases for T'.

(1) For a middle corner, a top-most corner with $\delta_\beta = 0$, or a bottom-most corner with $\delta_\alpha = 0$, we have $\mathrm{wt}(T') = \mathrm{wt}(T)$ and $\pi(T' \to T) = 1$. This is the inverse of Case 1 of Remark 4.1. This gives a contribution of $\mathrm{wt}(T)(C - 2 + (1 - \delta_\beta) + (1 - \delta_\alpha))$ to the right hand side (RHS) of the detailed balance equation.[2]

(2) For a top-most corner with $\delta_\beta = 1$ and $\delta_R = 0$, we have a transition involving the right-leg of T', so $\mathrm{wt}(T') = \frac{1}{\alpha}\mathrm{wt}(T)$ and $\pi(T' \to T) = \alpha$. This is the inverse of Case 2 of Remark 4.1. This gives a contribution of $\alpha\frac{1}{\alpha}\mathrm{wt}(T)\delta_\beta(1 - \delta_R)$ to the RHS of the detailed balance equation.

(3) For a bottom-most corner with $\delta_\alpha = 1$ and $\delta_L = 0$, we have a transition involving the left-leg of T', so $\mathrm{wt}(T') = \frac{1}{\beta}\mathrm{wt}(T)$ and $\pi(T' \to T) = \beta$. This is the inverse of Case 3 of Remark 4.1. This gives a contribution of $\beta\frac{1}{\beta}\mathrm{wt}(T)\delta_\alpha(1 - \delta_L)$ to the RHS of the detailed balance equation.

(4) For a top-most corner with $\delta_\beta = 1$ and $\delta_R = 1$, there are two possibilities. For the first, T' could fall into Case 2 of Remark 4.1, meaning that T' has a top-most corner containing a β, which is the usual transition with $\mathrm{wt}(T') = \mathrm{wt}(T)$. For the second possibility, T' could fall into Case 4 of Remark 4.1, meaning that T' has a top-most corner containing an α and the column containing it has length 1. In that case, $\mathrm{wt}(T') = \alpha\,\mathrm{wt}(T)$. In both situations, $\pi(T' \to T) = 1$. We obtain a contribution of $\mathrm{wt}(T)\delta_\beta\,(\delta_R + \alpha(1 - \delta_R))$ to the RHS of the detailed balance equation.

(5) For a bottom-most corner with $\delta_\alpha = 1$ and $\delta_L = 1$, there are two possibilities. For the first, T' could fall into Case 3 of Remark 4.1, meaning that T' has a bottom-most corner containing an α, which is the usual transition with $\mathrm{wt}(T') = \mathrm{wt}(T)$. For the second possibility, T' could fall into Case 5 of Remark 4.1, meaning that T' has a bottom-most corner containing a β and the row containing it has length 1. In that case, $\mathrm{wt}(T') = \beta\,\mathrm{wt}(T)$. In both situations, $\pi(T' \to T) = 1$. We obtain a contribution of $\mathrm{wt}(T)\delta_\alpha\,(\delta_L + \beta(1 - \delta_L))$ to the RHS of the detailed balance equation.

[2] Note that if $C < 2$, the formulae we give have some degeneracies. However, it is easy to verify that these do not cause any problems due to cancellation of all the degenerate terms.

We sum up the contributions to the RHS of the detailed balance equation to obtain

$$\sum_{T'\in\mathcal{T}} \mathrm{wt}(T')\pi(T'\to T) = \mathrm{wt}(T)(C-\delta_\beta-\delta_\alpha+\delta_\beta(1-\delta_R)+\delta_\alpha(1-\delta_L) \\ +\delta_\beta(\delta_R+\alpha(1-\delta_R))+\delta_\alpha(\delta_L+\beta(1-\delta_L))). \tag{4.3}$$

We see that after simplification, Equation 4.3 equals Equation 4.2, so indeed the desired Equation 4.1 holds for "most" T.

It remains to check a few degenerate cases for T, in particular, when $\lambda(T) = (0,\dots)$. However, those cases can only occur when type(T) contains zero $\textcolor{gray}{\bullet}$'s. Thus we refer to [3] for these details.

Condition (ii). In the definition of the Markov chain, the transitions on the corners of the tableau are set to have rate 1. These transitions, which occur on 2-0, 2-1, and 1-0 corners, precisely correspond to the respective transitions $X\bullet\circ Y \to X\circ\bullet Y$, $X\bullet\textcolor{gray}{\bullet}Y \to X\textcolor{gray}{\bullet}\bullet Y$, and $X\textcolor{gray}{\bullet}\circ Y \to X\circ\textcolor{gray}{\bullet}Y$ on the two-species TASEP that do not involve particles hopping on and off the boundary. On the two-species TASEP, such transitions have probability $\frac{1}{n+1}$, as desired.

Similarly, the transitions involving an empty column on the east end of the tableau have rate α, and they precisely correspond to the transition $\circ X \to \bullet X$ of the two-species TASEP, which has probability $\frac{\alpha}{n+1}$. Analogously, the transitions involving an empty row on the south end of the tableau have rate β, and they precisely correspond to the transition $X\bullet \to X\circ$ of the two-species TASEP, which has probability $\frac{\beta}{n+1}$.

Condition (iii). This condition holds by the definition of the Markov chain on the multi-Catalan tableaux. □

Theorem 4.5 and its proof imply the following corollary, which completes the proof of Theorem 2.1.

Corollary 4.6. *The stationary probability of a two-species TASEP state X is proportional to* weight(X).

Remark 4.2. The multi-Catalan tableaux can be given a certain binary tree structure in the flavor of *tree-like tableaux*, which are are in bijection with alternative tableaux, and were introduced in [2] based off theory developed by Viennot [11]. The tree structure on the multi-Catalan tableaux naturally generalizes the tree structure on the usual Catalan tableaux. The perspective of the tree structure gives an illuminating interpretation of the multi-Catalan tableaux, and in particular provides an easy way to visualize the Markov chain on the tableaux and prove it projects to the two-species TASEP. We leave this interpretation of the tableaux for future work.

5. Two-species tableaux at $q = 1$

The goal of this section is to give a combinatorial formula for the two-species ASEP at $q = 1$. To this end, we define *two-species alternative tableaux*. These tableaux are inspired by the alternative tableaux of Viennot. When the type of a two-species alternative tableau has zero ●'s, it reduces to a usual alternative tableau. Furthermore, if only the tableaux with no q's are considered, we recover the multi-Catalan tableaux. In Figure 12, we illustrate the rules given in the following definition.

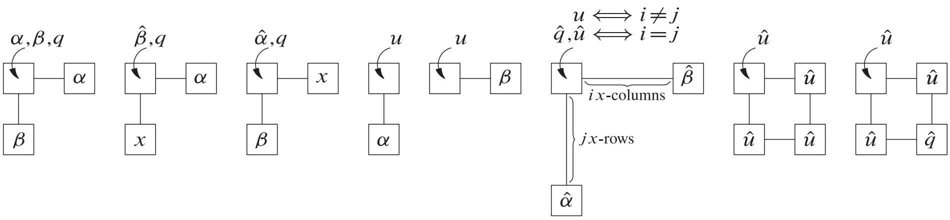

Figure 12. An illustration of the rules in Definition 5.1 for the filling of a two-species alternative tableau.

Definition 5.1. A *two-species alternative tableau* of *size* n is a filling of a Young diagram of shape $(n, n-1, \ldots, 1)$ with the symbols $\alpha, \hat{\alpha}, \beta, \hat{\beta}, q, \hat{q}, u, \hat{u}$ according to the rules below:

- every box on the anti-diagonal must contain an α, β, or x;
- a box that sees an α to its right and a β below must contain an α, β, or q;
- a box that sees an α to its right and a β below must contain an $\hat{\alpha}$ or q;
- a box that sees an α to its right and a β below must contain a $\hat{\beta}$ or q;
- every box in the same column and above an α must contain a u, and every box in the same row and left of a β must contain a u;

- for every pair of $\hat{\alpha}$'s and $\hat{\beta}$'s ($\hat{\alpha}$ left of $\hat{\beta}$) such that the number of x-rows and x-columns (i.e. rows and columns containing an x in the anti-diagonal box) between them *is not* equal, put a u at the intersection of the $\hat{\alpha}$ column and the $\hat{\beta}$ row;
- for every pair $\hat{\alpha}$'s and $\hat{\beta}$'s ($\hat{\alpha}$ left of $\hat{\beta}$) such that the number of x-rows and x-columns between them *is* equal, put either a $\hat{q}$ or a $\hat{u}$ at the intersection of the $\hat{\alpha}$ column and the $\hat{\beta}$ row;
- the placement of the $\hat{q}$ and $\hat{u}$ above must satisfy that there is no instance of $\begin{smallmatrix}\hat{q} & \hat{u}\\ \hat{u} & \hat{u}\end{smallmatrix}$ or $\begin{smallmatrix}\hat{q} & \hat{u}\\ \hat{u} & \hat{q}\end{smallmatrix}$.
- every other box must contain a u.

In these fillings, the u's are simply place-holders for the empty boxes, and the $\hat{u}$'s are place-holders that enforce valid placement of the $\hat{q}$'s. An easy way to construct these fillings is to place the Greek symbols and q's starting from the boxes closest to the anti-diagonal and moving inwards. Once these symbols are placed everywhere possible, we define an $\hat{\alpha}$-column to be the boxes directly above an $\hat{\alpha}$, and a $\hat{\beta}$-row to be the boxes directly to the left of a $\hat{\beta}$. We then identify the boxes that lie at the intersections of the $\hat{\alpha}$-columns and the $\hat{\beta}$-rows, and fill them appropriately with $\hat{q}$'s, $\hat{u}$'s, or u's. The rest of the tableau is automatically filled with u's.

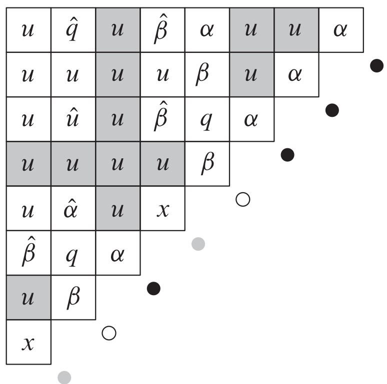

Figure 13. A two-species alternative tableau of type ●●●○●●○● and weight $\alpha^6\beta^6$. The shaded boxes are ones that are automatically empty due to the labels on the anti-diagonal.

Definition 5.2. The *type* of the two-species alternative tableau is read off of the anti-diagonal from top to bottom, by reading an α as ●, a β as ○, and an x as ●. The *weight* of the tableau is the product of the symbols in the filling in the form of a monomial in α and β, where we set $\hat{u} = u = 1$, $\hat{\alpha} = \alpha$, $\hat{\beta} = \beta$, and $\hat{q} = q = 1$.

Note that if we instead set $\hat{q} = q = 0$, erase the $\hat{u}$ and u, and replace $\hat{\alpha}$ and $\hat{\beta}$ with α and β respectively, we obtain once more the multi-Catalan tableaux.

The following conjecture is analogous to the main result of Section 2, Theorem 2.1.

Conjecture 5.1. Consider the two-species ASEP at $q = 1$, and let X be a state represented by a word in $\{\bullet, \circ, \textcolor{lightgray}{\bullet}\}^n$ with precisely r $\textcolor{lightgray}{\bullet}$'s. Then the steady state probability of X is

$$\mathrm{Prob}(X) = \frac{1}{Z^1_{n,r}} \sum_T \mathrm{wt}(T),$$

where the sum is over all two-species alternative tableaux T such that $\mathrm{type}(T) = X$, and where $Z^1_{n,r} = \sum_T \mathrm{wt}(T)$, for T ranging over all two-species alternative tableaux of size n whose type has exactly r $\textcolor{lightgray}{\bullet}$'s.

We have verified the above using SAGE for up to $n = 10$. However, so far the proof appears tedious.

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