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The arcsine law and an asymptotic behavior of orthogonal polynomials

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Abstract. Interacting Fock spaces connect the study of quantum probability theory, classical random variables, and orthogonal polynomials. They are pre-Hilbert spaces associated with creation, preservation, and annihilation processes. We prove that if three processes are asymptotically commutative, the arcsine law arises as the "large quantum number limits." As a corollary, it is shown that for many probability measures, the asymptotic behavior of orthogonal polynomials is described by the arcsine function. A weaker form of asymptotic commutativity provides us with a discretized arcsine law, which is described by the Bessel functions of the first kind.

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1. Introduction

In the quantum field theory, the pair of "annihilation" and "creation" operators plays a crucial role. The notion of interacting Fock spaces, which are pre-Hilbert spaces associated with "annihilation" and "creation" operators (and "preservation" operators) satisfying characteristic commutation relations, were discovered in a quantum-probabilistic approach to the mathematical analysis of interacting fields [2, 3].

Later Accardi and Bożejko [1] discovered a striking connection between interacting Fock spaces and the theory of orthogonal polynomials for probability measures on $\mathbb{R}$ whose moments are finite. Three-term recurrence relations for orthogonal polynomials provide a method of decomposition. The three-term recurrence relation implies that the multiplication of x on the polynomials is a summation of three operators (annihilation, creation and preservation). The method is called "quantum decomposition" [9, 10].

In terms of interacting Fock space and quantum decomposition, Saigo [14] introduced quite a simple combinatoric and algebraic method for the study of asymptotic behavior of Hermite polynomials. The sequence of orthogonal polynomials for the Gaussian measure

$$d\mu(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

is that of Hermite polynomials $\{H_n\}$. The measure $H_n^2\mu$ is nothing but the probability measure representing energy eigenstates of the quantum harmonic oscillator. The limit distribution of the normalization of $H_n^2\mu$ is

$$d\mu_{As}(x) = \frac{dx}{\pi\sqrt{2-x^2}}, \quad (-\sqrt{2} < x < \sqrt{2}).$$

The measure μ_{As} is called the *arcsine law*. It is especially known for the relationships to Brownian motion discovered by P. Lévy [13]. (See also [8, 15, 16] for example.) Since the arcsine law is the distribution of position distribution of classical harmonic oscillators, the algebraic method introduced in [14] can be considered as a new combinatoric approach to quantum-classical correspondence.

In the present paper, the methods and results in [14] is extended to the general interacting Fock spaces satisfying a relative asymptotic commutativity condition which we call (RAC1). As a corollary, the asymptotic behavior of a wide class of orthogonal polynomials (including polynomials of Hermite, Jacobi, Laguerre, etc.) can be characterized in terms of the arcsine law. (As basic references for orthogonal polynomials and their asymptotic behavior, see [5, 17] for example.) The proof is given in Section 5 following the preliminaries introduced in the first four sections. In the last section, we discuss a weaker form of relative asymptotic commutativity condition (RAC2) and identify the limit distribution which we call discrete arcsine law. The distribution is explicitly represented in terms of Bessel functions. It also appears in the context of quantum walks [12],

2. Basic notions

2.1. Algebraic probability space. For a classical probability space $(\Omega, \mathcal{F}, P)$, a pair of a complex algebra and its linear functional is defined. Such a pair is given by

$$\left(L^\infty(\Omega, \mathcal{F}, P), \int_\Omega \cdot \, dP \right),$$

and sometimes given by

$$\left(\bigcap_{1 \le p < \infty} L^p(\Omega, \mathcal{F}, P), \int_\Omega \cdot \, dP \right).$$

An algebraic probability space is a generalization of these pairs. In this generalization, we do not assume commutativity on the algebra.

Definition 2.1. A $*$-*algebra* $\mathcal{A}$ is a complex algebra equipped with a mapping $\mathcal{A} \ni x \mapsto x^* \in \mathcal{A}$ satisfying

$$(X^*)^* = X, \quad (\alpha X)^* = \bar{\alpha} X^*, \quad (X+Y)^* = X^* + Y^*, \quad (XY)^* = Y^* X^*,$$

for every $X, Y \in \mathcal{A}$ and $\alpha \in \mathbb{C}$.

Definition 2.2. Let $\mathcal{A}$ be a $*$-algebra having a unit element $1 \in \mathcal{A}$. We call a linear map $\varphi \colon \mathcal{A} \to \mathbb{C}$ a *state* on $\mathcal{A}$ if it satisfies

$$\varphi(1) = 1, \ \varphi(X^* X) \ge 0, \quad \text{for } X \in \mathcal{A}.$$

It is not necessary to assume an operator algebra structure on $\mathcal{A}$ in this paper.

Definition 2.3. A pair $(\mathcal{A}, \varphi)$ of a $*$-algebra and a state on $\mathcal{A}$ is called an *algebraic probability space*. An element of $\mathcal{A}$ is called an *algebraic random variable* of the algebraic probability space $(\mathcal{A}, \varphi)$. If an algebraic random variable $X \in \mathcal{A}$ satisfies $X^* = X$, then X is called a *self-adjoint algebraic random variable.*

We define a notation for a state $\varphi \colon \mathcal{A} \to \mathbb{C}$, a self-adjoint algebraic random variable $X \in \mathcal{A}$ and a probability distribution μ on $\mathbb{R}$ as follows.

Notation 2.4. We use the notation $X \sim_\varphi \mu$ when

$$\varphi(X^m) = \int_{\mathbb{R}} x^m \, d\mu(x) \quad \text{for all } m \in \mathbb{N}.$$

This stands for the identity between two moment sequences.

2.2. Interacting Fock space

Definition 2.5 (Jacobi sequence). A pair of sequences $(\{\omega_{n+1/2}\}, \{\alpha_n\})$ is called a *Jacobi sequence,*

- if $\{\omega_{n+1/2}\}$ are positive real numbers $0 < \omega_{1/2}, \omega_{3/2}, \omega_{5/2}, \dots$ labeled by half natural numbers, and
- if $\{\alpha_n\}$ are real numbers $\alpha_0, \alpha_1, \alpha_2, \dots$ labeled by natural numbers.

In other works as [10, Definition 1.24], the sequence $\{\omega_{n+1/2}\}$ is called a *Jacobi sequence of infinite type* and given different labels.

Definition 2.6 (interacting Fock space). Let $(\{\omega_{n+1/2}\}, \{\alpha_n\})$ be a Jacobi sequence. An *interacting Fock space* $\Gamma_{\omega,\alpha}$ is a complex pre-Hilbert space $\Gamma(\mathbb{C})$ equipped with the following additional structure $(\{\Phi_n\}_{n=0}^{\infty}, A, B, C)$:

- Fixed sequence of vectors $\{\Phi_n\}_{n=0}^{\infty} \subset \Gamma(\mathbb{C})$ satisfying
  - $\langle \Phi_n, \Phi_m \rangle = 0$ if $m \neq n$, and $\langle \Phi_n, \Phi_n \rangle = 1$,
  - $\Gamma(\mathbb{C})$ is a complex linear span of $\{\Phi_n\}$,
- $A, B, C\colon \Gamma(\mathbb{C}) \to \Gamma(\mathbb{C})$ are linear operators uniquely determined by
  - $A\Phi_0 = 0$, $A\Phi_n = \sqrt{\omega_{n-1/2}}\,\Phi_{n-1}$,
  - $B\Phi_n = \alpha_n \Phi_n$,
  - $C\Phi_n = \sqrt{\omega_{n+1/2}}\,\Phi_{n+1}$.

The sequence of vectors $\{\Phi_n\}_{n=0}^{\infty} \subset \Gamma(\mathbb{C})$ forms a orthonormal set of $\Gamma(\mathbb{C})$. The operator A is called the *annihilation* operator, B is called the *preservation* operator, and C is called the *creation* operator.

Definition 2.7. The summation $X = A + B + C$ is expressed by the symmetric tridiagonal matrix

$$X = \begin{pmatrix} \alpha_0 & \sqrt{\omega_{1/2}} & 0 & \cdots \\ \sqrt{\omega_{1/2}} & \alpha_1 & \sqrt{\omega_{3/2}} & \ddots \\ 0 & \sqrt{\omega_{3/2}} & \alpha_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

This is called the *Jacobi matrix.*

The sequence of real numbers $\langle X^m \Phi_0, \Phi_0 \rangle$ is called the *moments sequence* of the Jacobi matrix X. Accardi and Bożejko showed in [1, Section 5] that for every probability measure μ on $\mathbb{R}$ whose moments are finite, the moment sequence

$$M_m = \int_{\mathbb{R}} x^m d\mu(x)$$

can be realized as that of an interacting Fock space $\langle X^m \Phi_0, \Phi_0 \rangle$.

Let $\mathcal{A}$ be the complex algebra generated by the matrices A, B, C and by the identity matrix id. The multiplication and the linear structure are defined by the usual matrix calculations. The $*$-operation is given by the composition of transpose and complex conjugation. Since the generating set $\{A = C^*, B = B^*, C = A^*\} \subset \mathcal{A}$ is closed under the $*$-operation, the whole algebra $\mathcal{A}$ is also closed under the $*$-operation.

Recall that the operators A, B, C act on the linear space $\bigoplus_{n=0}^{\infty} \mathbb{C}\Phi_n$. Let φ_k be the state defined as $\varphi_k(\cdot) := \langle \cdot \Phi_k, \Phi_k \rangle$. Then the pairs $\{(\mathcal{A}, \varphi_k)\}_{k\in\mathbb{N}}$ are algebraic probability spaces labeled by k. The asymptotic behavior of the sequence $\{(\mathcal{A}, \varphi_k)\}_{k\in\mathbb{N}}$ is the subject of this paper.

2.3. Interacting Fock spaces and orthogonal polynomials. Theorems for interacting Fock spaces often have interesting interpretation in terms of orthogonal polynomials. To see this, we review the relation between interacting Fock spaces, probability measures and orthogonal polynomials. Let μ be a probability measure on $\mathbb{R}$ having finite moments. Then the space of polynomial functions is contained in the Hilbert space $L^2(\mathbb{R}, \mu)$. The Gram-Schmidt procedure which provides orthogonal polynomials depends only on the moment sequence.

Let $\{p_n(x)\}_{n=0,1,\ldots}$ be the monic orthogonal polynomials of μ such that the degree of p_n equals to n. Then a relation among consecutive three terms

$$\begin{aligned}
p_0(x) &= 1, \\
x p_0(x) &= p_1(x) + \alpha_0 p_0(x), \\
x p_n(x) &= p_{n+1}(x) + \alpha_n\, p_n(x) + \omega_{n-1/2}\, p_{n-1}(x), \quad n \geq 1
\end{aligned}$$

holds, if we appropriately choose the real numbers α_n, $\omega_{n-1/2}$. It is not hard to prove that $\omega_{n-1/2}$ is positive for every n, if the support of μ is an infinite set. Thus we obtain a Jacobi sequence $(\{\omega_{n+1/2}\}, \{\alpha_n\})$ out of the measure μ.

Let P_n denote the normalized orthogonal polynomial $p_n/\|p_n\|_2$. It has been proved that the isometry $U \colon \Gamma_{\omega,\alpha} \to L^2(\mathbb{R}, \mu) \colon \Phi_n \mapsto P_n$ satisfies that $U^* x U = A + B + C$, where x stands for the multiplication operator acting on $L^2(\mathbb{R}, \mu)$. See [10, Theorem 1.51] for the proof. This means that we can decompose a measure-theoretic random variable into the sum of non-commutative algebraic random variables. This crucial idea in algebraic probability theory is called “quantum decomposition” in [9] (see also [10, Section 1.5]). Through the equality $U^* x U = A + B + C$, we obtain an identity for the moments $A + B + C \sim_{\varphi_n} |P_n(x)|^2 \mu(dx)$.

Remark 2.8. For every algebraic probability space $(\mathcal{A}, \varphi)$ and every self-adjoint algebraic random variable $X \in \mathcal{A}$, it is known that there exists a probability measure μ on $\mathbb{R}$ which satisfies $X \sim_{\varphi} \mu$.

3. Quantum-classical correspondence for the Harmonic oscillator

The interacting Fock space corresponding to $\omega_{n+1/2} = n + 1, \alpha_n = 0$ is called the "quantum harmonic oscillator." For the quantum harmonic oscillator, it is well known that

$$X := A + B + C = A + C$$

represents the "position" and that

$$X \sim_{\varphi_0} \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big) dx.$$

That is, in the $n = 0$ case the distribution of the position is Gaussian.

The asymptotic behavior of the distributions of position is nontrivial. What is the "classical limit" of the quantum harmonic oscillator? This question, which is related to fundamental problems in quantum theory and asymptotic analysis [7, Section 3.6], was analyzed in [14, Section 3] from the viewpoint of non-commutative algebraic probability with quite a simple combinatorial argument. The answer is nothing but the arcsine law.

Theorem 3.1 ([14, Theorem 3.1]). *Let $\Gamma_{\omega,\alpha} := (\Gamma(\mathbb{C}), A, B \equiv 0, C)$ be the quantum harmonic oscillator, $X := A + C$ and μ_n be a probability distribution on $\mathbb{R}$ such that*

$$\frac{X}{\sqrt{2k+1}} \sim_{\varphi_k} \mu_k.$$

Then μ_n weakly converges to the arcsine law μ_{As}.

Here $\sqrt{2k+1}$ is the normalization factor to make the variance one, that is,

$$\Big\langle \Big(\frac{X}{\sqrt{2k+1}}\Big)^2 \Phi_k, \Phi_k \Big\rangle = 1.$$

Since it is easy to see that the arcsine law gives "time-averaged behavior" of the classical harmonic oscillator, the result can be viewed as "quantum-classical correspondence" for harmonic oscillators.

As the case for the quantum harmonic oscillator, we define the notion of classical limit distribution for interacting Fock spaces. It is a distribution to which the distribution for X under φ_n, after normalization, converges in moment.

Definition 3.2 (classical limit distribution). Let $\Gamma_{\omega,\alpha} := (\Gamma(\mathbb{C}), A, B, C)$ be an interacting Fock space and let X be $A+B+C$. Let μ_n be a probability distribution on $\mathbb{R}$ such that

$$\frac{X - \alpha_n}{\sqrt{\omega_{n+1/2} + \omega_{n-1/2}}} \sim_{\varphi_n} \mu_n.$$

A probability distribution μ on $\mathbb{R}$ is called a *classical limit distribution* of $\Gamma_{\omega,\alpha}$, if μ_n converges μ in moment.

By the normalizations $-\alpha_n$ and $\cdot/\sqrt{\omega_{n+1/2} + \omega_{n-1/2}}$, the measure μ_n has mean 0 and variance 1.

Remark 3.3. Existence of a classical limit distribution depends on the Jacobi sequence (ω, α). In many cases which historically attract attention, the limit exists. See Remark 5.4.

Uniqueness of the classical limit distribution relates to the moment problem. In the case that the limit distribution is a unique solution of a moment problem, moment convergence implies weak convergence [6, Theorem 4.5.5.].

Thus, a classical limit distribution of an interacting Fock space is also a weak limit of measures defined by square of orthogonal polynomials. For example, in the case of Gaussian distribution, Theorem 3.1 implies the following. Let P_k be the sequence of normalized Hermite polynomials. Then

$$|P_k(x)|^2 \frac{\exp(-x^2/2)}{\sqrt{2\pi}} dx$$

defines a sequence of probability measures whose second moment is $2k + 1$. The sequence of normalizations

$$\sqrt{2k+1}\left|P_k\left(\sqrt{2k+1}x\right)\right|^2 \frac{\exp(-(2k+1)x^2/2)}{\sqrt{2\pi}} dx$$

weakly converges to the arcsine law

$$\mu_{As}(dx) = \frac{dx}{\pi\sqrt{2-x^2}}.$$

4. Two-sided infinite Jacobi sequences

In this section, we set up a framework to analyze classical limit distributions. We introduce two-sided infinite Jacobi sequences.

Definition 4.1 (Two-sided Jacobi sequence). Let

$$\omega = \left\{\omega_m \geq 0 \;\middle|\; m = \dots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots\right\},$$

$$\alpha = \{\alpha_n \in \mathbb{R} \mid n = \dots, -2, -1, 0, 1, \dots\}$$

be two-sided infinite sequences of reals satisfying one of the following conditions (1) or (2):

(1) there exists a non-positive integer N such that

- if $m < N$, then $\omega_m = 0$,
- if $m > N$, then $\omega_m > 0$,
- and if $n < N$, then $\alpha_n = 0$;

(2) for every half integers $m = \dots, -3/2, -1/2, 1/2, 3/2, \dots$, we have $\omega_m > 0$.

We call the pair (ω, α) a *two-sided Jacobi sequence.*

Definition 4.2 (two-sided interacting Fock space). Let (ω, α) be a two-sided Jacobi sequence. An interacting Fock space $\Gamma_{\omega,\alpha}$ is a quadruple $(\Gamma(\mathbb{C}), A, B, C)$ consists of a pre-Hilbert space $\Gamma(\mathbb{C}) = \bigoplus_{n=-\infty}^{\infty} \mathbb{C}\Phi_n$ with inner product given by $\langle \Phi_n, \Phi_m \rangle = \delta_{n,m}$, and operators A, B, C defined as follows:

- A is the annihilation operator $A\Phi_n = \sqrt{\omega_{n-1/2}}\,\Phi_{n-1}$;
- B is the preservation operator $B\Phi_n = \alpha_n \Phi_n$;
- C is the creation operator $C\Phi_n = \sqrt{\omega_{n+1/2}}\,\Phi_{n+1}$.

Definition 4.3. The summation $X = A + B + C$ is expressed by the tridiagonal matrix $X = [X_{m,n}]_{m,n\in\mathbb{Z}}$ whose matrix coefficients are given by

$$X_{m,n} = \begin{cases} \sqrt{\omega_{n-1/2}} & m = n-1, \\ \alpha_n, & m = n, \\ \sqrt{\omega_{n+1/2}} & m = n+1, \\ 0, & |m-n| \geq 2. \end{cases}$$

This operator X is called the *two-sided Jacobi matrix* of (ω, α).

The matrix X is an algebraic random variable. Its moments with respect to the state $\langle \cdot \Phi_0, \Phi_0 \rangle$ can be described by the matrix entries as follows:

$$\langle X^1 \Phi_0, \Phi_0 \rangle = \alpha_0,$$

$$\langle X^2 \Phi_0, \Phi_0 \rangle = \omega_{-1/2} + \alpha_0^2 + \omega_{-1/2},$$

$$\langle X^3 \Phi_0, \Phi_0 \rangle = \omega_{-1/2}\alpha_{-1} + 2\omega_{-1/2}\alpha_0 + \alpha_0^3 + 2\omega_{1/2}\alpha_0 + \omega_{1/2}\alpha_1,$$

and so on.

Lemma 4.4. *The matrix coefficients of X^k are described by polynomials of $\{\sqrt{\omega_{n+1/2}}\} \cup \{\alpha_n\}$.*

Proof. The above claim holds for $X^0 = \mathrm{id}$ and for $X^1 = X$.

We prove the general case by induction. Supposing that the lemma holds for X^{k-1}, X^k, we prove that the matrix coefficients of X^{k+1} are described by polynomials of $\{\sqrt{\omega_{n+1/2}}\} \cup \{\alpha_n\}$.

Let m, n be arbitrary integers. The (m, n)-entry of X^{k+1} is described by $\langle X^{k+1}\Phi_n, \Phi_m \rangle$. Since the matrix expression of X is symmetric, we have

$$\begin{aligned}
&\langle X^{k+1}\Phi_n, \Phi_m \rangle \\
&\quad = \langle X^k \Phi_n, X\Phi_m \rangle \\
&\quad = \langle X^k \Phi_n, A\Phi_m + B\Phi_m + C\Phi_m \rangle \\
&\quad = \langle X^k \Phi_n, \sqrt{\omega_{n-1/2}}\Phi_{m-1} + \alpha_n \Phi_m + \sqrt{\omega_{n+1/2}}\Phi_{m+1} \rangle \\
&\quad = \sqrt{\omega_{n-1/2}}\langle X^k \Phi_n, \Phi_{m-1} \rangle + \alpha_n \langle X^k \Phi_n, \Phi_m \rangle + \sqrt{\omega_{n+1/2}}\langle X^k \Phi_n, \Phi_{m+1} \rangle.
\end{aligned}$$

The induction hypothesis, this can be expressed by a polynomial of $\{\sqrt{\omega_{n+1/2}}\} \cup \{\alpha_n\}$. □

In fact, the moments $\langle X^k \Phi_0, \Phi_0 \rangle$ of X is described by a polynomial of $\{\omega_{n+1/2}\} \cup \{\alpha_n\}$. The weaker claim in the above lemma suffices to imply the following.

Lemma 4.5. *Let $\{(\omega^{(k)}, \alpha^{(k)})\}_k$ be a sequence of two-sided Jacobi sequences and let (ω, α) be a two-sided Jacobi sequence. Let $X^{(k)}$ and X be the corresponding Jacobi matrices acting on $\bigoplus_{n=-\infty}^{\infty} \mathbb{C}\Phi_n$. If $\lim_{k\to\infty} \omega_{n+1/2}^{(k)} = \omega_{n+1/2}$ and $\lim_{k\to\infty} \alpha_n^{(k)} = \alpha_n$ for every integer n, then we have the following moment convergence: $\lim\limits_{k\to\infty} \langle (X^{(k)})^m \Phi_0, \Phi_0 \rangle = \langle X^m \Phi_0, \Phi_0 \rangle$.*

Proof. By Lemma 4.4, the m-th moment $\langle (X^{(k)})^m \Phi_0, \Phi_0 \rangle$ of $X^{(k)}$ is expressed by a polynomial of $\{\sqrt{\omega^{(k)}_{n+1/2}}\} \cup \{\alpha^{(k)}_n\}$. If $\lim_{k\to\infty} \omega^{(k)}_{n+1/2} = \omega_{n+1/2}$ and $\lim_{k\to\infty} \alpha^{(k)}_n = \alpha_n$ for every integer n, then the polynomial also converges. The limit is the m-th moment $\langle (X)^m \Phi_0, \Phi_0 \rangle$ of X. $\square$

5. The arcsine law as classical limit distribution

5.1. Relative asymptotic commutativity (RACl) . In this part, we propose a condition (RACl) for the one-sided interacting Fock space $\Gamma_{\omega,\alpha}$. The condition handles asymptotic behavior of creation C, preservation B, and annihilation A modulo standard variance.

Definition 5.1. The interacting Fock space is said to satisfy (RACl), if the commutators $[A, C]$ and $[A, B]$ are asymptotically zero in the following sense:

$$\lim_{n\to\infty} \frac{AC - CA}{\omega_{n+1/2} + \omega_{n-1/2}} \Phi_n = 0, \qquad \lim_{n\to\infty} \frac{AB - BA}{\omega_{n+1/2} + \omega_{n-1/2}} \Phi_n = 0.$$

Recall that $\langle \cdot\, \Phi_n, \Phi_n \rangle$ stands for the n-th state of the interacting Fock space. The denominator $\omega_{n+1/2}+\omega_{n-1/2}$ is the variance of the algebraic random variable $X = A + B + C$ with respect to the state $\langle \cdot\, \Phi_n, \Phi_n \rangle$.

Lemma 5.2. *The condition* (RACl) *is equivalent to*

$$\lim_{n\to\infty} \frac{\omega_{n+1/2}}{\omega_{n-1/2}} = 1, \qquad \lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\sqrt{\omega_{n+1/2} + \omega_{n-1/2}}} = 0.$$

Proof. The commutators $[A, C]$ and $[A, B]$ satisfy the following:

$$\frac{AC - CA}{\omega_{n+1/2} + \omega_{n-1/2}} \Phi_n = \frac{\omega_{n+1/2} - \omega_{n-1/2}}{\omega_{n+1/2} + \omega_{n-1/2}} \Phi_n,$$

$$\frac{AB - BA}{\omega_{n+1/2} + \omega_{n-1/2}} \Phi_n = \frac{\alpha_n - \alpha_{n-1}}{\omega_{n+1/2} + \omega_{n-1/2}} \sqrt{\omega_{n-1/2}} \Phi_{n-1}.$$

Thus we have

$$\frac{AC - CA}{\omega_{n+1/2} + \omega_{n-1/2}} \Phi_n = \frac{\dfrac{\omega_{n+1/2}}{\omega_{n-1/2}} - 1}{\dfrac{\omega_{n+1/2}}{\omega_{n-1/2}} + 1} \Phi_n,$$

$$\frac{AB - BA}{\omega_{n+1/2} + \omega_{n-1/2}} \Phi_n = \frac{\alpha_n - \alpha_{n-1}}{\sqrt{\omega_{n+1/2} + \omega_{n-1/2}}} \frac{1}{\sqrt{\dfrac{\omega_{n+1/2}}{\omega_{n-1/2}} + 1}} \Phi_{n-1}.$$

The latter conditions in the lemma imply that the right hand sides converges to 0. Then the condition (RAC1) follows.

Conversely, we suppose that (RAC1) holds. In this case, we have

$$\lim_{n\to\infty}\frac{\omega_{n+1/2}-\omega_{n-1/2}}{\omega_{n+1/2}+\omega_{n-1/2}}=0,\qquad \lim_{n\to\infty}\frac{\alpha_n-\alpha_{n-1}}{\omega_{n+1/2}+\omega_{n-1/2}}\sqrt{\omega_{n-1/2}}=0.$$

By the equality

$$\frac{2}{1-\dfrac{\omega_{n+1/2}-\omega_{n-1/2}}{\omega_{n+1/2}+\omega_{n-1/2}}}-1=\frac{\omega_{n+1/2}}{\omega_{n-1/2}},$$

the first condition of (RAC1) implies that

$$\lim_{n\to\infty}\frac{\omega_{n+1/2}}{\omega_{n-1/2}}=\frac{2}{1-0}-1=1.$$

The second condition of (RAC1) implies that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\alpha_n-\alpha_{n-1}}{\sqrt{\omega_{n+1/2}+\omega_{n-1/2}}}\\
&=\lim_{n\to\infty}\frac{\alpha_n-\alpha_{n-1}}{\omega_{n+1/2}+\omega_{n-1/2}}\sqrt{\omega_{n-1/2}}\lim_{n\to\infty}\sqrt{\frac{\omega_{n+1/2}+\omega_{n-1/2}}{\omega_{n-1/2}}}\\
&=0\sqrt{2}\\
&=0.
\end{aligned}$$

Now we obtain the conditions in the lemma. □

The quantum harmonic oscillator introduced in Section 3 satisfies the above condition. The following theorem is the main result in this paper, which generalizes Theorem 3.1.

Theorem 5.3. *Let* $\Gamma_{\omega,\alpha} := (\Gamma(\mathbb{C}), A, B, C)$ *be an interacting Fock space satisfying asymptotic commutativity* (RAC1)*. Then the classical limit distribution given in Definition* 3.2 *exists and is the arcsine law* $dx/(\pi\sqrt{2-x^2})$.

Proof. Let $(\{\omega_{n+1/2}\},\{\alpha_n\})$ be a one-sided Jacobi sequence. Suppose that (RAC1) holds. Consider the k-th state $\langle\,\cdot\,\Phi_k,\Phi_k\rangle$ and the normalized algebraic random variable

$$X^{(k)}=\frac{X-\alpha_k}{\sqrt{\omega_{k+1/2}+\omega_{k-1/2}}}$$

acting on $\bigoplus_{n=0}^{\infty} \mathbb{C}\Phi_n$. The matrix coefficients are described by

$$X^{(k)}_{m,n} = \begin{cases} \dfrac{\omega_{n-1/2}}{\sqrt{\omega_{k+1/2} + \omega_{k-1/2}}}, & m = n-1, \\ \dfrac{\alpha_n - \alpha_k}{\sqrt{\omega_{k+1/2} + \omega_{k-1/2}}}, & m = n, \\ \dfrac{\omega_{n+1/2}}{\sqrt{\omega_{k+1/2} + \omega_{k-1/2}}}, & m = n+1, \\ 0, & |m-n| \geq 2. \end{cases}$$

To study asymptotic behavior of $X^{(k)}$ acting on $\bigoplus_{n=0}^{\infty} \mathbb{C}\Phi_n$, we change the index $m, n = 0, 1, \ldots, k, \ldots$ to $m, n = -k, -k+1, \ldots, 0, \ldots$, and exploit two-sided interacting Fock space $\Gamma^{(k)} = \bigoplus_{n=-k}^{\infty} \mathbb{C}\Phi_n$. We now consider the state $\langle \cdot \Phi_0, \Phi_0 \rangle$ and the algebraic random variable $\widetilde{X^{(k)}}$ defined by

$$\widetilde{X^{(k)}}_{m,n} = \begin{cases} \dfrac{\omega_{n+k-1/2}}{\sqrt{\omega_{k+1/2} + \omega_{k-1/2}}}, & m = n-1, \\ \dfrac{\alpha_{n+k} - \alpha_k}{\sqrt{\omega_{k+1/2} + \omega_{k-1/2}}}, & m = n, \\ \dfrac{\omega_{n+k+1/2}}{\sqrt{\omega_{k+1/2} + \omega_{k-1/2}}}, & m = n+1, \\ 0, & |m-n| \geq 2. \end{cases}$$

By the first condition of Lemma 5.2, the neighboring ratio of $\{\omega_{n+k+1/2}\}_{n=-k}^{\infty}$ is 1. This implies that for every fixed integer n,

$$\lim_{k\to\infty} \widetilde{X^{(k)}}_{n-1,n} = \frac{1}{\sqrt{2}} = \lim_{k\to\infty} \widetilde{X^{(k)}}_{n+1,n}.$$

By the second condition of Lemma 5.2,

$$\lim_{k\to\infty} \frac{\alpha_k - \alpha_{k-1}}{\sqrt{\omega_{k+1/2} + \omega_{k-1/2}}} = 0.$$

Together with

$$\lim_{k\to\infty} \frac{\omega_{n+k+1/2}}{\omega_{k+1/2}} = 0,$$

this implies that for every n,

$$\lim_{k\to\infty} \widetilde{X^{(k)}}_{n,n} = 0.$$

Now we exploit Lemma 4.5. Let $\widetilde{X}$ be the two-sided infinite matrix

$$\widetilde{X} = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & 0 & 1/\sqrt{2} & & \\ & 1/\sqrt{2} & \mathbf{0} & 1/\sqrt{2} & \\ & & 1/\sqrt{2} & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

acting on $\ell^2(\mathbb{Z})$. The bold zero $\mathbf{0}$ stands for the position of the matrix coefficient $\langle \cdot \Phi_0, \Phi_0 \rangle$. By Lemma 4.5, we have

$$\lim_{k\to\infty} \langle (X^{(k)})^m \Phi_k, \Phi_k \rangle = \lim_{k\to\infty} \langle (\widetilde{X^{(k)}})^m \Phi_0, \Phi_0 \rangle = \langle (\widetilde{X})^m \Phi_0, \Phi_0 \rangle.$$

It turns out that the classical limit distribution does exist. For the rest of this proof, we calculate the limit distribution.

Now we exploit the the Fourier duality between $\mathbb{Z}$ and $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Let $\mathcal{F}\colon L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ denote the Fourier transform. We identify the characteristic function δ_n on $\{n\} \subset \mathbb{Z}$ with the vector Φ_n in the completion of the two-sided Fock space. We can describe $\mathcal{F}$ by $\mathcal{F}(z^n) = \Phi_n$, where z^n stands for the function $\mathbb{T} \ni z \mapsto z^n \in \mathbb{C}$. The restriction of $\mathcal{F}$ gives a surjective isometry between the polynomial functions on $\mathbb{T}$ and the two-sided Fock space.

Since the operator $\widetilde{X}$ maps Φ_n to $\Phi_{n-1}/\sqrt{2} + \Phi_{n+1}/\sqrt{2}$, its Fourier transform $\mathcal{F}^{-1}\widetilde{X}\mathcal{F}$ maps z^n to

$$\frac{z^{n-1}}{\sqrt{2}} + \frac{z^{n+1}}{\sqrt{2}} = \left(\frac{\bar{z}}{\sqrt{2}} + \frac{z}{\sqrt{2}}\right) z^n = \sqrt{2}\mathrm{Re}(z) z^n$$

This means that the operator $\mathcal{F}^{-1}\widetilde{X}\mathcal{F}$ is the multiplication operator by the function $\mathbb{T} \ni z \mapsto \sqrt{2}\mathrm{Re}(z) \in \mathbb{R}$

Thus we have

$$\begin{aligned} \langle (\widetilde{X})^m \Phi_0, \Phi_0 \rangle_{\ell^2(\mathbb{Z})} &= \langle (\sqrt{2}\mathrm{Re}(z))^m 1, 1 \rangle_{L^2(\mathbb{T})} \\ &= \int_{\mathbb{T}} (\sqrt{2}\mathrm{Re}(z))^m d(\text{Haar measure}) \\ &= \int_0^{2\pi} (\sqrt{2}\mathrm{Re}(e^{it}))^m \frac{dt}{2\pi} \\ &= \int_\pi^{2\pi} (\sqrt{2}\cos t)^m \frac{dt}{\pi}. \end{aligned}$$

Replacing $\sqrt{2}\cos t$ with x, we have

$$\lim_{k\to\infty} \langle (X^{(k)})^m \Phi_k, \Phi_k \rangle = \int_{-\sqrt{2}}^{\sqrt{2}} x^m \frac{dx}{\pi\sqrt{2-x^2}}. \qquad \square$$

Remark 5.4. The theorem means that the arcsine law is turned out to be the classical limit distribution in many cases. We pick up several examples.

(1) The interacting Fock spaces corresponding to the uniform distribution $\chi_{[-1,1]}dx/2$, are described by the Jacobi sequence

$$\omega_{n+1/2} = \frac{(n+1)^2}{(2n+1)(2n+3)}, \quad \alpha_n = 0.$$

(2) The quantum decomposition of the exponential distribution $\chi_{[0,\infty)}e^{-x}dx$ is given by the Jacobi sequence

$$\omega_{n+1/2} = (n+1)^2, \quad \alpha_n = 2n+1.$$

(3) q-Gaussians $(-1 < q \le 1)$ are probability measures on $\mathbb{R}$ given by the Jacobi sequence

$$\omega_{n+1/2} = 1 + q + q^2 + \cdots + q^n, \quad \alpha_n = 0.$$

The case of $q = 1$ corresponds to the Gaussian measure. The case of $q = 0$ corresponds to the semicircle law $(\sqrt{4-x^2}\,dx)/(2\pi)$ of Wigner.

By Lemma 5.2, these interacting Fock spaces satisfy (RAC1).

Remark 5.5. Since the arcsine law is the solution of a determinate moment problem, moment convergence implies weak convergence.

Remark 5.6. It is quite interesting to compare Kerov's theorem [11, Theorem 33]. It is well known that the roots of $(n-1)$-th and n-th orthogonal polynomials interlace. The mutual relationship of interlacing roots can be represented by a rectangular diagram (a continuous version of Young diagram, for more details see [11, Chapter 0, Section 4]). Kerov's theorem states that the asymptotic behavior of the rectangular diagrams obeys "the arcsine law" (which is different from the probability measure $dx/(\pi\sqrt{2-x^2})$ but closely related to it), for the orthogonal polynomials satisfying the condition corresponding to (RAC1).

Theorem 5.3 implies the following asymptotic behavior of orthogonal polynomials:

Corollary 5.7. *Let μ be a probability measure such that the corresponding Jacobi sequence $(\{\omega_n\}, \{\alpha_n\})$ satisfies* (RAC1). *Let P_n be the normalized orthogonal polynomial with degree n. The measure μ_n defined as*

$$\mu_n(dx) := |P_n(\sqrt{\omega_{n+1/2} + \omega_{n-1/2}}x)|^2 \mu(\sqrt{\omega_{n+1/2} + \omega_{n-1/2}}dx)$$

weakly converge to the arcsine law μ_{As}.

It turns out that many kinds of orthogonal polynomials such as Legendre polynomials, Laguerre polynomials or q-Hermite polynomials for $-1 < q \leq 1$ satisfy the above condition.

6. Weaker form of asymptotic commutativity and classical limits

It is reasonable to guess that we can obtain other types of classical limits assuming a weaker condition on the operators A, B, C. Relaxing the commutativity condition between A and B, we have discretized arcsine laws as classical limits.

Definition 6.1. The interacting Fock space is said to satisfy (RAC2), if the commutator $[A, C]$ is asymptotically zero and if $[A, B]$ is asymptotically a scalar multiple of A in the following sense:

- $\displaystyle\lim_{n\to\infty} \frac{AC - CA}{\omega_{n+1/2} + \omega_{n-1/2}} \Phi_n = 0$ and
- there exists a real number r satisfying

$$\lim_{n\to\infty} \frac{(AB - BA) - rA}{\omega_{n+1/2} + \omega_{n-1/2}} \Phi_n = 0.$$

Recall that $\omega_{n+1/2} + \omega_{n-1/2}$ is the variance of $X = A + B + C$ with respect to $\langle \cdot \Phi_n, \Phi_n \rangle$. Calculation on the matrix coefficients yields the following lemma. The proof is almost the same as that of Lemma 5.2.

Lemma 6.2. *The condition* (RAC2) *is equivalent to*

$$\lim_{n\to\infty} \frac{\omega_{n+1/2}}{\omega_{n-1/2}} = 1$$

and convergence of the sequence $\left\{\dfrac{\alpha_n - \alpha_{n-1}}{\sqrt{\omega_{n+1/2} + \omega_{n-1/2}}}\right\}_n.$

In the following subsection, we denote by c the limit of the latter sequence.

Example 6.3.
- An interacting Fock space with (RAC1) satisfies (RAC2).
- The one-sided interacting Fock space $\Gamma_{\omega,\alpha}$ defined by $\omega_{n+1/2} = 1/2$ and $\alpha_n = cn$ shares the property (RAC2). The infinite Jacobi matrix is given by

$$X = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 & \\ 1/\sqrt{2} & c & 1/\sqrt{2} & \ddots \\ 0 & 1/\sqrt{2} & 2c & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

6.1. Calculation of the classical limits. From now on, we consider the case that the interacting Fock space satisfy (RAC2). Let $X^{(k)}$ be the random variable

$$\frac{X - \alpha_k}{\sqrt{\omega_{k+1/2} + \omega_{k-1/2}}}.$$

Observing the Jacobi sequence, we obtain the following lemma.

Lemma 6.4. *For every integers m, n, we have*

$$\lim_{k\to\infty} \langle X^{(k)} \Phi_{k+m}, \Phi_{k+n} \rangle = \begin{cases} 1/\sqrt{2}, & m = n-1, \\ cn, & m = n, \\ 1/\sqrt{2}, & m = n+1, \\ 0, & |m-n| \geq 2. \end{cases}$$

Proof. The proof is similar to the first half of the proof of Theorem 5.3. By the definition of the operators A, B, C, the operator $X^{(k)}$ satisfies

$$\langle X^{(k)} \Phi_{k+m}, \Phi_{k+n} \rangle = \begin{cases} \dfrac{\omega_{n+k-1/2}}{\sqrt{\omega_{k+1/2} + \omega_{k-1/2}}}, & m = n-1, \\ \dfrac{\alpha_{n+k} - \alpha_k}{\sqrt{\omega_{k+1/2} + \omega_{k-1/2}}}, & m = n, \\ \dfrac{\omega_{n+k+1/2}}{\sqrt{\omega_{k+1/2} + \omega_{k-1/2}}}, & m = n+1, \\ 0, & |m-n| \geq 2. \end{cases}$$

The conditions in Lemma 6.2 imply the above lemma. $\square$

We grasp the asymptotic behavior of $X^{(k)}$ with respect to the state $\langle \cdot\, \Phi_k, \Phi_k\rangle$, using two-sided infinite tridiagonal matrices acting on the inner product space $\bigoplus_{k\in\mathbb{Z}} \mathbb{C}\Phi_k$. Putting the limit of the matrix coefficient $\langle X^{(k)}\Phi_{k+m}, \Phi_{k+n}\rangle$ at (m,n)-entry, we obtain the following tridiagonal operator:

$$\widetilde{X} = \begin{pmatrix} \ddots & \ddots & \ddots & & & & \\ \ddots & -2c & 1/\sqrt{2} & 0 & & & \\ \ddots & 1/\sqrt{2} & -c & 1/\sqrt{2} & 0 & & \\ & 0 & 1/\sqrt{2} & \mathbf{0} & 1/\sqrt{2} & 0 & \\ & & 0 & 1/\sqrt{2} & c & 1/\sqrt{2} & \ddots \\ & & & 0 & 1/\sqrt{2} & 2c & \ddots \\ & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where "$\mathbf{0}$" is at the position of $(0,0)$. By Lemma 4.5, convergence of the matrix coefficients implies the following moment convergence:

$$\lim_{n\to\infty} \langle (X^{(k)})^m \Phi_k, \Phi_k\rangle = \langle \widetilde{X}^m \Phi_0, \Phi_0\rangle. \tag{6.1}$$

To see the moment sequence $\{\langle \widetilde{X}^m \Phi_0, \Phi_0\rangle\}$ of $\widetilde{X}$, we study the densely defined operator $\widetilde{X}$ acting on $\ell^2(\mathbb{Z}) = \overline{\bigoplus_{k\in\mathbb{Z}} \mathbb{C}\Phi_k}$. Via the Fourier transform $\ell^2(\mathbb{Z}) \cong L^2(\mathbb{T})$, we may regard $\widetilde{X}$ as a densely defined symmetric operator acting on $L^2(\{e^{it}\})$. The space of Laurent polynomials of $z = e^{it}$ is the domain of $\widetilde{X}$.

For a bounded measurable function f on $\mathbb{T} = \{e^{it} \mid t \in \mathbb{R}\}$, we denote by $M[f]$ the multiplication operator $L^2(\mathbb{T}) \ni g \mapsto fg \in L^2(\mathbb{T})$. The operator $\widetilde{X}$ acts on the Laurent polynomials as follows:

- the annihilation part of $\widetilde{X}$ is identified with the multiplication operator $M[e^{-it}/\sqrt{2}]$,
- the diagonal part of $\widetilde{X}$ is identified with the differential operator $(c/i)d/dt$.
- the creation part of $\widetilde{X}$ is identified with the multiplication operator $M[e^{it}/\sqrt{2}]$,

In the case that $c \neq 0$, the summation is expressed by

$$M\Big[\frac{e^{-it}}{\sqrt{2}}\Big] + \frac{c}{i}\frac{d}{dt} + M\Big[\frac{e^{it}}{\sqrt{2}}\Big] = \frac{c}{i}\Big(\frac{d}{dt} + iM\Big[\frac{\sqrt{2}\cos t}{c}\Big]\Big),$$

We may further calculate

$$
\begin{aligned}
& M\Big[\frac{e^{-it}}{\sqrt{2}}\Big] + \frac{c}{i}\frac{d}{dt} + M\Big[\frac{e^{it}}{\sqrt{2}}\Big] \\
&= \frac{c}{i} M\Big[\exp\Big(-i\frac{\sqrt{2}\sin t}{c}\Big)\Big] \circ \frac{d}{dt} \circ M\Big[\exp\Big(i\frac{\sqrt{2}\sin t}{c}\Big)\Big] \\
&= M\Big[\exp\Big(-i\frac{\sqrt{2}\sin t}{c}\Big)\Big] \circ \Big(\frac{c}{i}\frac{d}{dt}\Big) \circ M\Big[\exp(i\frac{\sqrt{2}\sin t}{c})\Big].
\end{aligned}
$$

We can easily prove the above equation by hitting an arbitrary Laurent polynomial of e^{it}. We note that the absolute value of $\exp\big(i\frac{\sqrt{2}\sin t}{c}\big)$ is 1. Define $a_n(c)$ by the Fourier expansion

$$
\exp\Big(i\frac{\sqrt{2}\sin t}{c}\Big) = \sum_{n\in\mathbb{Z}} a_n(c)e^{int}.
$$

Definition 6.5. For $x \in \mathbb{R}$, we denote by δ_x the probability measure concentrated on x. The probability measure

$$
\mu_c = \sum_{n\in\mathbb{Z}} |a_n(c)|^2 \delta_{cn}
$$

on $\mathbb{R}$ is called a *discrete arcsine distribution.*

Theorem 6.6. *Suppose that the interacting Fock space $\Gamma_{\{\omega_n\},\{\alpha_n\}}$ satisfy the condition* (RAC2) *but does not satisfy* (RAC1). *Define a real number c by*

$$
\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\sqrt{\omega_{n-1/2} + \omega_{n+1/2}}}.
$$

Then for each natural number m, we have the following moment convergence:

$$
\lim_{k\to\infty} \Big\langle\Big(\frac{X-\alpha_k}{\sqrt{\omega_{k+1/2} + \omega_{k-1/2}}}\Big)^m \Phi_k, \Phi_k\Big\rangle = \int_{\mathbb{R}} x^m \mu_c(dx).
$$

Proof. By the equation (6.1), it suffices to show that

$$
\langle \widetilde{X}^k\, \Phi_0, \Phi_0\rangle = \int_{\mathbb{R}} x^m \mu_c(dx).
$$

Considering the Fourier transform, the left hand side is equal to

$$
\begin{aligned}
&\Big\langle \Big(M\Big[\frac{e^{-it}}{\sqrt{2}}\Big] + \frac{c}{i}\frac{d}{dt} + M\Big[\frac{e^{it}}{\sqrt{2}}\Big]\Big)^m 1, 1\Big\rangle_{L^2(\{e^{it}\})} \\
&\quad = \Big\langle \Big\{ M\Big[\exp\Big(-i\frac{\sqrt{2}\sin t}{c}\Big)\Big] \circ \Big(\frac{c}{i}\frac{d}{dt}\Big) \circ M\Big[\exp\Big(i\frac{\sqrt{2}\sin t}{c}\Big)\Big]\Big\}^m 1, 1\Big\rangle_{L^2(\{e^{it}\})} \\
&\quad = \Big\langle M\Big[\exp\Big(-i\frac{\sqrt{2}\sin t}{c}\Big)\Big] \circ \Big(\frac{c}{i}\frac{d}{dt}\Big)^m \circ M\Big[\exp\Big(i\frac{\sqrt{2}\sin t}{c}\Big)\Big] 1, 1\Big\rangle_{L^2(\{e^{it}\})} \\
&\quad = \Big\langle \Big(\frac{c}{i}\frac{d}{dt}\Big)^m \exp\Big(i\frac{\sqrt{2}\sin t}{c}\Big), \exp\Big(i\frac{\sqrt{2}\sin t}{c}\Big)\Big\rangle_{L^2(\{e^{it}\})}.
\end{aligned}
$$

By the Fourier expansion of $\exp\big(i\frac{\sqrt{2}\sin t}{c}\big)$, the above quantity is

$$
\begin{aligned}
&\Big\langle \Big(\frac{c}{i}\Big)^m \frac{d^m}{dt^m} \exp\Big(i\frac{\sqrt{2}\sin t}{c}\Big), \sum_{n\in\mathbb{Z}} a_n(c) e^{int}\Big\rangle_{L^2(\{e^{it}\})} \\
&\quad = \sum_{n\in\mathbb{Z}} \overline{a_n(c)} \Big\langle \Big(\frac{c}{i}\Big)^m \frac{d^m}{dt^m} \exp\Big(i\frac{\sqrt{2}\sin t}{c}\Big), e^{int}\Big\rangle_{L^2(\{e^{it}\})}.
\end{aligned}
$$

By iteration of partial integration, this is equal to

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}} \overline{a_n(c)} \Big\langle \exp\Big(i\frac{\sqrt{2}\sin t}{c}\Big), \Big(\frac{c}{i}\Big)^m \frac{d^m}{dt^m} e^{int}\Big\rangle_{L^2(\{e^{it}\})} \\
&\quad = \sum_{n\in\mathbb{Z}} (cn)^m \overline{a_n(c)} \Big\langle \exp\Big(i\frac{\sqrt{2}\sin t}{c}\Big), e^{int}\Big\rangle_{L^2(\{e^{it}\})} \\
&\quad = \sum_{n\in\mathbb{Z}} (cn)^m |a_n(c)|^2.
\end{aligned}
$$

This is nothing other than $\int_{\mathbb{R}} x^m \mu_c(dx)$. □

6.2. Calculation of the discrete arcsine law μ_c. To identify the discrete arcsine law μ_c, we have only to calculate the Fourier expansion of $\exp\big(i\frac{\sqrt{2}\sin t}{c}\big)$. By the Maclaurin expansion of the exponential function, we have

$$
\exp\Big(i\frac{\sqrt{2}\sin t}{c}\Big) = \exp\Big(\frac{e^{it} - e^{-it}}{\sqrt{2}c}\Big) = \sum_{k=0}^{\infty} \frac{1}{k!}\Big(\frac{e^{it} - e^{-it}}{\sqrt{2}c}\Big)^k.
$$

By the binomial theorem, we have

$$\exp\left(i\frac{\sqrt{2}\sin t}{c}\right) = \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{l=0}^{k}\frac{(-1)^l}{(\sqrt{2}c)^k}\binom{k}{l}e^{i(k-l)t}e^{-ilt}$$
$$= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(-1)^l}{(\sqrt{2}c)^k}\frac{1}{l!(k-l)!}e^{i(k-2l)t}.$$

It is not hard to check that this summation of the absolute values uniformly converges. Therefore it is possible to change the order of summation. Now we define n by $k-2l$. The condition $0 \le l \le k$ is described by $0 \le l \le n+2l$. This is equivalent to $\max\{0,-n\} \le l$. Then the Fourier expansion is described by

$$\exp\left(i\frac{\sqrt{2}\sin t}{c}\right) = \sum_{n=-\infty}^{\infty}\sum_{l=\max\{0,-n\}}^{\infty}\frac{(-1)^l}{(\sqrt{2}c)^{n+2l}}\frac{1}{l!(n+l)!}e^{int}.$$

For $n \ge 0$, we have

$$a_n(c) = \sum_{l=0}^{\infty}\frac{(-1)^l}{(\sqrt{2}c)^{n+2l}}\frac{1}{l!(n+l)!},$$
$$a_{-n}(c) = \sum_{l=n}^{\infty}\frac{(-1)^l}{(\sqrt{2}c)^{-n+2l}}\frac{1}{l!(-n+l)!} = \sum_{l=0}^{\infty}\frac{(-1)^{l+n}}{(\sqrt{2}c)^{n+2l}}\frac{1}{(n+l)!l!}.$$

Let J_n denote the n-th Bessel function of first kind

$$J_n(x) = \sum_{l=0}^{\infty}\frac{(-1)^l}{2^{2l+n}l!(l+n)!}x^{2l+n}.$$

The above Fourier coefficients are described as follows:

$$a_n(c) = J_n\left(\frac{\sqrt{2}}{c}\right), \quad a_{-n}(c) = (-1)^n J_n\left(\frac{\sqrt{2}}{c}\right).$$

Theorem 6.7. *The discrete arcsine law μ_c is a probability measure supported on $c\mathbb{Z}$. For $n = 0, 1, 2, \ldots$, the weights $\mu_c(\{cn\})$ and $\mu_c(\{-cn\})$ are given by the following*:

$$\mu_c(\{cn\}) = \mu_c(\{-cn\}) = \frac{1}{2^n c^{2n}}\left(\sum_{l=0}^{\infty}\frac{(-1)^l}{(\sqrt{2}c)^{2l}}\frac{1}{(n+l)!l!}\right)^2 = \left\{J_n\left(\frac{\sqrt{2}}{c}\right)\right\}^2.$$

Remark 6.8. We thank Prof. Marek Bożejko and Prof. Wojciech Młotkowski for pointing out the relationship between the discrete arcsine law and Bessel functions.

6.3. Remarks on the discrete arcsine law. Before closing this subsection, let us consider the limit of μ_c as $c \to 0$. The m-th moment of the discrete arcsine μ_c is given by

$$\Big\langle \Big(\frac{e^{-it}}{\sqrt{2}} + \frac{e^{it}}{\sqrt{2}} + \frac{c}{i}\frac{d}{dt}\Big)^m 1, 1\Big\rangle_{L^2(\{e^{it}\})}.$$

When c goes to 0, the moment converges to

$$\Big\langle \Big(\frac{e^{-it}}{\sqrt{2}} + \frac{e^{it}}{\sqrt{2}}\Big)^m 1, 1\Big\rangle_{L^2(\{e^{it}\})} = \int_{-\pi}^{\pi} (\sqrt{2}\cos t)^m \frac{dt}{2\pi} = \int_{-\sqrt{2}}^{\sqrt{2}} x^m \frac{dx}{\pi\sqrt{2-x^2}}.$$

This is the k-th moment of the arcsine law. Since the moment sequence of the arcsine law characterizes the measure, convergence in law implies weak convergence.

Theorem 6.9. *As $c \to 0$, the discrete arcsine law μ_c weakly converges to the arcsine law $dx/\pi\sqrt{2-x^2}$.*

If a measure on $\mathbb{R}$ has the same moment sequence as μ_c, it is identical to μ_c.

Theorem 6.10. *The discrete arcsine law μ_c is characterized by its moments.*

Proof. We exploit the Carleman's condition for the moment sequence

$$\Big\langle \Big(\frac{e^{-it}}{\sqrt{2}} + \frac{e^{it}}{\sqrt{2}} + \frac{c}{i}\frac{d}{dt}\Big)^m 1, 1\Big\rangle_{L^2(\{e^{it}\})}$$

of the discrete arcsine law. We may assume that $c > 0$, since $-c$ also gives the same moment sequence. Consider the Fourier expansion

$$\sum_n b_n^{(m)} e^{int} = \Big(\frac{e^{-it}}{\sqrt{2}} + \frac{e^{it}}{\sqrt{2}} + \frac{c}{i}\frac{d}{dt}\Big)^m 1.$$

Note that if $n \notin [-m, m]$ then $b_n^{(m)} = 0$. By the equality

$$b_n^{(m+1)} = \frac{b_{n-1}^{(m)}}{\sqrt{2}} + \frac{b_{n+1}^{(m)}}{\sqrt{2}} + \frac{c}{i} n b_n^{(k)},$$

we have

$$\sum_{n=-m-1}^{m+1} |b_n^{(m+1)}| \le \Big(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + c(m+1)\Big) \sum_{n=-m}^{m} |b_n^{(m)}|.$$

It is easy to show by induction that

$$\sum_{n=-m}^{m} |b_n^{(m)}| \leq (\sqrt{2} + cm)^m.$$

In particular the $(2m)$-th moment $b_0^{(2m)}$ is at most $(\sqrt{2} + 2cm)^{2m}$. Therefore we have

$$\sum_{m=0}^{\infty} \frac{1}{\sqrt[2m]{b_0^{(2m)}}} \geq \sum_{m=0}^{\infty} \frac{1}{\sqrt{2} + 2cm} = +\infty.$$

This means that the moment sequence of the discrete arcsine law satisfies the Carleman's condition, which is a sufficient condition for determinacy. For the Carleman's condition, we refer the readers to the book [4] by Akhiezer. □

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