Déviations pour les temps locaux d'auto-intersections

Rencontre organisée par :
Amine Asselah

06-10 décembre 2010

Dmitry Ioffe and Yvan Velenik

Random Walks in Attractive Potentials: The Case of Critical Drifts

Vol. 2, n^{o} 1 (2010), p. 11-13.

<http://acirm.cedram.org/item?id=ACIRM_2010__2_1_11_0>

Centre international de rencontres mathématiques
U.M.S. 822 C.N.R.S./S.M.F.
Luminy (Marseille) France

Actes des rencontres du C.I.R.M.
Vol. **2** n^o 1 (2010) 11-13

Random Walks in Attractive Potentials: The Case of Critical Drifts

Dmitry Ioffe and Yvan Velenik

Abstract

We consider random walks in attractive potentials - sub-additive functions of their local times. An application of a drift to such random walks leads to a phase transition: If the drift is small than the walk is still sub-ballistic, whereas the walk is ballistic if the drift is strong enough. The set of sub-critical drifts is convex with non-empty interior and can be described in terms of Lyapunov exponents (Sznitman, Zerner). Recently it was shown that super-critical drifts lead to a limiting speed. We shall explain that in dimensions $d \geq 2$ the transition is always of the first order. (Joint work with Y.Velenik)

1. Class of Models and Results

We consider nearest neighbour paths $\gamma = (\gamma(0), \ldots, \gamma(n))$ on $\mathbb{Z}^d$. The length of the path is denoted as $|\gamma| = n$ and its displacement is denoted as $\mathsf{X}(\gamma) = \gamma(n) - \gamma(0)$. Unless mentioned otherwise all the paths start at the origin, $\gamma(0) = 0$.

Paths γ are subject to a self-interacting potential $\Phi(\gamma)$ and to a drift $(h, \mathsf{X}(\gamma))$; $h \in \mathbb{R}^d$. The potential Φ is of the form:

$$\Phi(\gamma) = \sum_{x \in \mathbb{Z}^d} \phi\left(\ell_\gamma(x)\right),$$

where $\ell_\gamma(x)$ is the local time of γ at x. Here are our assumptions on ϕ:

A1. $\phi(1) > 0$ and $\phi(\ell)$ is non-decreasing in ℓ.

A2. $\phi(\ell + m) \leq \phi(\ell) + \phi(m)$.

A3. $\lim_{\ell\to\infty} \phi(\ell)/\ell = 0$.

The assumption **A2** means that Φ is a self-attractive potential. Assumption **A3** is just a normalization. Assumption **A1** ensures positivity of Lyapunov exponents (see below). The main example we have in mind is that of annealed random walks in random potentials,

$$\phi(\ell) = -\log \mathbb{E}\mathrm{e}^{-\ell V},$$

where V is a non-negative random variable with $0 \in \mathrm{supp}(V) \subseteq [0, \infty]$. Drifted Wiener sausage is a particular example. The n-step partition function is then given by

$$A_n^h = \sum_{|\gamma|=n} \left(\frac{1}{2d}\right)^{|\gamma|} \mathrm{e}^{-\Phi(\gamma)+(h,\mathsf{X}(\gamma))}.$$

Let $\mathbb{A}_n^h$ to denote the corresponding path measure. There are two competing contributions to $\mathbb{A}_n^h$: Because of the attractive nature of Φ paths prefer to collapse, whereas the drift h pulls them away. The following is known [2, 1]: Whichever h one chooses, the mean displacement $\mathsf{X}(\gamma)/n$ satisfies a large deviation principle under $\mathbb{A}_n^h$ with a convex rate function J^h. Moreover, there exists a critical set of drifts $\mathbf{K}_0$ - a compact convex subset of $\mathbb{R}^d$ with non-empty interiour $0 \in \mathrm{int}(\mathbf{K}_0)$, such that:

Text presented during the meeting "Excess Self-Intersections & Related Topics" organized by Amine Asselah. 06-10 décembre 2010, C.I.R.M. (Luminy).

Sub-ballistic drifts. If $h \in \mathrm{int}(\mathbf{K}_0)$, then J^h has a unique minimum at 0. In particular,

$$\lim_{n\to\infty} \mathbb{A}_n^h \left(\frac{\mathsf{X}(\gamma)}{n} \right) = 0.$$

Ballistic drifts. If $h \notin K_0$ then J^h has a unique minimum at some $v(h) \neq 0$. In particular,

$$\lim_{n\to\infty} \mathbb{A}_n^h \left(\frac{\mathsf{X}(\gamma)}{n} \right) = v(h).$$

Critical drifts. If $h \in \partial\mathbf{K}_0$, then $J^h(0) = 0$.

Our main result implies that in any dimension $d \geq 2$ the transition is of the first order:

Theorem A. *Let $h \in \partial\mathbf{K}_0$. Then, there exists $v(h) \neq 0$, such that the rate function J^h is zero on the segment $[0, v(h)]$ and strictly positive otherwise. Furthermore,*

$$\lim_{n\to\infty} \mathbb{A}_n^h \left(\frac{\mathsf{X}(\gamma)}{n} \right) = v(h).$$

Actually our proof of this result implies accompanying laws under $\mathbb{A}_n^h$:

Theorem B. *The set of critical drifts is regular. Namely, $\partial\mathbf{K}_0$ is locally analytic and has a uniformly positive Gaussian curvature. Let $h \in \partial\mathbf{K}_0$ and let $v(h) \neq 0$ be as above. Then,*

$$\lim_{n\to\infty} \mathbb{A}_n^h \left(\left| \frac{\mathsf{X}(\gamma)}{n} - v(h) \right| > \epsilon \right) = 0$$

for any $\epsilon > 0$. Moreover, there exists a non-degenerate covariance matrix Ξ, such that

$$\frac{\mathsf{X}(\gamma) - nv(h)}{\sqrt{n}} \Rightarrow \mathcal{N}(0, \Xi).$$

2. Lyapunov Exponents

The geometry of the problem is encoded in Lyapunov exponents: Given $x \in \mathbb{Z}^d$ and $\lambda \geq 0$ define

$$A_\lambda^x = \sum_{\mathsf{X}(\gamma)=x} \left(\frac{1}{2d} \right)^{|\gamma|} \mathrm{e}^{-\Phi(\gamma)-\lambda|\gamma|}.$$

Then,

$$a_\lambda(x) = -\lim_{N\to\infty} \frac{1}{N} \log A_\lambda^{\lfloor Nx \rfloor}.$$

It is easy to check that the limit is well defined for any $x \in \mathbb{R}^d$ and $\lambda \geq 0$. Moreover $a_\lambda(\cdot)$ is an equivalent norm;

$$0 < \frac{1}{c_\lambda} \leq \min_{x\neq 0} \frac{a_\lambda(x)}{|x|} \leq \max_{x\neq 0} \frac{a_\lambda(x)}{|x|} \leq c_\lambda,$$

for any $\lambda \geq 0$.

The set of critical drifts is related to a_0 as follows:

$$\mathbf{K}_0 = \{h \ : \ (h, x) \leq a_0(x) \ \forall x\}.$$

Alternatively, one can describe $\mathbf{K}_0$ as the closure of the domain of convergences of the series

$$h \mapsto \sum_{x\in\mathbb{Z}^d} \mathrm{e}^{(h,x)} A_0^x.$$

For any $x \in \mathbb{R}^d$ one can choose $h \in \partial\mathbf{K}_0$ such that

$$(h, x) = a_0(x) = \max_{g\in\partial\mathbf{K}_0} (g, x).$$

In the sequel we shall fix a small number $\delta > 0$ and use it in order to quantify the cone of good directions $\mathcal{C}_\delta(h)$ which is associated with a critical drift $h \in \partial\mathbf{K}_0$. Namely,

$$\mathcal{C}_\delta(h) = \left\{ x \in \mathbb{R}^d \ : \ (h, x) \geq (1-\delta) a_0(x) \right\}.$$

3. Notes on the Proof

Let $h \in \partial \mathbf{K}_0$, $x \in \mathbb{Z}^d$ and let $\gamma = (\gamma(0), \dots, \gamma(k), \dots, \gamma(m))$ be a path from 0 to x. We shall say that a point $u = \gamma(k)$ is an h-cone point of γ if

$$\gamma \subseteq (u - \mathcal{C}_\delta(h)) \cup (u + \mathcal{C}_\delta(h)) \,.$$

Here is the crucial result:

The Mass Gap Estimate. There exist $\delta, \eta, \nu > 0$ such that

$$\mathrm{e}^{(h,x)} A_0^x \left(\gamma \text{ has no } h\text{-cone points } \right) \leq \mathrm{e}^{-\nu |x|}$$

uniformly in $h \in \partial \mathbf{K}_0$ and in all $x \in \mathcal{C}_\eta(h)$ sufficiently large.

The Mass-Gap estimates sets up in motion the Ornstein-Zernike machinery developed in [1] and in references therein. An important new ingredient needed for the proof of the mass-gap is the following Lemma, which is used for controlling massless hairs of renormalized skeletons:

Lemma. *Let B_K be a Euclidean ball of radius K. Consider simple random walk paths $\gamma = (\gamma(0), \dots, \gamma(\tau_K))$ which are run up to the first exit time from B_K. This gives rise to a probability distribution $\mathbb{Q}_K$. For any path γ as above define $R_K = R_K(\gamma)$ to be the size of its range (number of different points visited by γ before τ_K). Then for every $c_1 > 0$ there exists $c_2 > 0$ such that*

$$\mathbb{Q}_K \left(R_K \leq c_1 K \right) \leq \mathrm{e}^{-c_2 K}$$

for all K sufficiently large.

References

[1] Dmitry Ioffe and Yvan Velenik. Ballistic phase of self-interacting random walks. In *Analysis and stochastics of growth processes and interface models*, pages 55–79. Oxford Univ. Press, Oxford, 2008.

[2] Martin P. W. Zerner. Directional decay of the Green's function for a random nonnegative potential on $\mathbf{Z}^d$. *Ann. Appl. Probab.*, 8(1):246–280, 1998.

Technion and Université de Genève