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**Determining Integer-Valued Polynomials From Their Image**

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# Determining Integer-Valued Polynomials From Their Image

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## Abstract

This note summarizes a presentation made at the Third International Meeting on Integer Valued Polynomials and Problems in Commutative Algebra. All the work behind it is joint with Scott T. Chapman, and will appear in [2]. Let  $\text{Int}(\mathbb{Z})$  represent the ring of polynomials with rational coefficients which are integer-valued at integers. We determine criteria for two such polynomials to have the same image set on  $\mathbb{Z}$ .

Let  $\text{Int}(\mathbb{Z}) = \{f(X) \mid f(X) \in \mathbb{Q}[X] \text{ with } f(z) \in \mathbb{Z} \text{ for all } z \in \mathbb{Z}\}$  represent the ring of integer-valued polynomials. Given  $f \in \text{Int}(\mathbb{Z})$ , we denote the image set of  $f$  on  $\mathbb{Z}$  as  $f(\mathbb{Z}) = \{f(x) \mid x \in \mathbb{Z}\}$ . Given  $f, g \in \text{Int}(\mathbb{Z})$  with  $f(\mathbb{Z}) = g(\mathbb{Z})$ , we propose to characterize the relationship between  $f$  and  $g$ . In fact, more generally we characterize this for  $f, g \in \mathbb{R}[X]$  with  $f(\mathbb{Z}) = g(\mathbb{Z})$ . This work is related to the notions of an interpolation domain (considered in [3] and [1]) and the parametrization of integral values of polynomials (considered in [4]).

We begin by defining an equivalence relation on  $\text{Int}(\mathbb{Z})$  (or, more generally,  $\mathbb{R}[x]$ ), setting  $f \sim g$  if there is some  $n \in \mathbb{Z}$  such that either  $f(X) = g(X - n)$  or  $f(X) = g(-X - n)$ . Certainly if  $f \sim g$  then  $f(\mathbb{Z}) = g(\mathbb{Z})$ . The converse does not hold, as demonstrated by Lemma 1.

**Lemma 1.** *Let  $f \in \text{Int}(\mathbb{Z})$  be such that  $f(-X) = f(X - k)$  for some odd integer  $k$ , and set  $h(X) = f(2X)$ . Then  $h(\mathbb{Z}) = f(\mathbb{Z})$ .*

Note that the condition  $f(-X) = f(X - k)$  in Lemma 1 is equivalent to the condition that  $f$  is symmetric about  $X = -\frac{k}{2}$ , which in turn implies that  $f$  is of even degree. Our main result is that the equivalence relation  $\sim$  together with this phenomenon suffice to provide a converse.

**Theorem 2.** *Let  $f, g \in \text{Int}(\mathbb{Z})$ . Then  $f(\mathbb{Z}) = g(\mathbb{Z})$  if and only if either:*

- (1)  $f \sim g$ , or
- (2) *One of  $f, g$  (say  $f$ ) is symmetric about  $X = \frac{k}{2}$  for some odd integer  $k$ ,  $g$  is symmetric about  $X = \frac{j}{4}$  for some odd integer  $j$ , and  $g \sim f(2X)$ .*

We now give a sketch of the proof of this theorem; the full details are in [2]. In both cases above,  $\deg(f) = \deg(g)$ . By the comments following Lemma 1, in case (2) this degree must be even. We assume henceforth that  $f(X) \in \text{Int}(\mathbb{Z})$  is unbounded above, in particular excluding constant  $f$ . For nonconstant  $f$ ,  $f(\mathbb{Z})$  is infinite; if it were bounded above, then it must be unbounded below, so to compare  $\{f, g\}$  we instead compare  $\{-f, -g\}$ , because  $(-f)(\mathbb{Z}) = (-g)(\mathbb{Z})$  is unbounded above. Hence the function  $\sigma(x) = \min\{y : y \in f(\mathbb{Z}), y > x\}$  is well-defined. By taking  $f(-X) \sim f(X)$  if necessary, we may also assume that the leading coefficient of  $f$  is positive. With this notation and assumptions, we define the following.

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**Definition 3.** Let  $f \in \text{Int}(\mathbb{Z})$ . We say that  $f$  is of:

Type 1: There exists  $A \in \mathbb{R}$ , and for all  $x \in \mathbb{Z}$  with  $x > A$ ,  $f(x+1) = \sigma(f(x))$ .

Type 2: There exists  $A \in \mathbb{R}$ , and for all  $x \in \mathbb{Z}$  with  $x > A$ ,  $f(x+1) = \sigma^2(f(x))$ .

**Lemma 4.** Let  $f \in \text{Int}(\mathbb{Z})$  be of odd degree. Then  $f$  is of type 1.

*Proof Sketch:* Odd-degree polynomials all have a positive branch going to  $+\infty$ , and a negative branch going to  $-\infty$ . Once  $x$  is sufficiently large,  $f(x)$  lies on the positive branch and hence  $f$  is of Type 1.  $\square$

**Lemma 5.** Let  $f \in \text{Int}(\mathbb{Z})$  be of even degree. Then  $f$  is of type 1 or 2.  $f$  is of type 1 if and only if there is some  $k \in \mathbb{Z}$  with  $f(X - k) = f(-X)$ .  $f$  is of type 2 if and only if there is some  $k \in \mathbb{Z}$  with  $f(x - k + 1) = \sigma(f(-x)) = \sigma^2(f(x - k))$  for all  $x > A$ .

*Proof Sketch:* Even-degree polynomials have both a positive and a negative branch going to  $+\infty$ . Once  $|x|$  is sufficiently large, either the image of the left and right branches (on  $\mathbb{Z}$ ) coincide, or they must alternate. If they coincide, then  $f$  is of Type 1; if they alternate, then  $f$  is of Type 2.  $\square$

Together, Lemmas 4 and 5 prove the following.

**Proposition 6.** Each  $f \in \text{Int}(\mathbb{Z})$  is of Type 1 or Type 2, but not both.

The next two lemmas use this Proposition strongly for  $f(\mathbb{Z}) = g(\mathbb{Z})$ . Once the types of  $f, g$  are known, then (after an appropriate transformation), we get polynomials that agree on an infinite consecutive sequence of integers, and therefore must be equal.

**Lemma 7.** Let  $f, g \in \text{Int}(\mathbb{Z})$  be of the same type with  $f(\mathbb{Z}) = g(\mathbb{Z})$ . Then  $f \sim g$ .

**Lemma 8.** Let  $f, g \in \text{Int}(\mathbb{Z})$  with  $f(\mathbb{Z}) = g(\mathbb{Z})$ . Suppose that  $f$  is of type 1 and  $g$  is of type 2. Then  $f(-X) = f(X - k)$  for some odd integer  $k$ , and  $g \sim h$  where  $h(X) = f(2X)$ .

This proves one direction of Theorem 2, namely that if  $f(\mathbb{Z}) = g(\mathbb{Z})$  then (1) or (2) must hold. It is clear that if (1) holds then  $f(\mathbb{Z}) = g(\mathbb{Z})$ . This last lemma provides the remaining part of the converse, namely that if (2) holds then  $f(\mathbb{Z}) = g(\mathbb{Z})$ .

**Lemma 9.** Let  $f \in \text{Int}(\mathbb{Z})$  be of even degree and of type 1. Suppose that  $f(X - k) = f(-X)$  for some  $k \in \mathbb{Z}$  with  $k$  odd. Let  $g \sim f(2X)$ . Then  $g \in \text{Int}(\mathbb{Z})$  is of type 2, symmetric about  $X = \frac{j}{4}$  for some odd integer  $j$ , and satisfies  $f(\mathbb{Z}) = g(\mathbb{Z})$ .

## REFERENCES

- [1] Paul-Jean Cahen, Jean-Luc Chabert, and Sophie Frisch, *Interpolation domains*, J. Algebra **225** (2000), no. 2, 794–803. MR 1741562 (2001b:13024)
- [2] Scott T. Chapman and Vadim Ponomarenko, *On image sets of integer-valued polynomials*, submitted.
- [3] Sophie Frisch, *Interpolation by integer-valued polynomials*, J. Algebra **211** (1999), no. 2, 562–577. MR 1666659 (99m:13016)
- [4] G. Peruginelli and U. Zannier, *Parametrizing over  $\mathbb{Z}$  integral values of polynomials over  $\mathbb{Q}$* , Comm. Algebra **38** (2010), no. 1, 119–130. MR 2597485 (2011e:11064)

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