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A basis for Numerical Functionals

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1. Introduction

In a recent paper [2] Buchstaber and Lazarev introduced the concept of *numerical functionals*. Their definition derives from a series of dualities, but the simplest case is the following.

Let $\mathbb{Z}\{\{T\}\}$ denote the Hurwitz ring over the integers, that is, the subring of the power series ring $\mathbb{Q}[[T]]$ consisting of series of the form

$$\sum_{n\geq 0} \frac{c_n}{n!} T^n, \quad \text{with } c_n \in \mathbb{Z}.$$

Let $\mathbb{Q}\mathbb{Z}$ denote the rational group-ring of the integers, that is, the set of finite linear combinations $\sum_j \lambda_j [a_j]$, where $\lambda_j \in \mathbb{Q}$ and $a_j \in \mathbb{Z}$. There is a linear map

$$\begin{aligned} \mathbb{Q}\mathbb{Z} &\to \mathbb{Q}[[T]] \\ [a] &\mapsto e^{aT} = \sum_{n\geq 0} \frac{a^n}{n!} T^n, \end{aligned}$$

in fact it is a $\mathbb{Q}$-algebra monomorphism. Buchstaber and Lazarev define the ring of numerical functionals $\underline{\mathrm{Num}}$ as the pull-back in the diagram

$$\begin{array}{ccc} \underline{\mathrm{Num}} & \longrightarrow & \mathbb{Z}\{\{T\}\} \\ \big\downarrow & & \big\downarrow \\ \mathbb{Q}\mathbb{Z} & \longrightarrow & \mathbb{Q}[[T]] \end{array}$$

Thus $\underline{\mathrm{Num}}$ is the intersection of $\mathbb{Z}\{\{T\}\}$ and $\mathbb{Q}\mathbb{Z}$ within $\mathbb{Q}[[T]]$.

A simple example of a non-trivial element of $\underline{\mathrm{Num}}$ is $([1] + [-1])/2$, which maps to

$$\cosh T = \frac{e^T + e^{-T}}{2} = \sum_{k\geq 0} \frac{1}{(2k)!} T^{2k} \in \mathbb{Z}\{\{T\}\}.$$

$\underline{\mathrm{Num}}$ is thus the ring of rational linear combinations of the e^{aT} (where $a \in \mathbb{Z}$) all of whose derivatives at the origin are integral. Since $\mathbb{Q}\mathbb{Z} \cong \mathbb{Q}[z, z^{-1}]$, where $z = [1] = e^T$, we may also think of $\underline{\mathrm{Num}}$ as the ring of rational Laurent polynomials $\sum_{r\in\mathbb{Z}} \lambda_r z^r$ such that $\sum_{r\in\mathbb{Z}} \lambda_r r^n \in \mathbb{Z}$ for all $n \geq 0$. For example, in the case of $\cosh T$ this condition is that $(1^n + (-1)^n)/2 \in \mathbb{Z}$ for all $n \geq 0$.

Buchstaber and Lazarev gave (in a more general context) a set of additive generators for $\underline{\mathrm{Num}}$. In this note we show how to construct a basis.

In a later paper we will generalise Buchstaber and Lazarev's definition, and our construction of a basis.

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2. Construction of a basis

Lemma 2.1. *For all $k \geq 0$ the series*

$$f_k := \frac{(e^T - 1)^k}{k!}$$

belongs to $\underline{\mathrm{Num}}$.

Proof. That $f_k \in \mathbb{Z}\{\{T\}\}$ follows from the result of Hurwitz [5, Satz I] that if $f \in \mathbb{Z}\{\{T\}\}$ has zero constant term, then$f^k/k! \in \mathbb{Z}\{\{T\}\}$. But this case one can be even more explicit, for it is a classical result (see, for example, [4, (7.49)]) that

$$f_k = \sum_{n \geq k} \frac{S(n,k)}{n!} T^n,$$

where $S(n,k)$ is the Stirling number of the second kind.

On the other hand, it is clear that f_k belongs to $\mathbb{Q}\mathbb{Z}$, being a rational linear combination of the e^{jT} for $j = 0, 1, \ldots, k$. □

To obtain a basis for $\underline{\mathrm{Num}}$ we must introduce negative powers of e^T. We do this in a manner similar to that used in [1] (see the end of the proof of Theorem 2.2) and [3, Corollary 6].

Proposition 2.2. *Let*

$$g_k = e^{-\lfloor k/2 \rfloor T} f_k = \frac{e^{-\lfloor k/2 \rfloor T}(e^T - 1)^k}{k!}.$$

Then the g_k, for $k \geq 0$, form an integral basis for $\underline{\mathrm{Num}}$.

Proof. We see from the lemma that $g_k \in \underline{\mathrm{Num}}$.

Suppose now that $h \in \underline{\mathrm{Num}}$. Since g_k is a rational linear combination of the e^{jT} for $j = -\lfloor k/2 \rfloor, \ldots, \lceil k/2 \rceil$, the g_k are a rational basis for $\mathbb{Q}\mathbb{Z}$, so we can write

$$h = \sum_{k=0}^{m} \lambda_k g_k$$

for some $\lambda_k \in \mathbb{Q}$.

On the other hand, since g_k has the form $\frac{T^k}{k!}$ + higher degree terms, the g_k form an integral topological basis for $\mathbb{Z}\{\{T\}\}$. Therefore there are unique $b_k \in \mathbb{Z}$ such that

$$h = \sum_{k=0}^{\infty} b_k g_k.$$

Choosing a positive integer N such that $N\lambda_k \in \mathbb{Z}$ for $k = 0, 1, \ldots, m$, the uniqueness of the expansion of Nh shows that $N\lambda_k = Nb_k$ for $0 \leq k \leq m$, with $b_k = 0$ for $k > m$. Hence $\lambda_k = b_k \in \mathbb{Z}$. □

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