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Generalized Ricci curvature and the geometry of graphs

Frank Bauer, Bobo Hua, Jürgen Jost, and Shiping Liu

1. Curvature

Originally, curvature was a concept of differential geometry developed with the purpose of describing the geometry of surfaces in space in a manner analogous to those of curves. Something is curved if it is not straight like a line or flat like a plane. Gauss [10] then realized that such a notion of curvature of surfaces in space confused two different aspects. One is concerned with how the surface bends in space, that is, how its normal direction changes when moving along the surface. The other, in contrast, is concerned with the inner geometry of that surface, that is, for instance, how slowly or fast geodesic curves emanating from the same point in different directions move away from each other. Riemann [18] then developed an intrinsic geometry on manifolds of arbitrary dimension built upon such an intrinsic curvature, see [12] for the current state of this important field of mathematics. The Riemann sectional curvature measures such a divergence of geodesics whose initial directions all lie in the same tangent plane. Averaging the curvatures over all such planes containing a given direction v then yields the Ricci curvature in the direction v. Finally, averaging over all directions v starting at the same point p yields the scalar curvature at p. In this contribution, we shall mainly be concerned with the properties of Ricci curvature. Ricci curvature characterizes the growth of the volume of distance balls as a function of their radius. More precisely, Ricci curvature controls the cost of transporting the mass of one distance ball to another one. When the Ricci curvature is large, the volumes of balls become smaller, but the relative volumes of the intersection of two balls become larger. Therefore, such a transport becomes less costly. Ricci curvature also yields lower bounds for the first nonzero eigenvalue of the Laplace operator on a compact Riemannian manifold.

While these curvature concepts were originally developed for Riemannian manifolds, that is, differentiable manifolds equipped with a smooth metric tensor, the characteristic properties of curvature just described are meaningful for more general metric spaces. Therefore, notions of generalized sectional or Ricci curvature have been developed that are meaningful for certain classes of metric spaces that are more general than Riemannian manifolds.

In particular, such concepts then also apply to graphs. For instance, one can consider an undirected and unweighted graph G as a metric space with each edge isometric to the unit interval, that is, of length one. For each vertex x, one also has a natural probability measure m_x on G that assigns the weight $\frac{1}{d_x}$ to every neighbor of x, where d_x is the degree of x, that is, the number of its neighbors, the vertices connected to x by an edge. All other vertices, including x itself, get the weight 0 under m_x. Again, we can ask for the cost of transporting m_x to m_y when x and y are neighbors. Ollivier [16] then defined Ricci curvature bounds for graphs in terms of such transportation costs.

In this contribution, we shall explain how the generalized Ricci curvature as defined by Ollivier relates to other characteristic properties of graphs, like the clustering coefficient [19] that is important for the analysis of social and other networks. We also show how this generalized Ricci

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curvature controls the smallest as well as the largest eigenvalue of the normalized graph Laplacian. In fact, we obtain nontrivial eigenvalue estimates for all graphs that are not bipartite. Our constructions utilize the concept of the neighborhood graph [4], a geometric representation of the concept of a random walk on a graph. Thereby, we see a natural link between Ricci curvature, eigenvalues, and stochastic analysis.

While these principles hold in more generality, here we only explore them for graphs.

2. Generalized Ricci curvature

Ollivier's [16, 17] definition of Ricci curvature depends on the L^1-Wasserstein distance.

Definition 2.1. Let (X, d) be a metric space equipped with its Borel sigma algebra, and let m_1, m_2 be probability measures on X. The L^1-Wasserstein or transportation distance between the probability measures m_1 and m_1 is

$$W_1(m_1, m_1) = \inf_{\xi^{x,y} \in \prod(m_1,m_2)} \sum_{(x',y') \in V \times V} d(x', y')\xi^{x,y}(x', y'), \tag{2.1}$$

where $\prod(m_1, m_2)$ is the set of probability measures $\xi^{x,y}$ that satisfy

$$\sum_{y' \in V} \xi^{x,y}(x', y') = m_1(x'), \ \sum_{x' \in V} \xi^{x,y}(x', y') = m_2(y'). \tag{2.2}$$

The conditions (2.2) mean that we start with the measure m_1 and end up with m_2. When we consider the distance $d(x', y')$ as the transportation cost from x' to y', then $W_1(m_1, m_2)$ is the minimal cost to transport the mass of m_1 to that of m_2. $\xi^{x,y}$ is considered as a transfer plan between m_1 and m_2, or a coupling of the two random walks governed by m_1 and m_2, respectively. Those $\xi^{x,y}$ which attain the infimum in (2.1) are called optimal couplings.

The transportation distance $W_1(m_1, m_2)$ can also expressed by the Kantorovich duality formula,

$$W_1(m_1, m_2) = \sup_{f:\mathrm{Lip}(f) \le 1} \left[\sum_{x' \in V} f(x')m_1(x') - \sum_{y' \in V} f(y')m_2(y') \right], \tag{2.3}$$

where $\mathrm{Lip}(f) := \sup_{x \neq y} \frac{|f(x)-f(y)|}{d(x,y)}$ is the Lipschitz seminorm of f.

Definition 2.2. Let (X, d) be a complete and separable metric space equipped with its Borel sigma algebra and a family of probability measueres $m_x, x \in M$ which depend measurably on x and which have finite first moments, i.e., $\int_M d(x, y)dm_x(y) < \infty$. For any two distinct points $x, y \in M$, the (Ollivier-) Ricci curvature of (X, d, m) then is defined as

$$\kappa(x, y) := 1 - \frac{W_1(m_x, m_y)}{d(x, y)}. \tag{2.4}$$

The probability measures m_x could also be interpreted as the probability densities associated to a random walk, that is, $m_x(y)$ is the probability that a random walker at x jumps to y in one time step.

Here, we shall restrict our attention to graphs considered as metric spaces with the measures m_x explained in Section 1.

3. Ricci curvature and the geometry of graphs

3.1. **Basic notions from graph theory.** We introduce some basic definitions and constructions from graph theory, including the (normalized) graph Laplacian, see [13] and the references given there.

We first consider a locally finite unweighted graph $G = (V, E)$. V is the vertex and E the edge set. We say that $x, y \in V$ are neighbors, and write $x \sim y$, when they are connected by an edge. The degree d_x of a vertex x is defined as the number of its neighbors.

We also assume that G is connected, that is, for every pair of distinct vertices $x, y \in V$, there exists a path between them, that is, a sequence $x = x_0, x_1, \ldots, x_m = y$ of distinct vertices such that $x_{\nu-1} \sim x_\nu$ for $\nu = 1, \ldots, m$. (Not connected graphs can simply be decomposed into their connected components.) A cycle in G is a closed path $x_0, x_1, \ldots, x_m = x_0$ for which all the vertices $x_1, \ldots, x_m$ are distinct. For $m = 3, 4, 5, \ldots$, we speak of a triangle, quadrangle, pentagon,... A graph without

cycles is called a tree. A graph is called bipartite if its vertex set can be decomposed into two disjoint components V_1, V_2 such that whenever $x \sim y$, then x and y are in different components. Any tree is bipartite. More generally, a graph is bipartite iff it has no cycles of odd length. In particular, it has no triangles.

To get a metric, for neighbors x, y, we put $d(x, y) = 1$. For arbitrary vertices x, y, $d(x, y)$ is the length of the shortest path connecting x and y, i.e. the minimal number of edges that needs to be traversed to get from x to y.

We next introduce the (normalized) graph Laplacian operating on L^2-functions on the vertex set V, see e.g. [8, 13]. Here, we use the scalar product

$$(v, u) := \sum_{x \in V} d_x v(x) u(x) \tag{3.1}$$

to define $L^2(G)$. We then put

$$\Delta : L^2(G) \to L^2(G)$$

$$\Delta v(x) := \frac{1}{d_x}\Big(\sum_{y, y \sim x} v(y) - d_x v(x)\Big) = \frac{1}{d_x} \sum_{y, y \sim x} v(y) - v(x). \tag{3.2}$$

This is a discrete analogue of the Laplace-Beltrami operator of a Riemannian manifold. We can also consider, for neighbors $x \sim y$, the discrete differential

$$Du(x, y) := u(y) - u(x), \tag{3.3}$$

a discrete analogue of the differential of a function. D can be considered as a map from functions on the vertices of D to functions on the edges of D. In order to make the latter space also an L^2-space, we introduce the product

$$(Du, Dv) := \sum_{e=(x,y)} (u(y) - u(x))(v(y) - v(x)). \tag{3.4}$$

Note that we are summing here over edges, and not over vertices. If we did the latter, we would need to put in a factor $1/2$ because each edge would then be counted twice. We then have

$$(\Delta u, v) = -(Du, Dv) \tag{3.5}$$

for all $u, v \in L^2(G)$.

We now list some basic properties of Δ.

(1) Δ is selfadjoint w.r.t. $(.,.)$:

$$(u, \Delta v) = (\Delta u, v) \tag{3.6}$$

for all $u, v \in L^2(G)$. This follows from (3.5).

(2) Δ is nonpositive:

$$(\Delta u, u) \leq 0 \tag{3.7}$$

for all u. This follows from the Cauchy-Schwarz inequality.

(3) $\Delta u = 0$ iff u is constant. In fact, when $\Delta u = 0$, there can neither be a vertex x with $u(x) \geq u(y)$ for all $y \sim x$ with strict inequality for at least one such y, since $\Delta u(x) = 0$ means that the value $u(x)$ is the average of the values at the neighbors of x. Since G is assumed to be connected, u then has to be a constant (if G were not connected, a solution of $\Delta u = 0$ would have to be constant on every connected component of G.) Of course, this is a discrete version of the standard maximum principle argument.

We are interested in the eigenvalues of the Laplacian, that is, in those λ with

$$\Delta u + \lambda u = 0 \tag{3.8}$$

for some nontrivial function $u \in L^2(G)$, called an eigenfunction for λ. From the properties of Δ just listed, we can infer some immediate consequences for the eigenvalues.

- All eigenvalues are real, because Δ is selfadjoint.
- All eigenvalues are nonnegative, because Δ is a nonpositive operator.

- The smallest eigenvalue is $\lambda_0 = 0$, with a constant eigenfunction. Since we assume that Γ is connected, this eigenvalue is simple. In other words,

$$\lambda_k > 0 \tag{3.9}$$

for $k > 0$ where we order the eigenvalues as

$$\lambda_0 = 0 < \lambda_1 \leq ... \leq \lambda_K$$

and put $K := N - 1$.
- The largest eigenvalue λ_{N-1} is 2 iff G is bipartite and is < 2 else. (See [4] for details and a systematic analysis of the highest eigenvalue.)

The eigenfunctions v_i, v_j for different eigenvalues λ_i, λ_j are orthogonal to each other,

$$(v_i, v_j) = 0. \tag{3.10}$$

In particular, since the constants are the eigenfunctions for the eigenvalue $\lambda_0 = 0$, for all $i > 0$, we then have

$$\sum_x m_x v_i(x) = 0. \tag{3.11}$$

3.2. **Ricci curvature and clustering.** In this section, we essentially describe the results of [14]. As explained, in order to define Ricci curvature, we need the probability measures from Section 1

$$m_x(y) = \begin{cases} \frac{1}{d_x} & \text{if } y \sim x; \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

We can interpret this in terms of a random walker that sits at x at time $t \in \mathbb{N}$ and then selects a neighbor of x with equal probability $\frac{1}{d_x}$ as the target of his walk at time $t + 1$. .

Theorem 3.1. *On a locally finite graph $G = (V, E)$, we have for any pair of neighboring vertices x, y,*

$$\kappa(x,y) \geq -\left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x,y)}{d_x \wedge d_y}\right)_+ - \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x,y)}{d_x \vee d_y}\right)_+ + \frac{\sharp(x,y)}{d_x \vee d_y},$$

where we have put

$$d_x \wedge d_y := \min\{d_x, d_y\}, \;\; d_x \vee d_y := \max\{d_x, d_y\}.$$

Remark: For the case where $\sharp(x, y) = 0$, this result was obtained in [15]. For our purposes, however, the key point is to understand how the presence of triangles in a graph improves the lower Ricci bound.

The proof of Theorem 3.1 depends on a careful transport plan, according to the definition of Ricci curvature. We do not present the details, but the following two figures illustrate the task.

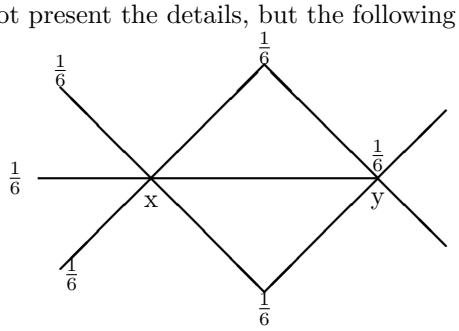

Figure 1. Starting configuration for the transport plan; mass 0 at all vertices without number attached.

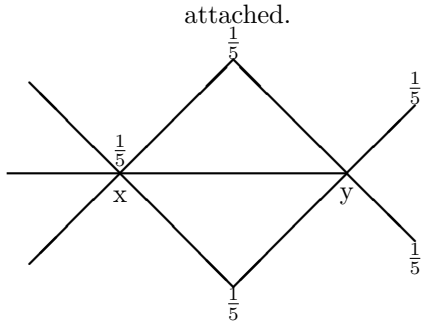

Figure 2. Target configuration for the transport plan

We can also recall the duality formula (2.3) and consider the following 1-Lipschitz function. From this function, we clearly see why triangles, that is common neighbors of the vertices x and y contribute to decreasing the transportation cost.

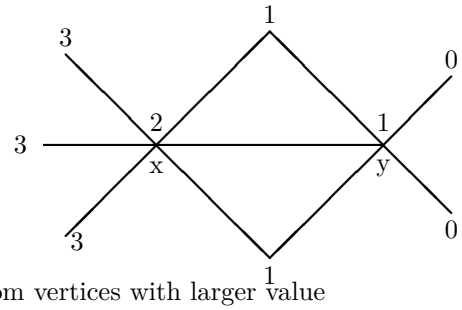

Figure 3. Mass moved from vertices with larger value

In fact, not only triangles, but also quadrangles and pentagons (but not polygons with more edges) influence Ricci curvature.

The lower bound of Theorem 3.1 is sharp both for complete graphs and for trees. On a complete graph $\mathcal{K}_n$ ($n \geq 2$) with n vertices, $\sharp(x,y) = n-2$ for any x, y. Hence the inequality

$$\kappa(x,y) \geq \frac{n-2}{n-1}$$

is sharp. That trees also attain the lower bound of Theorem 3.1, follows from the fact that on n a tree $T = (V, E)$, for any neighboring x, y,

$$\kappa(x,y) = -2\left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right)_+. \tag{3.13}$$

We can also relate this to the above heuristic discussion of the relation between Ricci curvature and the relative volume of the intersection of balls. In fact, $\sharp(x,y)/d_x \vee d_y$ is $m_x \wedge m_y(G) := m_x(G) - (m_x - m_y)_+(G)$, i.e. the intersection measure of m_x and m_y. The vertices x_1 that satisfy $x_1 \sim x$, $x_1 \sim y$ constitute the intersection of the unit metric spheres centered at x and y, resp.

We also have an easy upper bound for the Ricci curvature of a graph.

Theorem 3.2. *On a locally finite graph $G = (V, E)$, for any neighboring x, y, we have*

$$\kappa(x,y) \leq \frac{\sharp(x,y)}{d_x \vee d_y}. \tag{3.14}$$

We now consider the local clustering coefficient of Watts-Strogatz [19]

$$c(x) := \frac{1}{d_x(d_x-1)} \sum_{y, y\sim x} \sharp(x,y). \tag{3.15}$$

$c(x)$ measure the extent to which neighbors of x are directly connected. Expressed in words,

$$c(x) = \frac{\text{number of realized edges between neighbors of } x}{\text{number of possible edges between neighbors of } x}. \tag{3.16}$$

This clustering coefficient is an important quantity in network analysis. For instance, in social networks where the vertices represent individuals and the edges friendship relations, the question addressed by the clustering coefficient is "How many of the friends of my friends are also my friends?".

We may also consider this local clustering coefficient as an average over the $\sharp(x,y)$ for the neighbors of x. As such an average, we should also try to compare it to averaged Ricci curvature. In other words, we should consider the discrete version of scalar curvature,

$$\kappa(x) := \frac{1}{d_x} \sum_{y, y\sim x} \kappa(x,y). \tag{3.17}$$

This scalar curvature $\kappa(x)$ and the local clustering coefficient $c(x)$ then control each other. Indeed, from Theorems 3.1 and 3.2, with $D(x) := \max_{y,y\sim x} d_y$, we have

$$\frac{d_x - 1}{d_x} c(x) \geq \kappa(x) \geq -2 + \frac{d_x - 1}{d_x \vee D(x)} c(x).$$

3.2.1. *Stochastic processes on graphs.* We consider a graph with a lower Ricci bound

$$\kappa(x, y) \geq k \text{ for all } x \sim y, \tag{3.18}$$

or equivalently,

$$W_1(m_x, m_y) \leq (1 - k)d(x, y) = 1 - k \text{ for all } x \sim y. \tag{3.19}$$

We shall now interpret this in probabilistic terms as a path coupling criterion for random walks. This translates a lower bound of the Ollivier-Ricci curvature into a control on the expectation value of the distance between two coupled random walks.

By iteration, one may prove that when (3.18) and hence (3.19) holds, then for any t and any $\bar{x}, \bar{y}$, not necessarily neighbors,

$$W_1(\delta_{\bar{x}} P^t, \delta_{\bar{y}} P^t) \leq (1 - k)^t d(\bar{x}, \bar{y}). \tag{3.20}$$

In order to link this to Ricci curvature, we now consider two random walks $(\bar{X}_t, \bar{Y}_t)$ with distributions $\delta_{\bar{x}} P^t$, $\delta_{\bar{y}} P^t$ that are coupled in the sense that the joint probabilities satisfy

$$p(\bar{X}_t = \bar{x}', \bar{Y}_t = \bar{y}') = \xi_t^{\bar{x},\bar{y}}(\bar{x}', \bar{y}'),$$

where $\xi_t^{\bar{x},\bar{y}}(\cdot, \cdot)$ is the optimal coupling of $\delta_{\bar{x}} P^t$ and $\delta_{\bar{y}} P^t$ as in the definition of the Wasserstein distance W_1. The term $W_1(\delta_{\bar{x}} P^t, \delta_{\bar{y}} P^t)$ then becomes the expectation value of the distance $\mathbf{E}^{\bar{x},\bar{y}} d(\bar{X}_t, \bar{Y}_t)$ between the coupled random walks $\bar{X}_t$ and $\bar{Y}_t$.

Corollary 3.1. *If (3.18) holds, then for any $\bar{x}, \bar{y} \in V$,*

$$\mathbf{E}^{\bar{x},\bar{y}} d(\bar{X}_t, \bar{Y}_t) = W_1(\delta_{\bar{x}} P^t, \delta_{\bar{y}} P^t) \leq (1 - k)^t d(\bar{x}, \bar{y}). \tag{3.21}$$

3.2.2. *Weighted and neighborhood graphs.* Following [4], we now translate the properties of random walks into geometric structures, the neighborhood graphs. In Section 3.2.3, we shall then use this construct to derive eigenvalue bounds in terms of lower Ricci curvature bounds on graphs.

For this purpose, we shall need to consider weighted graphs, and also allow for the possibility of self-loops. That is, for any $x, y \in V$, not necessarily different, we have a symmetric, nonnegative connection weight

$$w_{xy} = w_{yx} \geq 0. \tag{3.22}$$

We can then declare x and y to be neighbors, $x \sim y$, iff $w_{xy} > 0$. Of course, the unweighted graphs that we have considered before constitute the special cases where $w_{xy} = 1$ iff $x \sim y$ and $w_{xy} = 0$ else. As mentioned, here, we also allow for the possibility of self-loops, that is, vertices x with $w_{xx} > 0$.

Remark: Of course, one could also allow for non-symmetric or negative weights. The spectrum of non-symmetric graphs was systematically investigated in [2], and some results on graphs with possibly negative connection weights can be found, for instance, in [3, 1]. For our present purposes, however, the class of weighted graphs satisfying (3.22) suffices.

The preceding constructions and results can be extended to weighted graphs. We now define the measure m_x by

$$m_x(y) := \frac{w_{xy}}{d_x}, \text{ where now } d_x := \sum_y w_{xy}. \tag{3.23}$$

We can again consider $m_x(y)$ as the probability that a random walker starting at x moves to y in one time step. Since now possibly $m_x(x) > 0$, because there might be a self-loop at x, the random walker might now be lazy and simply stay at x.

As before , the L^2-product is

$$(u, v) = \sum_x d_x u(x) v(x). \tag{3.24}$$

The Laplacian

$$\Delta v(x) = \frac{1}{d_x}\sum_y w_{xy}v(y) - v(x) = \sum_y m_x(y)v(y) - v(x) \tag{3.25}$$

is self-adjoint and nonpositive as before. Hence, the eigenvalues are nonnegative real numbers. We also have a version of Theorem 3.1 for weighted graphs, taken from [5].

Theorem 3.3. *On a weighted graph, we have for neighbors x, y*

$$\begin{aligned}\kappa(x,y) \geq &-\left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1\in N_{xy}} \frac{w_{x_1x}}{d_x} \vee \frac{w_{x_1y}}{d_y}\right)_+ \\ &-\left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1\in N_{xy}} \frac{w_{x_1x}}{d_x} \wedge \frac{w_{x_1y}}{d_y}\right)_+ \\ &+ \sum_{x_1\in N_{xy}} \frac{w_{x_1x}}{d_x} \wedge \frac{w_{x_1y}}{d_y} + \frac{w_{xx}}{d_x} + \frac{w_{yy}}{d_y}.\end{aligned}$$

Again, this inequality is sharp.

With the notation

$$\mu P(\cdot) = \sum_x \mu(x) m_x(\cdot),$$

the Dirac measure δ_x at x and $\delta_x P^1(\cdot) = \delta_x P(\cdot) = m_x(\cdot)$, the distribution of a t-step random walk starting at x with transition probability m_x becomes

$$\delta_x P^t(\cdot) = \sum_{x_1,\ldots,x_{t-1}} m_x(x_1) m_{x_1}(x_2)\cdots m_{x_{t-1}}(\cdot) \tag{3.26}$$

for $t > 1$. The probability that the random walker moves from x to y in t steps then is

$$\delta_x P^t(y) = \begin{cases} \sum_{x_1,\ldots,x_{t-1}} \frac{w_{xx_1}}{d_x}\frac{w_{x_1x_2}}{d_{x_1}}\cdots\frac{w_{x_{t-1}y}}{d_{x_{t-1}}}, & \text{if } t > 1; \\ \frac{w_{xy}}{d_x}, & \text{if } t = 1. \end{cases} \tag{3.27}$$

We now define a family of graphs $G[t]$ for $t \geq 1$ whose weights equal the transition probabilities of the t-step random walks on the graph G.

Definition 3.1. The neighborhood graph $G[t] = (V, E[t])$ of the graph $G = (V, E)$ of order $t \geq 1$ is the weighted graph with vertex set V and edge weights

$$w_{xy}[t] := \delta_x P^t(y) d_x \tag{3.28}$$

from (3.27).

Obviously, $G = G[1]$. Also, $w_{xy}[t] > 0$ if and only if there exists a path of length t between x and y in G.

We now describe the important properties of the neighborhood graph $G[t]$, its Laplacian $\Delta[t]$ and the eigenvalues $\lambda_i[t]$, see [4, 5].

Lemma 3.1. (i) *t even: $G[t]$ is connected iff G is not bipartite. $G[t]$ is not bipartite.*
(ii) *t odd: $G[t]$ is always connected and $G[t]$ is bipartite iff G is bipartite.*
(iii) *$d_x[t] = d_x$ for all $x \in V$, and the inner product (3.24) is the same on all the $G[t]$.*
(iv) *The Laplacian on $G[t]$ is*

$$\Delta[t] = -\mathrm{id} + (\mathrm{id} + \Delta)^t. \tag{3.29}$$

(v) *Therefore, for even t, the eigenvalues of $\Delta[t]$ satisfy*

$$0 = \lambda_0[t] \leq \lambda_1[t] \leq \ldots \leq \lambda_{N-1}[t] \leq 1. \tag{3.30}$$

(The smaller upper bound 1 as compared with the bound 2 discussed above stems here from the self-loops of $G[t]$.

(vi) *Let $d[t](x,y)$ be the distance on $G[t]$ defined as the smallest number of edges needed for a path connecting x and y (this is independent of the weights, except that vertices ξ and η are connected by an edge iff $w_{\xi\eta} > 0$). Then*

$$\frac{1}{t}d(x,y) \leq d[t](x,y), \tag{3.31}$$

with the convention $d[t](x,y) = \infty$ if $G[t]$ is not connected and x and y are in different components. Conversely, if $E \subseteq E[t]$, then

$$d[t](x,y) \leq d(x,y). \tag{3.32}$$

In [4], the relationship between the eigenvalues of the original graph G and those of its neighborhood graphs was analyzed.

Proposition 3.1. (i) *If $\lambda_1[t] \geq \mathcal{A}[t]$, then*

$$1-(1-\mathcal{A}[t])^{\frac{1}{t}} \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} \leq 1+(1-\mathcal{A}[t])^{\frac{1}{t}} \tag{3.33}$$

if t is even and

$$1-(1-\mathcal{A}[t])^{\frac{1}{t}} \leq \lambda_1 \tag{3.34}$$

if t is odd.

(ii) *If $\lambda_{N-1}[t] \leq \mathcal{B}[t]$, then all eigenvalues of Δ are contained in*

$$\left[0, 1-(1-\mathcal{B}[t])^{\frac{1}{t}}\right] \bigcup \left[1+(1-\mathcal{B}[t])^{\frac{1}{t}}, 2\right]$$

for even t, whereas

$$\lambda_{N-1} \leq 1-(1-\mathcal{B}[t])^{\frac{1}{t}}$$

for odd t.

Thus, eigenvalues bounds on $G[t]$ translate into eigenvalue bounds on the original graph G. This is a powerful principle for estimating the eigenvalues of G. As the neighborhood graphs constitute a geometric representation of the random walk on G, this can be seen as a scheme for translating properties of the random walk into eigenvalue bounds.

3.2.3. *Ricci curvature and eigenvalues of graphs.* In this section, we assume that the graph G is finite, that is, it has finitely many, say N, vertices, and then also finitely many edges. Here, we follow [5] to estimate the eigenvalues in terms of the Ricci curvature. Ollivier [16] showed

Theorem 3.4. *When we have a lower Ricci curvature bound*

$$\kappa(x,y) \geq k, \tag{3.35}$$

(in fact, it suffices to have this for all $x \sim y$), then

$$k \leq \lambda_1 \leq \ldots \leq \lambda_{N-1} \leq 2-k. \tag{3.36}$$

A problem with this estimate is that for most graphs, $k \leq 0$ in (3.35), so that (3.36) only yields a trivial estimate. We shall therefore develop an estimate of [5] which is nontrivial for all connected finite graphs that are not bipartite.

Lemma 3.2. *Let k be a lower bound of κ on G. If $E \subseteq E[t]$, then the curvature $\kappa[t]$ of the neighborhood graph $G[t]$ satisfies*

$$\kappa[t](x,y) \geq 1-t(1-k)^t, \ \ \forall x,y \in V. \tag{3.37}$$

We can now see the upper bound of the largest eigenvalue in Theorem 3.4. W.l.o.g. $k > 0$, in which case $E \subset E[t]$. From Lemma 3.2 and $\lambda_1 \geq k$, we know on $G[t]$,

$$\lambda_1[t] \geq 1-t(1-k)^t.$$

Then with Proposition 3.1 (i), for even t,

$$\lambda_{N-1} \leq 1+t^{\frac{1}{t}}(1-k).$$

Letting $t \to +\infty$ yields $\lambda_{N-1} \leq 2-k$, indeed.

The neighborhood graph technique then leads to the following generalization of Theorem 3.4, the main result of [5].

Theorem 3.5. *Let $k[t]$ be a lower bound of Ollivier-Ricci curvature of the neighborhood graph $G[t]$. Then for all $t \geq 1$ the eigenvalues of Δ on G satisfy*

$$1 - (1 - k[t])^{\frac{1}{t}} \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} \leq 1 + (1 - k[t])^{\frac{1}{t}}. \tag{3.38}$$

If G is not bipartite, then for all sufficiently large t, $k[t] > 0$, and hence (3.38) is nontrivial in the sense that the lower bound is positive and the upper bound is < 2.

3.3. **Other curvature notions for graphs.** We conclude this brief survey with some curvature notions for graphs other than Ricci curvature.

First, combinatorial curvature: we fill faces into the graph. We therefore assume that the (possibly infinite) graph G is embedded into a 2-manifold $S(G)$ such that each face is homeomorphic to a closed disk with finite edges as the boundary. For instance, G could be a planar graph, that is, a graph embedded into the plane. Therefore, we call such a $G = (V, E, F)$ that can be embedded into a 2-manifold a semiplanar graph. For each vertex $x \in V$, the combinatorial curvature at x is

$$\Phi(x) = 1 - \frac{d_x}{2} + \sum_{\sigma \ni x} \frac{1}{\deg(\sigma)}, \tag{3.39}$$

where, as before, d_x is the degree of the vertex x, whereas $\deg(\sigma)$ is the degree of the face σ. The sum is taken over all faces incident to x (i.e. $x \in \sigma$).

When we replace each face of G with a regular polygon of side lengths one and glue them along the common edges and equip the polygonal surface $S(G)$ with the resulting metric structure, then (3.39) simply measures the difference of 2π and the total angle Σ_x at the vertex x,

$$2\pi\Phi(x) = 2\pi - \Sigma_x. \tag{3.40}$$

Let $\chi(S(G))$ denote the Euler characteristic of the surface $S(G)$. We then have the Gauss-Bonnet formula of G of [9],

$$\sum_{x \in G} \Phi(x) \leq \chi(S(G)), \tag{3.41}$$

whenever $\Sigma_{x \in G : \Phi(x) < 0} \Phi(x)$ converges. Thus, the combinatorial curvature captures a topological property of semiplanar graphs.

We can also compare the combinatorial curvature with another version of curvature naturally obtained from the surface $S(G)$, its generalized sectional (Gaussian) curvature. It turns out that the semiplanar graph G has nonnegative combinatorial curvature precisely if the polygonal surface S(G) is an Alexandrov space with nonnegative sectional curvature, i.e. $\mathrm{Sec}\, S(G) \geq 0$ (or $\mathrm{Sec}(G) \geq 0$ for short). This principle is systematically explored in [11].

A metric space (X, d) on which each pair of points in X can be joined by a shortest path is called an Alexandrov space if locally satisfies the Toponogov triangle comparison. Essentially, nonnegative curvature in the present context means that the total angles of geodesic triangles are at least 2π. Monographs on Alexandrov spaces are [7, 6].

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