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The inviscid limit for density-dependent incompressible fluids[(*)]

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Abstract. — This paper is devoted to the study of smooth flows of density-dependent fluids in $\mathbb{R}^N$ or in the torus $\mathbb{T}^N$. We aim at extending several classical results for the standard Euler or Navier-Stokes equations, to this new framework.

Existence and uniqueness is stated on a time interval independent of the viscosity μ when μ goes to 0. A blow-up criterion involving the norm of vorticity in $L^1(0,T;L^\infty)$ is also proved. Besides, we show that if the density-dependent Euler equations have a smooth solution on a given time interval $[0,T_0]$, then the density-dependent Navier-Stokes equations with the same data and small viscosity have a smooth solution on $[0,T_0]$. The viscous solution tends to the Euler solution when the viscosity μ goes to 0. The rate of convergence in L^2 is of order μ.

An appendix is devoted to the proof of elliptic estimates in Sobolev spaces with positive or negative regularity indices, interesting for their own sake.

Résumé. — Cet article est consacré à l'étude des fluides incompressibles à densité variable dans $\mathbb{R}^N$ ou $\mathbb{T}^N$. On cherche à généraliser plusieurs résultats classiques pour les équations d'Euler et de Navier-Stokes incompressibles.

On établit un résultat d'existence et d'unicité sur un intervalle de temps indépendant de la viscosité μ du fluide ainsi qu'un critère d'explosion faisant intervenir la norme du tourbillon dans $L^1(0,T;L^\infty)$. On montre en outre que si les équations d'Euler ont une solution régulière sur un intervalle de temps $[0,T_0]$ donné alors les équations de Navier-Stokes avec mêmes données et petite viscosité ont une solution régulière sur le même intervalle de temps. De plus la solution visqueuse tend vers la solution d'Euler quand la viscosité tend vers 0. Le taux de convergence dans L^2 est de l'ordre de μ.

En appendice, on démontre des estimations a priori de type elliptique dans des espaces de Sobolev à indice positif ou négatif.

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0. Introduction

There is an important literature devoted to the mathematical study of the so called *incompressible Navier-Stokes equations*

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla \Pi = 0, \\ \operatorname{div} v = 0, \end{cases} \tag{NS_μ}$$

and of the limit system $(E) \stackrel{\text{def}}{=} (NS_0)$, called *incompressible Euler equations*:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla \Pi = 0, \\ \operatorname{div} v = 0. \end{cases} \tag{E}$$

Above, the parameter $\mu \geqslant 0$ denotes the viscosity and $v = v(t,x) \in \mathbb{R}^N$ (where $t \geqslant 0$ is the time and $x \in \mathbb{R}^N$ is the space variable) stands for the velocity field. The term $\nabla\Pi$ (the gradient of the pressure) may be seen as the Lagrange multiplier associated to the constraint $\operatorname{div} v = 0$.

Let us give a (non exhaustive) list of questions which have been addressed:

1. Local or global well-posedness for (E) and (NS_μ).

   Local well-posedness for both systems holds true in the Sobolev space H^s with $s > 1+N/2$ (see e.g. [12]). In the limit μ going to 0, estimates independent of the viscosity on a fixed time interval may be proved. In dimension $N = 2$, all these results are global in time.

2. Derivation of blow-up criteria.

   According to a celebrated paper by J. Beale, T. Kato and A. Majda (see [2]), no breakdown may occur at time T unless the vorticity becomes unbounded when the time tends to T.

3. Inviscid limit: A “weak result”.

   The construction of local solutions corresponding to smooth enough data combined with a result of convergence in L^2 norms gives “for free” the existence of a fixed interval $[0,T]$ on which the viscous solution v_μ tends strongly to the inviscid solution v when μ goes to 0 (see e.g. [12]). Moreover, if $u_0 \in H^s$ with s large enough, the rate of convergence in L^2 is of order μ (see e.g. [5]).

4. Inviscid limit: A “strong result”

   One can prove that, if the solution E remains smooth on some given interval $[0,T_0]$ then (NS_μ) with small μ has a solution on the same time interval. Besides, strong convergence holds true on $[0,T_0]$ (see e.g. [5]).

Though very exciting from a mathematical viewpoint, studying (NS_μ) and (E) is somewhat disconnected from physical applications. Indeed, a fluid is hardly homogeneous or incompressible. In the present paper, we are concerned with the generalization of results 1., 2., 3. and 4. to *incompressible inhomogeneous fluids*.

The fluid is now described by its velocity field $u = u(t,x)$ *and* its density $\rho = \rho(t,x) \in \mathbb{R}^+$ and satisfies the *density-dependent incompressible Navier-Stokes equations*:

$$\begin{cases} \partial_t \rho + \operatorname{div} \rho u = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u + \nabla \Pi = \rho f, \\ \operatorname{div} u = 0, \end{cases} \tag{INS_μ}$$

or the *density-dependent incompressible Euler equations* $(IE) \stackrel{\text{def}}{=} (INS_0)$:

$$\begin{cases} \partial_t \rho + \operatorname{div} \rho u = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla \Pi = \rho f, \\ \operatorname{div} u = 0. \end{cases} \tag{IE}$$

Equations (INS_μ) and (IE) are supplemented with initial conditions $(\rho, u)_{|t=0} = (\rho_0, u_0)$, and the term f (which represents external forces) is given. We shall assume throughout that the space variable x belongs to the torus $\mathbb{T}^N$ or to the whole space $\mathbb{R}^N$.

Few papers are devoted to density-dependent incompressible fluids. In the viscous case however, the existence of *global weak solutions* has been stated for long (see [1], [15] or [10] and the references therein). A few pages in the book by P.-L. Lions (see [15]) are devoted to the density-dependent Euler equations (IE). The study of smooth viscous solutions has been done by O. Ladyzhenskaya and V. Solonnikov in $W^{s,p}$ spaces with $p > N$, and by H. Okamoto in Sobolev spaces H^s (see [14] and [11]). In both papers, system (INS_μ) is considered in a smooth bounded domain with Dirichlet boundary conditions on the velocity.

In the present paper, we show that (IE) and (INS_μ) are locally well-posed for $u_0 \in H^s$, ρ_0 such that $\inf_x \rho_0(x) > 0$ and $(\rho_0 - c) \in H^s$ (where c is a positive constant which may be assumed to be 1 with no loss of generality), and $f \in L^1_{loc}(0,T;H^s)$ with $s > 1 + N/2$ (the limit case $s = 1 + N/2$ is also addressed). As for smooth enough solutions, one has $\inf_x \rho(t,x) = \inf_x \rho_0(x)$, one can define $a \stackrel{\text{def}}{=} \rho^{-1} - 1$ so that system (INS_μ) with data bounded away from zero rewrites

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \\ \partial_t u + u \cdot \nabla u + (1+a)(\nabla \Pi - \mu \Delta u) = f, \\ \operatorname{div} u = 0. \end{cases} \tag{$\widetilde{INS}_\mu$}$$

Let us introduce the functional framework needed in the statement of our main well-posedness result:

Definition 0.1. — *For $s\in\mathbb{R}$, $\mu\geqslant 0$ and $T>0$, we denote*

$$F^s_{T,\mu}\stackrel{\text{def}}{=}\Big\{(a,u,\nabla\Pi)\in\widetilde{C}_T(H^s)\times\Big(\widetilde{C}_T(H^s)\Big)^N\times\Big(\widetilde{L}^1_T(H^s)\Big)^N\mid \mu u\in\Big(\widetilde{L}^1_T(H^{s+2})\Big)^N\Big\},$$

endowed with the norm

$$\|(a,u,\nabla\Pi)\|_{F^s_{T,\mu}}\stackrel{\text{def}}{=}\|a\|_{\widetilde{L}^\infty_T(H^s)}+\|u\|_{\widetilde{L}^\infty_T(H^s)}+\mu\|u\|_{\widetilde{L}^1_T(H^{s+2})}+\|\nabla\Pi\|_{\widetilde{L}^1_T(H^s)}.$$

When $\mu=0$, we shall alternately denote $F^s_{T,\mu}$ by F^s_T.

Above, $\widetilde{L}^1_T(H^\sigma)$ is a functional space containing $L^1(0,T;H^\sigma)$ (but still rather close to $L^1(0,T;H^\sigma)$), the notation $\widetilde{L}^\infty_T(H^\sigma)$ stands for a (large) subspace of $L^\infty_T(H^\sigma)$ and $\widetilde{C}_T(H^\sigma)=\widetilde{L}^\infty_T(H^\sigma)\cap C([0,T];H^\sigma)$. The reader will find the rigorous definition in section 1. We shall also use the notation $\widetilde{L}^1_{loc}(H^\sigma)=\cap_{T>0}\widetilde{L}^1_T(H^\sigma)$.

Our main well-posedness result reads:

Theorem 0.2. — *Let $\gamma>0$, $u_0\in H^{\frac{N}{2}+1+\gamma}$ with $\operatorname{div}u_0=0$, $f\in\widetilde{L}^1_{loc}(H^{\frac{N}{2}+1+\gamma})$ and ρ_0 such that*

$$\underline{\rho}\stackrel{\text{def}}{=}\inf_x\rho_0(x)>0,\quad \overline{\rho}\stackrel{\text{def}}{=}\sup_x\rho_0(x)<\infty\quad\textit{and}\quad a_0\stackrel{\text{def}}{=}\rho_0^{-1}-1\in H^{\frac{N}{2}+1+\gamma}.$$

For all $\mu\geqslant 0$, there exists a positive T such that system (INS_μ) has a unique solution $(\rho,u,\nabla\Pi)$ on the time interval $[0,T]$ with $\underline{\rho}\leqslant\rho\leqslant\overline{\rho}$, $(a,u,\nabla\Pi)\in F^{\frac{N}{2}+1+\gamma}_{T,\mu}$ and $\|(a,u,\nabla\Pi)\|_{F^s_{T,\mu}}$ bounded independently of μ. Moreover, the energy equality is satisfied:

$$\|\sqrt{\rho}u(t)\|^2_{L^2}+2\mu\int_0^t\|\nabla u(\tau)\|^2_{L^2}\,d\tau=\|\sqrt{\rho_0}u_0\|^2_{L^2}+2\int_0^t\int(\rho f\cdot u)(\tau,x)\,dx\,d\tau.\tag{0.1}$$

The time T may be bounded from below by a constant depending only on γ, N, μ, $\underline{\rho}$, $\overline{\rho}$ and on the norm of the data. For small μ, this bound may be chosen independent of μ.

Remark 0.3. — In the case $\mu>0$, local well-posedness may also be proved under the assumption that f belongs to $\widetilde{L}^m_{loc}(H^{\frac{N}{2}-1+\gamma+\frac{2}{m}})$ for some $m\in[1,+\infty]$.

Remark 0.4. — The limit case $s = 1 + N/2$ may be handled by considering data in the *Besov space* $B^{1+\frac{N}{2}}_{2,1}$ rather than in $H^{1+\frac{N}{2}}$. A similar approach has been used by M. Vishik in [17] for the "standard" incompressible Euler equations (E). The reader is referred to section 7 for more details.

The proof of theorem 0.2 relies on estimates for an appropriate linearization of $(\widetilde{INS}_\mu)$. The first equation reduces to a mere transport equation, and the linearization of the momentum equation is a non-stationary Stokes equation which contains both a convective term and a second order term with variable coefficients (see section 3).

Taking advantage of theorem 0.2, one can prove that, for data satisfying the assumptions above, the solution $(\rho_\mu, u_\mu, \nabla\Pi_\mu)$ to (INS_μ) tends strongly to the corresponding solution $(\rho, u, \nabla\Pi)$ of (IE) with a rate of convergence of order (at least) μ in L^2 (see section 2). Therefore, results 1. and 3. extend to density-dependent fluids.

Let us now discuss the possible breakdown of solutions. For that, we first have to define what we mean by a smooth solution:

Definition 0.5. — *For data (a_0, u_0, f) in $(H^{\frac{N}{2}+1+\gamma})^N \times H^{\frac{N}{2}+1+\gamma} \times (\widetilde{L}^1_{loc}(H^{\frac{N}{2}+1+\gamma}))^N$ with $\operatorname{div} u_0 = 0$ and $(1 + a_0)^{-1} \geqslant \underline{\rho} > 0$, we say that $(\rho, u, \nabla\Pi)$ is a smooth solution of (INS_μ) on $[0, T)$ if $(a, u, \nabla\Pi)$ belongs to $F^{\frac{N}{2}+1+\gamma}_{T',\mu}$ for all $T' < T$ and satisfies $(\widetilde{INS}_\mu)$ on $[0, T)$ in the sense of distributions. The time*

$$T^\star \stackrel{\text{def}}{=} \sup\Big\{T > 0 \mid (a, u, \nabla\Pi) \quad \textit{is a smooth solution of} \quad (\widetilde{INS}_\mu) \quad \textit{on} \quad [0, T)\Big\}$$

is called lifespan of the solution $(\rho, u, \nabla\Pi)$.

Let us now state the generalization of property 2. to non-homogeneous fluids:

Proposition 0.6. — *Let $\gamma > 0$. Assume that ρ_0 is bounded away from 0, that $a_0, u_0 \in H^{\frac{N}{2}+1+\gamma}$ (with $\operatorname{div} u_0 = 0$) and that $f \in \widetilde{L}^1_{loc}(H^{\frac{N}{2}+1+\gamma})$. Let $(\rho, u, \nabla\Pi)$ be a smooth solution to (INS_μ) on $[0, T)$. If in addition*

$$\operatorname{curl} u \in L^1(0, T; L^\infty) \quad \textit{and}$$

$$\begin{cases} \nabla a \in \widetilde{L}^\infty_T(H^{\frac{N}{2}+\gamma}) & \textit{if} \quad \mu = 0, \\ \nabla a \in L^\infty(0, T; H^{\frac{N}{2}+\alpha-1}) \ \textit{for some}\ \alpha > 0 & \textit{if} \quad \mu > 0, \end{cases}$$

then $(\rho, u, \nabla\Pi)$ may be continued beyond T into a smooth solution of (INS_μ).

Remark 0.7. — The above criterion may be seen as an extension of the Beale-Kato-Majda criterion (see [2]) to density-dependent fluids. The condition on $\operatorname{curl} u$ is the same as for homogeneous fluids. Due to inhomogeneity however, an additional condition on ρ is required.

Combining proposition 0.6 with theorem 0.2, we get the following important result:

COROLLARY 0.8. — *Given H^∞ data with density bounded away from zero, systems (INS_μ) and (IE) have a (unique) local solution which belongs to $F^s_{T,\mu}$ for all $s \in \mathbb{R}$.*

Let us now focus on property 4. (i.e the strong result pertaining to the inviscid limit). Once again, it may be generalized to density-dependent fluids:

THEOREM 0.9. — *Let γ, ρ_0, u_0 and f satisfy the assumptions of theorem 0.2. Assume that the density-dependent Euler equations with data (ρ_0, u_0, f) have a unique solution $(\rho, u, \nabla\Pi)$ on $[0, T_0]$ with $(a, u, \nabla\Pi) \in F^{\frac{N}{2}+1+\gamma}_{T_0}$.*

There exists $\mu_0 > 0$ depending only on $\|(a, u, \nabla\Pi)\|_{F^{\frac{N}{2}+1+\gamma}_{T_0}}$, $\|f\|_{\widetilde{L}^1_{T_0}(H^{\frac{N}{2}+1+\gamma})}$, T_0, $\underline{\rho}$, $\overline{\rho}$, γ and N, and such that for all $\mu \in (0, \mu_0]$, system (INS_μ) has a unique solution $(\rho_\mu, u_\mu, \nabla\Pi_\mu)$ on $[0, T_0]$ with $(a_\mu, u_\mu, \nabla\Pi_\mu) \in F^{\frac{N}{2}+1+\gamma}_{T_0,\mu}$ and norm independent of μ. Moreover, $(a_\mu, u_\mu, \nabla\Pi_\mu)$ tends to $(a, u, \nabla\Pi)$ in

$$\widetilde{C}_{T_0}(H^{\frac{N}{2}+1+\gamma'}) \times \Big(\widetilde{C}_{T_0}(H^{\frac{N}{2}+1+\gamma'})\Big)^N \times \Big(\widetilde{L}^1_{T_0}(H^{\frac{N}{2}+1+\gamma'})\Big)^N \quad \textit{for all} \quad \gamma' < \gamma.$$

Let us conclude with a few remarks.

Remark 0.10. — For the sake of simplicity, we restricted ourselves to the framework of Sobolev spaces H^s. Our results may be easily carried out to Besov spaces $B^s_{2,r}$ with $1 \leqslant r \leqslant \infty$ and $s > 1 + N/2$. We also believe that most of the results presented here are not specific to spaces built on L^2 and may be generalized to the L^p framework.

Remark 0.11. — The final conclusion is that results 1., 2., 3. and 4. are true for density-dependent incompressible fluids, locally in time. Compare to homogeneous fluids however, we lack global results in dimension $N = 2$. Let us mention that in the viscous case, global existence of strong solutions holds true in dimension $N = 2$, and in dimension $N \geqslant 3$ for small data (see [7] and [8]). Constructing global solutions for (IE) is an open question.

Our paper is structured as follows.

Section 1 is devoted to the presentation of the functional tool box: Littlewood-Paley decomposition, product laws in Sobolev and Besov spaces, elementary results on paradifferential calculus, etc. In section 2, we focus on energy estimates associated to systems (INS_μ) and (IE). We get a weak-strong uniqueness result and state that the rate of convergence in L^2 norm for the inviscid limit pertaining to smooth enough solutions to (INS_μ) is of order μ (see corollary 2.4 and remark 6.2). The following section is devoted to the study of linearized equations associated to $(\widetilde{INS}_\mu)$. The proof of local well-posedness for (INS_μ) is postponed to section 4. In section 5, we give a blow-up criterion for smooth solutions. In section 6, we prove estimates for the difference between a viscous solution and an inviscid solution. This in particular yields theorem 0.9. The last section is devoted to the critical case $\gamma = 0$. Some technical lemmas are postponed in the appendix. There we prove new estimates in Sobolev spaces for the elliptic equation satisfied by the pressure, which are of independent interest.

Notation : Summation convention on repeated indices will be used.

Throughout the paper, C stands for a "harmless constant" whose precise meaning is clear from the context. We sometimes alternately use the notation $A \lesssim B$ instead of $A \leqslant CB$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. We denote $x \vee y = \min(x, y)$.

The notation $\mathcal{P}$ stands for the L^2 projector on solenoidal vector-fields, while $\mathcal{Q}$ stands for the L^2 projector on potential vector-fields. Of course, one has $\mathcal{P}u + \mathcal{Q}u = u$ whenever u is a vector-field with coefficients in L^2.

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1. The functional tool box

Most of the results presented in the paper rely on a *Littlewood-Paley decomposition*. Let us briefly explain how it may be defined in the case $x \in \mathbb{R}^N$ (for periodic boundary conditions, see e.g [6]).

Let (χ, φ) be a couple of C^∞ functions with

$$\operatorname{Supp}\chi \subset \{|\xi| \leqslant \frac{4}{3}\},\ \operatorname{Supp}\varphi \subset \{\frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\} \text{ and } \chi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = 1.$$

Let $\varphi_q(\xi) = \varphi(2^{-q}\xi)$, $h_q = \mathcal{F}^{-1}\varphi_q$ and $\check{h} = \mathcal{F}^{-1}\chi$. The dyadic blocks are defined by

$$\Delta_q u \stackrel{\text{def}}{=} 0 \quad \text{if} \quad q \leqslant -2, \quad \Delta_{-1} u \stackrel{\text{def}}{=} \chi(D)u = \int_{\mathbb{R}^N} \check{h}(y)u(x-y)\,dy,$$
$$\Delta_q u \stackrel{\text{def}}{=} \varphi(2^{-q}D)u = \int_{\mathbb{R}^N} h_q(y)u(x-y)\,dy \quad \text{if} \quad q \geqslant 0.$$

We also introduce the low-frequency cut-off $S_q u \stackrel{\text{def}}{=} \chi(2^{-q}D)u$. As $\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$, it is clear that we have

$$S_q u = \sum_{k \leqslant q-1} \Delta_k u.$$

We shall make an extensive use of the following obvious fact:

$$\Delta_k \Delta_q u \equiv 0 \quad \text{if} \quad |k-q| \geqslant 2 \text{ and } \Delta_k(S_{q-1}u\Delta_q u) \equiv 0 \quad \text{if} \quad |k-q| \geqslant 5. \tag{1.1}$$

A number of functional spaces may be characterized in terms of Littlewood-Paley decomposition. Let us give the definition of (non-homogeneous) Besov spaces:

Definition 1.1. — *For $s \in \mathbb{R}$, $(p,r) \in [1,+\infty]^2$ and $u \in \mathcal{S}'(\mathbb{R}^N)$, we set*

$$\|u\|_{B^s_{p,r}} \stackrel{\text{def}}{=} \left(\sum_{q \geqslant -1} 2^{rsq} \|\Delta_q u\|^r_{L^p} \right)^{\frac{1}{r}},$$

with the usual modification if $r = +\infty$.

We then define the Besov space $B^s_{p,r} = \left\{ u \in \mathcal{S}' \mid \|u\|_{B^s_{p,r}} < +\infty \right\}$.

The definition of $B^s_{p,r}$ does not depend on the choice of the Littlewood-Paley decomposition. One can further remark that H^s coincide with $B^s_{2,2}$.

Proposition 1.2. — *The following properties hold true:*

i) Derivatives: we have $\|\nabla u\|_{B^{s-1}_{p,r}} \lesssim \|u\|_{B^s_{p,r}}$.

ii) Sobolev embeddings: If $p_1 \leqslant p_2$ and $r_1 \leqslant r_2$ then $B^s_{p_1,r_1} \hookrightarrow B^{s-N(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.

If $s_1 > s_2$ and $1 \leqslant p, r_1, r_2 \leqslant +\infty$, then $B^{s_1}_{p,r_1} \hookrightarrow B^{s_2}_{p,r_2}$.

iii) Algebraic properties: for $s > 0$, $B^s_{p,r} \cap L^\infty$ is an algebra. So does H^s if $s > N/2$.

iv) Real interpolation: $(B^{s_1}_{p,r}, B^{s_2}_{p,r})_{\theta,r'} = B^{\theta s_2+(1-\theta)s_1}_{p,r'}$.

Let us recall some classical estimates in Sobolev spaces for the product of two functions.

PROPOSITION 1.3. — *The following estimates hold true:*

$$\|uv\|_{H^s} \lesssim \|u\|_{L^\infty} \|v\|_{H^s} + \|v\|_{L^\infty} \|u\|_{H^s} \quad \textit{if} \quad s > 0, \tag{1.2}$$

$$\|uv\|_{H^{s_1}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}} \quad \textit{if} \quad s_1 + s_2 > 0, \quad s_1 \leqslant \frac{N}{2} \textit{ and } s_2 > \frac{N}{2}, \tag{1.3}$$

$$\|uv\|_{H^{s_1+s_2-\frac{N}{2}}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}} \quad \textit{if} \quad s_1 + s_2 > 0, \textit{ and } s_1, s_2 < \frac{N}{2}, \tag{1.4}$$

$$\|uv\|_{H^s} \lesssim \|u\|_{H^s} \|v\|_{L^\infty \cap H^{\frac{N}{2}}} \quad \textit{if} \quad |s| < \frac{N}{2}. \tag{1.5}$$

More accurate results may be obtained by mean of (basic) paradifferential calculus, a tool which was introduced by J.-M. Bony in [3].

The paraproduct between f and g is defined by

$$T_f g \stackrel{\text{def}}{=} \sum_{q \in \mathbb{N}} S_{q-1} f \Delta_q g.$$

Denoting $R(f,g) \stackrel{\text{def}}{=} \sum_{q \geqslant -1} \Delta_q f \, \widetilde{\Delta}_q g$ with $\widetilde{\Delta}_q g \stackrel{\text{def}}{=} \Delta_{q-1} g + \Delta_q g + \Delta_{q+1} g,$ and $T'_f g \stackrel{\text{def}}{=} T_f g + R(f,g),$ we have the following so-called Bony's decomposition:

$$fg = T_f g + T_g f + R(f,g) = T'_f g + T_g f.$$

A bunch of continuity results for the paraproduct T and the remainder R are available. We have for instance the following results (see the proof in [16], section 4.4):

PROPOSITION 1.4. — *For all $s \in \mathbb{R}$, $\sigma > 0$ and $1 \leqslant p, r \leqslant +\infty$, the paraproduct is a bilinear continuous application from $B^{-\sigma}_{\infty,\infty} \times B^s_{p,r}$ to $B^{s-\sigma}_{p,r}$, and from $L^\infty \times B^s_{p,r}$ to $B^s_{p,r}$.*

The remainder is bilinear continuous from $B^{s_1}_{p,r} \times B^{s_2}_{p,\infty}$ to $B^{s_1+s_2-\frac{N}{p}}_{p,r}$ whenever $s_1 + s_2 > N \min(0, -1 + 2/p)$.

Remark 1.5. — According to (1.1), the paraproduct rewrites

$$T_u v = \sum_{q \geqslant 1} S_{q-1} u \, \Delta_q \Big((1-\chi)(D) v \Big).$$

Thus, the low frequencies of v do not matter in the bilinear estimates for the paraproduct. Therefore, one has for instance for all $s \in \mathbb{R}$,

$$\|T_u v\|_{B^s_{p,r}} \lesssim \|u\|_{L^\infty} \, \|\nabla v\|_{B^{s-1}_{p,r}}.$$

Remark 1.6. — By decomposing uv into

$$uv = T_{\widetilde{u}} v + T_v \widetilde{u} + R(\widetilde{u}, v) + v \Delta_{-1} u \quad \text{with} \quad \widetilde{u} \stackrel{\text{def}}{=} u - \Delta_{-1} u,$$

and combining proposition 1.4 and the above remark, one can also prove that

$$\|uv\|_{H^{\sigma \vee (\frac{N}{2}+\alpha)}} \lesssim \big(\|u\|_{L^\infty} + \|\nabla u\|_{H^{\frac{N}{2}+\alpha-1}} \big) \|v\|_{H^\sigma}$$

whenever $\sigma + \frac{N}{2} + \alpha > 0$.

The study of non stationary PDE's requires spaces of type $L^\rho_T(X) \stackrel{\text{def}}{=} L^\rho(0,T;X)$ for appropriate Banach spaces X. In our case, we expect X to be a Sobolev or a Besov space, so that it is natural to localize the equations through Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. That remark naturally leads to the following definition (introduced in [4]):

Definition 1.7. — *For $\rho \in [1, +\infty]$, $s \in \mathbb{R}$ and $T \in [0, +\infty]$, we set*

$$\|u\|_{\widetilde{L}^\rho_T(H^s)} \stackrel{\text{def}}{=} \bigg(\sum_{q \geqslant -1} 2^{2qs} \Big(\int_0^T \|\Delta_q u(t)\|^\rho_{L^2} \, dt \Big)^{\frac{2}{\rho}} \bigg)^{\frac{1}{2}}$$

and denote by $\widetilde{L}^\rho_T(H^s)$ the subset of distributions u of $\mathcal{S}'(0, T \times \mathbb{R}^N)$ (or $\mathcal{S}'(0, T \times \mathbb{T}^N)$) with finite $\|u\|_{\widetilde{L}^\rho_T(H^s)}$ norm. When $T = +\infty$, the index T will be omitted.

Of course, one can also define the spaces $\widetilde{L}^\rho_T(B^s_{p,r})$ pertaining to the Besov space $B^s_{p,r}$.

Let us remark that by virtue of Minkowski inequality, we have

$$\|u\|_{\widetilde{L}^\rho_T(H^s)} \leqslant \|u\|_{L^\rho_T(H^s)} \text{ if } \rho \leqslant 2 \quad \text{and} \quad \|u\|_{L^\rho_T(H^s)} \leqslant \|u\|_{\widetilde{L}^\rho_T(H^s)} \text{ if } \rho \geqslant 2,$$

and one can easily prove that, whenever $\epsilon > 0$,

$$\begin{aligned} &\|u\|_{\widetilde{L}^\rho_T(H^s)} \lesssim \|u\|_{L^\rho_T(H^{s+\epsilon})} \text{ if } \rho \geqslant 2 \\ \text{and} \quad &\|u\|_{L^\rho_T(H^s)} \leqslant \|u\|_{\widetilde{L}^\rho_T(H^{s+\epsilon})} \text{ if } \rho \leqslant 2. \end{aligned} \tag{1.6}$$

We will often use the following interpolation inequality:

$$\begin{aligned} &\|u\|_{\widetilde{L}^\rho_T(H^s)} \leqslant \|u\|^{\theta}_{\widetilde{L}^{\rho_1}_T(H^{s_1})} \|u\|^{1-\theta}_{\widetilde{L}^{\rho_2}_T(H^{s_2})} \\ \text{with } &\frac{1}{\rho} = \frac{\theta}{\rho_1} + \frac{1-\theta}{\rho_2} \text{ and } s = \theta s_1 + (1-\theta) s_2. \end{aligned} \tag{1.7}$$

Remark 1.8. — The product, the paraproduct and the remainder are continuous in a number of spaces $\widetilde{L}^\rho_T(B^s_{p,r})$. The indices s, p and r just behave like in propositions 1.3 and 1.4, and the indices pertaining to the time integrability behave according to Hölder inequality. For example inequality (1.2) becomes

$$\|uv\|_{\widetilde{L}^\rho_T(H^s)} \lesssim \|u\|_{L^{\rho_1}_T(L^\infty)} \|v\|_{\widetilde{L}^{\rho_2}_T(H^s)} + \|v\|_{L^{\rho_2}_T(L^\infty)} \|u\|_{\widetilde{L}^{\rho_1}_T(H^s)}$$

whenever $s > 0$, $1 \leqslant \rho, \rho_1, \rho_2 \leqslant +\infty$ and $1/\rho = 1/\rho_1 + 1/\rho_2$.

2. Energy estimates

This section is devoted to the proof of energy-type estimates for linearized versions of (INS_μ). As applications, we shall prove a weak-strong uniqueness result and bound the rate of convergence for the inviscid limit.

PROPOSITION 2.1. — *Let* $(\rho, u, \nabla\Pi)$ *solve the following linear system on* $[0,T]$*:*

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = \rho g, \\ \rho(\partial_t u + v \cdot \nabla u) - \mu \Delta u + \nabla \Pi = \rho f, \\ \operatorname{div} u = 0 \end{cases} \tag{2.1}$$

where v *is a conveniently smooth time-dependent solenoidal vector field.*

The following estimates hold true for $t \in [0,T]$*:*

$$\forall p \in [1,+\infty],\ \|\rho(t)\|_{L^p} \leqslant \|\rho_0\|_{L^p} + \int_0^t \|(\rho g)(\tau)\|_{L^p}\, d\tau, \tag{2.2}$$

$$e^{-\frac{1}{2}\int_0^t \|g(\tau)\|_{L^\infty} d\tau}\|(\sqrt{\rho}u)(t)\|_{L^2} \leqslant \|\sqrt{\rho_0}u_0\|_{L^2} + \int_0^t e^{-\frac{1}{2}\int_0^\tau \|g(\tau')\|_{L^\infty} d\tau'}\|(\sqrt{\rho}f)(\tau)\|_{L^2}\, d\tau. \tag{2.3}$$

Proof. — The proof of (2.2) for solenoidal Lipschitz vector field v is straightforward. It relies on the conservation of the measure by the flow of v. For proving (2.3), take the scalar product in $\mathbb{R}^N$ of the momentum equation with u. We get

$$\partial_t\Big(\rho\frac{|u|^2}{2}\Big) + \mathrm{div}\Big(\rho v\frac{|u|^2}{2}\Big) - \frac{|u|^2}{2}(\partial_t\rho + v\cdot\nabla\rho) - \mu u\cdot\Delta u + \nabla\Pi\cdot u = (\sqrt{\rho}f)\cdot(\sqrt{\rho}u).$$

Taking advantage of the first equation in (2.1) and integrating in space yields:

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u\|_{L^2}^2 + \mu\|\nabla u\|_{L^2}^2 = \int(\sqrt{\rho}f)\cdot(\sqrt{\rho}u)\,dx + \int \rho g\frac{|u|^2}{2}\,dx,$$

hence

$$\big\|(\sqrt{\rho}u)(t)\big\|_{L^2} \leqslant \big\|\sqrt{\rho}_0 u_0\big\|_{L^2} + \int_0^t \|(\sqrt{\rho}f)(\tau)\|_{L^2}\, d\tau + \frac{1}{2}\int_0^t \|g(\tau)\|_{L^\infty}\|(\sqrt{\rho}u)(\tau)\|_{L^2}\, d\tau,$$

so that Gronwall inequality completes the proof. □

As a corollary of the above proposition, we get the following result:

Proposition 2.2. — *Let* $(\rho_i, u_i, \nabla\Pi_i)$ $(i=1,2)$ *satisfy*

$$\begin{cases} \partial_t\rho_i + u_i\cdot\nabla\rho_i = \rho_i g_i, \\ \rho(\partial_t u_i + u_i\cdot\nabla u_i) - \mu\Delta u_i + \nabla\Pi_i = \rho f_i, \\ \mathrm{div}\, u_i = 0. \end{cases} \tag{2.4}$$

Denote $\delta\rho \stackrel{\text{def}}{=} \rho_2 - \rho_1$, $\delta u \stackrel{\text{def}}{=} u_2 - u_1$, $\delta f \stackrel{\text{def}}{=} f_2 - f_1$ *and* $\delta g \stackrel{\text{def}}{=} g_2 - g_1$. *The following estimate holds true:*

$$e^{-V_{1,2}(t)}\Big(\|\delta\rho(t)\|_{L^2} + \big\|(\sqrt{\rho_2}\delta u)(t)\big\|_{L^2}\Big) \leqslant \|\delta\rho(0)\|_{L^2} + \big\|(\sqrt{\rho_2}\delta u)(0)\big\|_{L^2} + \int_0^t e^{-V_{1,2}(\tau)}\Big(\|(\rho_1\delta g)(\tau)\|_{L^2} + \big\|(\sqrt{\rho_2}\delta f)(\tau)\big\|_{L^2}\Big)\,d\tau,$$

with

$$V_{1,2}(t) \stackrel{\text{def}}{=} \int_0^t \bigg(\|g_2(\tau)\|_{L^\infty} + \Big\|\frac{\nabla\rho_1}{\sqrt{\rho_2}}(\tau)\Big\|_{L^\infty} + \|\nabla u_1(\tau)\|_{L^\infty} + \Big\|\Big(\frac{\nabla\Pi_1 - \mu\Delta u_1}{\rho_1\sqrt{\rho_2}}\Big)(\tau)\Big\|_{L^\infty}\bigg)\,d\tau.$$

Proof. — As $\partial_t\delta\rho + u_2\cdot\nabla\delta\rho = -\delta u\cdot\nabla\rho_1 + \rho_1\delta g + \delta\rho\, g_2$, estimate (2.2) combined with Gronwall inequality yields:

$$e^{-\int_0^t \|g_2(\tau)\|_{L^\infty}\,d\tau}\,\|\delta\rho(t)\|_{L^2} \leqslant \|\delta\rho(0)\|_{L^2} + \int_0^t e^{-\int_0^\tau \|g_2(\tau')\|_{L^\infty}\,d\tau'}\,\|(\rho_1\delta g)(\tau)\|_{L^2}\,d\tau + \int_0^t e^{-\int_0^\tau \|g_2(\tau')\|_{L^\infty}\,d\tau'}\,\big\|(\sqrt{\rho_2}\delta u)(\tau)\big\|_{L^2}\Big\|\frac{\nabla\rho_1}{\sqrt{\rho_2}}(\tau)\Big\|_{L^\infty}\,d\tau. \quad (2.5)$$

On the other hand $(\rho_2, \delta u, \nabla\delta\Pi)$ solves

$$\begin{cases} \partial_t\rho_2 + u_2\cdot\nabla\rho_2 = \rho_2 g_2, \\ \rho_2(\partial_t\delta u + u_2\cdot\nabla\delta u) - \mu\Delta\delta u + \nabla\delta\Pi \\ \qquad\qquad = \rho_2\Big(\delta f - \delta u\cdot\nabla u_1 + \dfrac{\delta\rho}{\rho_1\rho_2}(\nabla\Pi_1 - \mu\Delta u_1)\Big), \\ \operatorname{div}\delta u = 0. \end{cases}$$

Applying inequality (2.3) yields

$$e^{-\int_0^t \frac{\|g_2(\tau)\|_{L^\infty}}{2}d\tau}\big\|(\sqrt{\rho_2}\delta u)(t)\big\|_{L^2} \leqslant \big\|(\sqrt{\rho_2}\delta u)(0)\big\|_{L^2} + \int_0^t e^{-\int_0^\tau \frac{\|g_2(\tau')\|_{L^\infty}}{2}d\tau'}\big\|(\sqrt{\rho_2}\delta f)(\tau)\big\|_{L^2}\,d\tau + \int_0^t e^{-\frac12\int_0^\tau \|g_2(\tau')\|_{L^\infty}\,d\tau'}\,\|\delta\rho(\tau)\|_{L^2}\Big\|\Big(\frac{\nabla\Pi_1 - \mu\Delta u_1}{\rho_1\sqrt{\rho_2}}\Big)(\tau)\Big\|_{L^\infty}\,d\tau + \int_0^t e^{-\frac12\int_0^\tau \|g_2(\tau')\|_{L^\infty}\,d\tau'}\,\|\nabla u_1(\tau)\|_{L^\infty}\big\|(\sqrt{\rho_2}\delta u)(\tau)\big\|_{L^2}\,d\tau.$$

Now, combining the above inequality with (2.5) and using Gronwall lemma yields the desired estimate. □

Corollary 2.3. — *Let* $(\rho_1, u_1, \nabla\Pi_1)$ *and* $(\rho_2, u_2, \nabla\Pi_2)$ *be two weak solutions of* (INS_μ) *or* (IE) *with the same initial data and external force. Assume that the density for both solution is bounded away from zero. If in addition* ∇u_1, $\nabla\rho_1$ *and* $\nabla\Pi_1 - \mu\Delta u_1$ *belong to* $L^1(0,T;L^\infty)$ *then* $(\rho_1, u_1, \nabla\Pi_1) \equiv (\rho_2, u_2, \nabla\Pi_2)$ *on* $[0,T]$.

Proof. — Apply proposition 2.2 with $f_1 = f_2 = f$, $g_1 = g_2 = 0$ and $(\rho_1(0), u_1(0)) = (\rho_2(0), u_2(0))$. □

Corollary 2.4. — *Let* $(\rho_\mu, u_\mu, \nabla\Pi_\mu)$ *be a solution to* (INS_μ) *and* $(\rho, u, \nabla\Pi)$ *be a solution to* (IE) *with the same external force and initial data. If in addition* $0 < \underline{\rho} \leqslant \rho_0 \leqslant \overline{\rho}$ *then the following estimate holds true:*

$$\|(\rho_\mu - \rho)(t)\|_{L^2} + \sqrt{\underline{\rho}}\,\|(u_\mu - u)(t)\|_{L^2} \leqslant \mu \frac{\sqrt{\overline{\rho}}}{\underline{\rho}}\, e^{V(t)} \int_0^t e^{-V(\tau)}\, \|\Delta u(\tau)\|_{L^2}\, d\tau,$$

with

$$V(t) \stackrel{\text{def}}{=} \int_0^t \Big(\underline{\rho}^{-\frac{1}{2}}\, \|\nabla\rho(\tau)\|_{L^\infty} + \|\nabla u(\tau)\|_{L^\infty} + \underline{\rho}^{-\frac{3}{2}}\, \|(\nabla\Pi - \mu\Delta u)(\tau)\|_{L^\infty} \Big)\, dt.$$

Proof. — Apply proposition 2.2 with viscosity μ and:

$$\begin{aligned} &(\rho_1, u_1, \nabla\Pi_1) \stackrel{\text{def}}{=} (\rho, u, \nabla\Pi), && g_1 \stackrel{\text{def}}{=} 0, \quad f_1 \stackrel{\text{def}}{=} f - \mu\rho^{-1}\Delta u,\\ &(\rho_2, u_2, \nabla\Pi_2) \stackrel{\text{def}}{=} (\rho_\mu, u_\mu, \nabla\Pi_\mu), && g_2 \stackrel{\text{def}}{=} 0, \quad f_2 \stackrel{\text{def}}{=} f. \end{aligned}$$

□

3. The linearized equations

This section is devoted to the proof of estimates and existence of solutions for linearized system $(\widetilde{INS}_\mu)$.

The first equation in $(\widetilde{INS}_\mu)$ is a mere transport equation for which the following proposition applies (see the proof in [9], Prop. 2.1).

Proposition 3.1. — *Let* $s > -1 - N/2$ *be such that* $s \neq 1 + N/2$. *Let* v *be a solenoidal vector field such that* ∇v *belongs to* $L^1(0,T; B^{\frac{N}{2}}_{2,\infty} \cap L^\infty)$ *if*

$|s| < 1 + N/2$ or to $L^1(0,T;H^s)$ if $s > 1 + N/2$. Suppose also that $a_0 \in H^s$, $g \in \widetilde{L}^1_T(H^s)$ and that $a \in L^\infty(0,T;H^s) \cap C([0,T];\mathcal{S}'(\mathbb{R}^N))$ solves

$$\begin{cases} \partial_t a + v \cdot \nabla a = g, \\ a_{|t=0} = a_0. \end{cases} \tag{3.1}$$

Then $a \in \widetilde{C}_T(H^s)$ and there exists a constant C depending only on s and N, and such that the following inequality holds on $[0,T]$:

$$\|a\|_{\widetilde{L}^\infty_t(H^s)} \leqslant e^{CV(t)} \left(\|a_0\|_{H^s} + \|g\|_{\widetilde{L}^1_t(H^s)} \right),$$

$$\textit{with} \quad V(t) = \begin{cases} \displaystyle\int_0^t \|\nabla v(\tau)\|_{B^{\frac{N}{2}}_{2,\infty} \cap L^\infty} \, d\tau & \textit{if} \quad |s| < 1 + N/2, \\ \displaystyle\int_0^t \|\nabla v(\tau)\|_{H^{s-1}} \, d\tau & \textit{if} \quad s > 1 + N/2. \end{cases}$$

Let us now focus on the study of the following linearization of the momentum equation:

$$\begin{cases} \partial_t u + v \cdot \nabla u + b(\nabla \Pi - \mu \Delta u) = f + g, \\ \operatorname{div} u = 0, \\ u_{|t=0} = u_0, \end{cases} \tag{M_μ}$$

where b, f, g, v and u_0 are given functions.

The reason why we introduce *two* types of external forces will appear in section 6 when studying the inviscid limit. We assume that there exist two positive constants $\underline{b}$ and $\overline{b}$ such that $\underline{b} \leqslant b \leqslant \overline{b}$ and that b tends to some positive constant (say 1 with no loss of generality) at infinity.

3.1. A priori estimates

In the present section, we aim at proving *a priori* estimates for (M_μ) in the framework of non-homogeneous Sobolev spaces and for arbitrary positive b such that $a \stackrel{\text{def}}{=} b - 1$ belongs to $\widetilde{L}^\infty_T(H^{\frac{N}{2}+\alpha})$ for some $\alpha > 0$. Before stating our results let us introduce the notation

$$\mathcal{A}_T \stackrel{\text{def}}{=} \begin{cases} \underline{b}^{-1} \Big(\overline{b} + \|\nabla a\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+\alpha-1})} \Big) & \text{if} \quad \alpha \neq 1, \\ \underline{b}^{-1} \Big(\overline{b} + \|\nabla a\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}}) \cap L^\infty_T(L^\infty)} \Big) & \text{if} \quad \alpha = 1. \end{cases} \tag{3.2}$$

We can now state a general estimate for (M_μ):

Proposition 3.2. — *Let $\mu \geqslant 0$, $m \geqslant 1$ ($m=1$ if $\mu=0$), $\alpha>0$ and $s \in (2-2/m, \alpha+N/2]$ with $s \neq 1+N/2$. Let u_0 be a solenoidal vector field with coefficients in H^s and f (resp. g) be a time dependent vector field with coefficients in $\widetilde{L}^1_T(H^s)$ (resp. $\widetilde{L}^m_T(H^{s-2+\frac{2}{m}})$). Assume that $a \in \widetilde{C}_T(H^{\frac{N}{2}+\alpha})$ (and also that $a \in L^\infty(0,T;\mathrm{Lip})$ if $\alpha=1$). Let v be a time dependent solenoidal vector field such that $\nabla v \in L^1(0,T;H^{\frac{N}{2}} \cap L^\infty)$ if $s<N/2+1$ and $\nabla v \in L^1(0,T;H^s)$ if $s>N/2+1$. Let $u \in \widetilde{L}^\infty_T(H^s)$ be a solution of (M_μ) on $[0,T]$ for some $\nabla\Pi \in \widetilde{L}^1_T(H^s)+\widetilde{L}^m_T(H^{s-2+\frac{2}{m}})$. Let $\alpha'>0$ satisfy*

$$\alpha' \leqslant \min\left(1,\alpha,\frac{s-2+\frac{2}{m}}{2}\right) \quad \textit{if} \quad \left[s<\frac{N}{2}+\alpha \quad \textit{or} \quad \left(s=\frac{N}{2}+\alpha \quad \textit{and} \quad (m>1 \quad \textit{or} \quad \alpha>1)\right)\right],$$

$$\alpha' \in (0,\alpha) \cap \left(0,\frac{s-2+\frac{2}{m}}{2}\right] \quad \textit{if} \quad \left[s=\frac{N}{2}+\alpha, \quad m=1 \quad \textit{and} \quad \alpha \leqslant 1\right].$$

There exists $C=C(s,N,\alpha,\alpha',m)$ such that

$$\|u\|_{\widetilde{L}^\infty_T(H^s)}+\underline{\mu}^{\frac{1}{m}}\|u\|_{\widetilde{L}^m_T(H^{s+\frac{2}{m}})} \leqslant Ce^{C\mathcal{A}_T^\kappa V(T)}\Big(\|u_0\|_{H^s}+\mathcal{A}_T^\kappa\Big(\|f\|_{\widetilde{L}^1_T(H^s)}+\underline{\mu}^{\frac{1}{m}-1}\|g\|_{\widetilde{L}^m_T(H^{s-2+\frac{2}{m}})} + \underline{\mu}^{\frac{1}{m}}\mathcal{A}_T\|u\|_{\widetilde{L}^m_T(H^{s+\frac{2}{m}-\alpha'})}\Big)\Big), \tag{3.3}$$

with $\underline{\mu} \stackrel{\text{def}}{=} \underline{b}\mu$, and $\kappa \stackrel{\text{def}}{=} s/\alpha'$, and

$$V(T) \stackrel{\text{def}}{=} \begin{cases} \displaystyle\int_0^T \|\nabla v\|_{H^{\frac{N}{2}} \cap L^\infty}\,dt & \textit{if} \quad s<\frac{N}{2}+1,\\ \displaystyle\int_0^T \|\nabla v\|_{H^{s-1}}\,dt & \textit{if} \quad s>\frac{N}{2}+1. \end{cases}$$

Moreover, we have

$$\underline{b}\|\nabla\Pi\|_{\widetilde{L}^1_T(H^s)+\widetilde{L}^m_T(H^{s-2+\frac{2}{m}})} \leqslant C\mathcal{A}_T^\kappa\Big(\|\mathcal{Q}f\|_{\widetilde{L}^1_T(H^s)}+\|\mathcal{Q}g\|_{\widetilde{L}^m_T(H^{s-2+\frac{2}{m}})} + \int_0^T V'(t)\|u(t)\|_{H^s}\,dt+\underline{\mu}\big(\mathcal{A}_T-\underline{b}/\overline{b}\big)\|u\|_{\widetilde{L}^m_T(H^{s+\frac{2}{m}})}\Big). \tag{3.4}$$

If $v=u$, the above estimates hold with $V(T)=\int_0^T \|\nabla u\|_{L^\infty}\,dt$ (even if $s=1+\frac{N}{2}$).

For proving proposition 3.2 in the case $b \equiv 1$, one can project (M_μ) on solenoidal vector-fields by making use of the Leray projector $\mathcal{P}$. Then system (M_μ) reduces to a convection-diffusion type equation which may be easily solved by mean of energy estimates. In our case where b is not assumed to be a constant, getting rid of the pressure will still be an appropriate strategy. This may be achieved by applying the operator div to (M_μ). Indeed, in doing so, we see that the pressure solves the elliptic equation

$$\operatorname{div}(b\nabla\Pi) = \operatorname{div} F \tag{3.5}$$

with $F = f + g + \mu a\Delta u - v \cdot \nabla u$.

Therefore, denoting by $\mathcal{H}_b$ the linear operator $F \mapsto \nabla\Pi$, system (M_μ) reduces to a linear ODE in Banach spaces.

Actually, due to the consideration of *two* forcing terms f and g with *different* regularities, the pressure has to be split into two parts, namely $\Pi = \Pi_1 + \Pi_2$ with

$$\operatorname{div}(b\nabla\Pi_1) = \operatorname{div} G \quad \text{and} \quad G \stackrel{\text{def}}{=} f - T_{\nabla u} v - T'_{\nabla v} u, \tag{3.6}$$

$$\operatorname{div}(b\nabla\Pi_2) = \operatorname{div} H \quad \text{and} \quad H \stackrel{\text{def}}{=} g + \mu a\Delta u. \tag{3.7}$$

Note that the expression of G has been obtained by making use of Bony's decomposition and by taking advantage of $\operatorname{div} u = \operatorname{div} v = 0$ which implies $\operatorname{div}(v \cdot \nabla u) = \operatorname{div}(u \cdot \nabla v)$.

Proof of proposition 3.2. — To simplify the presentation, we assume that $\alpha \neq 1$. The case $\alpha = 1$ may be handled by changing $\|a\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+\alpha})}$ into $\|a\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+1}) \cap L^\infty_T(\mathrm{Lip})}$.

Applying Δ_q to (M_μ), we get for all $q \geqslant -1$,

$$\partial_t \Delta_q u + v \cdot \nabla \Delta_q u + \Delta_q \nabla\Pi - \mu \operatorname{div}(b\Delta_q \nabla u) = \\ \Delta_q f + \Delta_q g + [v, \Delta_q] \cdot \nabla u - \Delta_q(a\nabla\Pi) + \mu R_q$$

with

$$R^j_q \stackrel{\text{def}}{=} \Delta_q(b\Delta u^j) - \operatorname{div}(b\Delta_q \nabla u^j).$$

Of course, we do not have to worry about R_q in the case $\mu = 0$.

Let $\widetilde{a} \stackrel{\text{def}}{=} a - \Delta_{-1} a$. Take the L^2-scalar product with $\Delta_q u$. As $\operatorname{div} u = 0$, using Bony's decomposition and performing an integration by parts yields

$$\Big|(\Delta_q(a\nabla\Pi)|\Delta_q u)\Big| \leqslant \Big|(\Delta_q T_{\nabla\widetilde{a}}\Pi|\Delta_q u)\Big| + \Big|(\Delta_q T'_{\nabla\Pi}\widetilde{a}|\Delta_q u)\Big| + \Big|(\Delta_q(\Delta_{-1}a\nabla\Pi)|\Delta_q u)\Big|. \tag{3.8}$$

Therefore, denoting $\underline{\mu} \stackrel{\text{def}}{=} \mu\underline{b}$, we get

$$\frac{1}{2}\frac{d}{dt}\|\Delta_q u\|_{L^2}^2 + \underline{\mu}\,\|\nabla\Delta_q u\|_{L^2}^2 \leqslant \|\Delta_q u\|_{L^2}\Big(\mu\,\|R_q\|_{L^2} + \|[v,\Delta_q]\cdot\nabla u\|_{L^2}$$
$$+\,\|\Delta_q T_{\nabla a}\Pi\|_{L^2} + \|\Delta_q T'_{\nabla\Pi}a\|_{L^2} + \|\Delta_q(\Delta_{-1}a\nabla\Pi)\|_{L^2} + \|\Delta_q\mathcal{P}f\|_{L^2} + \|\Delta_q\mathcal{P}g\|_{L^2}\Big).$$

According to Bernstein inequality, there exists $\kappa>0$ such that for all $q\geqslant 0$, we have $\|\Delta_q\nabla u\|_{L^2} \geqslant \sqrt{\kappa}2^q\,\|\Delta_q u\|_{L^2}$. Elementary computations thus yield (at least formally) :

$$e^{-\kappa\underline{\mu}2^{2q}t}\frac{d}{dt}\Big(e^{\kappa\underline{\mu}2^{2q}t}\,\|\Delta_q u\|_{L^2}\Big) \leqslant \mu\,\|R_q\|_{L^2} + \|[v,\Delta_q]\cdot\nabla u\|_{L^2} + \|\Delta_q T_{\nabla a}\Pi\|_{L^2}$$
$$+\,\|\Delta_q T'_{\nabla\Pi}a\|_{L^2} + \|\Delta_q(\Delta_{-1}a\nabla\Pi)\|_{L^2} + \|\Delta_q\mathcal{P}f\|_{L^2} + \|\Delta_q\mathcal{P}g\|_{L^2}\,. \tag{3.9}$$

When $q=-1$, a similar inequality holds true with $\kappa=0$.

Let us now focus on the pressure. As explained above, the pressure has to be split into two parts: $\nabla\Pi = \nabla\Pi_1 + \nabla\Pi_2$ where $\nabla\Pi_1$ and $\nabla\Pi_2$ have been defined in (3.6) and (3.7). Then proposition 1.4 combined with the embedding $L^1_T(H^s) \hookrightarrow \widetilde{L}^1_T(H^s)$ yields

$$\|\mathcal{Q}G\|_{\widetilde{L}^1_T(H^s)} \lesssim \|\mathcal{Q}f\|_{\widetilde{L}^1_T(H^s)} + \int_0^T V'(t)\|u(t)\|_{H^s}\,dt,$$

with

$$V'(t) \stackrel{\text{def}}{=} \begin{cases} \|\nabla v(t)\|_{H^{\frac{N}{2}}\cap L^\infty} & \text{if} \quad |s|<1+\frac{N}{2},\\ \|\nabla v(t)\|_{H^{s-1}} & \text{if} \quad s>1+\frac{N}{2},\\ \|\nabla u(t)\|_{L^\infty} & \text{if} \quad v=u \ \text{ and } \ s>-1.\end{cases} \tag{3.10}$$

Hence, in view of proposition 8.5 and provided that $0<\alpha'\leqslant\min(1,\alpha,s/2)$ and $s\leqslant\alpha+N/2$ (which is assumed in the statement of proposition 3.2), we get for $\alpha''=0$ or α',

$$\underline{b}\|\nabla\Pi_1\|_{\widetilde{L}^1_T(H^{s-\alpha''})} \lesssim \mathcal{A}_T^{\frac{s-\alpha''}{\alpha'}}\Big(\|\mathcal{Q}f\|_{\widetilde{L}^1_T(H^s)} + \int_0^T V'(t)\|u(t)\|_{H^s}\,dt\Big). \tag{3.11}$$

By virtue of proposition 1.4 combined with remarks 1.6 and 1.8, we have for $\alpha''=0,\alpha'$:

$$\|\mathcal{Q}H\|_{\widetilde{L}^m_T(H^{s-2+\frac{2}{m}-\alpha''})}\lesssim \|\mathcal{Q}g\|_{\widetilde{L}^m_T(H^{s-2+\frac{2}{m}})}+\mu\big(\|a\|_{L^\infty_T(L^\infty)}+\|\nabla a\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+\alpha-1})}\big)\|u\|_{\widetilde{L}^m_T(H^{s+\frac{2}{m}-\alpha''})}.$$

As $\alpha'\leqslant\min(1,\alpha,(s-2+2/m)/2)$, applying proposition 8.5 with $\alpha=\alpha'$, $\sigma=s-2+2/m$ or $\sigma=s-2+2/m-\alpha'$ (here comes $s>2-2/m$) yields for $\alpha''=0,\alpha'$,

$$\underline{b}\|\nabla\Pi_2\|_{\widetilde{L}^m_T(H^{s-2+\frac{2}{m}-\alpha''})}\lesssim\mathcal{A}^\kappa_T\Big(\|\mathcal{Q}g\|_{\widetilde{L}^m_T(H^{s-2+\frac{2}{m}})} +\mu\big(\|a\|_{L^\infty_T(L^\infty)}+\|\nabla a\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+\alpha-1})}\big)\|u\|_{\widetilde{L}^m_T(H^{s+\frac{2}{m}-\alpha''})}\Big).\quad(3.12)$$

Note that summing (3.11) and (3.12) with $\alpha''=0$ gives (3.4).

Let δ_{ij} stand for the Kronecker symbol on $\mathbb{Z}^2$. Performing a time integration in (3.9) and using convolution inequalities yields for all $q\geqslant-1$,

$$\begin{aligned}\|\Delta_q u\|_{L^\infty_T(L^2)}+\underline{\mu}^{\frac{1}{m}}2^{\frac{2q}{m}}\|\Delta_q u\|_{L^m_T(L^2)}\lesssim{}& \|u_0\|_{L^2}+\|\Delta_q\mathcal{P}f\|_{L^1_T(L^2)}+\delta_{-1q}\,\underline{\mu}^{\frac{1}{m}}2^{\frac{2q}{m}}\|\Delta_{-1}u\|_{L^m_T(L^2)}\\ &+\underline{b}^{-1}\underline{\mu}^{\frac{1}{m}}2^{q(\frac{2}{m}-2)}\|R_q\|_{L^m_T(L^2)}+\|[v,\Delta_q]\cdot\nabla u\|_{L^1_T(L^2)}+\|\Delta_q T_{\nabla\widetilde{a}}\Pi_1\|_{L^1_T(L^2)}\\ &+\|\Delta_q T'_{\nabla\Pi_1}\widetilde{a}\|_{L^1_T(L^2)}+\|\Delta_q(\Delta_{-1}a\nabla\Pi_1)\|_{L^1_T(L^2)}\\ &+\underline{\mu}^{\frac{1}{m}-1}2^{q(\frac{2}{m}-2)}\Big(\|\Delta_q T_{\nabla\widetilde{a}}\Pi_2\|_{L^m_T(L^2)}\\ &+\|\Delta_q T'_{\nabla\Pi_2}\widetilde{a}\|_{L^m_T(L^2)}+\|\Delta_q(\Delta_{-1}a\nabla\Pi_2)\|_{L^m_T(L^2)}+\|\Delta_q\mathcal{P}g\|_{L^m_T(L^2)}\Big),\end{aligned}$$

whence, multiplying both sides by 2^{qs} and summing on q, we get

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_T(H^s)}+\underline{\mu}^{\frac{1}{m}}\|u\|_{\widetilde{L}^m_T(H^{s+\frac{2}{m}})}\lesssim{}&\|u_0\|_{H^s}+\|\mathcal{P}f\|_{\widetilde{L}^1_T(H^s)}+\underline{\mu}^{\frac{1}{m}}\|\Delta_{-1}u\|_{L^m_T(L^2)}\\ &+\|T_{\nabla\widetilde{a}}\Pi_1\|_{\widetilde{L}^1_T(H^s)}+\|T'_{\nabla\Pi_1}\widetilde{a}\|_{\widetilde{L}^1_T(H^s)}+\|\Delta_{-1}a\nabla\Pi_1\|_{\widetilde{L}^1_T(H^s)}\\ &+\underline{\mu}^{\frac{1}{m}-1}\Big(\|T_{\nabla\widetilde{a}}\Pi_2\|_{\widetilde{L}^m_T(H^{s-2+\frac{2}{m}})}+\|T'_{\nabla\Pi_2}\widetilde{a}\|_{\widetilde{L}^m_T(H^{s-2+\frac{2}{m}})}\\ &+\|\Delta_{-1}a\nabla\Pi_2\|_{\widetilde{L}^m_T(H^{s-2+\frac{2}{m}})}+\|\mathcal{P}g\|_{\widetilde{L}^m_T(H^{s-2+\frac{2}{m}})}\Big)\\ &+\underline{b}^{-1}\underline{\mu}^{\frac{1}{m}}\Big(\sum_{q\geqslant-1}2^{2q(s-2+\frac{2}{m})}\|R_q\|^2_{L^m_T(L^2)}\Big)^{\frac{1}{2}}\\ &+\Big(\sum_{q\geqslant-1}2^{2qs}\|[v,\Delta_q]\cdot\nabla u\|^2_{L^1_T(L^2)}\Big)^{\frac{1}{2}}.\end{aligned}\quad(3.13)$$

With our assumptions on α, α' and s, the terms containing Π_1 may be bounded by

$$\big(\|a\|_{L^\infty_T(L^\infty)}+\|\nabla a\|_{\widetilde L^\infty_T(H^{\frac N2+\alpha-1})}\big)\|\nabla\Pi_1\|_{\widetilde L^1_T(H^{s-\alpha'})}$$

whereas those containing Π_2 may be bounded by

$$\big(\|a\|_{L^\infty_T(L^\infty)}+\|\nabla a\|_{\widetilde L^\infty_T(H^{\frac N2+\alpha-1})}\big)\|\nabla\Pi_2\|_{\widetilde L^m_T(H^{s-2+\frac2m-\alpha'})}.$$

Moreover, by virtue of lemmas 8.11 and 8.9 and using the notation (3.10), we have

$$\Big(\sum_{q\geqslant-1}2^{2qs}\|[v,\Delta_q]\cdot\nabla u\|^2_{L^1_T(L^2)}\Big)^{\frac12}\lesssim\int_0^T V'(t)\|u(t)\|_{H^s}\,dt,$$

$$\Big(\sum_{q\geqslant-1}2^{2q(s-2+\frac2m)}\|R_q\|^2_{L^m_T(L^2)}\Big)^{\frac12}\lesssim\|\nabla a\|_{\widetilde L^\infty_T(H^{\frac N2+\alpha-1})}\|\nabla u\|_{\widetilde L^m_T(H^{s-1+\frac2m-\alpha'})},$$

provided that α, s and α' satisfy the conditions of proposition 3.2.

Plugging all these inequalities in (3.13) eventually yields

$$\begin{aligned}\|u\|_{\widetilde L^\infty_T(H^s)}+\underline\mu^{\frac1m}\|u\|_{\widetilde L^m_T(H^{s+\frac2m})}\lesssim{}&\|u_0\|_{H^s}\\&+\underline\mu^{\frac1m}\|u\|_{\widetilde L^m_T(H^{s+\frac2m-\alpha'})}+\int_0^T V'(t)\|u(t)\|_{H^s}dt\\&+\underline b^{-1}\underline\mu^{\frac1m}\|a\|_{\widetilde L^\infty_T(H^{\frac N2+\alpha})}\|\nabla u\|_{\widetilde L^m_T(H^{s-1+\frac2m-\alpha'})}\\&+\|\mathcal Pf\|_{\widetilde L^1_T(H^s)}+\underline\mu^{\frac1m-1}\|\mathcal Pg\|_{\widetilde L^m_T(H^{s-2+\frac2m})}\\&+\big(\|a\|_{L^\infty_T(L^\infty)}+\|\nabla a\|_{\widetilde L^\infty_T(H^{\frac N2+\alpha-1})}\big)\big(\|\nabla\Pi_1\|_{\widetilde L^1_T(H^{s-\alpha'})}\\&\qquad+\underline\mu^{\frac1m-1}\|\nabla\Pi_2\|_{\widetilde L^m_T(H^{s-2+\frac2m-\alpha'})}\big).\end{aligned}$$

Appealing to (3.11) and (3.12) with $\alpha''=\alpha'$, we conclude that

$$\begin{aligned}\|u\|_{\widetilde L^\infty_T(H^s)}+\underline\mu^{\frac1m}\|u\|_{\widetilde L^m_T(H^{s+\frac2m})}\lesssim{}&\|u_0\|_{H^s}+\mathcal A_T^{\frac{s}{\alpha'}}\Big(\|f\|_{\widetilde L^1_T(H^s)}\\+\underline\mu^{\frac1m-1}\|g\|_{\widetilde L^m_T(H^{s-2+\frac2m})}+\int_0^T&V'(t)\|u(t)\|_{H^s}\,dt+\underline\mu^{\frac1m}\mathcal A_T\ \|u\|_{\widetilde L^m_T(H^{s+\frac2m-\alpha'})}\Big).\end{aligned}$$

Gronwall lemma completes the proof of the desired inequality for u. □

Remark 3.3. — According to remark 8.6, in the case $\alpha < 1$ and $s < \frac{N}{2} + \alpha$ one can take $\mathcal{A}_T = \underline{b}^{-1}\big(\overline{b} + \|\nabla a\|_{L^\infty_T(B^{\frac{N}{2}+\alpha-1}_{2,\infty})}\big)$. Note also that if $s < \frac{N}{2} + 1$ then the statement of proposition 3.2 holds with $V(T) = \int_0^T \|\nabla v(t)\|_{B^{\frac{N}{2}}_{2,\infty}\cap L^\infty}\,dt$. Besides, as the restriction $s \neq \frac{N}{2} + 1$ is due to the convective term only, it may be removed if $\nabla v \equiv 0$.

3.2. Global well-posedness for (M_μ)

PROPOSITION 3.4. — *Let $T > 0$. Let μ, m, s, α, u_0, f, g, a and v satisfy the assumptions of proposition 3.2. Then system (M_μ) has a unique solution $(u, \nabla\Pi)$ such that*

$$u \in \widetilde{C}_T(H^s), \quad \underline{\mu}^{\frac{1}{m}} u \in \widetilde{L}^m_T(H^{s+\frac{2}{m}}) \quad \textit{and} \quad \nabla\Pi \in \widetilde{L}^1_T(H^s) + \widetilde{L}^m_T(H^{s-2+\frac{2}{m}}).$$

Moreover $(u, \nabla\Pi)$ satisfies the estimates of proposition 3.2.

Proof. — Uniqueness is a consequence of the estimate given in proposition 3.2. Indeed, assuming (with no loss of generality) that $\alpha' \leqslant 2/m$, complex interpolation yields

$$\|u\|_{\widetilde{L}^m_t(H^{s+\frac{2}{m}-\alpha'})} \leqslant (\underline{\mu}t)^{\frac{1}{m}-\frac{1}{m'}} \|u\|^{1-\frac{m}{m'}}_{\widetilde{L}^\infty_t(H^s)} \big(\underline{\mu}^{\frac{1}{m}} \|u\|_{\widetilde{L}^m_t(H^{s+\frac{2}{m}})}\big)^{\frac{m}{m'}}$$
$$\text{with } \frac{1}{m'} = \frac{1}{m} - \frac{\alpha'}{2}. \qquad (3.14)$$

Hence the term $\|u\|_{\widetilde{L}^m_t(H^{s+\frac{2}{m}-\alpha'})}$ may be absorbed by the left-hand side of (3.3) in the limit t goes to 0. This yields uniqueness on a small interval $[0,\tau]$. Repeating the argument yields uniqueness on the whole interval $[0,T]$.

For proving existence, we use the fact that, owing to (3.6) and (3.7), system (M_μ) rewrites

$$\partial_t u = f + g + \mu b\Delta u - v\cdot\nabla u - b\mathcal{H}_b(f - T_{\nabla u}v - T'_{\nabla v}u) - b\mathcal{H}_b(g + \mu a\Delta u). \quad (\widetilde{M_\mu})$$

This latter system may be solved by using Friedrichs mollifiers: introduce the spectral cut-off $J_n \stackrel{\text{def}}{=} 1_{\{|D|\leqslant n\}}$. Let $f_n \stackrel{\text{def}}{=} J_n f$ and $g_n \stackrel{\text{def}}{=} J_n g$. The approximate equation

$$\partial_t u_n = f_n + g_n + \mu J_n(b\Delta J_n u_n) - J_n(v\cdot\nabla J_n u_n)$$
$$-J_n\Big(b\mathcal{H}_b(f_n - T_{\nabla J_n u_n}v - T'_{\nabla v}J_n u_n)\Big) - J_n\Big(b\mathcal{H}_b(g_n + \mu a\Delta J_n u_n)\Big)$$

with initial data $J_n u_0$ is a linear ODE in L^2. Using the integrability properties of v we can easily conclude that it has a unique solution u_n in $C^1([0,T];L^2)$.

As $J_n^2 = J_n$, we discover that $J_n u_n$ also satisfies the equation. We thus have $J_n u_n = u_n$. Because $\operatorname{div} J_n u_0 = 0$, elementary computations show that $\operatorname{div} u_n = 0$.

Next, going along the lines of the proof of proposition 3.2 and making an extensive use of $J_n u_n = u_n$, one can check that u_n satisfies (3.3) uniformly in n. Finally, combining (3.14) and Young inequality, we see that the bad term $\|u_n\|_{\widetilde{L}^m_T(H^{s+\frac{2}{m}-\alpha'})}$ may be absorbed by the left-hand side of (3.3) at small time. This provides a time $T^* \in (0,T]$ such that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $\widetilde{C}_{T^*}(H^s)\cap\widetilde{L}^m_{T^*}(H^{s+\frac{2}{m}})$. Note that T^* may be bounded by below in terms of $\underline{b}$, μ, α, N, $V(T)$ and $\mathcal{A}_T$.

By using the equation satisfied by u_n, it is now obvious that $(\partial_t u_n - f_n - g_n)_{n\in\mathbb{N}}$ is uniformly bounded in some space $\widetilde{L}^p_{T^*}(H^{-S})$ with $p>1$ and S suitably large. Taking advantage of compact embeddings in (local) Sobolev spaces, one can conclude to the convergence of a subsequence of $(u_n)_{n\in\mathbb{N}}$ to some distribution u. The uniform bounds for the sequence insure that in addition we have $u \in \widetilde{L}^\infty_{T^*}(H^s)\cap\widetilde{L}^m_{T^*}(H^{s+\frac{2}{m}})$. Interpolating between the results of convergence in small norm and the uniform bounds in large norm, it is now easy to show that u is indeed a solution to $(\widetilde{M}_\mu)$.

That u belongs to $C([0,T^*];H^s)$ may be obtained by using the properties of the standard heat kernel. Finally, proposition 8.5 yields the desired result on the pressure.

As T^* depends only on $\underline{b}$, μ, α, N, $V(T)$ and $\mathcal{A}_T$, the above argument may be repeated on $[T^*,2T^*]$, $[2T^*,3T^*]$, etc., until the whole interval $[0,T]$ is exhausted. $\square$

4. Existence of smooth solutions

This part is devoted to the proof of theorem 0.2.

First step: Construction of global approximate solutions

This may be done by induction. Set $a^0 \stackrel{\text{def}}{=} a_0$ and $u^0 \stackrel{\text{def}}{=} u_0$. Then, assuming that $(a^n, u^n, \nabla\Pi^n)$ is defined on $\mathbb{R}^+$ and belongs to $F^{\frac{N}{2}+1+\gamma}_{T,\mu}$ for all

$T > 0$, we define a^{n+1} as the global solution of the linear transport equation:

$$\begin{cases} \partial_t a^{n+1} + u^n \cdot \nabla a^{n+1} = 0, \\ a^{n+1}_{|t=0} = a_0. \end{cases} \tag{4.1}$$

Next, proposition 3.4 enables us to choose $(u^{n+1}, \nabla\Pi^{n+1})$ as the global solution to

$$\begin{cases} \partial_t u^{n+1} + u^n \cdot \nabla u^{n+1} + (1 + a^{n+1})(\nabla\Pi^{n+1} - \mu\Delta u^{n+1}) = f, \\ \operatorname{div} u^{n+1} = 0, \\ u^{n+1}_{|t=0} = u_0. \end{cases} \tag{4.2}$$

The results of the previous section insure that $(a^{n+1}, u^{n+1}, \nabla\Pi^{n+1})$ belongs to $F^{\frac{N}{2}+1+\gamma}_{T,\mu}$ for all $T > 0$. Besides, the energy equality (0.1) is satisfied (with $\rho^{n+1} = 1/(1 + a^{n+1})$), and we have $\underline{\rho} \leqslant \rho^{n+1}(t, x) \leqslant \overline{\rho}$ for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^N$.

Second step: Uniform bounds for the approximate solutions

On the one hand, according to proposition 3.1, we have for all $T \geqslant 0$,

$$\|a^{n+1}\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+1+\gamma})} \leqslant \|a_0\|_{H^{\frac{N}{2}+1+\gamma}} e^{C\int_0^T \|\nabla u^n(t)\|_{H^{\frac{N}{2}+\gamma}} dt}. \tag{4.3}$$

On the other hand, applying proposition 3.2 to (4.2) with $m = 1$, $s = N/2 + 1 + \gamma$, $\alpha = 1 + \gamma$ and $\alpha' = 1$ yields

$$\begin{aligned} &\|u^{n+1}\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+1+\gamma})} + \underline{\mu}\|u^{n+1}\|_{\widetilde{L}^1_T(H^{\frac{N}{2}+3+\gamma})} \\ &\qquad \leqslant C\mathcal{A}^\kappa_{T,n} e^{C\mathcal{A}^\kappa_{T,n}\int_0^T \|\nabla u^n(t)\|_{H^{\frac{N}{2}+\gamma}} dt}\Big(\|u_0\|_{H^{\frac{N}{2}+1+\gamma}} \\ &\qquad\qquad + \|f\|_{\widetilde{L}^1_T(H^{\frac{N}{2}+1+\gamma})} + \underline{\mu}\mathcal{A}_{T,n}\|u^{n+1}\|_{\widetilde{L}^1_T(H^{\frac{N}{2}+2+\gamma})}\Big), \end{aligned}$$

with $\kappa = N/2 + 1 + \gamma$, $\mathcal{A}_{T,n} \stackrel{\text{def}}{=} 1 + \overline{\rho}\|a^{n+1}\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+1+\gamma})}$ and $\underline{\mu} = \mu/\overline{\rho}$.

Let $U^{n+1}(T)$ be the left-hand side above and let
$U_0 \stackrel{\text{def}}{=} \|u_0\|_{H^{\frac{N}{2}+1+\gamma}} + \|f\|_{\widetilde{L}^1_{T_0}(H^{\frac{N}{2}+1+\gamma})}$ (for some large fixed T_0). Using (4.3), we gather

$$\begin{aligned} U^{n+1}(T) \leqslant C\mathcal{A}^\kappa_0 e^{C\int_0^T U^n(t)\,dt} e^{C\mathcal{A}^\kappa_0 \int_0^T U^n(t)\,dt\, \exp(C\int_0^T U^n(t)\,dt)} \Big(U_0 \\ + (\underline{\mu}T)^{\frac{1}{2}}\mathcal{A}_0 e^{C\int_0^T U^n(t)\,dt} U^{n+1}(T)\Big), \end{aligned}$$

with $\mathcal{A}_0 \stackrel{\text{def}}{=} 1+\overline{\rho}\|a_0\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+1+\gamma})}$, whence, using that $xe^x \leqslant e^{2x}-e^x$ for $x\geqslant 0$, we get up to an irrelevant change of the constant C,

$$U^{n+1}(T)\leqslant C\mathcal{A}_0^\kappa e^{C\mathcal{A}_0^\kappa \exp(C\int_0^T U^n(t)\,dt)}\Big(U_0+\sqrt{\underline{\mu}T}\mathcal{A}_0U^{n+1}(T)\Big). \tag{4.4}$$

Fix a positive T so that the following conditions are satisfied:

$$C\int_0^T U^n(t)\,dt\leqslant \log 2 \quad\text{and}\quad C\mathcal{A}_0^{\kappa+1}e^{2C\mathcal{A}_0^\kappa}\sqrt{\underline{\mu}T}\leqslant\frac{1}{2}\cdot \tag{$\mathcal{H}_n$}$$

Then (4.4) yields for all $t\in[0,T]$:

$$U^{n+1}(t)\leqslant 2CU_0\mathcal{A}_0^\kappa e^{2C\mathcal{A}_0^\kappa}. \tag{4.5}$$

Now, choosing for T the largest real number in $(0,T_0]$ such that

$$C^2T\mathcal{A}_0^\kappa U_0 e^{2C\mathcal{A}_0^\kappa}\leqslant\frac{\log 2}{2}\quad\text{and}\quad C^2\mu T\mathcal{A}_0^{2\kappa+2}e^{4C\mathcal{A}_0^\kappa}\leqslant\frac{1}{4}, \tag{4.6}$$

it may be shown by induction that $(\mathcal{H}_n)$ is satisfied, whence also (4.5). Next, combining the estimates of proposition 3.2 with uniform bounds for (a_n,u_n) provides uniform bounds for $\nabla\Pi^n$ in $\widetilde{L}^1_T(H^{\frac{N}{2}+1+\gamma})$. Hence, sequence $\{(a^n,u^n,\nabla\Pi^n)\}_{n\in\mathbb{N}}$ belongs to $F^{\frac{N}{2}+1+\gamma}_{T,\mu}$ and $\|(a^n,u^n,\nabla\Pi^n)\|_{F^{\frac{N}{2}+1+\gamma}_{T,\mu}}$ may be bounded independently of n.

Remark 4.1. — It is worth noting that for small enough μ the lifetime T does not depend on μ and that the bounds are independent of μ.

Third step: Convergence of the approximate solutions in the energy space

We claim that (a^n,u^n) is a Cauchy sequence in $C([0,T];L^2)$.

Let $\rho^n \stackrel{\text{def}}{=} 1/(1+a^n)$, $\delta\rho^n \stackrel{\text{def}}{=} \rho^{n+1}-\rho^n$, $\delta u^n \stackrel{\text{def}}{=} u^{n+1}-u^n$ and $\delta\Pi^n \stackrel{\text{def}}{=} \Pi^{n+1}-\Pi^n$. We have

$$\partial_t\delta\rho^n+u^n\cdot\nabla\delta\rho^n=-\delta u^{n-1}\cdot\nabla\rho^n,$$

whence, according to (2.2),

$$\|\delta\rho^n(t)\|_{L^2}\leqslant\int_0^t\underbrace{\Big\|(\rho^n)^{-\frac12}\nabla\rho^n)(\tau)\Big\|_{L^\infty}}_{C^n(\tau)}\big\|(\sqrt{\rho^n}\delta u^{n-1})(\tau)\big\|_{L^2}\,d\tau. \tag{4.7}$$

Since
$$\begin{cases} \partial_t \rho^{n+1} + u^n \cdot \nabla \rho^{n+1} = 0, \\ \rho^{n+1}(\partial_t \delta u^n + u^n \cdot \nabla \delta u^n) - \mu \Delta \delta u^n + \nabla \delta \Pi^n \\ \qquad\qquad = \dfrac{\delta \rho^n}{\rho^n}(\nabla \Pi^n - \mu \Delta u^n) - \rho^{n+1} \delta u^{n-1} \cdot \nabla u^n, \\ \operatorname{div} \delta u^n = 0, \end{cases}$$

inequality (2.3) yields

$$\left\| (\sqrt{\rho^{n+1}} \delta u^n)(t) \right\|_{L^2} \leqslant \int_0^t A^n(\tau) \left\| \delta \rho^n(\tau) \right\|_{L^2} d\tau \\ + \int_0^t B^n(\tau) \left\| (\sqrt{\rho^n} \delta u^{n-1})(\tau) \right\|_{L^2} d\tau, \tag{4.8}$$

with $A^n(t) \stackrel{\text{def}}{=} \left\| \left(\dfrac{\nabla \Pi^n - \mu \Delta u^n}{\rho^n \sqrt{\rho^{n+1}}} \right)(t) \right\|_{L^\infty}$ and $B^n(t) \stackrel{\text{def}}{=} \left\| \left(\sqrt{\dfrac{\rho^{n+1}}{\rho^n}} \nabla u^n \right)(t) \right\|_{L^\infty}$.
According to step two, for all $t \in [0, T]$, we have

$$K(t) \stackrel{\text{def}}{=} \sup_{n \in \mathbb{N}} \int_0^t A^n(\tau)\, d\tau < +\infty.$$

Therefore, adding (4.7) and (4.8) up, and using Gronwall lemma yields

$$X^n(t) \stackrel{\text{def}}{=} e^{-K(t)} \Big(\left\| \delta \rho^n(t) \right\|_{L^2} + \left\| \sqrt{\rho^{n+1}} \delta u^n(t) \right\|_{L^2} \Big) \\ \leqslant \int_0^t (B^n + C^n)(\tau) X^{n-1}(\tau)\, d\tau.$$

Now, step two insures that

$$L \stackrel{\text{def}}{=} \sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \Big(B^n(t) + C^n(t) \Big) < \infty$$

so that a straightforward induction yields

$$\sup_{t \in [0,T]} X^n(t) \leqslant \frac{L^n}{n!} \sup_{t \in [0,T]} X^0(t).$$

We conclude that $(\rho^n - \rho^0, u^n)$ (and thus (a^n, u^n)) is a Cauchy sequence in $C([0, T]; L^2)$.

Denoting by (a, u) its limit, the bounds of step two give $a \in \widetilde{L}^\infty_T(H^{\frac{N}{2}+1+\gamma})$, $u \in \widetilde{L}^\infty_T(H^{\frac{N}{2}+1+\gamma})$ and $\underline{\mu} u \in \widetilde{L}^1_T(H^{\frac{N}{2}+3+\gamma})$ uniformly in μ.

Now, interpolating with the results of convergence in $C([0,T];L^2)$, we gather that (a^n,u^n) tends to (a,u) in every intermediate space $\widetilde{C}_T(H^{\frac{N}{2}+1+\gamma'})$ with $\gamma'<\gamma$, and that $(u^n)_{n\in\mathbb{N}}$ tends to u in $\widetilde{L}^1_T(H^{\frac{N}{2}+3+\gamma'})$ if $\mu>0$.

As regards the convergence of the pressure, we remark that

$$\operatorname{div}\left(\frac{\nabla(\Pi^m-\Pi^n)}{\rho^m}\right)=\operatorname{div}\Big[\mu a^m\Delta(u^m-u^n)+(a^m-a^n)(\mu\Delta u^n-\nabla\Pi^n)$$
$$-(u^{m-1}-u^{n-1})\cdot\nabla u^m-u^{n-1}\cdot\nabla(u^m-u^n)\Big].$$

The previous results of convergence insure that the term between brackets tends to 0 in $L^1(0,T;L^2)$ when n,m go to infinity, hence, by virtue of proposition 8.2, $(\nabla\Pi^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^1(0,T;L^2)$. Denoting by $\nabla\Pi$ its limit in $L^1(0,T;L^2)$ and interpolating with the uniform estimates of step two, we conclude that $(a^n,u^n,\nabla\Pi^n)$ tends to $(a,u,\nabla\Pi)$ in every intermediate space $F^{\frac{N}{2}+1+\gamma'}_{T,\mu}$ with $\gamma'<\gamma$.

Fourth step: Checking that the limit is a solution

That $(a^n,u^n,\nabla\Pi^n)$ converges to $(a,u,\nabla\Pi)$ in $F^{\frac{N}{2}+1+\gamma'}_{T,\mu}$ with $\gamma'<\gamma$ suffices to pass to the limit in every nonlinear term of (4.1) and (4.2). Note besides that one can also pass to the limit in the energy equality (0.1).

Fifth step: Continuity with respect to time:

As a satisfies the transport equation $\partial_t a+u\cdot\nabla a=0$ with initial datum a_0 in $H^{\frac{N}{2}+1+\gamma}$ and $u\in L^1(0,T;H^{\frac{N}{2}+1+\gamma})$, proposition 3.1 entails that $a\in\widetilde{C}_T(H^{\frac{N}{2}+1+\gamma})$.

For the velocity, the same argument applies. Indeed u satisfies the transport equation

$$\partial_t u+u\cdot\nabla u=f-(1+a)\nabla\Pi+\mu(1+a)\Delta u$$

with right-hand side in $\widetilde{L}^1_T(H^{\frac{N}{2}+1+\gamma})$.

Last step: Uniqueness

Uniqueness in $F^{\frac{N}{2}+1+\gamma}_{T,\mu}$ is given by corollary 2.3. Indeed, the embedding $H^{\frac{N}{2}+\gamma}\hookrightarrow L^\infty$ ensures that $\nabla u,\nabla\rho$ belong to $L^\infty(0,T;L^\infty)$, and that $\nabla\Pi-\mu\Delta u$ belongs to $L^1(0,T;L^\infty)$. $\square$

5. A blow-up criterion

This section is devoted to the proof of proposition 0.6. It relies on estimates of section 3, logarithmic interpolation (see below) and on the following

Lemma 5.1 *Let* $\mu \geqslant 0$, $\gamma > 0$, $a_0 \in H^{\frac{N}{2}+1+\gamma}$ *with* $\underline{\rho} \leqslant \rho_0 \stackrel{\text{def}}{=} (1+a_0)^{-1} \leqslant \overline{\rho}$, $u_0 \in H^{\frac{N}{2}+1+\gamma}$ *with* $\operatorname{div} u_0 = 0$ *and* $f \in \widetilde{L}^1_{loc}(H^{\frac{N}{2}+1+\gamma})$. *Let* $(\rho, u, \nabla\Pi)$ *be a smooth solution of* (INS_μ) *on* $[0,T[$ *in the sense of definition* 0.5.

If in addition $a \in L^\infty_T(H^{\frac{N}{2}+1+\gamma})$ *and* $u \in L^\infty(0,T;H^{\frac{N}{2}+1+\gamma})$ *then there exists* $\eta > 0$ *such that* $(\rho, u, \nabla\Pi)$ *may be continued into a smooth solution of* (INS_μ) *on* $[0, T+\eta]$.

Proof. — Let

$$\eta \stackrel{\text{def}}{=} \frac{1}{4C^2} \min\left(\frac{e^{-2C\mathcal{A}_T^\kappa} \log 2}{\mathcal{A}_T^\kappa U_T}, \frac{e^{-4C\mathcal{A}_T^\kappa}}{2\mu \mathcal{A}_T^{2\kappa+2}} \right)$$

where C and κ are the constants appearing in (4.6), $\mathcal{A}_T \stackrel{\text{def}}{=} 1 + \overline{\rho}\|a\|_{L^\infty_T(H^{\frac{N}{2}+1+\gamma})}$ and $U_T \stackrel{\text{def}}{=} \|u\|_{L^\infty_T(H^{\frac{N}{2}+1+\gamma})} + \|f\|_{\widetilde{L}^1_T(H^{\frac{N}{2}+1+\gamma})}$.

Theorem 0.2 insures that (INS_μ) with data $\rho(T-\eta)$, $u(T-\eta)$ and $t \mapsto f(t+T-\eta)$ has a smooth solution $(\widetilde{\rho}, \widetilde{u}, \nabla\widetilde{\Pi})$ on $[0, 2\eta]$. By virtue of uniqueness, we have $(\widetilde{\rho}, \widetilde{u}, \nabla\widetilde{\Pi})(t) = (\rho, u, \nabla\Pi)(t+T-\eta)$ for $t \in [0,\eta)$. Hence, $(\widetilde{\rho}, \widetilde{u}, \nabla\widetilde{\Pi})$ provides the desired continuation. □

One can now state a first blow up criterion:

Proposition 5.2. — *Let* a_0, u_0, f *satisfy the hypotheses of proposition* 0.6 *and* $(\rho, u, \nabla\Pi)$ *be the corresponding smooth solution of* (INS_μ) *on* $[0,T)$. *If*

$$\nabla u \in L^1(0,T;L^\infty)$$

and

$$\begin{cases} \nabla a \in \widetilde{L}^\infty_T(H^{\frac{N}{2}+\gamma}) & \textit{if} \quad \mu = 0, \\ \nabla a \in L^\infty(0,T;H^{\frac{N}{2}+\alpha-1}) \ \textit{for some}\ \alpha > 0 & \textit{if} \quad \mu > 0, \end{cases}$$

then $(\rho, u, \nabla\Pi)$ *may be continued beyond* T *into a smooth solution of* (INS_μ).

Proof. — The mass conservation equation insures that for all $t \in [0, T)$, we have

$$\forall x \in \mathbb{R}^N, 0 < \inf_y \rho_0(y) \leqslant \rho(t,x) \leqslant \sup_y \rho_0(y) < \infty \text{ and } \|a(t)\|_{L^2} = \|a_0\|_{L^2}. \tag{5.1}$$

In the inviscid case, we have, by assumption, $\nabla a \in \widetilde{L}^\infty_T(H^{\frac{N}{2}+\gamma})$. Hence $a \in \widetilde{L}^\infty_T(H^{\frac{N}{2}+\gamma+1})$. Applying proposition 3.2 to $(a, u, \nabla\Pi)$ in the case $v = u$ thus yields $u \in \widetilde{L}^\infty_T(H^{\frac{N}{2}+1+\gamma})$. Lemma 5.1 finally insures that no blow-up may occur at time T.

Let us now focus on the viscous case. According to (5.1) and to (1.6), one can assume that $a \in \widetilde{L}^\infty_T(H^{\frac{N}{2}+\alpha})$ for some positive α. Therefore, proposition 3.2 yields

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+\alpha})} + \underline{\mu}\|u\|_{\widetilde{L}^1_T(H^{\frac{N}{2}+2+\alpha})} &\leqslant C e^{C\widetilde{\mathcal{A}}^\kappa_T \int_0^T \|\nabla u(t)\|_{L^\infty}\,dt} \\ &\times\Big(\|u_0\|_{H^{\frac{N}{2}+\alpha}} + \widetilde{\mathcal{A}}^\kappa_T\Big(\|f\|_{\widetilde{L}^1_T(H^{\frac{N}{2}+\alpha})} + \widetilde{\mathcal{A}}_T\underline{\mu}\|u\|_{\widetilde{L}^1_T(H^{\frac{N}{2}+2+\frac{\alpha}{2}})}\Big)\Big),\end{aligned}$$

with $\widetilde{\mathcal{A}}_T \stackrel{\text{def}}{=} 1 + \overline{\rho}\|a\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+\alpha})}$.

Interpolating between $\widetilde{L}^1_T(H^0)$ and $\widetilde{L}^1_T(H^{\frac{N}{2}+2+\alpha})$ and using Young inequality enables us to handle the last term. Up to a change of κ, we get

$$\begin{aligned}\|u\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+\alpha})} + \underline{\mu}\|u\|_{\widetilde{L}^1_T(H^{\frac{N}{2}+2+\alpha})} &\leqslant C\widetilde{\mathcal{A}}^\kappa_T e^{C\widetilde{\mathcal{A}}^\kappa_T \int_0^T \|\nabla u(t)\|_{L^\infty}\,dt} \\ &\times\Big(\|u_0\|_{H^{\frac{N}{2}+\alpha}} + \|f\|_{\widetilde{L}^1_T(H^{\frac{N}{2}+\alpha})} + \underline{\mu}T\|u\|_{L^\infty_T(L^2)}\Big).\end{aligned} \tag{5.2}$$

Now, energy inequality (2.3) insures that the last term $\|u\|_{L^\infty_T(L^2)}$ is finite, hence $u \in \widetilde{L}^\infty_T(H^{\frac{N}{2}+\alpha}) \cap \widetilde{L}^1_T(H^{\frac{N}{2}+2+\alpha})$. Note that in particular ∇u belongs to $L^1_T(H^{\frac{N}{2}+1+\frac{\alpha}{2}})$. Coming back to the transport equation, we can now prove that $a \in \widetilde{L}^\infty_T(H^{\frac{N}{2}+1+\min(\gamma,1+\frac{\alpha}{2})})$.

Then, one can use again the momentum equation to get additional regularity for u. Within a finite number of steps, one concludes that $a, u \in \widetilde{L}^\infty_T(H^{\frac{N}{2}+1+\gamma})$. Applying lemma 5.1 completes the proof. □

As for most first-order quasilinear hyperbolic equations, we claim that condition $\int_0^T \|\nabla u(t)\|_{L^\infty}\,dt < \infty$ may be replaced by a slightly weaker condition. Indeed we have:

Proposition 5.3. — *The conclusion of proposition 5.2 remains true if the assumption $\nabla u \in L^1(0,T;L^\infty)$ is replaced by*

$$\int_0^T \|\nabla u(t)\|_{\dot B^0_{\infty,\infty}}\,dt < +\infty \quad \textit{with} \quad \|\nabla u\|_{\dot B^0_{\infty,\infty}} \stackrel{\text{def}}{=} \sup_{q\in\mathbb{Z}} \left\|\varphi(2^{-q}D)\nabla u\right\|_{L^\infty}.$$

Proof. — Let us concentrate on the case $\mu = 0$, the case $\mu > 0$ being similar. The result stems from the following well known logarithmic interpolation inequality (see e.g [13]):

$$\|\nabla u\|_{L^\infty} \leqslant C\Big(1 + \|\nabla u\|_{\dot B^0_{\infty,\infty}} \log(e + \|\nabla u\|_{H^{\frac N2+\gamma}})\Big). \tag{5.3}$$

Now, in view of proposition 3.2, we have for all $t \in [0,T[$,

$$\|u(t)\|_{H^{\frac N2+1+\gamma}} \leqslant C_T e^{C_T \int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau}, \tag{5.4}$$

where C_T depends only on $\underline{\rho}$, $\overline{\rho}$, γ, N, $\|\nabla a\|_{\widetilde L^\infty_T(H^{\frac N2+\gamma})}$ and on the data.

Integrating (5.3) over $[0,t]$ and using (5.4) for bounding the term $\|\nabla u\|_{H^{\frac N2+\gamma}}$, we get (up to a change of C_T) for all $t \in [0,T)$,

$$\int_0^t \|\nabla u\|_{L^\infty}\,d\tau \leqslant C_T \int_0^t \Big(1 + \|\nabla u\|_{\dot B^0_{\infty,\infty}}\Big)\Big(1 + \int_0^\tau \|\nabla u\|_{L^\infty}\,d\tau'\Big)\,d\tau.$$

Applying Gronwall inequality completes the proof. □

Finally, as for solenoidal u the map $\operatorname{curl} u \mapsto \nabla u$ is a smooth homogeneous multiplier of degree 0, we have

$$\forall q \in \mathbb{Z},\ \left\|\varphi(2^{-q}D)\nabla u\right\|_{L^\infty} \leqslant C\left\|\varphi(2^{-q}D)\operatorname{curl} u\right\|_{L^\infty} \leqslant C\,\|\operatorname{curl} u\|_{L^\infty}.$$

This yields proposition 0.6.

6. The inviscid limit

This section is devoted to the proof of theorem 0.9.

Denote $\delta a \stackrel{\text{def}}{=} a_\mu - a$, $\delta u \stackrel{\text{def}}{=} u_\mu - u$ and $\delta\Pi \stackrel{\text{def}}{=} \Pi_\mu - \Pi$. The desired result of convergence stems from the following proposition.

Proposition 6.1. — *Let $\gamma \in (0,1)$ and $\underline{b} > 0$. Assume that $(\widetilde{INS}_\mu)$ (resp. $(\widetilde{INS}_0)$) has a solution $(a_\mu, u_\mu, \nabla\Pi_\mu) \in F^{\frac{N}{2}+\gamma}_{T_0,\mu}$ (resp. $(a, u, \nabla\Pi) \in F^{\frac{N}{2}+\gamma}_{T_0}$). If in addition*

$$\begin{gathered}\Delta u \in L^2(0,T_0; H^{\frac{N}{2}+\gamma-1}),\ \nabla\Pi,\ \nabla a,\\ \nabla u \in L^1(0,T_0; H^{\frac{N}{2}+\gamma}),\ 1+\min(a, a_\mu) \geqslant \underline{b},\end{gathered} \tag{6.1}$$

then there exist two positive constants X_0 and μ_0 depending only on $\|(a, u, \nabla\Pi)\|_{F^{\frac{N}{2}+\gamma}_{T_0}}$, N, γ, $\underline{b}$, T_0, and on the norms of the quantities appearing in (6.1) such that

$$\begin{aligned}\|\delta a\|_{\widetilde{L}^\infty_{T_0}(H^{\frac{N}{2}+\gamma})} + \|\delta u\|_{\widetilde{L}^\infty_{T_0}(H^{\frac{N}{2}+\gamma})} + \sqrt{\underline{\mu}}\|\delta u\|_{L^\infty_{T_0}(H^{\frac{N}{2}+\gamma})}&\\ +\|\nabla\delta\Pi\|_{\widetilde{L}^1_{T_0}(H^{\frac{N}{2}+\gamma})+L^2_{T_0}(H^{\frac{N}{2}-1+\gamma})} &\leqslant \sqrt{\underline{\mu}}X_0\end{aligned}$$

with $\underline{\mu} \stackrel{\text{def}}{=} \underline{b}\mu$ whenever $\mu \in [0, \mu_0]$.

Proof of theorem 0.9. — Let us admit for a while proposition 6.1.

We are given a solution $(\rho, u, \nabla\Pi)$ to (IE) with $(a, u, \nabla\Pi) \in F^{\frac{N}{2}+1+\gamma}_{T_0}$ and $1 + a \geqslant \underline{b}$. Throughout the proof, we assume that $\mu \leqslant \mu_0$ where μ_0 is the limit viscosity given by proposition 6.1 applied with $\check{\gamma} \stackrel{\text{def}}{=} \min(\frac{1}{2}, \gamma)$ instead of γ.

First step: Local existence

According to theorem 0.2, there exists a $\widetilde{T} > 0$ and a unique local solution $(\rho_\mu, u_\mu, \nabla\Pi_\mu)$ to (INS_μ) with $(a_\mu, u_\mu, \nabla\Pi_\mu) \in F^{\frac{N}{2}+1+\gamma}_{\widetilde{T},\mu}$ and $1 + a_\mu \geqslant \underline{b}$.

Second step: a lower bound for the existence time

Let T_μ be the lifespan for $(a_\mu, u_\mu, \nabla\Pi_\mu)$. Applying proposition 6.1 yields

$$\forall t \in [0, T_0] \cap [0, T_\mu),$$
$$\|\delta a\|_{\widetilde{L}^\infty_t(H^{\frac{N}{2}+\check{\gamma}})} + \|\delta u\|_{\widetilde{L}^\infty_t(H^{\frac{N}{2}+\check{\gamma}})} + \sqrt{\underline{\mu}}\|\delta u\|_{L^2_t(H^{\frac{N}{2}+1+\check{\gamma}})} \leqslant \sqrt{\underline{\mu}}X_0.$$

We notice that $\|a_\mu\|_{\widetilde{L}^\infty_t(H^{\frac{N}{2}+\gamma})}$ and $\|\nabla u_\mu\|_{L^1_t(L^\infty)}$ remain finite and bounded independently of $\mu \leqslant \mu_0$ whenever $t \leqslant T_0$ and $t < T_\mu$. Hence, by virtue of proposition 5.2, we must have $T_\mu > T_0$.

Third step: Uniform estimates in $F_{T,\mu}^{\frac{N}{2}+1+\gamma}$

Proposition 6.1 yields a bound independent of μ for $\|u_\mu\|_{L^2_{T_0}(H^{\frac{N}{2}+1+\check{\gamma}})}$. From it and proposition 3.1, we further gather a bound independent of μ for $\|a_\mu\|_{\widetilde{L}^\infty_{T_0}(H^{\frac{N}{2}+1+\check{\gamma}})}$. Next, applying proposition 3.2 with $m=1$, $\alpha=\check{\gamma}$, $s=\frac{N}{2}+1+\check{\gamma}$ and $\alpha'=\check{\gamma}/2$ supplies uniform bounds for u_μ in $\widetilde{L}^\infty_{T_0}(H^{\frac{N}{2}+1+\check{\gamma}})$, for μu_μ in $\widetilde{L}^1_{T_0}(H^{\frac{N}{2}+3+\check{\gamma}})$, and for $\nabla\Pi_\mu$ in $\widetilde{L}^1_{T_0}(H^{\frac{N}{2}+1+\check{\gamma}})$. (As usual, the term $\|u_\mu\|_{\widetilde{L}^1_{T_0}(H^{\frac{N}{2}+3+\gamma-\alpha'})}$ may be handled by interpolating between L^2 and $H^{\frac{N}{2}+3+\gamma}$, and by using the energy inequality (2.3).) Hence we have obtained uniform estimates in $F_{T,\mu}^{\frac{N}{2}+1+\check{\gamma}}$ (and thus in $F_{T,\mu}^{\frac{N}{2}+1+\gamma}$ if $\gamma\leqslant 1/2$).

If $\gamma>1/2$, starting from uniform estimates in $F_{T,\mu}^{\frac{N}{2}+1+\frac{1}{2}}$, the above argument may be repeated to get uniform estimates in $F_{T,\mu}^{\frac{N}{2}+1+\min(\frac{3}{2},\gamma)}$, then in $F_{T,\mu}^{\frac{N}{2}+1+\min(\frac{5}{2},\gamma)}$, etc.

Last step: Stronger results of convergence

In step two, convergence is shown to hold in

$$\widetilde{L}^\infty_{T_0}(H^{\frac{N}{2}+\min(\frac{1}{2},\gamma)})\times\widetilde{L}^\infty_{T_0}(H^{\frac{N}{2}+\min(\frac{1}{2},\gamma)})^N \times\Big(\widetilde{L}^1_{T_0}(H^{\frac{N}{2}+\min(\frac{1}{2},\gamma)})+L^2_{T_0}(H^{\frac{N}{2}-1+\min(\frac{1}{2},\gamma)})\Big)^N.$$

Interpolating with the uniform bounds of step three yields also convergence in

$$\widetilde{C}_{T_0}(H^{\frac{N}{2}+1+\gamma'})\times\Big(\widetilde{C}_{T_0}(H^{\frac{N}{2}+1+\gamma'})\Big)^N\times\Big(\widetilde{L}^1_{T_0}(H^{\frac{N}{2}+1+\gamma'})\Big)^N.$$

for any $\gamma'<\gamma$. □

Remark 6.2. — As a by-product of the proof, we get estimates independent of μ for

$$\|\Delta u_\mu\|_{L^1_{T_0}(L^2)},\quad \|\nabla u_\mu\|_{L^1_{T_0}(L^\infty)},\quad \|\nabla a_\mu\|_{L^1_{T_0}(L^\infty)}\quad\text{and}\quad \|\nabla\Pi_\mu\|_{L^1_{T_0}(L^\infty)}.$$

Applying proposition 2.2 with zero viscosity, $(\rho_1,u_1,\nabla\Pi_1)=(\rho,u,\nabla\Pi)$, $(\rho_2,u_2,\nabla\Pi_2)=(\rho_\mu,u_\mu,\nabla\Pi_\mu)$, $f_1=f$ and $f_2=f+\mu\rho^{-1}\Delta u$, thus provides a rate of convergence of order μ for the L^2 norm of a_μ and u_μ.

Proof of proposition 6.1. — Let us observe that it suffices to prove that the inequality of proposition 6.1 is satisfied by δa and δu. The result for the pressure term $\nabla\delta\Pi$ will follow from (3.4).

Let T denote a real number of $[0,T_0]$. As δa solves the following transport equation on $[0,T_0]$:

$$\partial_t\delta a+u_\mu\cdot\nabla\delta a=-\delta u\cdot\nabla a,\qquad \delta a_{|t=0}=0,$$

proposition 3.1 combined with the embedding $H^{\frac N2}\cap L^\infty\hookrightarrow H^{\frac N2+\frac\gamma2}$ entails that we have

$$\|\delta a\|_{\widetilde L^\infty_T(H^{\frac N2+\gamma})}\leqslant Ce^{CV_\mu(T)}\int_0^T\|\delta u\|_{H^{\frac N2+\gamma}}\|\nabla a\|_{H^{\frac N2+\gamma}}\,dt \tag{6.2}$$

with $V_\mu(T)\stackrel{\rm def}{=}\int_0^T\|\nabla u_\mu(t)\|_{H^{\frac N2+\frac\gamma2}}\,dt.$

Next, because $(\delta u,\nabla\delta\Pi)$ satisfies

$$\begin{cases}\partial_t\delta u+u_\mu\cdot\nabla\delta u+(1+a_\mu)(\nabla\delta\Pi-\mu\Delta\delta u)\\ \qquad\qquad\qquad=-\delta u\cdot\nabla u-\delta a\nabla\Pi+\mu(1+a_\mu)\Delta u,\\ \operatorname{div}\delta u=0,\qquad \delta u_{|t=0}=0,\end{cases} \tag{6.3}$$

applying proposition 3.2 with $m=2$, $s=\frac N2+\gamma$, $\alpha=\gamma$, $\alpha'=\frac\gamma2$, $f=-\delta u\cdot\nabla u-\delta a\nabla\Pi$ and $g=\mu(1+a_\mu)\Delta u$, and using that $H^{\frac N2}\cap L^\infty\hookrightarrow H^{\frac N2+\frac\gamma2}$ yields

$$\begin{aligned}&\|\delta u\|_{\widetilde L^\infty_T(H^{\frac N2+\gamma})}+\sqrt{\underline\mu}\|\delta u\|_{L^2_T(H^{\frac N2+\gamma+1})}\\ &\qquad\leqslant C\widetilde{\mathcal A}^\kappa_{T,\mu}e^{C\widetilde{\mathcal A}^\kappa_{T,\mu}V_\mu(T)}\Big(\underline b^{-1}\sqrt{\underline\mu}\|(1+a_\mu)\Delta u\|_{L^2_T(H^{\frac N2-1+\gamma})}\\ &+\int_0^T\Big(\|\delta u\cdot\nabla u\|_{H^{\frac N2+\gamma}}+\|\delta a\nabla\Pi\|_{H^{\frac N2+\gamma}}\Big)dt+\sqrt{\underline\mu}\widetilde{\mathcal A}_{T,\mu}\|\delta u\|_{L^2_T(H^{\frac N2+\frac\gamma2+1})}\Big)\end{aligned} \tag{6.4}$$

with $\widetilde{\mathcal A}_{T,\mu}=1+\underline b^{-1}\|a_\mu\|_{\widetilde L^\infty_T(H^{\frac N2+\gamma})}$ and $\kappa=2+N/\gamma$.

On one hand, according to proposition 1.3, we have

$$\|\delta u\cdot\nabla u\|_{H^{\frac N2+\gamma}}\leqslant C\|\delta u\|_{H^{\frac N2+\gamma}}\|\nabla u\|_{H^{\frac N2+\gamma}}, \tag{6.5}$$

$$\|\delta a\nabla\Pi\|_{H^{\frac N2+\gamma}}\leqslant C\|\delta a\|_{H^{\frac N2+\gamma}}\|\nabla\Pi\|_{H^{\frac N2+\gamma}}, \tag{6.6}$$

$$\|(1+a_\mu)\Delta u\|_{H^{\frac N2+\gamma-1}}\leqslant C\big(1+\|a\|_{H^{\frac N2+\gamma}}+\|\delta a\|_{H^{\frac N2+\gamma}}\big)\|\Delta u\|_{H^{\frac N2+\gamma-1}}. \tag{6.7}$$

On the other hand, combining Hölder inequality and interpolation, we have

$$\sqrt{\underline\mu}\|\delta u\|_{L^2_T(H^{\frac N2+\frac\gamma2+1})}\leqslant\sqrt{\underline\mu}T^{\frac\gamma4}\|\delta u\|_{\widetilde L_T^{\frac{4}{2-\gamma}}(H^{\frac N2+\frac\gamma2+1})},$$

$$\leqslant (\underline{\mu}T)^{\frac{\gamma}{4}}\big(\sqrt{\underline{\mu}}\|\delta u\|_{L^2_T(H^{\frac{N}{2}+\gamma+1})}\big)^{1-\frac{\gamma}{2}}\big(\|\delta u\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+\gamma})}\big)^{\frac{\gamma}{2}}. \quad (6.8)$$

Let $X_\mu(T) \stackrel{\text{def}}{=} \underline{\mu}^{-\frac{1}{2}}\big(\|\delta u\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+\gamma})}+\sqrt{\underline{\mu}}\|\delta u\|_{L^2_T(H^{\frac{N}{2}+\gamma+1})}+\underline{b}^{-1}\|\delta a\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+\gamma})}\big)$.

Adding (6.2) (divided by $\underline{b}$) to (6.4) and using (6.5), (6.6), (6.7) and (6.8) yields

$$X_\mu(T) \leqslant C\widetilde{\mathcal{A}}^{\kappa+1}_{T,\mu}e^{C\widetilde{\mathcal{A}}^\kappa_{T,\mu}V_\mu(T)}\bigg(\|\Delta u\|_{L^2_T(H^{\frac{N}{2}-1+\gamma})}+(\underline{\mu}T)^{\frac{\gamma}{4}}X_\mu(T)$$
$$+\int_0^T\Big(\|\nabla u(t)\|_{H^{\frac{N}{2}+\gamma}}+\underline{b}\|\nabla\Pi(t)\|_{H^{\frac{N}{2}+\gamma}}+\underline{b}^{-1}\|\nabla a(t)\|_{H^{\frac{N}{2}+\gamma}}\Big)X_\mu(t)dt\bigg).$$

Remark that we have

$$V_\mu(T) \leqslant V(T)+\sqrt{T}(T\underline{\mu})^{\frac{\gamma}{4}}X_\mu(T) \quad \text{with} \quad V(T) \stackrel{\text{def}}{=} \int_0^T \|\nabla u\|_{H^{\frac{N}{2}+\frac{\gamma}{2}}}\,dt,$$
$$\widetilde{\mathcal{A}}_{T,\mu} \leqslant \widetilde{\mathcal{A}}_T+\sqrt{\underline{\mu}}X_\mu(T) \quad \text{with} \quad \widetilde{\mathcal{A}}_T \stackrel{\text{def}}{=} 1+\underline{b}^{-1}\|a\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+\gamma})}.$$

Hence applying Gronwall lemma eventually leads to

$$X_\mu(T) \leqslant C\big(\widetilde{\mathcal{A}}_T+\sqrt{\underline{\mu}}X_\mu(T)\big)^{\kappa+1}e^{C\left(\widetilde{\mathcal{A}}_T+\sqrt{\underline{\mu}}X_\mu(T)\right)^\kappa\left(Z(T)+\sqrt{T}(T\underline{\mu})^{\frac{\gamma}{4}}X_\mu(T)\right)}$$
$$\times\Big(\|\Delta u\|_{L^2_T(H^{\frac{N}{2}-1+\gamma})}+(\underline{\mu}T)^{\frac{\gamma}{4}}X_\mu(T)\Big) \quad (6.9)$$

with $Z(T) \stackrel{\text{def}}{=} \int_0^T\Big(\|\nabla u(t)\|_{H^{\frac{N}{2}+\gamma}}+\underline{b}\|\nabla\Pi(t)\|_{H^{\frac{N}{2}+\gamma}}+\underline{b}^{-1}\|\nabla a(t)\|_{H^{\frac{N}{2}+\gamma}}\Big)dt.$

Let $X_0 \stackrel{\text{def}}{=} 3C\big(1+\widetilde{\mathcal{A}}_{T_0}\big)^{\kappa+1}e^{C\left(1+\widetilde{\mathcal{A}}_{T_0}\right)^\kappa\left(1+Z(T_0)\right)}\|\Delta u\|_{L^2_{T_0}(H^{\frac{N}{2}+\gamma-1})}$. Assume that μ is so small as to satisfy

$$\begin{cases} \max\big(\sqrt{T_0}(T_0\underline{\mu})^{\frac{\gamma}{4}},\sqrt{\underline{\mu}}\big)\,X_0 \leqslant 1, \\ C\big(1+\widetilde{\mathcal{A}}_{T_0}\big)^{\kappa+1}e^{C\left(1+\widetilde{\mathcal{A}}_{T_0}\right)^\kappa\left(1+Z(T_0)\right)}(\underline{\mu}T_0)^{\frac{\gamma}{4}} \leqslant \frac{1}{2}. \end{cases} \quad (6.10)$$

Then we claim that we have $X_\mu(T_0) \leqslant \frac{2}{3}\,X_0$.

Indeed, because $T \mapsto X_\mu(T)$ is a continuous nondecreasing function which vanishes at $T=0$, the set

$$E \stackrel{\text{def}}{=} \Big\{T\in[0,T_0] \;\big|\; X_\mu(T) \leqslant \frac{2}{3}\,X_0\Big\}$$

is a non-empty closed interval.

Now, if $T \in E$ is such that $T < T_0$ then assumption (6.10) insures that $X_\mu(T) < \frac{2}{3} X_0$. As X_μ is a continuous function, this means that T is not the supremum of E. Therefore $E = [0, T_0]$ and the proof of proposition 6.1 is complete. □

7. The critical case

In the present section, we investigate the limit case $\gamma = 0$. We shall see that most of the qualitative results of the case $\gamma > 0$ remain true provided that the initial data belong to the *Besov space* $B^{\frac{N}{2}+1}_{2,1}$. Let us first introduce the following functional space:

Definition 7.1. — *For $s \in \mathbb{R}$, $\mu \geqslant 0$ and $T > 0$, we denote*

$$G^s_{T,\mu} \stackrel{\text{def}}{=} \Big\{(a, u, \nabla\Pi) \in C([0,T]; B^s_{2,1}) \times \Big(C([0,T]; B^s_{2,1})\Big)^N \times \Big(L^1(0,T; B^s_{2,1})\Big)^N \mid \mu u \in \Big(L^1(0,T; B^{s+2}_{2,1})\Big)^N\Big\}.$$

Our well-posedness result reads:

Theorem 7.2. — *Let $u_0 \in B^{\frac{N}{2}+1}_{2,1}$ with $\operatorname{div} u_0 = 0$, $0 < \underline{\rho} \leqslant \rho_0 \leqslant \overline{\rho}$ with $a_0 \stackrel{\text{def}}{=} \rho_0^{-1} - 1 \in B^{\frac{N}{2}+1}_{2,1}$ and $f \in L^1(\mathbb{R}^+; B^{\frac{N}{2}+1}_{2,1})$. There exists a $T > 0$ such that systems (INS_μ) and (IE) have a unique solution $(\rho, u, \nabla\Pi)$ on $[0,T]$ with $(a, u, \nabla\Pi) \in G^{\frac{N}{2}+1}_{T,\mu}$.*

The time T may be bounded by below by a function depending only on γ, N, μ, $\underline{b}$ and on the norm of the data in $B^{\frac{N}{2}+1}_{2,1}$, and may be chosen independent of μ for vanishing μ. Moreover, the solution to (INS_μ) tends to the corresponding solution of (IE) when the viscosity tends to 0. The convergence holds true in every space $G^s_{T,0}$ with $s < 1 + N/2$.

As the proof of this theorem is very similar to the one of theorem 0.2, we only sketch it. It mainly lies on estimates in the space $B^{\frac{N}{2}+1}_{2,1}$ for the linearized system.

As regards (3.1), the following estimate is proved in [8]:

$$\|a\|_{L^\infty_T(B^{\frac{N}{2}+1}_{2,1})} \leqslant e^{CV(T)}\Big(\|a_0\|_{B^{\frac{N}{2}+1}_{2,1}} + \int_0^T e^{-CV(t)}\|g(t)\|_{B^{\frac{N}{2}+1}_{2,1}}\,dt\Big), \quad (7.1)$$

with $V(t) \stackrel{\text{def}}{=} \int_0^t \|\nabla v(\tau)\|_{B^{\frac{N}{2}}_{2,1}} d\tau.$

For (M_μ) with $g \equiv 0$, we have

Proposition 7.3. — *Let $\kappa = N/2 + 1$ if $N \geqslant 3$ ($\kappa = 2 + \epsilon$ for some $\epsilon > 0$ if $N = 2$). There exists a constant C depending only on N (and ϵ if $N = 2$) and such that*

$$\|u\|_{L^\infty_T(B^{\frac{N}{2}+1}_{2,1})} + \underline{\mu}\|u\|_{L^1_T(B^{\frac{N}{2}+3}_{2,1})} \leqslant Ce^{C\mathcal{A}_T^\kappa V(T)}$$
$$\times\Big(\|u_0\|_{B^{\frac{N}{2}+1}_{2,1}} + \mathcal{A}_T^\kappa\|f\|_{L^1_T(B^{\frac{N}{2}+1}_{2,1})} + \underline{\mu}\mathcal{A}_T^{1+\kappa}\|u\|_{L^1_T(B^{\frac{N}{2}+2}_{2,1})}\Big),$$

with $\underline{\mu} \stackrel{\text{def}}{=} \mu/\overline{\rho}$, $\mathcal{A}_T \stackrel{\text{def}}{=} 1 + \overline{\rho}\Big(\|a\|_{L^\infty_T(L^\infty)} + \|\nabla a\|_{L^\infty_T(B^{\frac{N}{2}}_{2,1})}\Big)$ and $V(T) \stackrel{\text{def}}{=} \int_0^T \|\nabla v\|_{B^{\frac{N}{2}}_{2,1}} dt.$

For the pressure, we have

$$\underline{b}\|\nabla\Pi\|_{L^1_T(B^{\frac{N}{2}+1}_{2,1})}$$
$$\lesssim \mathcal{A}_T^{\frac{N}{2}+1}\Big(\|\mathcal{Q}f\|_{L^1_T(B^{\frac{N}{2}+1}_{2,1})} + \underline{\mu}b(\mathcal{A}_T - 1)\|u\|_{L^1_T(B^{\frac{N}{2}+3}_{2,1})} + V(T)\|u\|_{L^\infty_T(B^{\frac{N}{2}+1}_{2,1})}\Big).$$

In the case $v = u$, the above estimate holds with $V(T) = \int_0^T \|\nabla u\|_{L^\infty} dt.$

Sketchy proof of proposition 7.3. — Starting from (3.9), we get

$$\|u\|_{L^\infty_T(B^{\frac{N}{2}+1}_{2,1})} + \underline{\mu}\|u\|_{L^1_T(B^{\frac{N}{2}+3}_{2,1})} \lesssim \|u_0\|_{B^{\frac{N}{2}+1}_{2,1}} + \|\mathcal{P}f\|_{L^1_T(B^{\frac{N}{2}+1}_{2,1})}$$
$$+\underline{\mu}\|\Delta_{-1}u\|_{L^1_T(L^2)} + \|T_{\nabla a}\Pi\|_{L^1_T(B^{\frac{N}{2}+1}_{2,1})} + \|T'_{\nabla\Pi}a\|_{L^1_T(B^{\frac{N}{2}+1}_{2,1})}$$
$$+\mu\int_0^T\Big(\sum_{q\geqslant -1} 2^{q(\frac{N}{2}+1)}\|R_q\|_{L^2}\Big)dt + \int_0^T\Big(\sum_{q\geqslant -1} 2^{q(\frac{N}{2}+1)}\|[v,\Delta_q]\cdot\nabla u\|_{L^2}\Big)dt.$$

According to lemma 8.8, we have

$$\sum_{q\geqslant -1} 2^{q(\frac{N}{2}+1)}\|[v,\Delta_q]\cdot\nabla u\|_{L^2} \lesssim \|\nabla v\|_{B^{\frac{N}{2}}_{2,\infty}\cap L^\infty}\|\nabla u\|_{B^{\frac{N}{2}}_{2,1}}.$$

On the other hand, according to remark 8.10,

$$\sum_{q\geqslant -1} 2^{q(\frac{N}{2}+1)}\|R_q\|_{L^2} \lesssim \|\nabla a\|_{B^{\frac{N}{2}}_{2,1}}\|\nabla u\|_{B^{\frac{N}{2}+1}_{2,1}},$$

and, arguing as in proposition 1.4 and remark 1.5, we get

$$\|T_{\nabla a}\Pi\|_{B^{\frac{N}{2}+1}_{2,1}} + \|T'_{\nabla\Pi}a\|_{B^{\frac{N}{2}+1}_{2,1}} \lesssim \Big(\|a\|_{L^\infty} + \|\nabla a\|_{B^{\frac{N}{2}}_{2,1}}\Big)\|\nabla\Pi\|_{B^{\frac{N}{2}}_{2,1}}.$$

Therefore,

$$\begin{aligned}\|u\|_{L^\infty_T(B^{\frac{N}{2}+1}_{2,1})} + \underline{\mu}\|u\|_{L^1_T(B^{\frac{N}{2}+3}_{2,1})} \lesssim \\ \|u_0\|_{B^{\frac{N}{2}+1}_{2,1}} + \|\mathcal{P}f\|_{L^1_T(B^{\frac{N}{2}+1}_{2,1})} + \underline{\mu}\|\Delta_{-1}u\|_{L^1_T(L^2)} \\ +\Big(\|a\|_{L^\infty_T(L^\infty)} + \|\nabla a\|_{L^\infty_T(B^{\frac{N}{2}}_{2,1})}\Big)\|\nabla\Pi\|_{L^1_T(B^{\frac{N}{2}}_{2,1})} \\ +\int_0^T \big(\mu\|\nabla a\|_{B^{\frac{N}{2}}_{2,1}} + \|\nabla v\|_{B^{\frac{N}{2}}_{2,1}}\big)\|\nabla u\|_{B^{\frac{N}{2}}_{2,1}}\,dt.\end{aligned}$$

The pressure may be eliminated by making use of proposition 8.4 with $s = N/2$. Indeed,

$$\operatorname{div}(b\nabla\Pi) = \partial_i(\mu a\Delta u^i - T_{\partial_j u^i}v^j - T'_{\partial_i v^j}u^j + f^i), \tag{7.2}$$

so that

$$\begin{aligned}\underline{b}\|\nabla\Pi\|_{L^1_T(B^{\frac{N}{2}}_{2,1})} \lesssim \mathcal{A}^\kappa_T\Big(\|\mathcal{Q}f\|_{L^1_T(B^{\frac{N}{2}}_{2,1})} + \int_0^T \|u\|_{B^{\frac{N}{2}}_{2,1}}\|\nabla v\|_{B^{\frac{N}{2}}_{2,1}}\,dt \\ +\Big(\|a\|_{L^\infty_T(L^\infty)} + \|\nabla a\|_{L^\infty_T(B^{\frac{N}{2}}_{2,1})}\Big)\|\Delta u\|_{L^1_T(B^{\frac{N}{2}}_{2,1})}\Big),\end{aligned} \tag{7.3}$$

with $\kappa = N/2$ if $N \geqslant 3$, and $\kappa = 1+\epsilon$ if $N = 2$.

Gronwall lemma yields the desired estimate for u. Next, applying proposition 8.4 with $s = N/2+1$ to (7.2) yields the desired inequality for $\nabla\Pi$.

Note that when $v = u$, we have

$$\sum_{q\geqslant -1} 2^{q(\frac{N}{2}+1)}\|[v,\Delta_q]\cdot\nabla u\|_{L^2} \lesssim \|\nabla u\|_{L^\infty}\|\nabla u\|_{B^{\frac{N}{2}}_{2,1}}$$

so that $\|\nabla v\|_{B^{\frac{N}{2}}_{2,1}}$ may be replaced by $\|\nabla u\|_{L^\infty}$ in the estimate for u. The same remark applies to the estimates pertaining to the pressure. □

Let us now prove estimates for the solutions to (INS_μ).

Assuming that $a \in L^\infty(0,T;B_{2,1}^{\frac{N}{2}+1})$, $u \in L^\infty(0,T;B_{2,1}^{\frac{N}{2}+1})$, $\mu u \in L^1(0,T;B_{2,1}^{\frac{N}{2}+3})$ and $\nabla\Pi \in L^1(0,T;B_{2,1}^{\frac{N}{2}+1})$, inequality (7.1) and proposition 7.3 yield

$$\|a\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}+1})} \leqslant \|a_0\|_{B_{2,1}^{\frac{N}{2}+1}} e^{C\|\nabla u\|_{L_T^1(B_{2,1}^{\frac{N}{2}})}},$$

$$\|u\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}+1})} + \underline{\mu}\|u\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}+3})} \leqslant$$
$$Ce^{C\mathcal{A}_T^\kappa\|\nabla u\|_{L_T^1(L^\infty)}}\Big(\|u_0\|_{B_{2,1}^{\frac{N}{2}+1}} + \mathcal{A}_T^\kappa\|f\|_{L_T^1(B_{2,1}^{\frac{N}{2}+1})} + \underline{\mu}\mathcal{A}_T^{\kappa+1}\|u\|_{L_T^1(B_{2,1}^{\frac{N}{2}+2})}\Big),$$

with $\mathcal{A}_T = 1+\overline{\rho}\|a\|_{L_T^\infty(B_{2,1}^{\frac{N}{2}+1})}$, $\kappa = 1+N/2$ if $N \geqslant 3$, and $\kappa = 1+\epsilon$ if $N=2$.

Next, going along the lines of the proof of the existence in theorem 0.2, we easily get estimates independent of μ on the time interval $[0,T]$ with

$$T = c\,\min\left(\frac{e^{-C\mathcal{A}_0^\kappa}}{\mathcal{A}_0^\kappa U_0}, \frac{e^{-2C\mathcal{A}_0^\kappa}}{\mu\mathcal{A}_0^{2\kappa+2}}\right) \tag{7.4}$$

for some constants c and C, $\mathcal{A}_0 \overset{\text{def}}{=} 1+\underline{b}^{-1}\|a_0\|_{B_{2,1}^{\frac{N}{2}+1}}$ and

$U_0 \overset{\text{def}}{=} \|u_0\|_{B_{2,1}^{\frac{N}{2}+1}} + \|f\|_{L^1(B_{2,1}^{\frac{N}{2}+1})}$.

Note that when μ goes to 0, the time T defined in (7.4) does not depend on μ and one can get estimates independent of μ of the solution. Hence applying proposition 2.2, one can prove that the viscous solution tends to the inviscid one in L^2 when the viscosity goes to 0. Interpolating with the uniform estimates above, we conclude that convergence holds true in

$$C([0,T];B_{2,1}^s) \times \Big(C([0,T];B_{2,1}^s)\Big)^N \times \Big(L^1(0,T;B_{2,1}^s)\Big)^N$$

for all $s < 1+N/2$.

Now, for data satisfying the assumptions of theorem 7.2, one can easily prove existence on the interval $[0,T]$ with T defined in (7.4): as a first step we smooth out the data (take $(S_n a_0, S_n u_0, S_n f)$ instead of (a_0,u_0,f)) and solve the corresponding initial value problem. According to corollary 0.8 we get a local solution $(a_n,u_n,\nabla\Pi_n)$ in H^∞. By taking advantage of the above calculations and of the continuation criterion given in proposition 5.2, the lifespan of $(a_n,u_n,\nabla\Pi_n)$ may be bounded from below according to (7.4).

Besides, one gets estimates in $G^{\frac{N}{2}+1}_{T,\mu}$ independent on n (and of μ for small μ) for $(a_n, u_n, \nabla\Pi_n)$. Convergence of the sequence together with uniqueness then readily stem from corollary 2.3. □

Remark 7.4. — A blow-up criterion involving the $L^1_T(L^\infty)$ norm of ∇u may also be proved. The details are left to the reader.

8. Appendix

8.1. Elliptic estimates

This section is devoted to the proof of new estimates (of independent interest) for the elliptic equation (3.5). The results we prove here are somewhat more general than needed in the present paper. In particular, we state estimates in Sobolev spaces with *negative* index of regularity, a result which has been of much use in a recent work (see [8]).

Let us first study the stationary case where F and b are independent of the time:

PROPOSITION 8.1. — *Assume that $F \in L^2$ and that b is bounded and satisfies $b \geqslant \underline{b} > 0$. There exists a unique distribution Π modulo the constants such that $\nabla\Pi \in L^2$ and Π solves (3.5) in the sense of distributions. Moreover, the linear operator $\mathcal{H}_b : F \mapsto \nabla\Pi$ is bounded in L^2 and satisfies*

$$\underline{b}\, \|\mathcal{H}_b(F)\|_{L^2} \leqslant \|\mathcal{Q}F\|_{L^2} \quad \textit{for all} \quad F \in L^2. \tag{8.1}$$

Proof. — For smooth Π, the proof of (8.1) is straightforward: take the L^2-scalar product of (3.5) with Π, integrate by parts in the left-hand side and use Hölder inequality to deal with the right-hand side. Density arguments yield the estimate in the general case. As for existence, it stems from Lax-Milgram theorem. □

For smoother b, one can get estimates in Sobolev spaces with positive or negative regularity index. This latter point comes from the positive regularity of b outweighing the negative regularity of Π:

PROPOSITION 8.2. — *Let $\alpha > 0$ and $\sigma \in \mathbb{R}$ satisfy $1\vee\alpha \leqslant |\sigma| \leqslant \alpha+N/2$. Then the operator $\mathcal{H}_b$ is a linear bounded operator in H^σ and the following estimate holds true:*

$$\underline{b}\|\nabla\Pi\|_{H^\sigma} \lesssim A^{\frac{|\sigma|}{1\vee\alpha}}\|\mathcal{Q}F\|_{H^\sigma}, \tag{8.2}$$

$$\textit{with } \nabla\Pi \stackrel{\text{def}}{=} \mathcal{H}_b(F) \textit{ and } A \stackrel{\text{def}}{=} \begin{cases} 1+\underline{b}^{-1}\|\nabla b\|_{H^{\frac{N}{2}+\alpha-1}} & \textit{if} \quad \alpha \neq 1, \\ 1+\underline{b}^{-1}\|\nabla b\|_{H^{\frac{N}{2}}\cap L^\infty} & \textit{if} \quad \alpha = 1. \end{cases}$$

Proof.— i) Case $\sigma \in [\alpha \vee 1, \alpha + N/2]$.

Apply Δ_q to (3.5) and proceed as in the L^2 case. We gather:

$$\underline{b}\,\|\Delta_q\nabla\Pi\|_{L^2}^2 \leqslant \Big|(\Delta_q\mathcal{Q}F|\Delta_q\nabla\Pi) + (\Delta_q\nabla\Pi\,|\,[b,\Delta_q]\nabla\Pi)\Big|, \tag{8.3}$$

whence

$$\underline{b}\Big(\sum_q 2^{2q\sigma}\,\|\Delta_q\nabla\Pi\|_{L^2}^2\Big)^{\frac12} \leqslant \Big(\sum_q 2^{2q\sigma}\,\|\Delta_q\mathcal{Q}F\|_{L^2}^2\Big)^{\frac12} + \Big(\sum \|[b,\Delta_q]\nabla\Pi\|_{L^2}\,2^{2q\sigma}\Big)^{\frac12}.$$

Assume $\alpha \neq 1$ to simplify the presentation. The commutator in the right-hand side may be bounded thanks to lemma 8.8 below with $p = r = 2$, $\check\sigma = \sigma - \alpha \vee 1$ and $\widetilde\sigma = \sigma$ (here comes the assumption $\sigma \leqslant \alpha + N/2$).

If in addition $\sigma \neq 1 \vee \alpha + N/2$, we end up with

$$\underline{b}\|\nabla\Pi\|_{H^\sigma} \lesssim \|\mathcal{Q}F\|_{H^\sigma} + \|\nabla b\|_{H^{\frac{N}{2}+\alpha-1}}\|\nabla\Pi\|_{H^{\sigma-\alpha\vee 1}}. \tag{8.4}$$

Since $\sigma \geqslant \alpha \vee 1$, complex interpolation between L^2 and H^σ yields

$$\|\nabla\Pi\|_{H^{\sigma-\alpha\vee1}} \leqslant \|\nabla\Pi\|_{H^\sigma}^{\frac{\sigma-\alpha\vee1}{\sigma}}\;\|\nabla\Pi\|_{L^2}^{\frac{\alpha\vee1}{\sigma}}. \tag{8.5}$$

Let us now remark that for all $a, b, c \geqslant 0$ and $\theta \in [0,1)$, we have

$$a\,b^\theta c^{1-\theta} \leqslant \theta b + (1-\theta)a^{\frac{1}{1-\theta}}\,c \tag{8.6}$$

Plugging (8.5) in (8.4), and using (8.6) and (8.1), we get (8.2).

In the limit case $\sigma = 1\vee\alpha + N/2$, we have to bound $\|\nabla\Pi\|_{H^{\frac{N}{2}}\cap L^\infty}$. This may be done by combining *real* interpolation with the embedding $B^{\frac{N}{2}}_{2,1} \hookrightarrow L^\infty$. We end up with

$$\|\nabla\Pi\|_{B^{\frac{N}{2}}_{2,1}} \lesssim \|\nabla\Pi\|_{H^{1\vee\alpha+N/2}}^{\frac{N/2}{1\vee\alpha+N/2}}\;\|\nabla\Pi\|_{L^2}^{\frac{\alpha\vee1}{1\vee\alpha+N/2}},$$

and we can conclude as before.

Actually, we did not prove that $\nabla\Pi \in H^\sigma$. Note that proposition 8.1 insures that $\nabla\Pi \in L^2$. Therefore, applying (8.4) with $\sigma = \alpha \vee 1$ yields $\nabla\Pi \in H^{\alpha\vee 1}$. Next, applying again (8.4) yields $\nabla\Pi \in H^{\sigma\vee(2\alpha\vee 2)}$, etc.

ii) Case $\sigma \in [-\alpha - N/2, -\alpha \vee 1]$.

Unsurprisingly, a duality method will do. Indeed

$$\begin{aligned} \|\nabla\Pi\|_{H^\sigma} &= \sup_{\|g\|_{H^{-\sigma}}\leqslant 1} \int g\cdot\nabla\Pi\,dx, \\ &= \sup_{\|g\|_{H^{-\sigma}}\leqslant 1} \int \Pi\,\mathrm{div}\,g\,dx. \end{aligned} \tag{8.7}$$

Since $-\sigma \geqslant \alpha\vee 1$, the results of case *i)* yield a unique $\nabla h_g \in H^{-\sigma}$ satisfying

$$\mathrm{div}(b\nabla h_g) = \mathrm{div}\,g$$

and, besides,

$$\underline{b}\|\nabla h_g\|_{H^{-\sigma}} \lesssim \mathcal{A}^{-\frac{\sigma}{1\vee\alpha}}\|\mathcal{Q}g\|_{H^{-\sigma}}. \tag{8.8}$$

On the other hand, according to the definition of h_g, integrating twice by parts in (8.7) and using (3.5), we get

$$\begin{aligned} \|\nabla\Pi\|_{H^\sigma} &= \sup_{\|g\|_{H^{-\sigma}}\leqslant 1} \int h_g\,\mathrm{div}\,F\,dx, \\ &= \sup_{\|g\|_{H^{-\sigma}}\leqslant 1} \int \mathcal{Q}F\cdot\nabla h_g\,dx, \end{aligned} \tag{8.9}$$

which, according to (8.8), completes the proof of (8.2). Of course, in order to make the above computations rigorous, one has to argue by density. □

Remark 8.3. — Actually, if $0<\alpha<1$ and $\alpha \leqslant |\sigma| < \frac{N}{2}+\alpha$, lemma 8.7 enables us to take $A = 1+\underline{b}^{-1}\|\nabla b\|_{B^{\frac{N}{2}+\alpha-1}_{2,\infty}}$.

Let us now state continuity results in *Besov spaces*, a result which is needed in section 7.

Proposition 8.4. — *Let* $s \in (1, \frac{N}{2}+1]$ *and* Π *satisfy* $\nabla\Pi \in B^s_{2,1}$ *and* $\mathrm{div}(b\nabla\Pi) = \mathrm{div}\,F$. *Denote* $A \stackrel{\text{def}}{=} 1+\underline{b}^{-1}\|\nabla b\|_{B^{\frac{N}{2}}_{2,1}}$. *The following estimate holds true:*

$$\underline{b}\|\nabla\Pi\|_{B^s_{2,1}} \lesssim A^s\|\mathcal{Q}f\|_{B^s_{2,1}}. \tag{8.10}$$

In the limit case $s = 1$, we have for all $\epsilon > 0$,

$$\underline{b}\|\nabla\Pi\|_{B^1_{2,1}} \lesssim A^{1+\epsilon}\|\mathcal{Q}f\|_{B^1_{2,1}}. \tag{8.11}$$

Proof. — Starting from (8.3), we get

$$\underline{b}\|\nabla\Pi\|_{B^s_{2,1}} \leqslant \|\mathcal{Q}f\|_{B^s_{2,1}} + \sum_{q\geqslant -1} 2^{qs}\, \|[b, \Delta_q]\nabla\Pi\|_{L^2}\,.$$

The commutator may be bounded thanks to lemma 8.8 with $\check{\sigma} = s - 1$, $\widetilde{\sigma} = s$, $\alpha = 1$, $p = 2$ and $r = 1$. We get

$$\underline{b}\|\nabla\Pi\|_{B^s_{2,1}} \leqslant \|\mathcal{Q}f\|_{B^s_{2,1}} + C\|\nabla a\|_{B^{\frac{N}{2}}_{2,1}}\|\nabla\Pi\|_{B^{s-1}_{2,1}}.$$

If $s > 1$, real interpolation between L^2 and $B^s_{2,1}$ yields

$$\|\nabla\Pi\|_{B^{s-1}_{2,1}} \lesssim \|\nabla\Pi\|_{L^2}^{\frac{1}{s}}\, \|\nabla\Pi\|_{B^s_{2,1}}^{\frac{s-1}{s}},$$

whence the desired result.

If $s = 1$, one can alternatively use the estimate

$$\sum_{q\geqslant -1} 2^q\, \|[b, \Delta_q]\nabla\Pi\|_{L^2} \lesssim \|\nabla a\|_{B^{\frac{N}{2}-\epsilon}_{2,1}}\|\nabla\Pi\|_{B^{\epsilon}_{2,1}}$$

then interpolate between L^2 and $B^1_{2,1}$. □

Let us finally study the non-stationary problem (3.5). We have the following

Proposition 8.5. — *Let $m \in [1, +\infty]$, $\epsilon > 0$, $\alpha > 0$ and σ satisfy $1\vee\alpha \leqslant |\sigma| \leqslant \alpha+N/2$. Then $\mathcal{H}_b$ is a bounded operator on $\widetilde{L}^m_T(H^\sigma)$. Besides, the following estimate holds:*

$$\underline{b}\|\nabla\Pi\|_{\widetilde{L}^m_T(H^\sigma)} \lesssim \mathcal{A}_T^{\frac{|\sigma|+\epsilon}{1\vee\alpha}}\, \|\mathcal{Q}F\|_{\widetilde{L}^m_T(H^\sigma)}, \tag{8.12}$$

with $\mathcal{A}_T$ defined in (3.2).

If $1 \leqslant m \leqslant 2$ and $\sigma \geqslant \alpha \vee 1$, or $2 \leqslant m \leqslant +\infty$ and $\sigma \leqslant -(\alpha \vee 1)$, the above inequality holds with $\epsilon = 0$.

Proof. — It is very similar to the one of proposition 8.2. We now have to use a non-stationary version of lemma 8.8, namely lemma 8.7.

Assuming for the sake of simplicity that $\alpha \neq 1$ and that $\alpha + N/2 > \sigma \geqslant \alpha \vee 1$, inequality (8.3) eventually leads to

$$\underline{b}\|\nabla\Pi\|_{\widetilde{L}^m_T(H^\sigma)} \lesssim \|\mathcal{Q}F\|_{\widetilde{L}^m_T(H^\sigma)} + \|\nabla a\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+\alpha-1})}\|\nabla\Pi\|_{\widetilde{L}^m_T(H^{\sigma-\alpha\vee 1})}.$$

If $m \leqslant 2$, Minkowski inequality yields

$$\|\nabla\Pi\|_{\widetilde{L}^m_T(H^0)} \leqslant \|\nabla\Pi\|_{L^m_T(L^2)}.$$

Therefore complex interpolation entails

$$\|\nabla\Pi\|_{\widetilde{L}^m_T(H^{\sigma-\alpha\vee 1})} \leqslant \|\nabla\Pi\|^{\frac{\alpha\vee 1}{\sigma}}_{L^m_T(L^2)}\|\nabla\Pi\|^{\frac{\sigma-\alpha\vee 1}{\sigma}}_{\widetilde{L}^m_T(H^\sigma)} \tag{8.13}$$

whence, according to (8.1) and Young inequality (8.6),

$$\underline{b}\|\nabla\Pi\|_{\widetilde{L}^m_T(H^\sigma)} \lesssim \|\mathcal{Q}F\|_{\widetilde{L}^m_T(H^\sigma)} + \left(\frac{\|\nabla b\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+\alpha-1})}}{\underline{b}}\right)^{\frac{\sigma}{\alpha\vee 1}}\|\mathcal{Q}F\|_{L^m_T(L^2)}.$$

Since $\sigma > 0$, we have

$$\|\mathcal{Q}F\|_{\widetilde{L}^m_T(L^2)} \lesssim \|\mathcal{Q}F\|_{\widetilde{L}^m_T(H^\sigma)}$$

which yields (8.12).

If $m > 2$, the embedding $L^m_T(L^2) \hookrightarrow \widetilde{L}^m_T(L^2)$ fails so that we are induced to interpolate between $H^{-\epsilon}$ and H^σ. We eventually get

$$\underline{b}\|\nabla\Pi\|_{\widetilde{L}^m_T(H^\sigma)} \lesssim \|\mathcal{Q}F\|_{\widetilde{L}^m_T(H^\sigma)} + \left(\frac{\|\nabla b\|_{\widetilde{L}^\infty_T(H^{\frac{N}{2}+\alpha-1})}}{\underline{b}}\right)^{\frac{\sigma+\epsilon}{\alpha\vee 1}}\|\nabla\Pi\|_{\widetilde{L}^m_T(H^{-\epsilon})}.$$

As $\|\nabla\Pi\|_{\widetilde{L}^m_T(H^{-\epsilon})} \lesssim \|\nabla\Pi\|_{L^m_T(L^2)}$, we can now conclude as before.

The limit cases $\alpha = 1$ or $\sigma = \alpha + N/2$ are left to the reader.

Note that we actually did not prove that $\mathcal{H}_b$ is bounded in $\widetilde{L}^m_T(H^\sigma)$. This may be achieved by using that, according to proposition 8.1 (combined with time mollifiers), problem (3.5) has a unique solution $\nabla\Pi \in L^m(0,T;L^2)$ whenever $F \in L^m(0,T;L^2)$. Next, arguing like in proposition 8.2, one can show that $\nabla\Pi$ belongs to $\widetilde{L}^m_T(H^\sigma)$.

The case $\sigma \leqslant -\alpha \vee 1$ stems from duality arguments. □

Remark 8.6. — If $0 < \alpha < 1$ and $\alpha \leqslant \sigma < \frac{N}{2} + \alpha$, lemma 8.7 enables us to take $A_T = 1 + \underline{b}^{-1}\|\nabla b\|_{\widetilde{L}^\infty_T(B^{\frac{N}{2}+\alpha-1}_{2,\infty})}$.

8.2. Commutator estimates

The following lemma was needed in the proof of proposition 8.5:

LEMMA 8.7 *Let* $m \in [1,+\infty]$, $1 \leqslant p, r \leqslant +\infty$, $\alpha \neq 1$ *and* $\check{\sigma} \in \mathbb{R}$ *satisfy*

$$\alpha + \check{\sigma} + N \min(1/p, 1/p') > 0 \quad \textit{and} \quad \alpha + N/p > 0.$$

Let $\widetilde{\sigma} \leqslant \min(\check{\sigma} + \alpha \vee 1, \alpha + N/p)$. *The following inequalities hold true:*

$$\Big(\sum_{q \geqslant -1} \|[a, \Delta_q]w\|^r_{L^m_T(L^p)} 2^{rq\widetilde{\sigma}}\Big)^{\frac{1}{r}} \lesssim \|\nabla a\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}+\alpha-1}_{p,\infty})} \|w\|_{\widetilde{L}^m_T(B^{\check{\sigma}}_{p,r})} \quad \textit{if} \quad \check{\sigma} < N/p,$$

$$\Big(\sum_{q \geqslant -1} \|[a, \Delta_q]w\|^r_{L^m_T(L^p)} 2^{rq\widetilde{\sigma}}\Big)^{\frac{1}{r}} \lesssim \|\nabla a\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}+\alpha-1}_{p,r})} \|w\|_{\widetilde{L}^m_T(B^{\check{\sigma}}_{p,r})} \quad \textit{if} \quad \check{\sigma} \geqslant N/p.$$

In the limit case $\alpha = 1$, *the term* $\|\nabla a\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,\infty})}$ *(resp.* $\|\nabla a\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,r})}$*) has to be replaced by* $\|\nabla a\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,\infty}) \cap L^\infty_T(L^\infty)}$ *(resp.* $\|\nabla a\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,r}) \cap L^\infty_T(L^\infty)}$*).*

Proof. — Let $\widetilde{a} \stackrel{\text{def}}{=} a - \Delta_{-1}a$. Decompose $[a, \Delta_q]w$ as follows:

$$[a, \Delta_q]w = \underbrace{[T_{\widetilde{a}}, \Delta_q]w}_{R^1_q} + \underbrace{T'_{\Delta_q w}\widetilde{a}}_{R^2_q} - \underbrace{\Delta_q T_w \widetilde{a}}_{R^3_q} - \underbrace{\Delta_q R(\widetilde{a}, w)}_{R^4_q} + \underbrace{[\Delta_{-1}a, \Delta_q]w}_{R^5_q}. \tag{8.14}$$

The term R^1_q may be bounded by combining (1.1) and first order Taylor's formula. We get

$$\left\|R^1_q\right\|_{L^p} \lesssim 2^{-q} \sum_{|q'-q| \leqslant 4} \|\nabla S_{q'-1}\widetilde{a}\|_{L^\infty} \|\Delta_{q'} w\|_{L^p}.$$

Hence,

$$\Big(\sum_{q \geqslant -1} \Big(2^{q(\alpha+\check{\sigma})} \|R^1_q\|_{L^m_T(L^p)}\Big)^r\Big)^{\frac{1}{r}} \lesssim \|\nabla \widetilde{a}\|_{L^\infty_T(B^{\alpha-1}_{\infty,\infty})} \|w\|_{\widetilde{L}^m_T(B^{\check{\sigma}}_{p,r})} \quad \text{if} \quad \alpha < 1, \tag{8.15}$$

$$\Big(\sum_{q \geqslant -1} \Big(2^{q(1+\check{\sigma})} \|R^1_q\|_{L^m_T(L^p)}\Big)^r\Big)^{\frac{1}{r}} \lesssim \|\nabla \widetilde{a}\|_{L^\infty_T(L^\infty)} \|w\|_{\widetilde{L}^m_T(B^{\check{\sigma}}_{p,r})} \quad \text{if} \quad \alpha \geqslant 1. \tag{8.16}$$

Taking advantage of (1.1) and of Hölder inequality, R_q^2 may be bounded as follows:

$$2^{q(\alpha+\check\sigma)}\|R_q^2\|_{L_T^m(L^p)} \lesssim \sum_{q'\geqslant q-2} 2^{(q-q')(\alpha+\frac{N}{p})}\Big(2^{q'(\alpha+\frac{N}{p})}\|\Delta_{q'}\widetilde{a}\|_{L_T^\infty(L^p)}\Big)\Big(2^{q(\check\sigma-\frac{N}{p})}\|\Delta_q w\|_{L_T^m(L^\infty)}\Big).$$

If $\alpha+N/p>0$, convolution inequalities for series yield

$$\bigg(\sum_{q\geqslant -1}\Big(2^{q(\alpha+\check\sigma)}\|R_q^2\|_{L_T^m(L^p)}\Big)^r\bigg)^{\frac{1}{r}} \lesssim \|\widetilde{a}\|_{L_T^\infty(B_{p,\infty}^{\frac{N}{p}+\alpha})}\|w\|_{\widetilde{L}_T^m(B_{\infty,r}^{\check\sigma-\frac{N}{p}})}. \qquad (8.17)$$

For bounding the third term in (8.14), one can further decompose it into

$$\Delta_q T_w\widetilde{a} = \sum_{|q''-q|\leqslant 4}\ \sum_{q'\leqslant q''-2}\Delta_q\Big(\Delta_{q'}w\Delta_{q''}\widetilde{a}\Big).$$

Now, denoting $\sigma_1=\check\sigma-\frac{N}{p}$ and $\sigma_2=\alpha+\frac{N}{p}$, we have

$$2^{q(\sigma_1+\sigma_2)}\|\Delta_q T_w\widetilde{a}\|_{L_T^m(L^p)} \lesssim \sum_{\substack{|q''-q|\leqslant 4\\ q'\leqslant q''-2}} 2^{(q''-q')\sigma_1}\Big(2^{q'\sigma_1}\|\Delta_{q'}w\|_{L_T^m(L^\infty)}\Big)\Big(2^{q''\sigma_2}\|\Delta_{q''}\widetilde{a}\|_{L_T^\infty(L^p)}\Big),$$

so that, if $\sigma_1<0$,

$$\bigg(\sum_{q\geqslant -1}\Big(2^{q(\check\sigma+\alpha)}\|R_q^3\|_{L_T^m(L^p)}\Big)^r\bigg)^{\frac{1}{r}} \lesssim \|w\|_{\widetilde{L}_T^m(B_{\infty,r}^{\sigma_1})}\|\widetilde{a}\|_{L_T^\infty(B_{p,\infty}^{\sigma_2})}. \qquad (8.18)$$

Remark that one can also prove that

$$\bigg(\sum_{q\geqslant -1}\Big(2^{q\sigma_2}\|R_q^3\|_{L_T^m(L^p)}\Big)^r\bigg)^{\frac{1}{r}} \lesssim \|w\|_{L_T^m(L^\infty)}\|\widetilde{a}\|_{\widetilde{L}_T^\infty(B_{p,r}^{\sigma_2})}. \qquad (8.19)$$

A straightforward adaptation of proposition 1.4 to non-stationary spaces yields

$$\|R(\widetilde{a},w)\|_{\widetilde{L}_T^m(B_{p,r}^{\alpha+\check\sigma})} \lesssim \|w\|_{\widetilde{L}_T^m(B_{p,r}^{\check\sigma})}\|\widetilde{a}\|_{L_T^\infty(B_{p,\infty}^{\alpha+\frac{N}{p}})} \quad \text{if} \quad \alpha+\check\sigma>-N\min(1/p,1/p'). \qquad (8.20)$$

The term R_q^5 may be treated by arguing like in the proof of (8.16). One ends up with

$$\left(\sum_{q\geqslant -1}\left(2^{q(1+\check\sigma)}\|[\Delta_{-1}a,\Delta_q]w\|_{L_T^m(L^p)}\right)^r\right)^{\frac{1}{r}} \lesssim \|\nabla\Delta_{-1}a\|_{L_T^\infty(L^\infty)}\|w\|_{\widetilde{L}_T^m(B^{\check\sigma}_{p,r})}. \tag{8.21}$$

Of course, since $\widetilde{a}$ has no low frequencies, we have for all $r'\in[1,+\infty]$,

$$\|\widetilde{a}\|_{\widetilde{L}_T^\infty(B^{\frac{N}{p}+\alpha}_{p,r'})} \lesssim \|\nabla a\|_{\widetilde{L}_T^\infty(B^{\frac{N}{p}+\alpha-1}_{p,r'})}. \tag{8.22}$$

Therefore, plugging (8.15) or (8.16), (8.17), (8.18) or (8.19), (8.20) and (8.21) in (8.14), we conclude to lemma 8.7. □

Let us also give a stationary statement of lemma 8.7:

LEMMA 8.8. — *Let p, r, α, $\widetilde{\sigma}$ and $\check\sigma$ be as in the statement the previous lemma. The following inequalities hold true:*

$$\left(\sum_{q\geqslant -1}\|[a,\Delta_q]w\|^r_{L^p}2^{rq\widetilde{\sigma}}\right)^{\frac{1}{r}} \lesssim \|\nabla a\|_{B^{\frac{N}{p}+\alpha-1}_{p,\infty}}\|w\|_{B^{\check\sigma}_{p,r}} \quad \text{if} \quad \check\sigma < N/p,$$

$$\left(\sum_{q\geqslant -1}\|[a,\Delta_q]w\|^r_{L^p}2^{rq\widetilde{\sigma}}\right)^{\frac{1}{r}} \lesssim \|\nabla a\|_{B^{\frac{N}{p}+\alpha-1}_{p,r}}\|w\|_{B^{\check\sigma}_{p,r}} \quad \text{if} \quad \check\sigma \geqslant N/p.$$

In the limit case $\alpha=1$, the term $\|\nabla a\|_{B^{\frac{N}{p}}_{p,\infty}}$ (resp. $\|\nabla a\|_{B^{\frac{N}{p}}_{p,r}}$) has to be replaced by $\|\nabla a\|_{B^{\frac{N}{p}}_{p,\infty}\cap L^\infty}$ (resp. $\|\nabla a\|_{B^{\frac{N}{p}}_{p,r}\cap L^\infty}$).

LEMMA 8.9. — *Let $(r,m)\in[1,+\infty]^2$. Assume that $\alpha>1-N/2$, $\alpha\neq 1$, and $\sigma\in(1-N/2,1+\alpha+N/2)$. Let $R_q \stackrel{\text{def}}{=} \Delta_q(a\,\mathrm{div}\,w)-\mathrm{div}(a\Delta_q w)$. Then we have*

$$\left(\sum_{q\geqslant -1}\left(2^{q(\sigma-1)}\|R_q\|_{L_T^m(L^2)}\right)^r\right)^{\frac{1}{r}} \lesssim \|\nabla a\|_{L_T^\infty(B^{\frac{N}{2}+\alpha-1}_{2,\infty})}\|w\|_{\widetilde{L}_T^m(B^{\sigma-\alpha\vee 1}_{2,r})}.$$

with the usual modifications if $r=+\infty$ or $\alpha=1$.

In the limit case $\sigma = 1+\alpha+N/2$, we have

$$\Big(\sum_{q\geqslant -1}\Big(2^{q(\frac{N}{2}+\alpha)}\|R_q\|_{L^m_T(L^2)}\Big)^r\Big)^{\frac1r}\lesssim \|\nabla a\|_{\widetilde L^\infty_T(B^{\frac N2+\alpha-1}_{2,r})}\|w\|_{\widetilde L^m_T(B^{\frac N2+1}_{2,r})\cap L^m_T(\mathrm{Lip})}\qquad \textit{if}\quad \alpha<1,$$

$$\lesssim \|\nabla a\|_{\widetilde L^\infty_T(B^{\frac N2}_{2,r})\cap L^\infty_T(L^\infty)}\|w\|_{\widetilde L^m_T(B^{\frac N2+1}_{2,r})\cap L^m_T(\mathrm{Lip})}\qquad \textit{if}\quad \alpha=1,$$

$$\lesssim \|\nabla a\|_{\widetilde L^\infty_T(B^{\frac N2+\alpha-1}_{2,r})}\|w\|_{\widetilde L^m_T(B^{\frac N2+\alpha}_{2,r})}\qquad \textit{if}\quad \alpha>1.$$

Proof. — Let $\widetilde a \overset{\mathrm{def}}{=} a-\Delta_{-1}a$. Using Bony's decomposition, R_q rewrites

$$R_q=\underbrace{\partial_i[\Delta_q,T_{\widetilde a}]w^i}_{R_q^1}-\underbrace{\Delta_q T_{\partial_i\widetilde a}w^i}_{R_q^2}+\underbrace{\Delta_q T'_{\mathrm{div}\,w}\widetilde a}_{R_q^3}-\underbrace{\partial_i T'_{\Delta_q w^i}\widetilde a}_{R_q^4}+\underbrace{[\Delta_q,\Delta_{-1}a]\,\mathrm{div}\,w}_{R_q^5}-\underbrace{\nabla\Delta_{-1}a\cdot\Delta_q w}_{R_q^6}.$$

In view of (1.1), R_q^1 further decomposes into

$$R_q^1=\sum_{|q'-q|\leqslant 4}\partial_i[\Delta_q,S_{q'-1}\widetilde a]\Delta_{q'}w^i.$$

Now, combining Bernstein inequality with the first order Taylor's formula yields

$$\|\partial_i[\Delta_q,S_{q'-1}\widetilde a]\Delta_{q'}w\|_{L^2}\lesssim 2^{q'-q}\,\|\nabla S_{q'-1}\widetilde a\|_{L^\infty}\,\|\Delta_{q'}w\|_{L^2}\,,$$

whence,

$$\Big(\sum_{q\geqslant -1}\Big(2^{q(\sigma-1)}\|R_q^1\|_{L^m_T(L^2)}\Big)^r\Big)^{\frac1r}\lesssim\|\nabla\widetilde a\|_{L^\infty_T(B^{\alpha-1}_{\infty,\infty})}\|w\|_{\widetilde L^m_T(B^{\sigma-\alpha}_{2,r})}\qquad \text{if } \alpha<1,\tag{8.23}$$

$$\Big(\sum_{q\geqslant -1}\Big(2^{q(\sigma-1)}\|R_q^1\|_{L^m_T(L^2)}\Big)^r\Big)^{\frac1r}\lesssim\|\nabla\widetilde a\|_{L^\infty_T(L^\infty)}\|w\|_{\widetilde L^m_T(B^{\sigma-1}_{2,r})}\qquad \text{if } \alpha\geqslant 1.\tag{8.24}$$

Applying proposition 1.4 combined with remark 1.8 yields

$$\Big(\sum_{q\geqslant -1}\Big(2^{q(\sigma-1)}\|R_q^2\|_{L_T^m(L^2)}\Big)^r\Big)^{\frac{1}{r}} \lesssim \|\nabla\widetilde{a}\|_{L_T^\infty(B^{\alpha-1}_{\infty,\infty})}\|w\|_{\widetilde{L}_T^m(B^{\sigma-\alpha}_{2,r})} \quad \text{if} \quad \alpha<1, \tag{8.25}$$

$$\Big(\sum_{q\geqslant -1}\Big(2^{q(\sigma-1)}\|R_q^2\|_{L_T^m(L^2)}\Big)^r\Big)^{\frac{1}{r}} \lesssim \|\nabla\widetilde{a}\|_{L_T^\infty(L^\infty)}\|w\|_{\widetilde{L}_T^m(B^{\sigma-1}_{2,r})} \quad \text{if} \quad \alpha\geqslant 1. \tag{8.26}$$

By virtue of (8.18), (8.19) and (8.20), one has also, if $1-\frac{N}{2}<\sigma<1+\alpha+\frac{N}{2}$,

$$\Big(\sum_{q\geqslant -1}\Big(2^{q(\sigma-1)}\|R_q^3\|_{L_T^m(L^2)}\Big)^r\Big)^{\frac{1}{r}} \lesssim \|\widetilde{a}\|_{L_T^\infty(B^{\frac{N}{2}+\alpha}_{2,\infty})}\|\operatorname{div} w\|_{\widetilde{L}_T^m(B^{\sigma-1-\alpha}_{2,r})} \tag{8.27}$$

and

$$\Big(\sum_{q\geqslant -1}\Big(2^{q(\frac{N}{2}+\alpha)}\|R_q^3\|_{L_T^m(L^2)}\Big)^r\Big)^{\frac{1}{r}} \lesssim \|\widetilde{a}\|_{\widetilde{L}_T^\infty(B^{\frac{N}{2}+\alpha}_{2,r})}\|\operatorname{div} w\|_{\widetilde{L}_T^m(B^{\frac{N}{2}}_{2,\infty})\cap L_T^m(L^\infty)}. \tag{8.28}$$

If $\alpha>1$ and $\sigma=1+\alpha+N/2$, one can further use that $\widetilde{L}_T^m(B^{\sigma-2}_{2,r})\hookrightarrow \widetilde{L}_T^m(B^{\frac{N}{2}}_{2,r})\cap L_T^m(L^\infty)$.

In view of (1.1) and according to Bernstein inequality, we have

$$2^{q(\sigma-1)}\|R_q^4\|_{L_T^m(L^2)} \lesssim \sum_{q'\geqslant q-2} 2^{(q-q')(\alpha+\frac{N}{2}-1)}\Big(2^{q'(\alpha+\frac{N}{2})}\|\Delta_{q'}\widetilde{a}\|_{L_T^\infty(L^2)}\Big)\Big(2^{q(\sigma-\alpha-\frac{N}{2})}\|\Delta_q w\|_{L_T^m(L^\infty)}\Big).$$

Using that convolution maps $\ell^1\star\ell^r$ onto ℓ^r, we easily conclude that, if $\alpha-1+N/2>0$,

$$\Big(\sum_{q\geqslant -1}\Big(2^{q(\sigma-1)}\|R_q^4\|_{L_T^m(L^2)}\Big)^r\Big)^{\frac{1}{r}} \lesssim \|\widetilde{a}\|_{L_T^\infty(B^{\frac{N}{2}+\alpha}_{2,\infty})}\|w\|_{\widetilde{L}_T^m(B^{\sigma-\alpha-\frac{N}{2}}_{\infty,r})}. \tag{8.29}$$

For bounding R_q^5, we use the decomposition

$$R_q^5=\sum_{|q'-q|\leqslant 4}[\Delta_q,\Delta_{-1}a]\Delta_{q'}\operatorname{div} w,$$

whence, by virtue of the first order Taylor's formula,

$$\Big(\sum_{q\geqslant -1} 2^{rq(\sigma-1)}\|R_q^5\|_{L_T^m(L^2)}^r\Big)^{\frac1r}\lesssim \|\nabla\Delta_{-1}a\|_{L_T^\infty(L^\infty)}\|w\|_{\widetilde L_T^m(B^{\sigma-1}_{2,r})}. \tag{8.30}$$

Finally, it is straightforward that

$$\Big(\sum_{q\geqslant -1} 2^{rq(\sigma-1)}\|R_q^6\|_{L_T^m(L^2)}^r\Big)^{\frac1r}\lesssim \|\nabla\Delta_{-1}a\|_{L_T^\infty(L^\infty)}\|w\|_{\widetilde L_T^m(B^{\sigma-1}_{2,r})}. \tag{8.31}$$

Combining (8.23), (8.25) or 8.26), (8.27) or (8.28), (8.29), (8.30), (8.31) and (8.22) yields the desired inequality. □

Remark 8.10. — Under the same assumptions on s and α, one can easily prove the following stationary estimate:

$$\Big\|2^{q(\sigma-1)}\|\Delta_q(a\,\mathrm{div}\,w)-\mathrm{div}(a\Delta_q w)\|_{L^2}\Big\|_{\ell^r}\lesssim\|\nabla a\|_{B^{\frac N2+\alpha-1}_{2,r}}\|w\|_{B^{\sigma-\alpha\vee 1}_{2,r}}$$

with the same modifications in the endpoint cases as in proposition 8.9.

Lemma 8.11. — *Let v be a solenoidal vector field. There exists a constant $C=C_{\sigma,N}$ such that the following estimates hold true:*

$$\begin{aligned}\Big(\textstyle\sum_q 2^{2q\sigma}\|[v^j,\Delta_q]\partial_j u\|_{L_T^1(L^2)}^2\Big)^{\frac12} &\lesssim \int_0^T\|\nabla v(t)\|_{L^\infty\cap B^{\frac N2}_{2,\infty}}\|\nabla u(t)\|_{H^{\sigma-1}}\,dt\\ &\quad\text{if}\quad |\sigma|<1+\tfrac N2,\\ \Big(\textstyle\sum_q 2^{2q\sigma}\|[v^j,\Delta_q]\partial_j u\|_{L_T^1(L^2)}^2\Big)^{\frac12} &\lesssim \int_0^T\|\nabla v(t)\|_{H^{\sigma-1}}\|\nabla u(t)\|_{H^{\sigma-1}}\,dt\\ &\quad\text{if}\quad \sigma>1+\tfrac N2.\end{aligned}$$

Besides, if $v=u$, for all $\sigma>-1$ holds

$$\Big(\sum_q 2^{2q\sigma}\|[u^j,\Delta_q]\partial_j u\|_{L_T^1(L^2)}^2\Big)^{\frac12}\lesssim\int_0^T\|\nabla u(t)\|_{L^\infty}\|\nabla u(t)\|_{H^{\sigma-1}}\,dt.$$

Proof. — We proceed as for proving lemma A.1 in [8]. Let $\widetilde u\overset{\text{def}}{=}u-\Delta_{-1}u$ and $\widetilde v\overset{\text{def}}{=}v-\Delta_{-1}v$. Note that for all $\tau\in\mathbb R$, $1\leqslant p,r\leqslant+\infty$, we have

$$\|\widetilde u\|_{B^\tau_{p,r}}\lesssim\|\nabla u\|_{B^{\tau-1}_{p,r}}\quad\text{and}\quad\|\widetilde v\|_{B^\tau_{p,r}}\lesssim\|\nabla v\|_{B^{\tau-1}_{p,r}}. \tag{8.32}$$

Split the commutator into six parts: $[v^j, \Delta_q]\partial_j u = \sum_{i=1}^6 R_q^i$ with

$$
\begin{array}{rclcl}
R_q^1 & = & [T_{\widetilde{v}^j}, \Delta_q]\partial_j u & = & \sum_{|q'-q|\leqslant 4}[S_{q'-1}\widetilde{v}^j, \Delta_q]\Delta_{q'}\partial_j u,\\
R_q^2 & = & T'_{\Delta_q\partial_j u}\widetilde{v}^j & = & \sum_{q'\geqslant q-2} S_{q'+2}\Delta_q\partial_j u\; \Delta_{q'}\widetilde{v}^j,\\
R_q^3 & = & -\Delta_q T_{\partial_j u}\widetilde{v}^j & = & \sum_{|q'-q|\leqslant 4}\Delta_q\big(S_{q'-1}\partial_j u\; \Delta_{q'}\widetilde{v}^j\big),\\
R_q^4 & = & -\Delta_q\partial_j R(\widetilde{u}, \widetilde{v}^j) & = & \sum_{q'\geqslant q-3}\Delta_q\partial_j\Big(\Delta_{q'}\widetilde{u}\,\widetilde{\Delta}_{q'}\widetilde{v}^j\Big),\\
R_q^5 & = & [\Delta_{-1}v^j, \Delta_q]\partial_j u & = & \sum_{|q'-q|\leqslant 2}[\Delta_{-1}v^j, \Delta_q]\partial_j\Delta_{q'}u,\\
R_q^6 & = & -\Delta_q R(\partial_j\Delta_{-1}u, \widetilde{v}^j) & = & \sum_{q'\leqslant 1}\Delta_q\Big(\partial_j\widetilde{\Delta}_{q'}\Delta_{-1}u\; \Delta_{q'}\widetilde{v}^j\Big),
\end{array}
$$

and denote

$$
\mathcal{R}^i \overset{\text{def}}{=} \Big(\sum_q 2^{2q\sigma}\|R_q^i\|^2_{L^1_T(L^2)}\Big)^{\frac{1}{2}}.
$$

In the light of first order Taylor's formula, we have

$$
\mathcal{R}^1 \lesssim \sum_{i=-4}^{4}\Bigg[\sum_q 2^{2q(\sigma-1)}\Big(\int_0^T \big\|\nabla S_{q+i-1}\widetilde{v}^j\big\|_{L^\infty}\|\Delta_{q+i}\partial_j u\|_{L^2}\, dt\Big)^2\Bigg]^{\frac{1}{2}}.
$$

Now, as $\big\|\nabla S_{q+i-1}\widetilde{v}^j\big\|_{L^\infty} \lesssim \|\nabla v\|_{L^\infty}$, Minkowski inequality entails

$$
\mathcal{R}^1 \lesssim \int_0^T \|\nabla v(t)\|_{L^\infty}\|\nabla u(t)\|_{H^{\sigma-1}}\, dt. \tag{8.33}
$$

We have

$$
\mathcal{R}^2 \lesssim \Bigg[\sum_q\Big(\int_0^T\sum_{q'\geqslant q-2} 2^{q'(\frac{N}{2}+1)}\|\Delta_{q'}\widetilde{v}\|_{L^2}\; 2^{q(\sigma-\frac{N}{2}-1)}\|\Delta_q\nabla u\|_{L^\infty}\; 2^{(q-q')(\frac{N}{2}+1)}dt\Big)^2\Bigg]^{\frac{1}{2}}.
$$

Minkowski and convolution inequalities enable us to get

$$
\mathcal{R}^2 \lesssim \int_0^T \|\nabla v(t)\|_{B^{\frac{N}{2}}_{2,\infty}}\|\nabla u(t)\|_{B^{\sigma-1-\frac{N}{2}}_{\infty,2}}\, dt. \tag{8.34}
$$

In the particular case $u = v$, using Bernstein inequality, one can rather write that

$$
\mathcal{R}^2 \lesssim \Bigg[\sum_q\Big(\int_0^T\sum_{q'\geqslant q-2} 2^{q-q'}\, 2^{q(\sigma-1)}\,\|\Delta_q\nabla u\|_{L^2}\; 2^{q'\sigma}\,\|\Delta_{q'}\nabla\widetilde{u}\|_{L^\infty}\, dt\Big)^2\Bigg]^{\frac{1}{2}}
$$

whence

$$
\mathcal{R}^2 \lesssim \int_0^T \|\nabla u(t)\|_{B^0_{\infty,\infty}}\|\nabla u(t)\|_{H^{\sigma-1}}\, dt. \tag{8.35}
$$

Next, we have

$$\mathcal{R}^3 \lesssim \left[\sum_q \left(\sum_{q'\leqslant q-2} \int_0^T 2^{(q-q')(\sigma-\frac{N}{2}-1)} 2^{q'(\sigma-\frac{N}{2}-1)} \|\Delta_{q'}\partial_j u\|_{L^\infty} \, 2^{q(\frac{N}{2}+1)} \left\|\Delta_q \widetilde{v}^j\right\|_{L^2} dt\right)^2\right]^{\frac{1}{2}}.$$

If $\sigma < 1 + N/2$, Minkowski and convolution inequalities yield

$$\mathcal{R}^3 \lesssim \int_0^T \|\nabla v(t)\|_{B^{\frac{N}{2}}_{2,\infty}} \|\nabla u(t)\|_{B^{\sigma-1-\frac{N}{2}}_{\infty,2}} \, dt. \tag{8.36}$$

In the case $\sigma > 1 + N/2$ or $u = v$, one can alternately get

$$\mathcal{R}^3 \lesssim \int_0^T \|\nabla v(t)\|_{H^{\sigma-1}} \|\nabla u(t)\|_{L^\infty} \, dt. \tag{8.37}$$

For bounding $\mathcal{R}^4$, we first use Bernstein inequality, which yields

$$\mathcal{R}^4 \lesssim \left[\sum_q \left(\sum_{q'\geqslant q-3} \int_0^T 2^{(q-q')(\frac{N}{2}+\sigma+1)} \, 2^{q'\sigma} \|\Delta_{q'}\widetilde{u}\|_{L^2} \, 2^{q'(\frac{N}{2}+1)} \left\|\widetilde{\Delta}_{q'}\widetilde{v}\right\|_{L^2} \, dt\right)^2\right]^{\frac{1}{2}},$$

hence, if $\sigma > -\frac{N}{2} - 1$,

$$\mathcal{R}^4 \lesssim \int_0^T \|\nabla v(t)\|_{B^{\frac{N}{2}}_{2,\infty}} \|\nabla u(t)\|_{H^{\sigma-1}} \, dt. \tag{8.38}$$

If $u = v$ and $\sigma > -1$, one can rather write

$$\mathcal{R}^4 \leqslant \left[\sum_q \left(\sum_{q'\geqslant q-3} \int_0^T 2^{(q-q')(\sigma+1)} \, 2^{q'\sigma} \|\Delta_{q'}\widetilde{u}\|_{L^2} \, 2^{q'} \left\|\widetilde{\Delta}_{q'}\widetilde{u}\right\|_{L^\infty} \, dt\right)^2\right]^{\frac{1}{2}},$$

whence, in view of Minkowski inequality and (8.32),

$$\mathcal{R}^4 \lesssim \int_0^T \|\nabla u(t)\|_{B^0_{\infty,\infty}} \|\nabla u(t)\|_{H^{\sigma-1}} \, dt. \tag{8.39}$$

Next, according to first order Taylor's formula, we have

$$\left\|[\Delta_{-1}v^j, \Delta_q]\Delta_{q'}\partial_j u\right\|_{L^2} \lesssim 2^{-q} \|\nabla\Delta_{-1}v\|_{L^\infty} \|\nabla\Delta_{q'}u\|_{L^2}.$$

Therefore, in the light of Minkowski inequality,

$$\mathcal{R}^5 \lesssim \int_0^t \|\Delta_{-1}\nabla v(t)\|_{L^\infty} \|\nabla u(t)\|_{H^{\sigma-1}} \, dt. \tag{8.40}$$

Finally, as R_q^6 vanishes for $q > 3$, we easily get

$$\mathcal{R}^6 \lesssim \int_0^t \|\nabla v(t)\|_{B^0_{\infty,\infty}} \|\Delta_{-1}\nabla u(t)\|_{L^2}\, dt. \tag{8.41}$$

Combining inequalities (8.33), (8.34), (8.36), (8.38), (8.40) and (8.41) with elementary embeddings yields the desired estimates. If in addition $u = v$, one can use inequalities (8.33), (8.35), (8.37), (8.39), (8.40) and (8.41). The proof of lemma 8.11 is complete. □

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