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Classical Poincaré metric pulled back off singularities using a Chow-type theorem and desingularization[(*)]

Caroline Grant Melles[(1)], Pierre Milman[(2)]

Abstract. — We construct complete Kähler metrics on the nonsingular set of a subvariety X of a compact Kähler manifold. To that end, we develop (i) a constructive method for replacing a sequence of blow-ups along smooth centers, with a single blow-up along a product of coherent ideals corresponding to the centers and (ii) an explicit local formula for a Chern form associated to this 'singular' blow-up. Our metrics have a particularly simple local formula of a sum of the original metric and of the pull back of the classical Poincaré metric on the punctured disc by a 'size-function' S_I of a coherent ideal I used to resolve the singularities of X by a 'singular' blow-up, where $(S_I)^2 := \sum_{j=1}^{r} \mid f_j \mid^2$ and the f_j's are the local generators of the ideal I . Our proof of (i) makes use of our generalization of Chow's theorem for coherent ideals. We prove Saper type growth for our metric near the singular set and local boundedness of the gradient of a local generating function for our metric, motivated by results of Donnelly-Fefferman, Ohsawa, and Gromov on the vanishing of certain L_2-cohomology groups.

Résumé. — Nous construisons des métriques complètes Kähleriennes sur le lieu non-singulier d'une sous-variété X d'une variété compacte Kählerienne lisse. A cet effet, nous développons : (i) une méthode constructive pour le remplacement d'une suite d'éclatements le long des centres lisses par un seul éclatement le long d'un produit d'idéaux cohérents et (ii) une formule locale explicite pour une forme de Chern associée à cet éclatement. Nos métriques sont décrites par une formule locale particulièrement simple comme la somme de la métrique de départ et le tire-en-arrière de la métrique de Poincaré classique sur le disque épointé par une

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'fonction de grandeur' S_I de l'idéal cohérent I utilisé pour la résolution des singularités de X, ou $(S_I)^2 := \sum_{j=1}^{r} | f_j |^2$ et les f_j sont des générateurs locaux de I. Notre preuve de (i) utilise notre généralisation du théorème de Chow pour les idéaux cohérents. Nous montrons que la vitesse de croissance de notre métrique près du lieu singulier est de type Saper ainsi que le fait que le gradient d'une fonction génératrice locale de notre métrique est borné. Cela est motivé par les résultats de Donnelly-Fefferman, Ohsawa, et Gromov sur l'annulation de certains groupes de cohomologie L_2.

0. Introduction

Let X be a singular subvariety of a compact Kähler manifold M. In [GM1] we showed how to construct a particular type of complete Kähler metric on the nonsingular set of X. These metrics grow less rapidly than Poincaré metrics near the singular set X_{sing} of X (cf. Example 9.8), and are of interest because in certain cases it is known that their L_2-cohomology equals the intersection cohomology of X, while the L_2-cohomology of a Poincaré metric is not equal to the intersection cohomology of X, but rather to the cohomology of the desingularization of X ([Zu1], [Zu2]). We called our metrics Saper-type or modified Saper metrics after Leslie Saper, who first drew our attention to this subject. Saper proved that on any variety with isolated singularities there is a complete Kähler metric whose L_2-cohomology equals its intersection cohomology (see [Sa1], [Sa2]). We show that our metrics are locally quasi-isometric to metrics satisfying a boundedness condition of Ohsawa's: the gradient of a generating function is locally bounded with respect to the metric. Our construction requires no restriction on the type of singularities and relates directly the desingularizations of X by means of a 'singular' blow-up with certain complete Kähler metrics of Saper-type growth near X_{sing} and satisfying Ohsawa's boundedness condition for a local generating function. Below we introduce these metrics explicitly by a particularly simple formula involving the pull back of the Poincaré metric on the punctured disc by a 'size-function' of a coherent ideal used to resolve the singularities of X by a 'singular' blow-up.

The construction of Saper-type metrics in [GM1] used the geometry of a finite sequence of blow-ups along smooth centers which resolves the singularities of X. In this paper we show how to replace a finite sequence of blow-ups along smooth centers by a single blow-up along one center (perhaps singular), which we describe in terms of its coherent sheaf of ideals $\mathcal{I}$. Hironaka and Rossi proved in [HR] that there is such an ideal sheaf. We give a constructive proof, using our version of Chow's Theorem for ideals, which we prove using the Direct Image Theorem (for a blow-up along a smooth center). Blowing up M along $\mathcal{I}$ desingularizes X. The support of $\mathcal{I}$ is the singular locus X_{sing} of X. The ideal sheaf $\mathcal{I}$ is a product of coherent ideals $\mathcal{I}_j$ corresponding to the smooth centers C_j. Each $\mathcal{I}_j$ is the direct image on M of a product of the ideal sheaf of C_j with a sufficiently high power of the exceptional ideal of the previous blow-ups. In practice, the calculation of $\mathcal{I}$ may be quite explicit: see for example the algorithm of [GM2] for combinatorial blow-ups, which may be applied to desingularization of toric varieties by [BM2]. We then give a simple and explicit construction of a Chern form associated to the blow-up along $\mathcal{I}$, in terms of local generators of $\mathcal{I}$. Finally we use this Chern form to obtain a simpler and more explicit expression for our Saper-type metrics. We also give an example in which we compute $\mathcal{I}$ explicitly in a neighborhood of a singular point.

The Saper-type metric which we obtain can be described in terms of its Kähler (1,1)-form as

$$\omega_S = \omega - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\log F)^2,$$

where ω is the Kähler (1,1)-form of a Kähler metric on M, and F is a C^∞ function on M, vanishing on X_{sing}. We first construct local C^∞ functions F_α, on small open sets U_α in M, by setting

$$F_\alpha = \sum_{j=1}^{r} \mid f_j \mid^2$$

where $f_1, ..., f_r$ are local holomorphic generating functions on U_α for the coherent ideal sheaf $\mathcal{I}$ described above. To construct a global metric on $M - X_{\text{sing}}$ (and consequently on $X - X_{\text{sing}}$), we patch with a C^∞ partition of unity on M. It is crucial that this patching takes place on M, rather than on a blow-up of M (cf. [GM1]), which might add unwanted elements to the L_2-cohomology.

In an appendix we give a simple constructive proof of a valuation criterion due to M. Lejeune and B. Teissier.

We are grateful to the two referees who have read this paper and given us helpful suggestions. A preliminary version of this paper appeared as [GM3].

Table of contents

1. Outline and main results

In sections 2 and 3 we give some background and basic results about coherent sheaves of ideals and blow-ups. We begin by describing the direct and inverse images of sheaves, and in particular, direct and inverse images of coherent sheaves of ideals. Then we describe the blow-up $\pi : \tilde{M} \to M$ of a complex manifold M along a coherent sheaf of ideals $\mathcal{I}$. The analytic subset

$C = V(\mathcal{I})$ of M determined by $\mathcal{I}$ is called the center of the blow-up. If C is smooth and of codimension at least 2, then $\tilde{M}$ is smooth. The blow-up map π is proper and is a biholomorphism except along its exceptional divisor $E = \pi^{-1}(C)$. Even though the direct image of an ideal sheaf may not be an ideal sheaf in general, the direct image of an ideal sheaf under a blow-up map **is** an ideal sheaf.

Section 4 is devoted to a proof of our version of the Chow's Theorem that we state below, using the Direct Image Theorem (for the blow-up map of $U \times \mathbb{C}^{n+1}$ along the smooth center $U \times \{0\}$), which states that the direct image of a coherent sheaf under a proper map is coherent. Section 5 contains some corollaries for blow-up maps which are useful in constructing single-step blow-ups from a sequence of blow-up maps.

Chow's Theorem for Ideals. — *Let U be an open neighborhood of* 0 *in $\mathbb{C}^m$ and let X be an analytic subset of $U \times \mathbb{P}^n$. Let $\mathcal{J}$ be a coherent sheaf of ideals on X. Then $\mathcal{J}$ is* ***relatively algebraic*** *in the following sense: $\mathcal{J}$ is generated (after shrinking U if necessary) by a finite number of homogeneous polynomials in homogeneous $\mathbb{P}^n$-coordinates, with analytic coefficients in U-coordinates.*

Chow's Theorem for Ideals helps to describe the relatively algebraic structure of blow-ups. Most useful for the purposes of this paper is the following corollary, which shows that, even though the inverse image of the direct image of an ideal sheaf may not be the original ideal sheaf in general, on a blow-up of a compact complex manifold we can ensure that the two are equal by first multiplying by a high enough power of the ideal sheaf $\mathcal{I}_E$ of the exceptional divisor. This result was proved by Hironaka and Rossi in [HR] but our proof is constructive in nature and is substantially simpler, being more explicit in the methods used. We also describe the relationship between local generators of the sheaves $\mathcal{J}_2$ and $\mathcal{J}_2\mathcal{I}_E^d$ on $\tilde{M}$. We go on to apply this corollary repeatedly to get an explicit description of a coherent sheaf for single-step blow-ups, as a product of coherent sheaves corresponding to a sequence of blow-ups along smooth centers.

Corollary 1.1. — *Let $\pi : \tilde{M} \to M$ be the blow-up of a compact complex manifold M along a coherent sheaf of ideals $\mathcal{J}_1$ and let E be the exceptional divisor of π. Let $\mathcal{J}_2$ be a coherent sheaf of ideals on $\tilde{M}$. Then there exists an integer d_0 such that*

$$\pi^{-1}\pi_*(\mathcal{J}_2\mathcal{I}_E^d) = \mathcal{J}_2\mathcal{I}_E^d$$

for all $d \geqslant d_0$.

We end section 5 with a simple proof of a valuation criterion due to M. Lejeune and B. Teissier, which illustrates the methods developed in sections 4 and 5 and applied similarly in section 6.

For the purposes of this paper and to apply Hironaka's theorem on embedded resolution of singularities, we need only to consider blow-ups of smooth spaces. If the blow-up $\tilde{M}$ (of M along $\mathcal{J}_1$) is smooth, the blow-up of $\tilde{M}$ along $\mathcal{J}_2$ is isomorphic to the blow-up of $\tilde{M}$ along $\mathcal{J}_2\mathcal{I}_E^d$. Furthermore, the blow-up of $\tilde{M}$ along $\mathcal{J}_2\mathcal{I}_E^d$ is isomorphic to the blow-up of the base space M along $\mathcal{J}_1\pi_*(\mathcal{J}_2\mathcal{I}_E^d)$. Thus we can replace the pair of blow-ups, first along $\mathcal{J}_1$ and then along $\mathcal{J}_2$, by a single blow-up along $\mathcal{J}_1\pi_*(\mathcal{J}_2\mathcal{I}_E^d)$. Repeating this procedure for a finite sequence of smooth centers enables us to construct a coherent sheaf of ideals $\mathcal{I}$ on M such that blowing up M along $\mathcal{I}$ is equivalent to blowing up successively along smooth centers. Section 6 contains a more detailed version of the proof of the following proposition, which was proved by Hironaka and Rossi in [HR]. This result is also related to Theorem II.7.17 of [Ha1]. The method of construction of $\mathcal{I}$ is of interest in itself, because in practice it may be quite explicit and algorithmic, as for example, for combinatorial blow-ups for desingularization of toric varieties (see [GM2] and [BM2]).

Proposition 1.2 (Single-Step Blow-ups). — *Let M be a compact complex manifold and let*

$$M_m \xrightarrow{\pi_m} M_{m-1} \to \dots \to M_2 \xrightarrow{\pi_2} M_1 \xrightarrow{\pi_1} M_0 = M$$

be a finite sequence of blow-ups along smooth centers $C_j \subset M_{j-1}$ of codimension at least 2. Then there is a coherent sheaf of ideals $\mathcal{I}$ on M such that the blow-up of M along $\mathcal{I}$ is isomorphic to the blow-up of M along the sequence of smooth centers C_j. Furthermore, we may construct $\mathcal{I}$ to be of the form

$$\mathcal{I} = \mathcal{I}_1\mathcal{I}_2 \dots \mathcal{I}_m,$$

where each $\mathcal{I}_j$ is a coherent sheaf of ideals on M and

i. *$\mathcal{I}_j$ is the direct image on M of the ideal sheaf of C_j multiplied by a high enough power of the ideal sheaf of the exceptional divisor of the first $j-1$ blow-ups,*

ii. *the inverse image of $\mathcal{I}_j$ on M_{j-1} is the ideal sheaf of C_j multiplied by the same power of the exceptional ideal sheaf as in (i), and*

iii. *the blow-up of M_{j-1} along the inverse image of $\mathcal{I}_j$ is isomorphic to the blow-up of M_{j-1} along C_j.*

We are particularly interested in the case of a sequence of blow-ups along smooth centers which resolves the singularities of a singular subvariety X of M. In this case, the proposition gives us a coherent ideal sheaf $\mathcal{I}$ on M, supported on the singular locus of X, such that blowing up along $\mathcal{I}$ desingularizes X, and also gives a factorization of $\mathcal{I}$ in terms of ideals corresponding to the original sequence of blow-ups. This factorization of $\mathcal{I}$ is essentially unique for curves ([ZS], Appendix 5).

In section 7 we give a simple and explicit construction of a Chern form associated to a blow-up. Suppose that $\pi : \tilde{M} \to M$ is the blow-up of a complex manifold M along a coherent sheaf of ideals $\mathcal{I}$ such that $\tilde{M}$ is smooth. Let E be the exceptional divisor and L_E the line bundle on $\tilde{M}$ associated to E. Let $f_1, ..., f_r$ be local holomorphic generating functions for $\mathcal{I}$ on a small open set $U \subset M$. We show that there is a hermitian metric h, on the restriction of L_E to $\tilde{U} = \pi^{-1}(U)$, whose Chern form may be constructed as the pullback of the negative of a Fubini-Study form on projective space. This Chern form is given on $\tilde{U} - \tilde{U} \cap E$ by

$$c_1(L_E, h) = \pi^*(-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\sum_{j=1}^{r} \mid f_j \mid^2).$$

If M is compact, we may patch together local Chern forms using a C^∞ partition of unity on M, in such a way that the negativity on fibres is preserved.

Now consider in more detail a singular subvariety X of a compact Kähler manifold M. Hironaka's famous theorem on embedded desingularization (and its canonical version in [BM1]) tell us that the singularities of X may be resolved by a finite sequence of blow-ups of M along smooth centers, such that the total exceptional divisor of the composite of all the blow-ups is a normal crossings divisor D in $\tilde{M}$ which has normal crossings with the desingularization $\tilde{X}$ in $\tilde{M}$ and such that $\tilde{M} - D \cong M - X_{\text{sing}}$ and $\tilde{X} - \tilde{X} \cap D \cong X - X_{\text{sing}}$. By the Single-Step Blow-up Proposition, we may resolve the singularities of X by blowing up M along a single coherent sheaf of ideals $\mathcal{I}$ on M, whose blow-up is isomorphic to the blow-up obtained using the sequence of smooth centers. The inverse image ideal sheaf of $\mathcal{I}$ in the blow-up $\tilde{M}$ determines the normal crossings divisor D and the support of $\mathcal{I}$ in M is X_{sing}. We construct a Chern form for the line bundle L_D corresponding to the blow-up along $\mathcal{I}$, using local holomorphic generating functions of $\mathcal{I}$ as above and patching with a C^∞ partition of unity on M. We show that subtracting this Chern form from the Kähler (1,1)-form of a Kähler metric on M gives the (1,1)-form of a Kähler metric on $\tilde{M}$, our "desingularizing metric." The completion of $X - X_{\text{sing}}$ with respect to this metric is nonsingular.

We say that two metrics on an open set U are quasi-isometric if their fundamental $(1,1)$-forms ω_A and ω_B satisfy $c\omega_A \leqslant \omega_B \leqslant C\omega_A$ on U, for some positive constants c and C. We call a metric on $\tilde{M}-D$ a modified Saper or Saper-type metric if it is quasi-isometric to a metric with fundamental $(1,1)$-form

$$l\pi^*\omega - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\log || \; s \; ||^2)^2,$$

where ω is the fundamental $(1,1)$-form of a metric on M, l is a positive integer, s is a global holomorphic section of the line bundle L_D whose vanishing set is D, and $|| \; s \; ||$ is the norm of s with respect to a metric h on L_D. The corresponding metric on $M - X_{\text{sing}} \cong \tilde{M} - D$ and its restriction to $X - X_{\text{sing}}$ are also called Saper-type or modified Saper. In [GM1], a more general class of modified Saper metrics is discussed and the growth rates of such metrics are studied in more detail.

Our main theorem on desingularizing and Saper-type metrics is proved in section 8.

Theorem 1.3. — *Let X be a singular subvariety of a compact Kähler manifold M and let ω be the Kähler (1,1)-form of a Kähler metric on M. Then there exists a C^∞ function F on M, vanishing on X_{sing}, such that for k a large enough positive integer,*

i. *the (1,1)-form*

$$\tilde{\omega} = k\omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log F$$

is the Kähler form of a desingularizing Kähler metric for X, i.e. the completion of $X - X_{\text{sing}}$ with respect to $\tilde{\omega}$ is a desingularization of X and

ii. *the (1,1)-form*

$$\omega_S = \omega - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\log F)^2$$

is the Kähler form of a complete Kähler Saper-type metric on $M - X_{\text{sing}}$ and hence on $X - X_{\text{sing}}$.

Furthermore, the function F may be constructed to be of the form

$$F = \prod_\alpha F_\alpha^{\rho_\alpha},$$

where $\{\rho_\alpha\}$ *is a* C^∞ *partition of unity subordinate to an open cover* $\{U_\alpha\}$ *of* M*,* F_α *is a function on* U_α *of the form*

$$F_\alpha = \sum_{j=1}^{r} \mid f_j \mid^2,$$

and $f_1, ..., f_r$ *are holomorphic functions on* U_α*, vanishing exactly on* $X_{\text{sing}} \cap U_\alpha$*. More specifically,* $f_1, ..., f_r$ *are local holomorphic generators of a coherent ideal sheaf* $\mathcal{I}$ *on* M *such that blowing up* M *along* $\mathcal{I}$ *desingularizes* X*,* $\mathcal{I}$ *is supported on* X_{sing}*, and the exceptional divisor in the blow-up* $\tilde{M}$ *along* $\mathcal{I}$ *has normal crossings and is also normal crossings with the strict transform* $\tilde{X}$ *of* X *in* $\tilde{M}$ *(the so-called embedded desingularization of* X*).*

The coherent ideal sheaf $\mathcal{I}$ is constructed as a product $\mathcal{I}_1\mathcal{I}_2...\mathcal{I}_m$ of coherent ideal sheaves corresponding to a sequence of blow-ups along smooth centers C_j which resolves the singularities of X. This factorization of $\mathcal{I}$ gives a corresponding factorization of F_α, as essentially a product of distances to the centers,

$$F_\alpha = \prod_{j=1}^{m} \sum_{i=1}^{r_j} \mid v_{ji} \mid^2$$

where, for each j, the functions $\{v_{ji}\}$ are local holomorphic functions on U_α whose pullbacks to the preimage of U_α under the first $j-1$ blow-ups generate an ideal sheaf with the same blow-up as C_j.

The idea behind the metric constructions in this paper is to first find simple and explicit formulas locally on M, then patch by C^∞ partitions of unity on M. We wish to avoid formulas which are local only on blow-ups of M and we also wish to avoid introducing C^∞ partition-of-unity functions on the blow-ups (unlike in our previous work [GM1]).

We prove that, locally on M, our Saper-type metrics are quasi-isometric to metrics satisfying a boundedness criterion of Ohsawa's: the gradient of a generating function of the fundamental (1,1)-form of the metric is locally bounded with respect to the metric. In view of results of Donnelly-Fefferman [DF], Ohsawa [O], and Gromov [Gro] on vanishing of certain L_2-cohomology groups, we hope (and expect) that this property would allow one to apply Goresky-MacPherson's work on the axiomatic characterization of intersection cohomology for the purpose of identification of the latter (for the middle perversity) with the L_2-cohomology groups for our Saper-type metrics.

We conclude, in section 10, by constructing $\mathcal{I}$ for the cuspidal cubic $y^2 - x^3$. The method used generalizes to the case of any singular locally toric complex analytic variety (see [GM2] and [BM2]).

2. Direct and inverse images of coherent sheaves of ideals

Coherent Sheaves

We first review the important concept of coherence (see e.g. [GrR1], [GuR]).

Let M be a complex space and let $\mathcal{S}$ be an analytic sheaf on M, i.e. a sheaf of $\mathcal{O}_M$-modules. For example, consider an ideal sheaf of $\mathcal{O}_M$ or the sheaf of holomorphic sections of a holomorphic vector bundle on M.

Definition 2.1. — *The sheaf $\mathcal{S}$ is* ***of finite type at*** $x \in M$ *if there exists an open set U of x such that the restriction $\mathcal{S}_U$ of $\mathcal{S}$ to U is generated by a finite number of sections of $\mathcal{S}$ over U. This means that there exist sections $s_1, ..., s_r$ of $\mathcal{S}$ over U such that for each point $y \in U$ and for each germ $g_y \in \mathcal{S}_y$, there exist $a_{1y}, ..., a_{ry} \in \mathcal{O}_{M,y}$ such that*

$$g_y = \sum_{i=1}^{r} a_{iy} s_{iy}.$$

The sheaf $\mathcal{S}$ is ***of finite type on*** M *if $\mathcal{S}$ is of finite type at x for all $x \in M$.*

Remark 2.2. — Note that if s and t are sections of $\mathcal{S}$ on a neighborhood of a point y such that $s_y = t_y$ (i.e. they have the same germs at y), then $s = t$ in an open neighborhood of y, by fundamental properties of sheaves. In particular, in the definition above, if $g_y, a_{1y}, ..., a_{ry}$ are the germs of $g, a_1, ..., a_r$ at y then there exists a neighborhood $V \subset U$ of y such that

$$g = \sum_{i=1}^{r} a_i s_i$$

on V.

Each finite collection $s = (s_1, ..., s_r)$ of sections of $\mathcal{S}$ over U determines a sheaf homomorphism

$$\psi_s : \mathcal{O}_U^r \to \mathcal{S}_U$$

given by

$$(f_1, ..., f_r) \mapsto \sum_{i=1}^{r} f_i s_i.$$

Definition 2.3. — *The sheaf $\mathcal{S}$ is* ***of relation finite type at*** $x \in M$ *if $\ker \psi_s$ is of finite type at x for all finite collections s of sections of $\mathcal{S}$ over an open neighborhood U of x. $\mathcal{S}$ is of* ***relation finite type on*** M *if $\mathcal{S}$ is of relation finite type at x for all $x \in M$.*

DEFINITION 2.4. — *The sheaf $\mathcal{S}$ is* ***coherent on*** *M if*

1. *$\mathcal{S}$ is of finite type on M, and*
2. *$\mathcal{S}$ is of relation finite type on M.*

Since coherent sheaves are always finite type, by definition, it follows that if $\mathcal{S}$ is a coherent sheaf on a complex space X and $s_1, ..., s_r$ are sections of $\mathcal{S}$ on a neighborhood U of a point x such that the germs $s_{1x}, ..., s_{rx}$ generate $\mathcal{S}_x$, then there exists a neighborhood $V \subset U$ of x such that $s_1, ..., s_r$ generate $\mathcal{S}_V$.

We refer the reader to [F], [GrR1], [GrR2], [GuR], and [W] for background on the following and other fundamental properties of coherent sheaves:

i. The sheaf $\mathcal{O}_M$ is coherent.

ii. A subsheaf of a coherent sheaf is coherent if and only if it is of finite type. In particular, an ideal sheaf of $\mathcal{O}_M$ is coherent if and only if it is of finite type.

iii. A coherent ideal sheaf $\mathcal{I}$ on a complex space determines a closed complex analytic subspace $V(\mathcal{I})$, and the ideal sheaf $\mathcal{I}_Y$ of a closed complex analytic subspace Y of a complex space is coherent.

LEMMA 2.5. — *If $\mathcal{I}_1$ and $\mathcal{I}_2$ are coherent sheaves of ideals on a complex space M, then the product ideal sheaf $\mathcal{I}_1\mathcal{I}_2$ is also coherent.*

Proof. — Since both $\mathcal{I}_1$ and $\mathcal{I}_2$ are of finite type, their product is of finite type and is thus coherent. □

We define direct images and inverse images of coherent sheaves of ideals, and give some conditions under which these sheaves are themselves coherent ideal sheaves (in general they may be only sheaves of modules). We show that direct and inverse images of composite maps are composites of the direct and inverse image maps (functoriality). We also show that the inverse image of a product of ideals is the product of the inverse image ideals. Direct and inverse images of ideal sheaves under blow-up maps are discussed in Lemmas 3.9 and 5.7.

Direct images

DIRECT IMAGE. — Let $f : M \to N$ be a holomorphic map of complex spaces and let $\mathcal{S}$ be a sheaf on M. The direct image sheaf $f_*\mathcal{S}$ on N is the sheaf associated with the presheaf given by $f_*\mathcal{S}(U) = \mathcal{S}(f^{-1}(U))$, for U any open set in N.

If $\mathcal{S}$ is coherent, the direct image $f_*\mathcal{S}$ is not necessarily coherent. However $f_*\mathcal{S}$ is coherent if f is proper, by the Direct Image Theorem. We recall the Direct Image Theorem in our context (see e.g. [GrR1], pp 207, 227, and 36).

Direct Image Theorem .— *Let $f : M \to N$ be a holomorphic map of complex spaces and let $\mathcal{S}$ be a coherent sheaf on M. If f is proper then $f_*\mathcal{S}$ is coherent.*

In particular, if f is a blow-up map (see section 3), then f is proper and $f_*\mathcal{S}$ is coherent if $\mathcal{S}$ is.

If $\mathcal{J}$ is a sheaf of ideals on M, then $f_*\mathcal{J}$ is a sheaf of $\mathcal{O}_N$-modules but not, in general, an ideal sheaf on N. We will show (Lemma 3.9) that if f is a blow-up map then $f_*\mathcal{J}$ is an ideal sheaf.

Inverse images

Once again, let $f : M \to N$ be a holomorphic map of complex spaces. Let $\mathcal{S}$ be a sheaf of $\mathcal{O}_N$-modules.

Topological Inverse Image. — We define the topological inverse image $f'\mathcal{S}$ to be the fibre product $\mathcal{S} \times_N M$, i.e. the stalk of $f'\mathcal{S}$ over a point $m \in M$ is the stalk of $\mathcal{S}$ over $f(m) \in N$:

$$(f'\mathcal{S})_m = \mathcal{S}_{f(m)}.$$

Note that $f'\mathcal{S}$ is a sheaf of $f'\mathcal{O}_N$-modules. If $\mathcal{S}$ is coherent then so is $f'\mathcal{S}$.

Pullback Sheaf .— We define the pullback sheaf as

$$f^*\mathcal{S} = f'\mathcal{S} \otimes_{f'\mathcal{O}_N} \mathcal{O}_M.$$

Note that $f^*\mathcal{S}$ is a sheaf of $\mathcal{O}_M$-modules and once again, if $\mathcal{S}$ is coherent then so is $f^*\mathcal{S}$. Also

$$f^*\mathcal{O}_N = f'\mathcal{O}_N \otimes_{f'\mathcal{O}_N} \mathcal{O}_M = \mathcal{O}_M.$$

If $\mathcal{I}$ is an ideal sheaf on N, we have an exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_N.$$

Since tensoring is not in general left exact, the induced map

$$f^*\mathcal{I} \to f^*\mathcal{O}_N = \mathcal{O}_M$$

is not necessarily injective, so $f^*\mathcal{I}$ is not necessarily an *ideal sheaf* on M. The *image* of $f^*\mathcal{I}$ in $\mathcal{O}_M$ is an ideal sheaf, which we call the inverse image ideal sheaf and will describe in more detail later in this section.

Flat Maps. — A holomorphic map $f : M \to N$ of complex spaces is flat if

$$\mathcal{O}_{M,m} \text{ is } \mathcal{O}_{N,f(m)}\text{-flat}$$

for all $m \in M$. Equivalently, f is flat if for every exact sequence

$$0 \to S_1 \to S_2$$

of $\mathcal{O}_{N,f(m)}$-modules, the induced sequence

$$0 \to S_1 \otimes_{\mathcal{O}_{N,f(m)}} \mathcal{O}_{M,m} \to S_2 \otimes_{\mathcal{O}_{N,f(m)}} \mathcal{O}_{M,m}$$

is also exact.

There are many references on flat maps, e.g. ([F], p. 147 and p. 155).

Example 2.6. — If X and Y are complex spaces, the canonical projection $X \times Y \to Y$ is flat. Every locally trivial holomorphic map is flat. In particular, if $f : L \to X$ is a line bundle over a complex space X (or more generally, a vector bundle), then f is flat.

Lemma 2.7. — *If $f : M \to N$ is a flat holomorphic map of complex spaces and $0 \to \mathcal{S}_1 \to \mathcal{S}_2$ is an exact sequence of sheaves of $\mathcal{O}_N$-modules, then $0 \to f^*\mathcal{S}_1 \to f^*\mathcal{S}_2$ is an exact sequence of sheaves of $\mathcal{O}_M$-modules.*

Proof. — Suppose that

$$0 \to \mathcal{S}_1 \to \mathcal{S}_2$$

is an exact sequence of sheaves of $\mathcal{O}_N$-modules, i.e.

$$0 \to \mathcal{S}_{1,n} \to \mathcal{S}_{2,n}$$

is an exact sequence of $\mathcal{O}_{N,n}$-modules for each $n \in N$. Then in particular,

$$0 \to \mathcal{S}_{1,f(m)} \to \mathcal{S}_{2,f(m)}$$

is an exact sequence of $\mathcal{O}_{N,f(m)}$-modules for all $m \in M$. If $f : M \to N$ is flat, then

$$0 \to S_{1,f(m)} \otimes_{\mathcal{O}_{N,f(m)}} \mathcal{O}_{M,m} \to S_{2,f(m)} \otimes_{\mathcal{O}_{N,f(m)}} \mathcal{O}_{M,m}$$

is exact for all $m \in M$, i.e.

$$0 \to (f'\mathcal{S}_1)_m \otimes_{(f'\mathcal{O}_N)_m} \mathcal{O}_{M,m} \to (f'\mathcal{S}_2)_m \otimes_{(f'\mathcal{O}_N)_m} \mathcal{O}_{M,m}$$

is exact for all $m \in M$. These tensor products can be rewritten as

$$0 \to (f'\mathcal{S}_1 \otimes_{f'\mathcal{O}_N} \mathcal{O}_M)_m \to (f'\mathcal{S}_2 \otimes_{f'\mathcal{O}_N} \mathcal{O}_M)_m,$$

showing that

$$0 \to f'\mathcal{S}_1 \otimes_{f'\mathcal{O}_N} \mathcal{O}_M \to f'\mathcal{S}_2 \otimes_{f'\mathcal{O}_N} \mathcal{O}_M$$

is exact. By the definition of f^*, this means that

$$0 \to f^*\mathcal{S}_1 \to f^*\mathcal{S}_2$$

is exact. □

Lemma 2.8. — *If $\mathcal{L}$ is the sheaf of holomorphic sections of a line bundle (or more generally of a vector bundle) over a complex space M, and*

$$0 \to \mathcal{S}_1 \to \mathcal{S}_2$$

is an exact sequence of sheaves of $\mathcal{O}_M$-modules, then

$$0 \to \mathcal{S}_1 \otimes \mathcal{L} \to \mathcal{S}_2 \otimes \mathcal{L}$$

is also exact.

Proof. — A finitely generated module over a local noetherian ring is flat if and only if it is free ([Ma], Proposition 3.G, p. 21). Therefore $\otimes_{\mathcal{O}_{M,m}} \mathcal{L}_m$ preserves exact sequences. □

Inverse Image Ideal . — Let $f : M \to N$ be a holomorphic map of complex spaces. If $\mathcal{I}$ is an ideal sheaf on N, the image of $f^*\mathcal{I}$ in $\mathcal{O}_M$ is an ideal sheaf which we define to be the inverse image ideal sheaf $f^{-1}\mathcal{I}$.

The ideal sheaf $f^{-1}\mathcal{I}$ is sometimes written $f^*\mathcal{I} \cdot \mathcal{O}_M$ or $f^{-1}\mathcal{I} \cdot \mathcal{O}_M$. If $\mathcal{I}$ is coherent, then $f^{-1}\mathcal{I}$ is also coherent.

If $\mathcal{I}$ is a coherent ideal, the subscheme of M determined by $f^{-1}\mathcal{I}$ is the inverse image scheme of the subscheme of N determined by $\mathcal{I}$, i.e.

$$V(f^{-1}\mathcal{I}) = f^{-1}(V(\mathcal{I})).$$

Lemma 2.9. — *If $f : M \to N$ is a flat holomorphic map of complex spaces and $\mathcal{I}$ is an ideal sheaf on N, then $f^{-1}\mathcal{I} \cong f^*\mathcal{I}$.*

Proof. — By Lemma 2.7 above, if f is flat, then the map $f^*\mathcal{I} \to f^*\mathcal{O}_N = \mathcal{O}_M$ is injective. $\square$

Corollary 2.10. — *If $f : L \to X$ is a line bundle (or more generally a vector bundle) and $\mathcal{I}$ is an ideal sheaf on X, then $f^{-1}\mathcal{I} = f^*\mathcal{I}$.*

Proof. — As noted in the discussion of flat maps above, the projection of a line bundle (or vector bundle) onto its base space is a flat map. $\square$

Composites

Next we describe the behavior of direct and inverse images under composites. The proofs are straightforward, using the definitions above.

Lemma 2.11 (The Composite of Direct Images is the Direct Image of the Composite). — *Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be holomorphic maps of complex spaces and let $\mathcal{S}$ be a sheaf on M_1. Then*

$$g_*(f_*\mathcal{S}) \cong (g \circ f)_*\mathcal{S}.$$

Proof. — Let U be an open set in M_3. Then

$$\begin{aligned} g_*(f_*\mathcal{S})(U) &= (f_*\mathcal{S})(g^{-1}(U)) \\ &= \mathcal{S}(f^{-1}g^{-1}(U)) \\ &= \mathcal{S}((g \circ f)^{-1}(U)) \\ &= (g \circ f)_*(U). \qquad \square \end{aligned}$$

Lemma 2.12 (The Composite of Topological Inverse Images is the Topological Inverse Image of the Composite). — *Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be holomorphic maps of complex spaces and let $\mathcal{S}$ be a sheaf on M_3. Then*

$$f'(g'\mathcal{S}) \cong (g \circ f)'\mathcal{S}.$$

Proof. — We will prove the statement on stalks. Let m be a point in M_1. Then

$$\begin{aligned} (f'(g'\mathcal{S}))_m &= (g'\mathcal{S})_{f(m)} \\ &= \mathcal{S}_{g \circ f(m)} \\ &= ((g \circ f)'\mathcal{S})_m. \qquad \square \end{aligned}$$

Lemma 2.13 (The Composite of Pullbacks is the Pullback of the Composite). — *Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be holomorphic maps of complex spaces and let $\mathcal{S}$ be a sheaf on M_3. Then*

$$f^*(g^*\mathcal{S}) \cong (g \circ f)^*\mathcal{S}.$$

Proof. — For convenience, let $\mathcal{O}_i$ represent $\mathcal{O}_{M_i}$ for $i = 1, 2, 3$. Recall that

$$g^*\mathcal{S} = g'\mathcal{S} \otimes_{g'\mathcal{O}_3} \mathcal{O}_2.$$

Similarly

$$\begin{aligned} f^*(g^*\mathcal{S}) &= f'(g^*\mathcal{S}) \otimes_{f'\mathcal{O}_2} \mathcal{O}_1 \\ &= f'(g'\mathcal{S} \otimes_{g'\mathcal{O}_3} \mathcal{O}_2) \otimes_{f'\mathcal{O}_2} \mathcal{O}_1. \end{aligned}$$

Looking at stalks over $m \in M_1$ we have

$$\begin{aligned} (f^*(g^*\mathcal{S}))_m &= f'(g'\mathcal{S} \otimes_{g'\mathcal{O}_3} \mathcal{O}_2)_m \otimes_{(f'\mathcal{O}_2)_m} \mathcal{O}_{1,m} \\ &= (g'\mathcal{S} \otimes_{g'\mathcal{O}_3} \mathcal{O}_2)_{f(m)} \otimes_{\mathcal{O}_{2,f(m)}} \mathcal{O}_{1,m} \\ &= (g'\mathcal{S})_{f(m)} \otimes_{(g'\mathcal{O}_3)_{f(m)}} \mathcal{O}_{2,f(m)} \otimes_{\mathcal{O}_{2,f(m)}} \mathcal{O}_{1,m} \\ &= \mathcal{S}_{g(f(m))} \otimes_{\mathcal{O}_{3,g(f(m))}} \mathcal{O}_{2,f(m)} \otimes_{\mathcal{O}_{2,f(m)}} \mathcal{O}_{1,m} \\ &= \mathcal{S}_{g(f(m))} \otimes_{\mathcal{O}_{3,g(f(m))}} \mathcal{O}_{1,m} \\ &= ((g \circ f)'\mathcal{S})_m \otimes_{((g\circ f)'\mathcal{O}_3)_m} \mathcal{O}_{1,m} \\ &= ((g \circ f)^*\mathcal{S})_m. \qquad \square \end{aligned}$$

The following lemma is more naturally understood in terms of subschemes determined by coherent sheaves of ideals. Its interpretation in term of subschemes is that the inverse image subscheme under a composite map is the composite of the inverse images. Briefly, f^{-1} is functorial on ideals and their corresponding subschemes.

Lemma 2.14 (The Composite of Inverse Images is the Inverse Image of the Composite). — *Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ be holomorphic maps of complex spaces and let $\mathcal{I}$ be a sheaf of ideals on M_3. Then*

$$f^{-1}(g^{-1}\mathcal{I}) \cong (g \circ f)^{-1}\mathcal{I}.$$

Proof. — As in the previous proof, let $\mathcal{O}_i = \mathcal{O}_{M_i}$. Recall that $g^{-1}\mathcal{I}$ is defined to be the image of $g^*\mathcal{I}$ in $\mathcal{O}_2$, so there is a surjective map

$$g^*\mathcal{I} \mapsto g^{-1}\mathcal{I}.$$

The map of topological inverse images

$$f'g^*\mathcal{I} \mapsto f'g^{-1}\mathcal{I}$$

is also surjective.

Tensoring over $f'\mathcal{O}_2$ by $\mathcal{O}_1$ we obtain the map

$$f^*g^*\mathcal{I} \mapsto f^*g^{-1}\mathcal{I},$$

which is surjective since tensoring is right exact.

Finally we note that

$$\begin{array}{rcll} f^{-1}g^{-1}\mathcal{I} & = & \text{image of } f^*g^{-1}\mathcal{I} \text{ in } \mathcal{O}_1 & \text{by definition} \\ & = & \text{image of } f^*g^*\mathcal{I} \text{ in } \mathcal{O}_1 & \text{by surjectivity} \\ & = & \text{image of } (g\circ f)^*\mathcal{I} \text{ in } \mathcal{O}_1 & \text{by Lemma 2.13} \\ & = & (g\circ f)^{-1}\mathcal{I} & \text{by definition} \end{array}$$ □

Products of ideals

The following lemma is also more naturally understood in terms of subschemes determined by coherent sheaves of ideals. The subscheme of M determined by $(f^{-1}\mathcal{I}_1)(f^{-1}\mathcal{I}_2)$ is the union of the subschemes determined by $f^{-1}\mathcal{I}_1$ and $f^{-1}\mathcal{I}_2$, which are the inverse images of the subschemes determined by $\mathcal{I}_1$ and $\mathcal{I}_2$. The subscheme of M determined by $f^{-1}(\mathcal{I}_1\mathcal{I}_2)$ is the inverse image of the union of the subschemes determined by $\mathcal{I}_1$ and $\mathcal{I}_2$, which is the same as the union of the inverse images.

Lemma 2.15 (The Inverse Image Ideal of a Product of Ideal Sheaves is the Product of the Inverse Image Ideal Sheaves). *Let $f : M \to N$ be a holomorphic map of complex spaces and let $\mathcal{I}_1$ and $\mathcal{I}_2$ be sheaves of ideals on N. Then*

$$(f^{-1}\mathcal{I}_1)(f^{-1}\mathcal{I}_2) \cong f^{-1}(\mathcal{I}_1\mathcal{I}_2).$$

Proof. — Note that both $f^{-1}(\mathcal{I}_1\mathcal{I}_2)$ and $(f^{-1}\mathcal{I}_1)(f^{-1}\mathcal{I}_2)$ are generated as ideals in $\mathcal{O}_M$ by products of the form $f^*w_1 f^*w_2$ where w_1 is a germ in $\mathcal{I}_1$ and w_2 a germ in $\mathcal{I}_2$. □

The direct image of a product of ideal sheaves is not necessarily equal to the product of the direct images, but we will show later (Lemma 5.7) that the two are equal if the map is a blow-up of a smooth center and the ideal sheaves are first multiplied by a high enough power of the ideal sheaf of the exceptional divisor.

3. Blowing up a complex manifold along a Coherent sheaf of ideals

Let M be a complex manifold and let $\mathcal{I}$ be a coherent sheaf of ideals on M. Here and throughout the paper we will always assume that $\mathcal{I}$ is not the zero sheaf. Since $\mathcal{I}$ is coherent, for each point $p \in M$ we may choose a coordinate neighborhood U, centered at p, such that $\mathcal{I}(U)$ is generated by a finite number of global sections over U. We first define the blow-up of M along $\mathcal{I}$ locally over such an open set U, using a collection of generators of $\mathcal{I}(U)$. We then show that the result is independent of the collection of generators chosen, so that the blow-up may be defined globally over M.

Blow-ups may also be defined for singular complex spaces but we do not need such generality here.

Local description of blow-ups

Let M be a complex manifold and $\mathcal{I}$ a coherent sheaf of ideals on M as above. Let U be a small enough coordinate neighborhood in M that $I = \mathcal{I}(U)$ is generated by a finite collection of global sections $f_1, ..., f_r$ on U. Set

$$V(I) = \{z \in U : h(z) = 0 \quad \text{for all} \quad h \in I\}.$$

We define a map

$$\phi_f : U - V(I) \to \mathbb{P}^{r-1}$$

by setting $\phi_f(z) = [f_1(z) : ... : f_r(z)]$. Let $\Gamma(\phi_f)$ be the graph of ϕ_f in $U \times \mathbb{P}^{r-1}$, i.e.

$$\begin{aligned}\Gamma(\phi_f) &= \{(z, [\xi]) : z \in U - V(I) \quad \text{and} \quad [\xi] = [f_1(z) : ... : f_r(z)]\} \\ &= \{(z, [\xi]) : z \in U - V(I) \quad \text{and} \quad f_i(z)\xi_j = f_j(z)\xi_i, \quad 1 \leqslant i, j \leqslant r\}.\end{aligned}$$

We define $\tilde{U}_f$ to be the smallest reduced complex analytic subspace of $U \times \mathbb{P}^{r-1}$ containing the graph $\Gamma(\phi_f)$. The support of $\tilde{U}_f$ is the closure of $\Gamma(\phi_f)$ in the usual topology.

The **blow-up map** of U along $\mathcal{I}$ is the projection $\pi : \tilde{U}_f \to U$, which is a proper map.

We will now show that the complex space $\tilde{U}_f$ is independent of the generators f chosen for I.

Lemma 3.1. — *If $\{f_1, ..., f_r\}$ and $\{g_1, ..., g_s\}$ are two collections of generators of $\mathcal{I}$ on U then*

$$\tilde{U}_f \cong \tilde{U}_g.$$

Proof. — Define a map $\psi : \Gamma(\phi_f) \to \Gamma(\phi_g)$ by

$$\psi(z, [\xi]) = (z, [g_1(z) : ... : g_s(z)].$$

The map ψ is well-defined because $g_1(z), ..., g_s(z)$ are not all 0 for $z \in U - V(I)$, (since $g_1, ..., g_s$ are generators of I). Furthermore ψ^{-1} exists and is given by

$$\psi^{-1}(z, [\zeta]) = (z, [f_1(z) : ... : f_r(z)].$$

Both ψ and ψ^{-1} are clearly holomorphic so $\Gamma(\phi_f) \cong \Gamma(\phi_g)$. We will now show that they extend to holomorphic maps on $\tilde{U}_f$ and $\tilde{U}_g$.

Since $\{f_1, ..., f_r\}$ and $\{g_1, ..., g_s\}$ both generate I, there exist $\alpha_{ij}, \beta_{ij} \in \mathcal{O}(U)$ such that

$$g_i(z) = \sum_{j=1}^{r} \alpha_{ij}(z) f_j(z)$$

and

$$f_i(z) = \sum_{j=1}^{s} \beta_{ij}(z) g_j(z)$$

for all z in U. Briefly,

$$f(z) = \beta(z)g(z) = \beta(z)\alpha(z)f(z) \tag{$*$}$$

for all $z \in U$. The functions α and β might not define maps on all of $\mathbb{P}^{r-1}$ and $\mathbb{P}^{s-1}$ but they do define maps on $\Gamma(\phi_f)$ and $\Gamma(\phi_g)$.

Suppose that $(z', [\xi']) \in U \times \mathbb{P}^{r-1}$ is the limit of points $(z_\gamma, [\xi_\gamma]) \in \Gamma(\phi_f)$, i.e. there is a sequence of points $\{z_\gamma\} \in U$ such that

$$z_\gamma \to z' \qquad \text{and} \qquad [\xi_\gamma] = [f_1(z_\gamma) : ... : f_r(z_\gamma)] \to [\xi'].$$

Some component of $[\xi']$ is nonzero, say the first component, so that we may assume that $\xi' = (1, \xi'_2, ...\xi'_r)$. Then we may also assume that the sequence $\{z_\gamma\}$ has the property that $f_1(z_\gamma) \neq 0$ for all γ and that the sequence ξ_γ is of the form

$$\xi_\gamma = (1, \xi_{\gamma 2}, ..., \xi_{\gamma r}) = \left(1, \frac{f_2(z_\gamma)}{f_1(z_\gamma)}, ..., \frac{f_r(z_\gamma)}{f_1(z_\gamma)}\right) \tag{$**$}$$

where

$$\xi_\gamma \to \xi'.$$

We will use this description to show that $\alpha(z')\xi' \neq 0$. We have

$$\begin{aligned}
\beta(z_\gamma)\alpha(z_\gamma)\xi_\gamma &= \beta(z_\gamma)\alpha(z_\gamma)\frac{f(z_\gamma)}{f_1(z_\gamma)} && \text{by (**)}\\
&= \frac{f(z_\gamma)}{f_1(z_\gamma)} && \text{by (*)}\\
&= \xi_\gamma && \text{by (**).}
\end{aligned}$$

Thus

$$\begin{aligned}
\beta(z')\alpha(z')\xi' &= \lim_{\gamma\to\infty}\beta(z_\gamma)\alpha(z_\gamma)\xi_\gamma\\
&= \lim_{\gamma\to\infty}\xi_\gamma\\
&= \xi'
\end{aligned}$$

by continuity of α and β. In particular, $\alpha(z')\xi' \neq 0$ so $[\zeta] = [\alpha(z')\xi']$ exists as a point of $\mathbb{P}^{s-1}$ (and is independent of the choices of representatives ξ' and ξ_γ).

We define ψ on $(z', [\xi'])$ to be

$$\psi(z', [\xi']) = (z', [\alpha(z')\xi']).$$

The definition of ψ^{-1} is similar. Clearly these extensions of ψ and ψ^{-1} to the closures of $\Gamma(\phi_f)$ and $\Gamma(\phi_g)$ are holomorphic and their compositions are the identity, so we obtain the required isomorphism $\tilde{U}_f \cong \tilde{U}_g$. □

Blow-ups Locally. — From the preceding lemma we see that it makes sense to define the blow-up of U along $\mathcal{I}$ as $\mathrm{Bl}_{\mathcal{I}}U = \tilde{U} = \tilde{U}_f$ for any set of generators f.

If $\mathcal{I}$ is the ideal of a **smooth** subspace C of U then $\tilde{U}$ is also smooth. The set C is called the **center** of the blow-up. If $\mathcal{I}$ is the ideal of a singular subset of U then $\tilde{U}$ may be singular.

Lemma 3.2. — *Let $\mathcal{I}$ and $\mathcal{J}$ be nonzero coherent ideal sheaves on U which are generated by global sections on U. Suppose that $\mathcal{J}$ is principal, i.e. generated by a single function on U. Then*

$$Bl_{\mathcal{I}\mathcal{J}}U \cong Bl_{\mathcal{I}}U.$$

Proof. — Suppose that $\mathcal{J}$ is generated locally by the single function h. Then

$$[hf_1 : ... : hf_r] = [f_1 : ... : f_r]$$

on $U - V(\mathcal{I}\mathcal{J})$. □

Global description of blow-ups

Let $\mathcal{I}$ be a coherent sheaf of ideals on a complex manifold M. By Lemma 3.1, we may extend the local definition of the blow-up canonically, to define a global blow-up

$$\pi : \tilde{M} = \mathrm{Bl}_{\mathcal{I}} M \to M.$$

The blow-up map π is proper and the restriction of π from $\tilde{M} - \pi^{-1}(V(\mathcal{I}))$ to $M - V(\mathcal{I})$ is biholomorphic.

If $\mathcal{I}$ is the ideal sheaf of a **smooth** submanifold C of M, then $\tilde{M}$ is smooth.

Ideals, divisors, line bundles, and sections

Let M be a complex manifold and let D be a divisor on M. We denote by L_D or $[D]$ the corresponding line bundle on M. Let $\mathcal{L}_D$ be the invertible sheaf of holomorphic sections of $[D]$.

Let s_D be a meromorphic section of $[D]$ whose divisor (s_D) is D. Such a section always exists: if D is defined on an open covering $\{U_i\}$ of M by meromorphic functions $\{f_i\}$, the functions $\{f_i\}$ themselves define such a section s_D.

If s is any other meromorphic section of $[D]$ then $\frac{s}{s_D}$ is a meromorphic function on M. Let $\mathcal{K}_M$ be the sheaf of meromorphic functions on M. We may embed $\mathcal{L}_D$ into $\mathcal{K}_M$ by the map

$$s \mapsto \frac{s}{s_D},$$

i.e. if U is any open set in M and $s \in \mathcal{L}_D(U)$, we map s to $\frac{s}{s_D} \in \mathcal{K}_M(U)$.

Now suppose that Y is an effective divisor (codimension one subscheme) of M with ideal sheaf $\mathcal{I}_Y$, and that Y is given on an open cover $\{U_i\}$ of M by holomorphic functions $\{f_i\}$. Let s_Y be the corresponding holomorphic section of $[Y]$. Then $\frac{1}{s_Y}$ is a meromorphic section of $[-Y]$. We may embed $\mathcal{L}_{-Y}$ into $\mathcal{K}_M$ by the map

$$s \mapsto s s_Y.$$

The image of $\mathcal{L}_{-Y}$ in $\mathcal{K}_M$ is just the ideal $\mathcal{I}_Y$ in $\mathcal{O}_M \subset \mathcal{K}_M$. Therefore

$$\mathcal{L}_{-Y} \cong \mathcal{I}_Y.$$

Suppose that $\mathcal{I}$ is any coherent ideal in $\mathcal{O}_M$. Tensoring the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_M$$

by $\mathcal{L}_{-Y}$ gives an exact sequence

$$0 \to \mathcal{I} \otimes \mathcal{L}_{-Y} \to \mathcal{O}_M \otimes \mathcal{L}_{-Y} = \mathcal{L}_{-Y}$$

by Lemma 2.8 above. The image of $\mathcal{I} \otimes \mathcal{L}_{-Y}$ in $\mathcal{L}_{-Y}$ is just $\mathcal{I}\mathcal{L}_{-Y}$ (see e.g. [Ma], p. 18). The image of $\mathcal{I}\mathcal{L}_{-Y}$ under the embedding $\mathcal{L}_{-Y} \hookrightarrow \mathcal{K}_M$ is then $\mathcal{I}\mathcal{I}_Y$. Therefore

Lemma 3.3. — *Let $\mathcal{I}$ be a coherent sheaf of ideals on a complex manifold M and let Y be an effective divisor on M. Then*

$$\mathcal{I} \otimes \mathcal{L}_{-Y} \cong \mathcal{I}\mathcal{I}_Y.$$

Lemma 3.4. — *If $\mathcal{I}$ is a coherent sheaf of ideals on a complex manifold M and Y is an effective divisor on M, then the blow-up of M along $\mathcal{I}$ is biholomorphic to the blow-up of M along $\mathcal{I}\mathcal{I}_Y \cong \mathcal{I} \otimes \mathcal{L}_{-Y}$.*

Proof. — Apply Lemma 3.2, since $\mathcal{I}_Y$ is principal. □

Lemma 3.5. — *Let M be a complex manifold and let $\mathcal{I}$ be a coherent sheaf of ideals on M. Let $\pi : \tilde{M} = Bl_{\mathcal{I}}M \to M$ be the blow-up of M along $\mathcal{I}$. Then $\pi^{-1}\mathcal{I}$ is a sheaf of principal ideals on M (i.e. an invertible sheaf). The complex subspace of $\tilde{M}$ corresponding to $\pi^{-1}\mathcal{I}$ is a hypersurface.*

Proof. — Suppose that $\mathcal{I}$ is generated locally on an open set U in M by $f_1, ..., f_r$. Since $\tilde{U}$ is contained in the subset of $U \times \mathbb{P}^{r-1}$ given by the equations $f_i(z)\xi_j = f_j(z)\xi_i$, it is enough to prove that the inverse image ideal of $\mathcal{I}$ on this set is principal. But this is clear since on the set $U_i = \{\xi_i \neq 0\}$, we have

$$f_j = \frac{\xi_j}{\xi_i} f_i$$

so f_i generates the inverse image ideal of $\mathcal{I}$ on U_i. □

Exceptional divisors of blow-ups

The hypersurface in $\tilde{M}$ corresponding to $\pi^{-1}\mathcal{I}$, described in Lemma 3.5 above, is called the **exceptional divisor** E of π, i.e.

$$E = V(\pi^{-1}(\mathcal{I})) = \pi^{-1}V(\mathcal{I}).$$

The proof of Lemma 3.5 above gives us a local description of E. Suppose that $f_1, ..., f_r$ generate $\mathcal{I}$ on an open set U in M. Cover $\tilde{U} \subset U \times \mathbb{P}^{r-1}$ by sets $U_i = \{\xi_i \neq 0\}$. Then E is given on U_i by $f_i = 0$.

The map $\pi : \tilde{M} \to M$ is a proper map which is biholomorphic from $\tilde{M} - E$ to $M - V(\mathcal{I})$. If $\mathcal{I}$ is the ideal sheaf of a smooth center C, i.e. $\mathcal{I} = \mathcal{I}_C$, then $\tilde{M}$ is smooth, $E = \pi^{-1}(C)$ is a smooth submanifold of $\tilde{M}$, and for each $p \in C$ the inverse image $E_p = \pi^{-1}(p)$ is biholomorphic to $\mathbb{P}^{k-1}$, where k is the codimension of C in M.

Exceptional line bundles of blow-ups

Corresponding to the exceptional divisor E on $\tilde{M}$ is an exceptional line bundle $L_E = [E]$. Both E and L_E are independent of the local generators of $\mathcal{I}$ used to construct the blow-up.

In terms of local generators $f_1, ..., f_r$ of $\mathcal{I}$, transition functions for L_E are

$$g_{ij} = \frac{f_i}{f_j} = \frac{\xi_i}{\xi_j},$$

i.e. if s is a holomorphic section of L_E over $\tilde{U}$ then s is represented by holomorphic functions s_i on $U_i = \{\xi_i \neq 0\}$ with

$$s_i = g_{ij} s_j \qquad \text{on } U_i \cap U_j.$$

Since local transition functions for L_E on the set $\tilde{U}$ are of the form $g_{ij} = \frac{\xi_i}{\xi_j}$, the line bundle L_E on $\tilde{U}$ is the restriction of the universal bundle $\mathcal{O}(-1)$ on $U \times \mathbb{P}^{r-1}$. More precisely, let $\sigma_1 : U \times \mathbb{P}^{r-1} \to U$ and $\sigma_2 : U \times \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ be the first and second projection maps, as shown below.

$$\begin{array}{ccccc} \mathrm{Bl}_I U = \tilde{U} & \longrightarrow & U \times \mathbb{P}^{r-1} & \xrightarrow{\sigma_2} & \mathbb{P}^{r-1} \\ & & \sigma_1 \downarrow & & \\ & & U & & \end{array}$$

Let $\mathcal{O}_{\mathbb{P}^{r-1}}(-1)$ be the universal bundle on $\mathbb{P}^{r-1}$. Then the restriction to $\tilde{U}$ of the line bundle $\sigma_2^* \mathcal{O}_{\mathbb{P}^{r-1}}(-1)$ is L_E on $\tilde{U}$.

We may interpret the fibre of L_E over $(z, [\xi]) \in \tilde{U}$ as the line through ξ in $\mathbb{C}^r$.

Universal property of blow-ups

Lemma 3.6 (Universal Property of Blow-ups). — *Let M be a complex manifold and let $\mathcal{I}$ be a coherent sheaf of ideals on M. Let $\pi : \tilde{M} = Bl_{\mathcal{I}} M \to M$ be the blow-up of M along $\mathcal{I}$. Suppose that $\phi : N \to M$ is a* *holomorphic map of a complex space N to M, such that the inverse image ideal $\phi^{-1}\mathcal{I}$ is principal (i.e. an invertible sheaf). Then there exists a unique holomorphic lifting*

$$\tilde{\phi} : N \to \tilde{M}$$

such that $\pi \circ \tilde{\phi} = \phi$.

Proof. — Suppose that $f_1, ..., f_r$ are generators for $\mathcal{I}$ over a small open set $U \subset M$. Then $f_1 \circ \phi, ..., f_r \circ \phi$ are generators for $\phi^{-1}\mathcal{I}$ over $\phi^{-1}(U)$ in N. Since $\phi^{-1}\mathcal{I}$ is assumed to be a principal ideal sheaf, all of the functions $f_i \circ \phi$ are multiples of one of them, so we have a well-defined map

$$\tilde{\phi} : \phi^{-1}(U) \to U \times \mathbb{P}^{r-1}$$

given by

$$v \mapsto (\phi(v), [f_1 \circ \phi(v) : ... : f_r \circ \phi(v)]).$$

By construction, the image of $\tilde{\phi}$ lies in the blow-up $\tilde{U}$ in $U \times \mathbb{P}^{r-1}$ and $\pi \circ \tilde{\phi}(v) = \phi(v)$.

By an argument similar to the proof of Lemma 3.1 above, which showed that the blow-up $\tilde{U}$ is independent of the collection of generators $\{f_i\}$ used to construct it, we see that the map $\tilde{\phi}$ is independent of the generators $\{f_i\}$. Thus we can extend our local construction to a well-defined holomorphic map $\tilde{\phi} : N \to \tilde{M}$.

Finally we check the uniqueness of $\tilde{\phi}$. Suppose that $\tilde{\phi}'$ is any holomorphic map from N to $\tilde{M}$ such that $\pi \circ \tilde{\phi}' = \phi = \pi \circ \tilde{\phi}$. Since π is a biholomorphism away from the exceptional set, $\tilde{\phi}'$ and $\tilde{\phi}$ must agree on $\phi^{-1}(M - V(\mathcal{I})) = N - V(\phi^{-1}\mathcal{I})$. But $\phi^{-1}\mathcal{I}$ was assumed to be a principal ideal, so $V(\phi^{-1}\mathcal{I})$ is a hypersurface in N. This means that $\tilde{\phi}'$ and $\tilde{\phi}$ agree on a dense set of N, so they must agree everywhere. □

Blow-up of a product of ideals

We will show that the blow-up of a product of two ideals is isomorphic to the composite of two blow-ups. Since we have defined blow-ups only for smooth manifolds, we will restrict ourselves to the case in which the blow-up along one ideal is smooth, for example if that ideal is the ideal of a smooth submanifold.

PROPOSITION 3.7. — *Let M be a complex manifold and $\mathcal{I}_1$ and $\mathcal{I}_2$ coherent sheaves of ideals on M. Let $\pi : Bl_{\mathcal{I}_1}M \to M$ be the blow-up of M along $\mathcal{I}_1$ and suppose that the blow-up space $Bl_{\mathcal{I}_1}M$ is smooth. Then*

$$Bl_{\mathcal{I}_1\mathcal{I}_2}M \cong Bl_{\pi^{-1}\mathcal{I}_2}Bl_{\mathcal{I}_1}M,$$

i.e. the blow-up of M along the product ideal $\mathcal{I}_1\mathcal{I}_2$ is isomorphic to the blow-up of M along $\mathcal{I}_1$ followed by the blow-up along the inverse image ideal of $\mathcal{I}_2$.

Proof. — We will apply the universal mapping property of blow-ups (Lemma 3.6). Let $N = \mathrm{Bl}_{\pi^{-1}\mathcal{I}_2}\mathrm{Bl}_{\mathcal{I}_1}M$ and let $\phi : N \to M$ be the composite of the blow-up maps. Then $\phi^{-1}\mathcal{I}_1$ and $\phi^{-1}\mathcal{I}_2$ are principal ideal sheaves on N so $\phi^{-1}(\mathcal{I}_1\mathcal{I}_2)$ is also principal. By the universal mapping property, ϕ lifts to a holomorphic map $\tilde{\phi} : N \to \mathrm{Bl}_{\mathcal{I}_1\mathcal{I}_2}M$. This map is a biholomorphism away from the exceptional sets.

Similarly, if $\psi : \mathrm{Bl}_{\mathcal{I}_1\mathcal{I}_2}M \to M$ is the blow-up of M along $\mathcal{I}_1\mathcal{I}_2$, then $\psi^{-1}\mathcal{I}_1$ is a principal ideal sheaf on $\mathrm{Bl}_{\mathcal{I}_1\mathcal{I}_2}M$ and we can lift ψ to a map $\psi_1 : \mathrm{Bl}_{\mathcal{I}_1\mathcal{I}_2}M \to \mathrm{Bl}_{\mathcal{I}_1}M$. Next we check that $\psi_1^{-1}(\pi^{-1}\mathcal{I}_2)$ is again a principal ideal sheaf, so that we can lift ψ_1 to a map $\tilde{\psi} : \mathrm{Bl}_{\mathcal{I}_1\mathcal{I}_2}M \to \mathrm{Bl}_{\pi^{-1}\mathcal{I}_2}\mathrm{Bl}_{\mathcal{I}_1}M = N$.

Since the maps $\tilde{\psi}$ and $\tilde{\phi}$ are holomorphic everywhere and are inverses of each other on open dense sets, they must be inverses of each other everywhere. □

Corollary 3.8. — *Let M be a complex manifold, C a smooth center in M, and $\mathcal{I}_C$ the ideal sheaf of C. Then the blow-up of M along $\mathcal{I}_C$ is isomorphic to the blow-up along $\mathcal{I}_C^d$ for any integer $d > 1$, i.e.*

$$Bl_{\mathcal{I}_C}M \cong Bl_{\mathcal{I}_C^d}M.$$

Proof. — Apply Proposition 3.7, noting that $\pi^{-1}\mathcal{I}_C$ is principal and that blowing-up along a principal ideal sheaf leaves a space unchanged. □

Direct images under blow-up maps

We conclude section 3 by showing that the direct image of an ideal sheaf under a blow-up map is an ideal sheaf. As always, we assume that the ideal sheaf $\mathcal{I}$ for our blow-up is not the zero sheaf, so that $C = V(\mathcal{I})$ has codimension at least 1.

Lemma 3.9. — *Let $\pi : \tilde{M} \to M$ be the blow-up of a complex manifold M along a coherent sheaf of ideals $\mathcal{I}$ on M. Let $\mathcal{J}$ be a sheaf of ideals on $\tilde{M}$. Then the direct image $\pi_*\mathcal{J}$ is a sheaf of ideals on M. If $\mathcal{J}$ is coherent then so is $\pi_*\mathcal{J}$.*

Proof. — We wish to define a map $\pi_*\mathcal{J} \to \mathcal{O}_M$ and show that it is injective. To define a sheaf map $\pi_*\mathcal{J} \to \mathcal{O}_M$, it is enough to define presheaf maps $\pi_*\mathcal{J}(U) \to \mathcal{O}_M(U)$ for all open sets U in M. To show that a map of sheaves $\pi_*\mathcal{J} \to \mathcal{O}_M$ is injective, it is enough to show that $\pi_*\mathcal{J}(U) \to \mathcal{O}_M(U)$ is injective for all open sets U in M.

Recall that $\pi_*\mathcal{J}(U) = \mathcal{J}(\tilde{U})$, where $\tilde{U} = \pi^{-1}(U)$. If U does not intersect $C = V(\mathcal{I})$, then $\tilde{U} \cong U$ and $\pi_*\mathcal{J}(U)$ may be identified naturally as an ideal in $\mathcal{O}_M(U)$. Now suppose that U does intersect C and consider $g \in \pi_*\mathcal{J}(U) = \mathcal{J}(\tilde{U})$. Let E be the exceptional divisor of π in $\tilde{M}$. Since

$$\tilde{U} - \tilde{U} \cap E \cong U - U \cap C,$$

we may define a holomorphic function G on $U - U \cap C$ whose pullback to $\tilde{U} - \tilde{U} \cap E$ is g. For each $p \in U \cap C$, the fibre $\pi^{-1}(p)$ is compact, since π is proper. Therefore g is constant on $\pi^{-1}(p)$ and bounded on a neighborhood of $\pi^{-1}(p)$ in $\tilde{U}$. Thus the function G is locally bounded in U, so G extends uniquely to a holomorphic function on U by Riemann's Removable Singularity Theorem. Since π^*G and g are holomorphic on $\tilde{U}$ and equal on the dense set $\tilde{U} - \tilde{U} \cap E$, they must be equal on all of $\tilde{U}$, i.e. $\pi^*G = g$ on $\tilde{U}$. For each $g \in \mathcal{J}(\tilde{U})$ there is a unique such $G \in \mathcal{O}_M(U)$, so we have a well-defined map

$$\pi_*\mathcal{J}(U) \to \mathcal{O}_M(U).$$

Clearly G is identically zero if and only if g is identically zero, so the map is injective.

By the Direct Image Theorem, $\pi_*\mathcal{J}$ is coherent if $\mathcal{J}$ is, since π is proper (used only in the proof of Lemma 4.5, where indicated). □

4. Chow's theorem for ideals

This section is devoted to the proof of our version of Chow's Theorem, using the Direct Image Theorem (for a blow-up along a smooth center). References for the 'usual' Chow's theorem are [F] and [M]. In section 5 we will state some applications to blow-ups.

THEOREM 4.1 CHOW'S THEOREM FOR IDEALS. — *Let U be an open neighborhood of $\{0\}$ in $\mathbb{C}^r$ and let X be an analytic subset of $U \times \mathbb{P}^n$. Let $\mathcal{I}$ be a coherent sheaf of ideals on X. Then $\mathcal{I}$ is* ***relatively algebraic*** *in the following sense: $\mathcal{I}$ is generated (after shrinking U if necessary) by a finite number of homogeneous polynomials in homogeneous $\mathbb{P}^n$-coordinates, with analytic coefficients in U-coordinates.*

Since a sheaf on $X \subset U \times \mathbb{P}^n$ may be considered as a sheaf on $U \times \mathbb{P}^n$, we will ignore X and prove the theorem for a coherent sheaf of ideals $\mathcal{I}$ on $U \times \mathbb{P}^n$. Although we have assumed that U is an open neighborhood of $\{0\}$ in $\mathbb{C}^r$, the same methods could be used for any complex space U. When we say that $\mathcal{I}$ is generated by homogeneous polynomials in homogeneous $\mathbb{P}^n$-coordinates, we mean that the dehomogenizations of these polynomials generate the ideal locally. We will show at the end of this section that we may choose all the polynomial generators of $\mathcal{I}$ to be of the same degree d, for d sufficiently large.

The usual Chow's theorem as well as its generalization in [F] follow directly from Theorem 4.1: if Y is an analytic subset of $U \times \mathbb{P}^n$ and $\mathcal{I} = \mathcal{I}_Y$ is the ideal sheaf of Y on $X = U \times \mathbb{P}^n$, then (after shrinking U if necessary) Y is cut out by a finite number of homogeneous polynomials in $\mathbb{P}^n$-coordinates with analytic coefficients in U-coordinates.

Outline of Proof of Chow's Theorem for Ideals. Let $\tilde{\mathbb{C}}^{n+1}$ be the blow-up of $\mathbb{C}^{n+1}$ at the origin and let σ_1 and σ_2 be the two projection maps of $U \times \tilde{\mathbb{C}}^{n+1}$ as shown:

$$\begin{array}{ccc} U \times \tilde{\mathbb{C}}^{n+1} & \xrightarrow{\sigma_2} & U \times \mathbb{P}^n \\ \sigma_1 \downarrow & & \\ U \times \mathbb{C}^{n+1} & & \end{array}$$

The map σ_2 is flat since $U \times \tilde{\mathbb{C}}^{n+1}$ is a line bundle over $U \times \mathbb{P}^n$, the product of the identity on U with the universal line bundle on $\mathbb{P}^n$. Thus $\sigma_2^{-1}\mathcal{I} = \sigma_2^*\mathcal{I}$ (Lemma 2.9). This inverse image ideal sheaf is coherent (see facts on inverse image ideals, section 2). The sheaf $\mathcal{J} = \sigma_{1*}(\sigma_2^{-1}\mathcal{I})$ is a sheaf of ideals on $U \times \mathbb{C}^{n+1}$, not merely a sheaf of modules, since σ_1 is a blow-up (Lemma 3.9). Furthermore, the map σ_1 is proper, so the direct image $\mathcal{J}$ is also coherent, by the Direct Image Theorem. We will show (Lemmas 4.2 - 4.5) that $\mathcal{J}$ is generated by homogeneous polynomials in $\mathbb{C}^{n+1}$-coordinates on a neighborhood of $(0,0)$, and that the corresponding polynomials in homogeneous $\mathbb{P}^n$-coordinates generate $\mathcal{I}$.

More specifically, let $x = (x_1, ..., x_r)$ and $y = (y_0, ..., y_n)$ be coordinates for U and $\mathbb{C}^{n+1}$. If $F(x,y)$ is a holomorphic function in a neighborhood of $(0,0)$ in $U \times \mathbb{C}^{n+1}$ and $\lambda \in \mathbb{C}^*$, let $F^{(\lambda)}$ be the holomorphic function given by

$$F^{(\lambda)}(x,y) = F(x, \lambda y).$$

We first show (Lemma 4.2) that

$$F \in \mathcal{J}_{(0,0)} \Leftrightarrow F^{(\lambda)} \in \mathcal{J}_{(0,0)} \qquad \forall \lambda \in \mathbb{C}^*.$$

We use a corollary of Krull's Theorem to show that if $F^{(\lambda)} \in \mathcal{J}_{(0,0)}$ for all $\lambda \in \mathbb{C}^*$ then each homogeneous term in y of $F(x,y)$ is in $\mathcal{J}_{(0,0)}$ (Lemma 4.3).

It follows from Lemma 4.3 that $\mathcal{J}_{(0,0)}$ is generated by a collection of homogeneous polynomials in y with analytic coefficients in x. We then show that $\mathcal{J}_{(0,0)}$ is generated by a finite number of these homogeneous polynomials (Lemma 4.4). By the Direct Image Theorem, the latter generators also generate $\mathcal{J} = \sigma_{1*}(\sigma_2^{-1}\mathcal{I})$ over a neighborhood of $(0,0)$ in $U \times \mathbb{C}^{n+1}$. Finally we check that these polynomials generate $\mathcal{I}$ over a neighborhood of $\{0\} \times \mathbb{P}^n \subset U \times \mathbb{P}^n$ (Lemma 4.5). □

We will now prove Lemmas 4.2 - 4.5 to complete the proof of Chow's Theorem for Ideals. As above, let $x = (x_1, ..., x_r)$ and $y = (y_0, ..., y_n)$ be coordinates for $U \subset \mathbb{C}^r$ and $\mathbb{C}^{n+1}$, and let $F^{(\lambda)}(x,y) = F(x, \lambda y)$.

Lemma 4.2. — *A holomorphic function F is a section of $\mathcal{J} = \sigma_{1*}(\sigma_2^{-1}\mathcal{I})$ on a neighborhood of $(0,0) \in U \times \mathbb{C}^{n+1}$ if and only if $F^{(\lambda)}$ is a section of $\mathcal{J}$ in a neighborhood of $(0,0)$ for each $\lambda \in \mathbb{C}^*$.*

Proof. — A holomorphic function is a section of $\mathcal{J} = \sigma_{1*}(\sigma_2^{-1}\mathcal{I})$ on a neighborhood of $(0,0)$ in $U \times \mathbb{C}^{n+1}$ if and only if its pullback by σ_1 is a section of $\sigma_2^{-1}\mathcal{I}$ on a neighborhood of $\sigma_1^{-1}(0,0) \cong \{0\} \times \mathbb{P}^n$ in $U \times \tilde{\mathbb{C}}^{n+1}$. Suppose that F is a section of $\mathcal{J}$ on a neighborhood of $(0,0)$. To show that $F^{(\lambda)}$ is a section of $\mathcal{J}$ on a neighborhood of $(0,0)$, it is enough to show that $\sigma_1^* F^{(\lambda)}$ is a section of $\sigma_2^{-1}\mathcal{I}$ on a neighborhood of p for each $p \in \sigma_1^{-1}(0,0)$. This reduces the proof to a simple calculation in local coordinates near p and $q = \sigma_2(p)$.

Choose homogeneous coordinates $[\xi_0 : ... : \xi_n]$ on $\mathbb{P}^n$ such that the point $q = \sigma_2(p)$ in $U \times \mathbb{P}^n$ is given by $q = (0, [1 : 0 : ... : 0])$. Let $W \subset \{\xi_0 \neq 0\} \subset \mathbb{P}^n$ be a neighborhood of $[1 : 0 : ... : 0]$ and let $w_i = \frac{\xi_i}{\xi_0}$, for $1 \leqslant i \leqslant n$, be nonhomogeneous coordinates for W. The preimage $\sigma_2^{-1}(U \times W) \cong U \times \mathbb{C} \times W$ is a neighborhood of p in $U \times \tilde{\mathbb{C}}^{n+1}$ with coordinates $(x, y_0, w) = (x_1, ..., x_r, y_0, w_1, ..., w_n)$ in which $p = (0,0,0)$. The maps σ_1 and σ_2 are given by

$$\sigma_1(x, y_0, w) = (x, y_0, y_0 w) \qquad \text{and} \qquad \sigma_2(x, y_0, w) = (x, w).$$

Since the ideal sheaf $\mathcal{I}$ is coherent, $\mathcal{I}$ is generated on a neighborhood of q by a finite collection of holomorphic functions $G_1, ..., G_s$. The pullbacks $\sigma_2^* G_1, ..., \sigma_2^* G_s$ generate $\sigma_2^{-1}\mathcal{I}$ on a neighborhood of p. Since $\sigma_1^* F$ is a section of $\sigma_2^{-1}\mathcal{I}$ on a neighborhood of p, there exist holomorphic functions $A_1, ..., A_s$

on a neighborhood of p such that

$$\sigma_1^* F(x, y_0, w) = \sum_{i=1}^{s} A_i(x, y_0, w)\sigma_2^* G_i(x, y_0, w).$$

Fix $\lambda \neq 0$. Then for y_0 close enough to 0, $(x, \lambda y_0, w)$ is in the domain of the functions $\sigma_1^* F$ and $A_1, ..., A_s$ and

$$\begin{aligned}
\sigma_1^* F^{(\lambda)}(x, y_0, w) &= \sigma_1^* F(x, \lambda y_0, w) \\
&= \sum_{i=1}^{s} A_i(x, \lambda y_0, w)\sigma_2^* G_i(x, \lambda y_0, w) \\
&= \sum_{i=1}^{s} A_i(x, \lambda y_0, w) G_i(x, w) \\
&= \sum_{i=1}^{s} A_i(x, \lambda y_0, w)\sigma_2^* G_i(x, y_0, w).
\end{aligned}$$

Let $A_i^{(\lambda)}(x, y_0, w) = A_i(x, \lambda y_0, w)$ for $1 \leqslant i \leqslant s$. Then each $A_i^{(\lambda)}$ is holomorphic on a neighborhood of p and

$$\sigma_1^* F^{(\lambda)}(x, y_0, w) = \sum_{i=1}^{s} A_i^{(\lambda)}(x, y_0, w)\sigma_2^* G_i(x, y_0, w),$$

i.e. $\sigma_1^* F^{(\lambda)}$ is a section of $\sigma_2^{-1}\mathcal{I}$ on a neighborhood of p. $\square$

Lemma 4.3. — *If $F^{(\lambda)}(x, y)$ is a section of $\mathcal{J}$ on a neighborhood of $(0, 0) \subset U \times \mathbb{C}^{n+1}$ for all $\lambda \in \mathbb{C}^*$, then each homogeneous term in y of $F(x, y)$ is a section of $\mathcal{J}$ on a neighborhood of* $(0, 0)$.

Proof. — For any holomorphic function F on a neighborhood of $(0, 0)$, let

$$F(x, y) = \sum_{\alpha} a_\alpha(x) y^\alpha$$

be the expansion of $F(x, y)$ in terms of monomials $y^\alpha = y_0^{\alpha_0} y_1^{\alpha_1} ... y_n^{\alpha_n}$ in y with analytic coefficients $a_\alpha(x)$ in x. Let $\mid \alpha \mid = \alpha_0 + \alpha_1 + ... + \alpha_n$. The homogeneous term in y of degree k in F is

$$F_k(x, y) = \sum_{|\alpha|=k} a_\alpha(x) y^\alpha.$$

Then

$$F = \sum_{k=0}^{\infty} F_k \qquad \text{and} \qquad F^{(\lambda)} = \sum_{k=0}^{\infty} \lambda^k F_k.$$

We wish to show that if F is a section of $\mathcal{J}$ on a neighborhood of $(0,0)$, then each F_k is also a section of $\mathcal{J}$ on a neighborhood of $(0,0)$. To minimize the use of subscripts, we will also use F and F_k to represent the germs of these functions at $(0,0)$.

Let $A = \mathcal{O}_{U\times\mathbb{C}^{n+1},(0,0)}$ (a Noetherian local ring), $(y) = (y_0, ..., y_n)$ (an ideal contained in the unique maximal ideal in A), and $J = \mathcal{J}_{(0,0)}$ (also an ideal in A). Let

$$\mathrm{Jet}_m(F) = \sum_{k=0}^{m} F_k$$

be the m-jet of F with respect to y. Note that $F - \mathrm{Jet}_m(F) \in (y)^{m+1}$.

By a corollary of Krull's Theorem (see e.g. [K], Corollary 5.7, p. 151),

$$J = \cap_{m\geqslant 0}(J + (y)^m),$$

where $(y)^0$ is defined to be A. Since

$$A = J + (y)^0 \supset J + (y)^1 \supset J + (y)^2 \supset ...$$

it follows that

$$J = \cap_{m\geqslant m_0}(J + (y)^m)$$

for any $m_0 \geqslant 0$.

Suppose that $F^{(\lambda)} \in J$ for all $\lambda \in \mathbb{C}^*$. Then since

$$F^{(\lambda)} - \mathrm{Jet}_m(F^{(\lambda)}) \in (y)^{m+1}$$

we have

$$\mathrm{Jet}_m(F^{(\lambda)}) \in J + (y)^{m+1}$$

for all $\lambda \in \mathbb{C}^*$. Since $\mathrm{Jet}_m(F^{(\lambda)}) = \sum_{k=0}^{m} \lambda^k F_k$ for all $\lambda \in \mathbb{C}^*$, by taking $m+1$ distinct values of λ it follows that

$$F_k \in J + (y)^{m+1}$$

for $0 \leqslant k \leqslant m$. Fixing k, we have

$$F_k \in J + (y)^{m+1} \qquad \text{for } m \geqslant k$$

or

$$F_k \in J + (y)^{m} \qquad \text{for } m \geqslant k+1,$$

i.e.

$$F_k \in \cap_{m\geqslant k+1}(J + (y)^m).$$

By the corollary of Krull's Lemma mentioned above, $F_k \in J$ for all k. $\square$

Lemma 4.4. — *If $\mathcal{J}_{(0,0)}$ is generated by a collection of elements of $\mathcal{O}_{U,0}[y_0,...,y_n]$ which are homogeneous in y, then $\mathcal{J}_{(0,0)}$ is generated by a* ***finite*** *collection of elements of $\mathcal{O}_{U,0}[y_0,...,y_n]$ which are homogeneous in y.*

Proof. — This lemma is a simple consequence of the fact that the ring $\mathcal{O}_{U\times\mathbb{C}^{n+1},(0,0)}$ is Noetherian, since whenever an ideal in a Noetherian ring is generated by an infinite set of elements it is automatically also generated by a finite subset of this set of elements. $\square$

The homogeneous elements of $\mathcal{O}_{U,0}[y_0,...,y_n]$ in y are the homogeneous polynomials in y with analytic coefficients in x as analytic germs at $(0,0)$.

Lemma 4.5. — *The same polynomials that generate $\mathcal{J}_{(0,0)}$ also generate $\mathcal{J}$ over a neighborhood of $(0,0)$ in $U\times\mathbb{C}^{n+1}$ and generate $\mathcal{I}$ over a neighborhood of $\{0\}\times\mathbb{P}^n$ in $U\times\mathbb{P}^n$.*

Proof. — Suppose that $\mathcal{J}$ is generated in a neighborhood of $(0,0)$ by $F_1(x,y),...,F_s(x,y)$, where $F_i(x,y)$ is a homogeneous polynomial of degree d_i in y with analytic coefficients in x. We will show that $\mathcal{I}$ is generated on a neighborhood of $\{0\}\times\mathbb{P}^n$ in $U\times\mathbb{P}^n$ by the corresponding polynomials $F_i(x,\xi)$, where $[\xi]=[\xi_0:...:\xi_n]$ are homogeneous coordinates for $\mathbb{P}^n$. More precisely, we will show that $\mathcal{I}$ is generated on a neighborhood of any point $q\in\{0\}\times\mathbb{P}^n$ by dehomogenizations of $F_1,...,F_s$ near q.

Choose homogeneous coordinates ξ on $\mathbb{P}^n$ such that $q=(0,[1:0:...:0])$. Nonhomogeneous coordinates on the set $W=\{\xi_0\neq 0\}\subset\mathbb{P}^n$ are $w_i=\frac{\xi_i}{\xi_0}$ for $1\leqslant i\leqslant n$. We will check that $\mathcal{I}$ is generated in a neighborhood of q by the polynomials

$$\frac{F_i(x,\xi)}{\xi_0^{d_i}}=F_i\left(x,\frac{\xi}{\xi_0}\right)=F_i(x,1,w_1,...,w_n).$$

First we look at the maps σ_1 and σ_2 in local coordinates. We may use (x,y_0,w) as local coordinates in $\sigma_2^{-1}(U\times W)\cong U\times\mathbb{C}\times W$. Local coordinates for $U\times\mathbb{C}^{n+1}$ are $(x,y_0,y_1,...,y_n)$, where $y_i=y_0w_i$ for $1\leqslant i\leqslant n$. The maps σ_1 and σ_2 are given by

$$\sigma_1(x,y_0,w)=(x,y_0,y_0w)\qquad\text{and}\qquad\sigma_2(x,y_0,w)=(x,w).$$

Suppose that G is a holomorphic section of $\mathcal{I}$ on a neighborhood of q in $U\times\mathbb{P}^n$. Then σ_2^*G is a holomorphic section of $\sigma_2^{-1}\mathcal{I}$ in a neighborhood of $\sigma_2^{-1}(q)=\{(0,y_0,0):y_0\in\mathbb{C}\}$. The homogeneous polynomials $F_1,...,F_s$ that generate $\mathcal{J}_{(0,0)}$ also generate $\mathcal{J} = \sigma_{1*}(\sigma_2^{-1}\mathcal{I})$ on a neighborhood of $(0,0) \in U \times \mathbb{C}^{n+1}$ (since $\mathcal{J}$ is coherent, by the Direct Image Theorem), so their pullbacks $\sigma_1^* F_1, ..., \sigma_1^* F_s$ generate $\sigma_2^{-1}\mathcal{I}$ on a neighborhood of $\sigma_1^{-1}(0,0) \in U \times \tilde{\mathbb{C}}^{n+1}$. In particular, there exist holomorphic functions $A_1, ..., A_s$ on a neighborhood of the point $(x = 0, y_0 = 0, w = 0)$ in $U \times \tilde{\mathbb{C}}^{n+1}$ such that

$$\sigma_2^* G(x, y_0, w) = \sum_{i=1}^{s} A_i(x, y_0, w)\sigma_1^* F_i(x, y_0, w)$$

on that neighborhood. But $\sigma_2^* G(x, y_0, w) = G(x, w)$ is independent of the value of y_0 and $\sigma_1^* F_i(x, y_0, w) = F_i(x, y_0, y_0 w) = y_0^{d_i} F_i(x, 1, w)$ since F_i is homogeneous of degree d_i in y. Therefore

$$G(x, w) = \sum_{i=1}^{s} A_i(x, y_0, w) y_0^{d_i} F_i(x, 1, w).$$

Choose some fixed nonzero value of y_0, close enough to 0 that (x, y_0, w) is in the domain of all the functions A_i for x and w close enough to 0. Define

$$a_i(x, w) = A_i(x, y_0, w) {y_0}^{d_i}.$$

Then

$$G(x, w) = \sum_{i=1}^{s} a_i(x, w) F_i(x, 1, w).$$

Since the functions a_i are holomorphic on a neighborhood of the point $q = (x = 0, w = 0)$, and the functions $F_i(x, 1, w)$ are the local dehomogenizations of the homogeneous polynomials $F(x, \xi)$, we are done. $\square$

This completes the proof of Chow's Theorem for Ideals. We now show that the homogeneous polynomial generators of the ideal sheaf $\mathcal{I}$ can be chosen to be of the same degree d, for large enough d.

Corollary 4.6. — *Let U be an open neighborhood of $\{0\}$ in $\mathbb{C}^r$ and let X be an analytic subset of $U \times \mathbb{P}^n$. Let $\mathcal{I}$ be a coherent sheaf of ideals on X. Then (possibly after shrinking U) there exists a positive integer d_0 such that for all $d \geqslant d_0$ the ideal $\mathcal{I}$ is generated by a finite number of degree d homogeneous polynomials in homogeneous $\mathbb{P}^n$-coordinates with analytic coefficients in U-coordinates.*

Proof. — As before, we may treat $\mathcal{I}$ as a sheaf on $U \times \mathbb{P}^n$. By Chow's Theorem for Ideals, we may choose a finite collection of homogeneous polynomials generating $\mathcal{I}$. We wish to show that we can choose homogeneous polynomials which are all of the same degree. Suppose that $F_1, ..., F_s$ are homogeneous polynomials of degrees $d_1, ..., d_s$ generating $\mathcal{I}$ on $U \times \mathbb{P}^n$. Let d_0 be any integer at least as large as the largest of $d_1, ..., d_s$. Then replace each F_i with the set of all $\xi^\alpha F_i$ as ξ^α runs through all degree $d_0 - d_i$ monomials in homogeneous coordinates $[\xi] = [\xi_0 : ... : \xi_n]$ on $\mathbb{P}^n$, i.e. use all monomials of the form $\xi_0^{\alpha_0}\xi_1^{\alpha_1}...\xi_n^{\alpha_n}$ where $\alpha_0+\alpha_1+...+\alpha_n = d_0 - d_i$. At every point in $U \times \mathbb{P}^n$, the dehomogenizations of the polynomials $\xi^\alpha F_i$ generate the same ideal as the dehomogenization of the polynomial F_i. $\square$

Degree d homogeneous polynomials on $\mathbb{P}^n$ may be viewed as sections of $\mathcal{O}(d)$, the sheaf of holomorphic sections of the dth power of the hyperplane bundle on $\mathbb{P}^n$. By abuse of notation, we will also use $\mathcal{O}(d)$ to refer to the corresponding sheaf on $U \times \mathbb{P}^n$, obtained by pullback from $\mathbb{P}^n$ under the projection map $U \times \mathbb{P}^n \to \mathbb{P}^n$. If $\mathcal{I}$ is a coherent sheaf of ideals on $U \times \mathbb{P}^n$, holomorphic sections of $\mathcal{I} \otimes \mathcal{O}(d)$ may be represented by homogeneous polynomials of degree d in homogeneous $\mathbb{P}^n$-coordinates with analytic coefficients in U-coordinates, whose local dehomogenizations are sections of $\mathcal{I}$.

We can thus restate Corollary 4.6 as follows.

COROLLARY 4.7. — *Let U be an open neighborhood of $\{0\}$ in $\mathbb{C}^r$ and let X be an analytic subset of $U \times \mathbb{P}^n$. Let $\mathcal{I}$ be a coherent sheaf of ideals on X. Then (possibly after shrinking U) there exists a positive integer d_0 such that for all $d \geqslant d_0$ the ideal $\mathcal{I} \otimes \mathcal{O}(d)$ is generated by a finite number global sections on $X \subset U \times \mathbb{P}^n$.*

5. Chow's theorem for ideals and an application to blow-ups

In this section we consider consider some consequences of Chow's Theorem for Ideals for blow-ups.

Consider a coherent sheaf of ideals $\mathcal{J}$ on $\tilde{M}$. Corollary 4.7 tells us that if U is a small enough open set in M and d is a large enough positive integer, the sheaf $\mathcal{J} \otimes \mathcal{O}(d)$ is generated by a finite number of global sections on $\tilde{U} \subset U \times \mathbb{P}^{r-1}$. Recall from section 3 that the restriction of $\mathcal{O}(d)$ to $\tilde{U}$ is just $\mathcal{L}^d_{-E}$, the sheaf of holomorphic sections of the dth power of the dual of the exceptional line bundle. From this observation and from Lemma 3.3, we have

$$\mathcal{J} \otimes \mathcal{O}(d) \cong \mathcal{J} \otimes \mathcal{L}^d_{-E} \cong \mathcal{J}\mathcal{I}^d_E.$$

COROLLARY 5.1. — *Let $\pi : \tilde{M} \to M$ be the blow-up of a complex manifold M along a coherent sheaf of ideals $\mathcal{I}$ and let E be the exceptional divisor of π. Let $\mathcal{J}$ be a coherent sheaf of ideals on $\tilde{M}$. Then for each point p in*

M there exists a neighborhood U of p in M, an embedding of $\tilde{U} = \pi^{-1}(U)$ into $U \times \mathbb{P}^{r-1}$, for some r, and an integer d_0 such that the ideal $\mathcal{J}\mathcal{I}_E^d$ is generated by a finite number of global sections on $\tilde{U}$ for all $d \geqslant d_0$.

Proof. — Construct an embedding $\tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$ using local generators of $\mathcal{I}$, as usual. Then use Corollary 4.7 of Chow's Theorem for Ideals, with $X = \tilde{U}$ and the coherent sheaf of ideals $\mathcal{J}\mathcal{I}_E^d$ on $\tilde{U}$. □

Alternatively, the existence of these global generators over $\tilde{U}$ can be proved using the positivity of the line bundle L_E^{-1} along fibres of the map from E to its image in M, as in Hironaka and Rossi [HR], using results of Grauert. Except for the use of the Direct Image Theorem (for the blow-up map of $U \times \mathbb{C}^{n+1}$ along the smooth center $U \times \{0\}$), our method is more explicit (cf. explicit calculations in the toric case in [GM2]).

We show not only that global sections exist on $\tilde{U}$, but how they are related to homogeneous polynomials in $\mathbb{P}^{r-1}$-coordinates generating $\mathcal{I}$ locally.

In the special case of compact projective manifolds, these constructions can be made global, using an ample line bundle on the original manifold.

Applying the previous corollary and noting that homogeneous polynomials on $U \times \mathbb{P}^{r-1}$ determine hypersurfaces of $\tilde{U}$, we obtain the following.

Corollary 5.2. — *Let $\pi : \tilde{M} \to M$ be the blow-up of a complex manifold M along a coherent sheaf of ideals $\mathcal{I}$ and let $\mathcal{J}$ be a coherent sheaf of ideals on $\tilde{M}$. Then for each point p in M there exists a neighborhood U of p in M, such that the complex space $V(\mathcal{J})$ determined by $\mathcal{J}$ is cut out by a finite number of hypersurfaces in $\tilde{U} = \pi^{-1}(U)$. In particular, if C is a smooth center in $\tilde{M}$ and $\mathcal{J} = \mathcal{I}_C$, then C is cut out by hypersurfaces, not only locally in $\tilde{M}$, but over the pre-images $\tilde{U}$ of small open sets U in M.*

The next corollary will be instrumental in constructing single-step blow-ups. This result is proved in [HR] by other methods.

Corollary 5.3. — *Let $\pi : \tilde{M} \to M$ be the blow-up of a* ***compact*** *complex manifold M along a coherent sheaf of ideals $\mathcal{I}$ and let E be the exceptional divisor of π. Let $\mathcal{J}$ be a coherent sheaf of ideals on $\tilde{M}$. Then there exists an integer d_0 such that*

$$\pi^{-1}\pi_*(\mathcal{J}\mathcal{I}_E^d) = \mathcal{J}\mathcal{I}_E^d$$

for all $d \geqslant d_0$.

Proof. — By compactness it is enough to prove the statement locally over neighborhoods of points in M. By Corollary 5.1, for each point p in M there exists a neighborhood U, an embedding $\tilde{U} \hookrightarrow U \times \tilde{\mathbb{P}}^{r-1}$, for some r, and an integer d_0 such that $\mathcal{J}\mathcal{I}_E^d$ is generated by a finite number of global sections on $\tilde{U}$, for $d \geqslant d_0$. These sections are holomorphic functions, vanishing on E for $d > 0$. By the Riemann Extension Theorem, they determine holomorphic functions on U. These functions on U generate $\pi_*(\mathcal{J}\mathcal{I}_E^d)$ and their pullbacks to $\tilde{U}$ generate $\pi^{-1}\pi_*(\mathcal{J}\mathcal{I}_E^d)$. Therefore $\pi^{-1}\pi_*(\mathcal{J}\mathcal{I}_E^d) = \mathcal{J}\mathcal{I}_E^d$. □

Remark 5.4. — Using local coordinates and local generators of $\mathcal{I}$, we can describe more concretely the relationship between homogeneous polynomials generating $\mathcal{J}$ over $\tilde{U}$ and holomorphic functions generating $\mathcal{J}\mathcal{I}_E^d$ over $\tilde{U}$.

Since $\mathcal{I}$ is coherent, $\mathcal{I}$ is generated by a finite collection of holomorphic functions $f_1, ..., f_r$ on U, for U small enough. Let z represent U-coordinates and $[\xi] = [\xi_1 : ... : \xi_r]$ homogeneous $\mathbb{P}^{r-1}$-coordinates. By Chow's Theorem for Ideals, $\mathcal{J}$ is generated by a finite collection of homogeneous polynomials $F(z, \xi)$ (homogeneous in ξ and analytic in z). The ideal sheaf $\mathcal{I}_E$ of the exceptional divisor is generated by the pullbacks of $f_1, ..., f_r$ to $\tilde{U}$. For simplicity we will also refer to these pullbacks as $f_1, ..., f_r$. The sheaf $\mathcal{I}_E^d$ is generated by all monomials of degree d in $f_1, ..., f_r$. The sheaf $\mathcal{J}\mathcal{I}_E^d$ is generated by all products of the form $f^\alpha F(z, \xi)$, where f^α represents a degree d monomial in $f_1, ..., f_r$. The function $F(z, \xi)$ is of the form

$$F(z, \xi) = \sum_\beta c_\beta(z) \xi^\beta$$

where ξ^β is a monomial of degree d in $\xi_1, ..., \xi_r$ and $c_\beta(z)$ is a holomorphic function of z. Then

$$\begin{aligned} f^\alpha F(z, \xi) &= \sum_\beta c_\beta(z) \xi^\beta f^\alpha \\ &= \sum_\beta c_\beta(z) \xi^\alpha f^\beta \qquad \text{since } f_i \xi_j = f_j \xi_i. \end{aligned}$$

Thus

$$f^\alpha F(z, \xi) = \xi^\alpha F(z, f).$$

The sheaf $\mathcal{J}\mathcal{I}_E^d$ is generated by all such products as ξ^α ranges over all degree d monomials in $\xi_1, ..., \xi_r$. Since these monomials in ξ cannot all be zero simultaneously, the collection $\{\xi^\alpha F(z, f)\}_\alpha$ is generated by $F(z, f)$.

We now see explicitly the holomorphic generators of $\mathcal{J}\mathcal{I}_E^d$ described in the previous corollary - they are the functions $F(z, f)$. These functions are holomorphic on $\tilde{U}$ and vanish on E for $d > 0$, so they define holomorphic functions on U. As functions on U, they generate $\pi_*(\mathcal{J}\mathcal{I}_E^d)$. Their pullbacks to $\tilde{U}$ generate $\pi^{-1}\pi_*(\mathcal{J}\mathcal{I}_E^d)$ and are once again the functions $F(z, f)$.

Example 5.5. — Let $\mathcal{I}$ be ideal sheaf of the origin in $\mathbb{C}^3$ (i.e. $V(\mathcal{I}) = C = \{Z_1 = Z_2 = Z_3 = 0\}$), let $\pi : \tilde{\mathbb{C}}^3 \to \mathbb{C}^3$ be the blow-up along $\mathcal{I}$, and let $E = \pi^{-1}(C)$ be the exceptional divisor. Let $\mathcal{J}$ be the ideal on $\tilde{\mathbb{C}}^3$ generated by the homogeneous polynomial $\xi_1\xi_2 - \xi_3^2$. Let $F(Z) = Z_1Z_2 - Z_3^2$ be the corresponding polynomial on $\mathbb{C}^3$. Then π^*F is a holomorphic section of $\mathcal{J}\mathcal{I}_E^2$. We have

$$\mathcal{J} \supset \mathcal{J}\mathcal{I}_E \supset \mathcal{J}\mathcal{I}_E^2 \supset ...$$

and

$$\pi^{-1}\pi_*(\mathcal{J}\mathcal{I}_E^d) = \begin{cases} \mathcal{J}\mathcal{I}_E^2 & d < 2 \\ \mathcal{J}\mathcal{I}_E^d & d \geqslant 2. \end{cases}$$

Note that although we refer to $\xi_1\xi_2 - \xi_3^2$ as a generator of $\mathcal{J}$, it is not a function on $\tilde{\mathbb{C}}^3$. If U is any neighborhood of 0 in $\mathbb{C}^3$, the only nonzero holomorphic sections of $\mathcal{J}$ on $\tilde{U} = \pi^{-1}(U)$ are those generated by homogeneous polynomials of degree at least 2, which must be vanishing on E to degree at least 2.

Once again, the next result is proved by other methods in [HR]. In the algebraic setting it could be proved using ample line bundles. We restrict ourselves to the case in which the blow-up $\tilde{M}$ is smooth, since this is the only case we require and since we have defined the blow-up of $\tilde{M}$ along $\mathcal{J}$ only in the case in which $\tilde{M}$ is smooth.

Corollary 5.6. — *Let $\pi : \tilde{M} \to M$ be the blow-up of a compact complex manifold along a coherent sheaf of ideals $\mathcal{I}$ such that $\tilde{M}$ is smooth, and let E be the exceptional divisor of π. Let $\mathcal{J}$ be a coherent sheaf of ideals on $\tilde{M}$. Then there exists an integer d_0 such that the blow-up of $\tilde{M}$ along $\mathcal{J}$ is isomorphic to the blow-up of $\tilde{M}$ along $\pi^{-1}\pi_*(\mathcal{J}\mathcal{I}_E^d)$ for all $d \geqslant d_0$.*

Proof. — By Corollary 5.3 there exists a d_0 such that $\pi^{-1}\pi_*(\mathcal{J}\mathcal{I}_E^d) = \mathcal{J}\mathcal{I}_E^d$ for all $d \geqslant d_0$. By Lemma 3.4, the blow-up along $\mathcal{J}$ is isomorphic to the blow-up along $\mathcal{J}\mathcal{I}_E^d$. □

The direct image of a product is not always the product of the direct images. In the next lemma we give a condition under which products of ideal sheaves behave well under direct images of blow-up maps.

Lemma 5.7. — *Let $\pi : \tilde{M} \to M$ be the blow-up of a compact complex manifold M along a coherent sheaf of ideals $\mathcal{I}$ and let E be the exceptional divisor. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be coherent sheaves of ideals on $\tilde{M}$. Then for d_1 and d_2 large enough,*

$$\pi_*(\mathcal{J}_1\mathcal{J}_2\mathcal{I}_E^{d_1+d_2}) = \pi_*(\mathcal{J}_1\mathcal{I}_E^{d_1})\pi_*(\mathcal{J}_2\mathcal{I}_E^{d_2}).$$

Proof. — Since M is compact, it is enough to prove the lemma locally, on a blow-up $\pi : \tilde{U} \to U$ of an open set U. We use the notation of remark 5.4 above. By Corollary 4.6, if $\mathcal{J}$ is a coherent sheaf of ideals on $\tilde{U}$, then for d large enough and possibly after shrinking U, the ideal $\mathcal{J}$ is generated on $\tilde{U} \subset U \times \mathbb{P}^{r-1}$ by a finite number of degree d homogeneous polynomials $F(z,\xi)$ in homogeneous coordinates ξ on $\mathbb{P}^{r-1}$. As was shown in remark 5.4, the functions $F(z,f)$ generate the direct image $\pi_*(\mathcal{J}\mathcal{I}_E^d)$.

If a finite collection $\{F(z,\xi)\}$ of degree d_1 polynomials generates $\mathcal{J}_1$ and a finite collection $\{G(z,\xi)\}$ of degree d_2 polynomials generates $\mathcal{J}_2$, then the collection $\{F(z,f)\}$ generates $\pi_*(\mathcal{J}_1\mathcal{I}_E^{d_1})$ and the collection $\{G(z,f)\}$ generates $\pi_*(\mathcal{J}_2\mathcal{I}_E^{d_2})$. The collection of all products $F(z,f)G(z,f)$ generates $\pi_*(\mathcal{J}_1\mathcal{I}_E^{d_1})\pi_*(\mathcal{J}_2\mathcal{I}_E^{d_2})$. Similarly, the collection of all products $F(z,\xi)G(z,\xi)$ generates $\mathcal{J}_1\mathcal{J}_2$, and since these products are degree d_1+d_2 homogeneous polynomials in ξ, the collection of all products $F(z,f)G(z,f)$ generates $\pi_*(\mathcal{J}_1\mathcal{J}_2\mathcal{I}_E^{d_1+d_2})$. Thus

$$\pi_*(\mathcal{J}_1\mathcal{J}_2\mathcal{I}_E^{d_1+d_2}) = \pi_*(\mathcal{J}_1\mathcal{I}_E^{d_1})\pi_*(\mathcal{J}_2\mathcal{I}_E^{d_2}). \qquad \square$$

Remark 5.8. — To see that the direct image of a product is not always the product of the direct images, we refer to Example 5.5. In that example, we described a sheaf of ideals $\mathcal{J}$ on $\tilde{\mathbb{C}}^3$ generated by a degree 2 homogeneous polynomial and such that

$$\pi^{-1}\pi_*(\mathcal{J}\mathcal{I}_E^d) = \begin{cases} \mathcal{J}\mathcal{I}_E^2 & d < 2 \\ \mathcal{J}\mathcal{I}_E^d & d \geqslant 2. \end{cases}$$

Suppose that $\pi_*(\mathcal{J}\mathcal{I}_E^d) = (\pi_*\mathcal{J})(\pi_*\mathcal{I}_E^d)$. Then

$$\begin{aligned} \pi^{-1}\pi_*(\mathcal{J}\mathcal{I}_E) &= (\pi^{-1}\pi_*\mathcal{J})(\pi^{-1}\pi_*\mathcal{I}_E) \qquad \text{by Lemma 2.15} \\ &= (\mathcal{J}\mathcal{I}_E^2)\mathcal{I}_E \\ &= \mathcal{J}\mathcal{I}_E^3 \end{aligned}$$

which is impossible since

$$\pi^{-1}\pi_*(\mathcal{J}\mathcal{I}_E) = \mathcal{J}\mathcal{I}_E^2$$

by the example.

Appendix 5.A: A valuation criterion of Lejeune - Teissier

In the remainder of this section we provide, as an illustration of the methods developed in sections 4 and 5, a simple constructive proof of the following variant of the 'valuation criterion' due to M. Lejeune and B. Teissier [LT].

THEOREM 5.A.1. — *Let M be a complex manifold and let a be a point in M. For any ideal J in $\mathcal{O}_{M,a}$ the following 'closures' of J coincide:*

(1) the 'arc-closure' $J^{arc} := \{f : \text{for any arc } \gamma(t) \quad ord_t J(\gamma(t)) \leqslant ord_t f(\gamma(t))\}$.

(2) the integral closure $\overline{J} := \{f : f \cdot N \subset J \cdot N$ for a finite $\mathcal{O}_{M,a}$-module $N\} = \{f : f^m + \sum_j c_j f^{m-j} = 0$, for some $c_j \in (J)^j, j = 1, ..., m\}$.

(3) the set $J_\pi := \pi_*(\pi^{-1}J)$, where $\pi : \tilde{M} \to (M, a)$ is any desingularization of J, i.e. π is a composite of blow-ups with smooth centers and $\pi^{-1}J$ is a normal crossings divisor on $\tilde{M}$.

(4) the set $J^\phi := \{f \in \mathcal{O}_{M,a} : \phi^* f \in \phi^{-1}J\}$, where $\phi : \tilde{M} \to (M, a)$ is any proper dominating morphism from a smooth source $\tilde{M}$, such that $\phi^{-1}J$ is a normal crossings divisor on $\tilde{M}$.

Moreover, assuming that $\mathcal{J}$ is a coherent sheaf of ideals in $\mathcal{O}_M$ and K is a compact set in M, we construct 'explicitly' a coherent sheaf of ideals $\mathcal{N}$ in $\mathcal{O}_M$, such that for any $a \in K$ and f in the stalk J_π at a of $\mathcal{J}_\pi = \pi_*(\pi^{-1}\mathcal{J})$ from (3) above, the inclusion $f \cdot N \subset J \cdot N$ holds (as in (2) above), with J and N being the stalks at a of $\mathcal{J}$ and $\mathcal{N}$.

Proof. — Obviously $\overline{J} \subset J^{arc} \subset J_\pi$. (Note that $J_\pi = J^\pi$.) We will now show that $J^{arc} = J^\phi$. The ideal $\phi^{-1}J$ is generated by $\phi^* g$ for $g \in J$ and all arcs γ in (M, a) are the images of the arcs $\tilde{\gamma}$ in $\tilde{M}$. Therefore $f \in J^{arc}$ if and only if $\phi^* f \in (\phi^{-1}J)_b^{arc}$ for all $b \in \phi^{-1}(a)$. Assuming $\phi^{-1}J$ is a normal crossings divisor on $\tilde{M}$, it follows that $(\phi^{-1}J)_b^{arc} = (\phi^{-1}J)_b$ for all $b \in \phi^{-1}(a)$. Therefore $f \in J^{arc}$ if and only if $\phi^* f \in \phi^{-1}J$, as required in (4).

It remains to show that $J_\pi \subset \overline{J}$. This inclusion follows directly from Lemma 5.A.2 below applied inductively on the number of blow-ups, similarly to the application of sections 4 and 5 in section 6.

The last statement of the theorem also follows from Lemma 5.A.2 (see Remarks 5.A.3 below). □

LEMMA 5.A.2. — *Let U be a coordinate chart in a complex manifold. Let $\sigma : \tilde{U} \to U$ be a blow-up with smooth center C of codimension r, let* *b be a point in U, let $h \in O_{U,b}$ and set $\tilde{h} = \sigma^* h$. Let $E = \sigma^{-1}C$ be the exceptional divisor, and let $\mathcal{I}_C \subset \mathcal{O}_U$ and $\mathcal{I}_E \subset \mathcal{O}_{\tilde{U}}$ be the coherent sheaves of ideals corresponding to C and E, respectively. Suppose that there are coherent sheaves of ideals $\mathcal{I} \subset \mathcal{O}_U$, $\tilde{\mathcal{N}} \subset \mathcal{O}_{\tilde{U}}$, and $\tilde{\mathcal{I}} = \sigma^{-1}\mathcal{I}$, such that in germs at the points of $\sigma^{-1}(b)$ the inclusion $\tilde{h} \cdot \tilde{\mathcal{N}} \subset \tilde{\mathcal{I}} \cdot \tilde{\mathcal{N}}$ holds (our main inductive assumption). Then in germs at b the inclusion $h \cdot \mathcal{N} \subset \mathcal{I} \cdot \mathcal{N}$ holds, where $\mathcal{N} = \sigma_*(\tilde{\mathcal{N}} \cdot \mathcal{I}_E^s) \cdot \mathcal{I}_C^m$ and s and m are large enough integers.*

Remarks 5.A.3.. — (a) Assuming $g \in J_\pi \subset \mathcal{O}_{M,a}$, for a desingularization map π, we take the element h in Lemma 5.A.2 to be the appropriate pullback of g, which is therefore well defined at all points b of the respective inverse image of a, and also is well defined as a germ of a 'global' section at the whole inverse image of $a \in M$. Respectively there is a 'global' version of Lemma 5.A.2, i.e. replacing a single point b by the whole appropriate inverse image of $a \in M$. The version of Lemma 5.A.2 as stated above and proved below implies this 'global' version by a compactness argument (as in 5.3) and a choice of sufficiently large s and m. It is this 'global' version that we apply (see (b) below) inductively to conclude that $g \in \overline{J}$, as required.

(b) Suppose that $\mathcal{J}$ is a coherent sheaf of ideals on a neighborhood of $a \in \mathcal{O}_M$ such that J is the stalk of $\mathcal{J}$ at a. Note that if h and $\mathcal{I}$ are the pullbacks of the original g and $\mathcal{J}$ to the level preceding the blow-up σ in the desingularization 'tower' in (3), we then may continue the inductive process of 'descent' (one blow-up down in the desingularization 'tower') with $\mathcal{N}$ in the place of $\tilde{\mathcal{N}}$. Starting with $\mathcal{I}_0 := \pi^{-1}\mathcal{J}$, the equality $\mathcal{I}_0 = \mathcal{I}_0^{arc}$ is direct and easy, which allows us to start the inductive process. We have $\tilde{\mathcal{N}}_0 = \mathcal{O}_{\tilde{M}}$ and $\pi^* g \in \mathcal{I}_0$ in germs at points of $\pi^{-1}(a)$ at this top level.

Proof of Lemma. — The claim of Lemma 5.A.2. is in terms of germs in $O_{U,b}$, which allows to assume in the proof below that U is a coordinate chart such that $\mathcal{I}_C$ is generated over $\mathcal{O}_U$ by functions $f_1, ..., f_r$. Let z represent U-coordinates and let ξ represent homogeneous $\mathbb{P}^{r-1}$-coordinates.

The blow-up $\sigma : \tilde{U} \to U$ is a composite of an inclusion $i : \tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$ and the natural projection $p : U \times \mathbb{P}^{r-1} \to U$. Below we freely identify $g \in \mathcal{O}_U$ and $p^* g \in \mathcal{O}_{U \times \mathbb{P}^{r-1}}$ as functions independent of variables along $\mathbb{P}^{r-1}$ and via this natural convention (and by a slight abuse of notation) write g for $p^* g$, e.g. $\tilde{h} = h$. Similarly, $\mathcal{I}$ as a subset of $\mathcal{O}_{U \times \mathbb{P}^{r-1}}$ stands for $\{p^* g : g \in \mathcal{I}\}$.

We can view our sheaves of ideals on $\tilde{U}$ as sheaves of ideals on $U \times \mathbb{P}^{r-1}$, generated by the generators of the sheaves on $\tilde{U}$ plus the generators $\tilde{G}_{ij}(z, \xi) = f_j(z)\xi_i - f_i(z)\xi_j$ of the ideal $\mathcal{I}_{\tilde{U}}$ on $U \times \mathbb{P}^{r-1}$ [cf. 5.4], e.g. $\tilde{\mathcal{I}}$ in

$\mathcal{O}_{U\times\mathbb{P}^{r-1}}$ is generated by $\mathcal{I}$ and $\tilde{G}_{ij}$'s. Multiplying our main assumption of the lemma by $\mathcal{I}_E^s$ gives $\tilde{h}\cdot\tilde{\mathcal{N}}\cdot\mathcal{I}_E^s \subset \tilde{\mathcal{I}}\cdot\tilde{\mathcal{N}}\cdot\mathcal{I}_E^s$ (in $\mathcal{O}_{U\times\mathbb{P}^{r-1}}$ modulo the ideal $\mathcal{I}_{\tilde{U}}$ of $\tilde{U}$). By Theorem 4.1, we may assume that the ideals $\tilde{\mathcal{N}}\cdot\mathcal{I}_E^s$ and $\tilde{\mathcal{I}}\cdot\tilde{\mathcal{N}}\cdot\mathcal{I}_E^s$ on $\tilde{U}$ are generated by finite collections of homogeneous-in-ξ polynomials $\{H_j(z,\xi)\}$ and $\{G_k(z,\xi)\}$, respectively, of the same degree $d = s$ (as in 4.6 and 4.7), with analytic coefficients in U-coordinates z. The validity of the latter is the only meaning of 's is large enough' in the assumptions. When we view these sheaves of ideals as sheaves on $U \times \mathbb{P}^{r-1}$, we include the additional generators $\tilde{G}_{ij}$, as described above.

Recall from the end of the proof of 4.6, that if we start with a collection of homogeneous generators of differing degrees, we may obtain a collection of homogeneous generators which are all of the same degree by multiplying by suitable monomials in ξ. Thus, we may assume that the generators $\{G_k(z,\xi)\}$ of the ideal $\tilde{\mathcal{I}}\cdot\tilde{\mathcal{N}}\cdot\mathcal{I}_E^s$ in $\mathcal{O}_{U\times\mathbb{P}^{r-1}}$ (modulo $\mathcal{I}_{\tilde{U}}$) are products of generators of $\mathcal{I}$ (functions of z only) times generators of $\tilde{\mathcal{N}}\cdot\mathcal{I}_E^s$ (homogeneous in ξ by 4.1) times suitable monomials in ξ. This allows us to 'view' our main assumption in $\mathcal{O}_{U\times\mathbb{P}^{r-1}}$ as follows. Pick any H_j and let $H = h\cdot H_j$. Then our main assumption means that H is in the ideal generated by the G_k's (and $\tilde{G}_{ij}$'s) over $\mathcal{O}_{U\times\mathbb{P}^{r-1}}$ (in stalks at the points of $U\times\mathbb{P}^{r-1}$).

Now consider a point c in the coordinate chart $V_i = \{\xi_i \neq 0\} \subset \mathbb{P}^{r-1}$. Set $\xi_i = 1$ and let $(y_1, ..., y_{r-1})$ represent the $r-1$ entries of ξ other than ξ_i on $V_i \cong \mathbb{C}^{r-1}$. Let $H'(z,y)$, $G'_k(z,y)$, and $\tilde{G}'_{ij}(z,y)$ be the restrictions of H, G_k, and $\tilde{G}_{ij}$, respectively, to the chart $U_i = U\times V_i$. Then there exist elements $A_{k,c}$ and $B_{ij,c}$ in the local analytic ring on U_i at the point (b,c) such that

$$H' = \sum_k A_{k,c}G'_k + \sum_{ij} B_{ij,c}\tilde{G}'_{ij}.$$

We will show that the coefficients $A_{k,c}$ and $B_{ij,c}$ may actually be chosen to be quotients of polynomials in y with analytic coefficients in z, with common denominators D'_c, depending on c, which do not vanish at the point (b,c). The proof is by a standard application of Krull's Theorem: given a noetherian ring A with completion $\hat{A}$ at its maximal ideal (completion in Krull topology) then for any ideal $\mathbb{G}$ in A, the intersection of A with the ideal $\hat{\mathbb{G}} = \mathbb{G}\cdot\hat{A}$ is $\mathbb{G}$ (in [M] stated on the page preceeding Ch.1, follows from [ZS] p. 262 Theorem 9 and p. 257 Corollary 2). We may assume, without loss of generality, that $z = b = 0$ and $y = c = 0$. Let $B = \mathcal{O}_{U,0}[y_1, ..., y_{r-1}]$, and let A be the localization of B at the ideal generated by y's. (Thus A is a local noetherian ring and its completion in the Krull topology is the ring $\hat{A} = \hat{C}$, where $C = \mathcal{O}_{U\times\mathbb{C}^r,(0,0)}$.) Let $\mathbb{G}$ be the ideal of A generated by the polynomials G'_k (and $\tilde{G}'_{ij}$'s). Then $\mathbb{G} = A\cap(\mathbb{G}\cdot\hat{A})$ by Krull's Theorem.

Therefore, since $H' = \sum_k A_{k,c}G'_k + \sum_{ij} B_{ij,c}\tilde{G}'_{ij}$ at the point (b, c) with coefficients $A_{k,c}$, $B_{ij,c}$ in $\hat{A}$ and G'_k, $\tilde{G}'_{ij}$ in $\mathbb{G} \subset A$, and $H' \in A$, then also $H' \in \mathbb{G}$, i.e. $H' = \sum_k Q_{k,c}G'_k + \sum_k \tilde{Q}_{ij,c}\tilde{G}'_{ij}$ with $Q_{k,c}$ and $\tilde{Q}_{ij,c}$ in A (hence quotients of polynomials in y, with analytic coefficients in z, and common denominators D'_c, depending on $c \in V_i$, that do not vanish at (b, c)). Therefore

$$D'_c H' = \sum_k P'_{k,c}G'_k + \sum_k \tilde{P}'_{ij,c}\tilde{G}'_{ij},$$

where $P'_{k,c}$'s and $\tilde{P}'_{ij,c}$'s are polynomials in y with coefficients analytic germs in z (and D'_c's are as above).

Now, we replace the coordinates y by the $r - 1$ entries of ξ other than ξ_i, and then replace ξ_j, for $j \neq i$, by ξ_j/ξ_i. Multiplying both sides of the latter equality of polynomials (in affine coordinates ξ of the chart $U_i = \{\xi : \xi_i = 1\}$) by ξ_i^l for a sufficiently large integer l yields an equality of homogeneous-in-ξ polynomials: $\xi_i^n D_c H = \sum_k \xi_i^{n-d_k} P_{k,c} G_k \pmod{\mathcal{I}_{\tilde{U}}}$, where the polynomials D_c and $P_{k,c}$ are the 'homogenizations' of the polynomials D'_c and $P'_{k,c}$, the integers d_k are the differences of the degrees in ξ of $P_{k,c}$'s and D_c, and the integer n is large enough that $n - d_k \geqslant 0$ for all k (and all polynomials H and G_k are of the same degree d in ξ by our original assumption about the H_j's and the G_k's). The analogous argument yields the same conclusion for every ξ_i for $i = 1, ..., r$. The polynomials D_c are homogeneous in $\xi \in \mathbb{C}^r$, with coefficients which are analytic germs in z at b, $c \in \mathbb{P}^{r-1}$, and we view them as analytic germs at $(b, 0)$ in (z, ξ)-coordinates on $U \times \mathbb{C}^r$) which have common zeroes only for $\xi = 0$. Therefore, making use of the complex analytic Nullstellensatz (see e.g. [Lo, p. 196]), it follows that for a sufficiently large integer m the ideal generated by all 'denominators $\xi_i^n D_c$' in the ring of the germs at $z = b$, $\xi = 0 \in \mathbb{C}^r$ of complex analytic functions in z, ξ contains all monomials ξ^α of order m, and since the denominators are homogeneous in ξ yields a representation of each monomial (of a sufficiently high order) as a finite linear combination of 'denominators $\xi_i^n D_c$' with coefficients which are polynomials in ξ and analytic (germs at $z = b$) in z. The latter implies (for all sufficiently large integers m) an equality (modulo ideal $\mathcal{I}_{\tilde{U}}$) of homogeneous-in-ξ polynomials: $\xi^\alpha H = \sum_k P_{k,\alpha} G_k \pmod{\mathcal{I}_{\tilde{U}}}$, where all monomials ξ^α and polynomials $P_{k,\alpha}$ are of order m in ξ (and all polynomials H and G_k's are of the same degree d in ξ). Validity of the latter suffices for the 'm is large enough' in the assumptions of the Lemma 5.A.2.

Thus for sufficiently large integers m, all monomials ξ^α of order m, and for all j,

$$\xi^\alpha \tilde{h}(z) H_j(z, \xi) \in (\xi)^m(G) \pmod{\mathcal{I}_{\tilde{U}}},$$

where (G) stands for the ideal generated by the G_k's in the ring of polynomials in ξ with coefficients that are in $\mathcal{O}_{U,b}$. Letting $\xi_i = f_i(z)$, where the functions $f_1, ..., f_r$ are the local defining equations for the smooth center $C \subset U$, (similarly to the argument in 5.4) it follows, by making use of the third and fourth paragraphs of our proof, that

$$f^\alpha h(z) H_j(z, f) \in \mathcal{I} \cdot (\mathcal{I}_C)^m \cdot (\sigma_*(\tilde{\mathcal{N}} \cdot \mathcal{I}_E^s))$$

for all sufficiently large integers m. In other words in germs at b the inclusion $h \cdot \mathcal{N} \subset \mathcal{I} \cdot \mathcal{N}$ holds, with $\mathcal{N} = (\mathcal{I}_C)^m \cdot (\sigma_*(\tilde{\mathcal{N}} \cdot \mathcal{I}_E^s))$, as required. □

LEMMA 5.A.4. — *Let U be a coordinate chart in a complex manifold. Let $\sigma : \tilde{U} \to U$ be a blow-up with smooth center C of codimension r. Let $E = \sigma^{-1}C$ be the exceptional divisor, and let $\mathcal{I}_C \subset \mathcal{O}_U$ and $\mathcal{I}_E \subset \mathcal{O}_{\tilde{U}}$ be the coherent sheaves of ideals corresponding to C and E, respectively. Suppose that there are coherent sheaves of ideals $\mathcal{I} \subset \mathcal{O}_U$, $\tilde{\mathcal{N}} \subset \mathcal{O}_{\tilde{U}}$, and $\tilde{\mathcal{I}} = \sigma^{-1}\mathcal{I}$, such that the blow-up $\tilde{\sigma}$ of $\tilde{U}$ along $\tilde{\mathcal{N}}$ is a desingularization of $\tilde{\mathcal{I}}$, i.e., $\tilde{\sigma}$ is equivalent to a composite of a finite number of blow-ups with smooth centers and $\tilde{\sigma}^{-1}\tilde{\mathcal{I}}$ is a normal crossings divisor. Then (after shrinking U if necessary) the blow-up of U along $\mathcal{N}$ desingularizes $\mathcal{I}$, where $\mathcal{N} = \sigma_*(\tilde{\mathcal{N}} \cdot \mathcal{I}_E^s) \cdot \mathcal{I}_C^m$, $m \geqslant 1$ is an integer and s is a large enough integer.*

Proof. — The blow-up of U along $\mathcal{N}$ is equivalent to the blow-up of U along C followed by the blow-up of $\tilde{U}$ along $\sigma^{-1}\sigma_*(\tilde{\mathcal{N}} \cdot \mathcal{I}_E^s)$, by Proposition 3.7 and Corollary 3.8. But if s is large enough that $\tilde{\mathcal{N}} \cdot \mathcal{I}_E^s$ is generated on $\tilde{U}$ by global sections (after shrinking U if necessary), then $\sigma^{-1}\sigma_*(\tilde{\mathcal{N}} \cdot \mathcal{I}_E^s) = \tilde{\mathcal{N}} \cdot \mathcal{I}_E^s$ (see Corollaries 5.3 and 4.6). The blow-up of $\tilde{U}$ along $\tilde{\mathcal{N}} \cdot \mathcal{I}_E^s$ is equivalent to the blow-up of $\tilde{U}$ along $\tilde{\mathcal{N}}$, by Lemma 3.2. □

COROLLARY 5.A.5 *In Theorem 5.A.1, the blow-up of M along the coherent sheaf of ideals $\mathcal{N}$ which we construct using Lemma 5.A.3 is a desingularizing blow-up for $\mathcal{J}$.*

Proof. — Suppose that our desingularizing tower of blow-ups along smooth centers is $\tilde{M} = M_n \xrightarrow{\sigma} M_{n-1} \to ... \to M_0 = M$, where σ is a blow-up along smooth center C with exceptional divisor E. In the first step of our proof of 5.A.1 by inductive descent using Lemma 5.A.3, we construct $\mathcal{N}_1$ on M_{n-1} to be of the form $\mathcal{N}_1 = \sigma_*(\mathcal{O}_{\tilde{M}} \cdot \mathcal{I}_E^s) \cdot \mathcal{I}_C^m$, i.e., $\mathcal{N}_1$ is of the form $\mathcal{I}_C^k$ for k a large enough integer. We now apply Lemma 5.A.4 repeatedly. □

6. Replacing a sequence of blow-ups by a single blow-up

Let X be a singular subvariety of a compact complex manifold M. In this section we show how to replace a sequence of blow-ups along smooth centers, which gives embedded desingularization of X, by a single blow-up of M along a coherent sheaf of ideals $\mathcal{I}$, which is a product of coherent ideals corresponding to the centers. The support of $\mathcal{I}$ is the singular locus of X, the proper transform $\tilde{X}$ of X in the blow-up of M along $\mathcal{I}$ is nonsingular, and the exceptional divisor of the blow-up along $\mathcal{I}$ has normal crossings and is also normal crossings with the embedded desingularization $\tilde{X}$. This result is proved by other methods in [HR] (or its canonical version in [BMI]).

Proposition 6.1. — *Let M be a compact complex manifold and let*

$$M'' \xrightarrow{\pi'} M' \xrightarrow{\pi} M$$

be a sequence of blow-ups such that

a. *$\pi : M' \to M$ is the blow-up of M along a coherent sheaf of ideals $\mathcal{I}$ such that M' is smooth and $V(\mathcal{I})$ has codimension at least 2 and*

b. *$\pi' : M'' \to M'$ is the blow-up of M' along a smooth center C of codimension at least 2.*

Let E be the exceptional divisor of π in M'. Then the sequence of blow-ups $M'' \to M' \to M$ is equivalent to a single blow-up along a coherent sheaf of ideals $\mathcal{J}$ on M given by

$$\mathcal{J} = \mathcal{I}\mathcal{I}'$$

where $\mathcal{I}' = \pi_*(\mathcal{I}_C\mathcal{I}_E^d)$ and d is a large enough positive integer so that $\pi^{-1}\pi_*(\mathcal{I}_C\mathcal{I}_E^d) = \mathcal{I}_C\mathcal{I}_E^d$. Furthermore

i. *the blow-up of M' along $\pi^{-1}\mathcal{I}' = \mathcal{I}_C\mathcal{I}_E^d$ is isomorphic to the blow-up along C, i.e. the blow-up of M' along $\pi^{-1}\mathcal{I}'$ is isomorphic to M'', and*

ii. *the complex space $V(\mathcal{J})$ determined by $\mathcal{J}$ has codimension at least 2 in M.*

Proof. — By Corollary 5.3

$$\pi^{-1}\pi_*(\mathcal{I}_C\mathcal{I}_E^d) = \mathcal{I}_C\mathcal{I}_E^d$$

for all sufficiently large d. We apply Proposition 3.7 to $\mathcal{J} = \mathcal{I}\mathcal{I}' = \mathcal{I}\,\pi_*(\mathcal{I}_C\mathcal{I}_E^d)$ to show that blowing up M along $\mathcal{J}$ is equivalent to first blowing up M along $\mathcal{I}$ to obtain M', and then blowing up M' along $\mathcal{I}_C\mathcal{I}_E^d$. But the blow-up along $\mathcal{I}_C\mathcal{I}_E^d$ is equivalent to the blow-up along $\mathcal{I}_C$ by Lemma 3.4.

Finally we note that

$$\begin{aligned} V(\mathcal{J}) &= V(\mathcal{I}) \cup V(\pi_*(\mathcal{I}_C\mathcal{I}_E^d)) \\ &= V(\mathcal{I}) \cup \pi(V(\mathcal{I}_C) \cup V(\mathcal{I}_E^d)) \\ &= V(\mathcal{I}) \cup \pi(C) \end{aligned}$$

which has codimension at least 2. □

We apply Proposition 6.1 inductively to obtain

Proposition 6.2. — *Let M_0 be a compact complex manifold and let*

$$M_m \overset{\pi_m}{\to} M_{m-1} \overset{\pi_{m-1}}{\to} \dots \overset{\pi_2}{\to} M_1 \overset{\pi_1}{\to} M_0$$

be a sequence of blow-ups along smooth centers $C_j \subset M_{j-1}$ of codimension at least 2. Then the composite $\pi_1 \circ \dots \circ \pi_m : M_m \to M_0$ is equivalent to a single blow-up along a coherent sheaf of ideals

$$\mathcal{I} = \mathcal{I}_1\mathcal{I}_2 \dots \mathcal{I}_m$$

where $\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_m$ are coherent sheaves of ideals on M such that

i. *the blow-up of M_{j-1} along the inverse image ideal of $\mathcal{I}_j$ on M_{j-1} is isomorphic to the blow-up of M_{j-1} along C_j, and*

ii. *the complex space $V(\mathcal{I})$ has codimension at least 2 in M_0.*

Proof. — We construct the ideal sheaves $\mathcal{I}_1, \dots, \mathcal{I}_m$ inductively, using Proposition 6.1, and noting that all the spaces M_j are smooth, since the centers of the blow-ups are smooth. We may construct an ideal sheaf $\mathcal{I}_j$ from $\mathcal{I}_{C_j}$ either step-by-step, going down one level at a time, or all in one step, using the composite of the first $j-1$ blow-ups. We use the second method in this proof, because it is notationally simpler. The first method is computationally simpler, so we use it in our example in section 10.

Start by letting $\mathcal{I}_1 = \mathcal{I}_{C_1}$, the ideal sheaf of the first center C_1, and construct $\mathcal{I}_2$ as in Proposition 6.1. The blow-up of M_1 along $\pi_1^{-1}\mathcal{I}_2$ is isomorphic to M_2 and the complex space $V(\mathcal{I}_1\mathcal{I}_2)$ has codimension at least 2. Next suppose that we have constructed $\mathcal{I}_1, \dots, \mathcal{I}_{j-1}$ satisfying condition (i), and such that $V(\mathcal{I}_1...\mathcal{I}_{j-1})$ has codimension at least 2 in M_0. Condition (i) implies that the blow-up of M_0 along the product $\mathcal{I}_1...\mathcal{I}_{j-1}$ is isomorphic to M_{j-1}. Let

$$\tau = \pi_1 \circ ... \circ \pi_{j-1} : M_{j-1} \to M_0$$

be this blow-up map and let D be the exceptional divisor of τ in M_{j-1}. Pick d large enough such that $\tau^{-1}\tau_*(\mathcal{I}_j\mathcal{I}_D^d) = \mathcal{I}_j\mathcal{I}_D^d$ and set

$$\mathcal{I}_j = \tau_*(\mathcal{I}_j\mathcal{I}_D^d).$$

Then apply Proposition 6.1. □

Using Hironaka's theorem on the existence of embedded resolutions of singularities we obtain

Corollary 6.3. — *Let M be a compact complex manifold and let X be a singular subvariety of M. Let*

$$M_m \overset{\pi_m}{\to} M_{m-1} \overset{\pi_{m-1}}{\to} ... \overset{\pi_2}{\to} M_1 \overset{\pi_1}{\to} M_0 = M$$

be a sequence of blow-ups along smooth centers $C_j \subset M_{j-1}$ of codimension at least 2 which gives embedded resolution of X. Then there exists a coherent sheaf of ideals $\mathcal{I}$ on M of the form

$$\mathcal{I} = \mathcal{I}_1\mathcal{I}_2...\mathcal{I}_m$$

such that for each j, the blow-up map of M along $\mathcal{I}_1\mathcal{I}_2...\mathcal{I}_j$ is equivalent to the composite map $\pi_1 \circ \pi_2 \circ ... \circ \pi_j : M_j \to M_0$. In particular,

i. *the proper transform $\tilde{X}$ of X in the blow-up $\tilde{M}$ of M along $\mathcal{I}$ is nonsingular,*

ii. *$V(\pi^{-1}\mathcal{I})$ is a normal crossings divisor in $\tilde{M}$ which has normal crossings with $\tilde{X}$, and*

iii. *the support of $\mathcal{I}$ is the singular locus of X in M.*

7. Chern forms for exceptional line bundles

Let $\pi : \tilde{M} \to M$ be the blow-up of a compact complex manifold M along a coherent sheaf of ideals $\mathcal{I}$ such that $\tilde{M}$ is smooth. Let E be the exceptional divisor of π and $L_E = [E]$ the associated line bundle on $\tilde{M}$. In this section we describe explicitly the construction of a Chern form on L_E.

We first consider sets of the form $\tilde{U} = \pi^{-1}(U)$, where U is a small open set in M. We construct a local Chern form on $\tilde{U}$ associated with an embedding $\iota_f : \tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$ determined by holomorphic functions $f_1, ..., f_r$ which generate $\mathcal{I}$ on U. The embedding ι_f induces a local metric and local Chern form on the line bundle L_E over $\tilde{U}$, using the Fubini-Study form on $\mathbb{P}^{r-1}$. Different embeddings of $\tilde{U}$ corresponding to different choices of local generators of $\mathcal{I}$ may give different Chern forms in the same Chern class. This type of local Chern form has a particularly simple formula in terms of the local generators of $\mathcal{I}$ and is negative semi-definite on $\tilde{U}$, since it is the pullback of the negative of the Fubini-Study form on $\mathbb{P}^{r-1}$. If $\mathcal{I}$ is the sheaf of ideals of a smooth center of codimension $k \geqslant 2$, so that the fibers of the map $E \to C$ are isomorphic to $\mathbb{P}^{k-1}$, then this Chern form is negative definite on the fibres of the map $E \to C$. We patch globally using C^∞ partitions of unity on M, to obtain global metrics and Chern forms for L_E.

Chern forms on line bundles

We begin with some background material on Chern forms. Let $L \to N$ be a holomorphic line bundle on a complex manifold N. Choose a cover of N by open sets V_i such that L is trivial on V_i. A holomorphic section s of L over N may be given by a collection of holomorphic functions s_i on V_i which transform on $V_i \cap V_j$ by the rule

$$s_i = g_{ij} s_j,$$

where g_{ij} is a holomorphic transition function on $V_i \cap V_j$. A hermitian metric h on L may be described by a collection of positive C^∞ functions h_i on V_i such that the norm of s is given on V_i by

$$\| s \|^2 = | s_i |^2 h_i.$$

The functions h_i transform by the rule

$$h_j = | g_{ij} |^2 h_i.$$

Local description of a Chern form. — The Chern form of L with respect to h is given on V_i by

$$c_1(L, h) = -\frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log h_i.$$

Note that this formula gives a well-defined (1,1)-form on N, because

$$\partial\overline{\partial} \log h_j = \partial\overline{\partial} \log | g_{ij} |^2 h_i$$

$$= \partial\overline{\partial}(\log g_{ij} + \log \overline{g}_{ij} + \log h_i)$$
$$= \partial\overline{\partial}\log h_i \qquad\qquad \text{since } g_{ij} \text{ is holomorphic.}$$

Formula for a Chern form in terms of the norm of a section s. — For convenience, we often write the Chern form of L in terms of the norm of a nonzero section s with respect to h, as

$$c_1(L,h) = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log \|\, s\,\|^2.$$

We note that $c_1(L,h)$ is well-defined and independent of the section s used to calculate it, since locally on an open set V_i minus the zero locus of s we have $\partial\overline{\partial}\log\|\, s\,\|^2 = \partial\overline{\partial}\log h_i$.

Chern forms via pullbacks. — Chern forms behave well under pullbacks. Suppose that $\phi : N_1 \to N_2$ is a holomorphic map of complex manifolds and L is a line bundle on N_2 with metric h. Then ϕ^*L is a line bundle on N_1 with an induced metric ϕ^*h, and the Chern form of ϕ^*L with respect to ϕ^*h is the pullback by ϕ of the Chern form of L with respect to h, i.e.

$$c_1(\phi^*L, \phi^*h) = \phi^*c_1(L,h).$$

Local Chern forms for blow-ups

Let U be an open set in $\mathbb{C}^n$ and let $\pi : \tilde{U} \to U$ be the blow-up of U along a coherent sheaf of ideals $\mathcal{I}$ such that $\tilde{U}$ is smooth. We will assume that U is small enough that $\mathcal{I}$ is generated by global sections $f_1, ..., f_r$ over U. Let E be the exceptional divisor and L_E the associated line bundle on $\tilde{U}$.

If $\mathcal{I}$ is generated by a single function over U, then the sheaf $\mathcal{I}$ is principal, the line bundle L_E is trivial on $\tilde{U}$, and we may choose a metric h on L_E such that $c_1(L_E, h) = 0$.

We assume from now on that the blow-up is non-trivial, i.e. that $\mathcal{I}$ is not principal on U and has support of codimension at least 2 in U. In this case $r > 1$ and the generators $f_1, ..., f_r$ of $\mathcal{I}$ give an embedding

$$\iota_f : \tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1},$$

as described in section 3. Let $[\xi_1 : ... : \xi_r]$ be homogeneous coordinates for $\mathbb{P}^{r-1}$. The blow-up $\tilde{U}$ is covered by open sets

$$\tilde{U}_i = \tilde{U} \cap \{\xi_i \neq 0\}$$

on which L_E is trivial. Transition functions for L_E on the intersections $\tilde{U}_i \cap \tilde{U}_j$ are the functions $g_{ij} = \frac{\xi_i}{\xi_j}$. To distinguish between a generating function f_i on U and its pullback to $\tilde{U} \subset U \times \mathbb{P}^{r-1}$, we will let

$$\tilde{f}_i = \pi^* f_i.$$

The exceptional divisor E is given on $\tilde{U}_i$ by $\tilde{f}_i = 0$. The collection of functions $\tilde{f}_i$ on the sets $\tilde{U}_i$ determines a section s of L_E over $\tilde{U}$, vanishing exactly on E.

Lemma 7.1. — *Let U be an open set in $\mathbb{C}^n$ and let $\pi : \tilde{U} \to U$ be the blow-up of U along a coherent sheaf of ideals $\mathcal{I}$ which is generated by global sections $f_1, ..., f_r$ on U. Suppose that the blow-up is non-trivial and that $\tilde{U}$ is smooth. Let E be the exceptional divisor and L_E the associated line bundle on $\tilde{U}$. Then the embedding $\iota_f : \tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$ induces a metric h on L_E and associated Chern form $c_1(L_E, h)$ such that $c_1(L_E, h)$ is negative semi-definite on $\tilde{U}$ and is given on $\tilde{U} - E$ by*

$$c_1(L_E, h) = \pi^*\left(-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\sum_{j=1}^{r} \mid f_j \mid^2\right).$$

Remark 7.2. — If $\mathcal{I}$ is the sheaf of ideals of a smooth center of codimension $k \geqslant 2$, so that the fibers of the map $E \to C$ are isomorphic to $\mathbb{P}^{k-1}$, then this Chern form is negative definite on the fibres of the map $E \to C$.

Proof. — We will construct the Chern form by pullback, without using an explicit formula for the metric h. For an explicit local formula for h, see the remark following this proof.

Let σ_2 be the second projection map $\sigma_2 : U \times \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ and ι_f the inclusion map $\iota_f : \tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$. Recall that the exceptional line bundle L_E on $\tilde{U} \subset U \times \mathbb{P}^{r-1}$ is the pullback of the universal bundle $\mathcal{O}_{\mathbb{P}^{r-1}}(-1)$. The Fubini-Study form $\omega_{\mathrm{Fub-St}}$ on $\mathbb{P}^{r-1}$ gives a Chern form for $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ and $-\omega_{\mathrm{Fub-St}}$ gives a Chern form for $\mathcal{O}_{\mathbb{P}^{r-1}}(-1)$. Pulling back to $\tilde{U}$, we obtain a Chern form for L_E (with respect to an induced metric h) given by

$$c_1(L_E, h) = \iota_f^* \sigma_2^*(-\omega_{\mathrm{Fub-St}}).$$

The negativity properties of $c_1(L_E, h)$ stated in the lemma follow directly from the fact that $\omega_{\mathrm{Fub-St}}$ is positive on $\mathbb{P}^{r-1}$.

Now recall the formula for the Fubini-Study form on projective space. If $\xi_1, ..., \xi_r$ are homogeneous coordinates on $\mathbb{P}^{r-1}$, then $w_{ij} = \frac{\xi_j}{\xi_i}$ for $j \neq i$ are nonhomogeneous coordinates on $U_i = \{\xi_i \neq 0\} \subset U \times \mathbb{P}^{r-1}$. The pullback of the Fubini-Study form $\omega_{\mathrm{Fub-St}}$ is given on U_i by

$$\begin{aligned}\sigma_2^*\omega_{\mathrm{Fub-St}} &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(1+\sum_{j\neq i} \mid w_{ij}\mid^2)\\ &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(1+\sum_{j\neq i}\left|\frac{\xi_j}{\xi_i}\right|^2).\end{aligned}$$

We continue to use the notation $\tilde{f}_i = \pi^* f_i$ to distinguish between the function f_i on U and its pullback to $\tilde{U}$. On $\tilde{U}_i = \tilde{U}\cap U_i$ we have $\frac{\xi_j}{\xi_i} = \frac{\tilde{f}_j}{\tilde{f}_i}$ which gives us

$$\begin{aligned}c_1(L_E,h) &= \iota_f^*\sigma_2^*(-\omega_{\mathrm{Fub-St}})\\ &= -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(1+\sum_{j\neq i}\left|\frac{\tilde{f}_j}{\tilde{f}_i}\right|^2)\\ &= -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\frac{\sum_{j=1}^r \mid \tilde{f}_j\mid^2}{\mid \tilde{f}_i\mid^2}.\end{aligned}$$

On $\tilde{U}_i - \tilde{U}_i \cap E$ we have $\tilde{f}_i(z) \neq 0$ so

$$\begin{aligned}c_1(L_E,h) &= -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(\log\sum_{j=1}^r \mid \tilde{f}_j\mid^2 - \log\mid \tilde{f}_i\mid^2)\\ &= -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\sum_{j=1}^r \mid \tilde{f}_j(z)\mid^2\\ &= \pi^*(-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\sum_{j=1}^r \mid f_j(z)\mid^2).\end{aligned}$$

This formula is independent of i, so is valid on all of $\tilde{U} - E$. □

Remark 7.3. — Local defining functions for the metric h on L_E induced from the embedding $\tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$ may also be given explicitly. Let s be the section of L_E given on $\tilde{U}_i$ by $\tilde{f}_i = 0$. The norm of s under the metric h is given by

$$\| s \|^2 = \sum_{j=1}^r \mid \tilde{f}_j \mid^2.$$

The metric h is described locally by positive C^∞ functions h_i on $\tilde{U}_i$ satisfying

$$\| s \|^2 = \mid \tilde{f}_i \mid^2 h_i.$$

Thus

$$h_i = \frac{\sum_{j=1}^{r} \mid \tilde{f}_j \mid^2}{\mid \tilde{f}_i \mid^2} = 1 + \sum_{j \neq i} \mid w_{ij} \mid^2,$$

where $w_{ij} = \frac{\xi_j}{\xi_i}$ for $j \neq i$ are nonhomogeneous coordinates on $\tilde{U}_i$.

Global Chern forms for blow-ups

We next use a C^∞ partition of unity on M to patch local metrics and Chern forms to obtain a global metric and Chern form for L_E on M.

Remark 7.4. — Throughout sections 7 and 8 we will refer to a C^∞ function F on M which we define by the formula

$$F = \prod_\alpha F_\alpha^{\rho_\alpha},$$

where $\{\rho_\alpha\}$ is a C^∞ partition of unity subordinate to an open cover $\{U_\alpha\}$ of M and F_α is a function on U_α. More precisely, we mean that we extend each function $F_\alpha^{\rho_\alpha}$ on U_α to a global function G_α on M by setting

$$G_\alpha(x) = \begin{cases} F_\alpha(x)^{\rho_\alpha(x)}, & \text{if } x \in U_\alpha \\ 1, & \text{otherwise} \end{cases}$$

and note that G_α is C^∞ because ρ_α is 0 on a neighborhood of the complement of U_α, by the definition of a partition of unity subordinate to $\{U_\alpha\}$. We set $F = \prod_\alpha G_\alpha$. We use a similar convention when defining a metric h of L_E and the norm-squared $|| \ s \ ||^2$ of a section s of L_E with respect to h.

Using compactness of M, we construct F so that $F < 1$ on M, in order that the expression $\log(\log F)^2$, which will occur in our Saper-type metrics, is always defined.

Proposition 7.5. — *Let $\pi : \tilde{M} \to M$ be the blow-up of a compact complex manifold M along a coherent sheaf of ideals $\mathcal{I}$ such that $\tilde{M}$ is smooth. Let E be the exceptional divisor and L_E the associated line bundle.*

There is a metric h on L_E and holomorphic section s of L_E, vanishing exactly on E, such that the norm-squared of s with respect to h is of the form

$$|| \ s \ ||^2 = \pi^* F,$$

where F is a real C^∞ function on M, vanishing on the support of $\mathcal{I}$, with $F < 1$ on M. Furthermore, the function F may be constructed to be of the form

$$F = \prod_\alpha F_\alpha^{\rho_\alpha},$$

where $\{\rho_\alpha\}$ is a C^∞ partition of unity subordinate to an open cover $\{U_\alpha\}$ of M, F_α is a function on U_α of the form

$$F_\alpha = \sum_{j=1}^r \mid f_j \mid^2,$$

and $f_1, ..., f_r$ are local holomorphic generators of the coherent ideal sheaf $\mathcal{I}$ on U_α. The Chern form $c_1(L_E, h)$ of h on $\tilde{M}$ is given on $\tilde{M} - E$ by

$$c_1(L_E, h) = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log || \, s \, ||^2 = \pi^*(-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log F).$$

Proof. — Let $\{U_\alpha\}$ be a finite open cover of M by open sets small enough that $\mathcal{I}$ is generated by global sections on each U_α. If the support of $\mathcal{I}$ does not intersect some U_α or if $\mathcal{I}$ is generated by a single generator on U_α, then L_E is trivial on the set $\tilde{U}_\alpha = \pi^{-1}(U_\alpha)$. In this case we may choose F_α to be a constant and the local Chern form will be 0. Otherwise, in the nontrivial case, suppose that $f_1, ..., f_r$ are local generating functions for $\mathcal{I}$ on U_α and let

$$F_\alpha = \sum_{j=1}^r \mid f_j \mid^2 \qquad \text{and} \qquad \tilde{F}_\alpha = \pi^* F_\alpha.$$

By Lemma 7.1, there is a local C^∞ metric h_α for L_E on $\tilde{U}_\alpha$ whose associated Chern form is given on $\tilde{U}_\alpha - \tilde{U}_\alpha \cap E$ by

$$c_1(L_E, h_\alpha) = \pi^*(-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log F_\alpha) = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log \tilde{F}_\alpha.$$

Now choose a C^∞ partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$ and let $\tilde{\rho}_\alpha$ be the pullback of ρ_α to $\tilde{M}$. Then $\{\tilde{\rho}_\alpha\}$ is a partition of unity on $\tilde{M}$ subordinate to the open sets $\{\tilde{U}_\alpha\}$.

We define a global C^∞ metric for L_E as follows. For any section s of L_E, let $|| \, s \, ||_\alpha^2$ be the norm-squared of s with respect to the metric h_α on $\tilde{U}_\alpha$ and let

$$|| \, s \, ||^2 = \prod_\alpha || \, s \, ||_\alpha^{2\tilde{\rho}_\alpha} .$$

Let $\{V_i\}$ be a finite open cover of $\tilde{M}$ by open sets on which L_E is trivial, and let $h_{\alpha i}$ be the positive C^∞ function representing h_α on $\tilde{U}_\alpha \cap V_i$. Then the positive C^∞ function for h on V_i is

$$h_i = \prod_\alpha h_{\alpha i}^{\tilde{\rho}_\alpha}.$$

If s is given on V_i by the holomorphic function s_i, then on $U_\alpha \cap V_i$ we have

$$\| s \|_\alpha^2 = | s_i |^2 h_{\alpha i}$$

and on V_i,

$$\| s \|^2 = | s_i |^2 h_i.$$

The global form $c_1(L_E, h)$ associated with this metric is given on V_i by

$$\begin{aligned} c_1(L_E,h) &= -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h_i \\ &= -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log \prod_\alpha h_{\alpha i}^{\tilde{\rho}_\alpha} \\ &= -\frac{\sqrt{-1}}{2\pi}\sum_\alpha \partial\overline{\partial}\tilde{\rho}_\alpha \log h_{\alpha i}. \end{aligned}$$

Let s be a global holomorphic section of L_E on $\tilde{M}$ whose associated divisor is E. Such a section always exists - just choose local holomorphic defining equations of E to determine s locally. For example, on $\tilde{U}_{\alpha i} = \tilde{U}_\alpha \cap \{\xi_i \neq 0\} \subset U_\alpha \times \mathbb{P}^{r-1}$, take $s_{\alpha i} = \tilde{f}_i = \pi^* f_i$, where $f_1, ..., f_r$ are local holomorphic generators of $\mathcal{I}$ on U_α.

Then

$$\| s \|_\alpha^2 = \sum_{j=1}^r | \tilde{f}_j |^2 = \tilde{F}_\alpha$$

and

$$\| s \|^2 = \prod_\alpha \tilde{F}_\alpha^{\tilde{\rho}_\alpha} = \pi^*(\prod_\alpha F_\alpha^{\rho_\alpha}).$$

Thus the Chern form $c_1(L_E, h)$ is given on $\tilde{M} - E$ by

$$c_1(L_E,h) = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log \| s \|^2 = \pi^*(-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log F),$$

where

$$F = \prod_\alpha F_\alpha^{\rho_\alpha}.$$

We may also choose the functions f_i so that $F < 1$ on M. For $p \in X_{\text{sing}}$, we can choose a neighborhood U_α of p and local holomorphic generators $f_1, ..., f_r$ of $\mathcal{I}$ on U_α, such that $F_\alpha < 1$. For $p \notin X_{\text{sing}}$, we can choose a neighborhood U_β of p such that U_β does not intersect X_{sing}. Then $\mathcal{I}$ is generated by any nonzero constant on U_β, so we can set F_β equal to any nonzero constant less than 1. Choosing a finite subcover of M by such sets U_α and U_β and defining F_α and F_β this way ensures that $F < 1$ on M. $\square$

8. Construction of Saper-type metrics

Let X be a singular subvariety of a compact Kähler manifold M and let X_{sing} be the singular locus of X. We define what is meant by Saper-type (or modified Saper) metrics. We then construct Saper-type metrics on $M - X_{\text{sing}}$, first locally, then globally using a C^∞ partition of unity on M. We also show that our local and global Saper-type metrics are locally quasi-isometric. Our global Saper-type metrics are complete Kähler metrics on $M - X_{\text{sing}}$ which grow less rapidly than Poincaré metrics near the singular locus. More details on the growth rate of Saper-type metrics and their relationship to intersection cohomology may be found in [GM1], [Sa1], and [Sa2].

We also construct a non-complete Kähler metric on $M - X_{\text{sing}}$ with the property that the completion of $X - X_{\text{sing}}$ with respect to this metric is a desingularization of X. We call this metric a "desingularizing metric" for X.

The constructions of both metrics are based on resolution of singularities using a single coherent ideal sheaf $\mathcal{I}$ on M (see Corollary 6.3) and the explicit formula for a Chern form for the blow-up of M along $\mathcal{I}$ given in Proposition 7.5.

Quasi-Isometry. — *We call two metrics h_A and h_B* ***quasi-isometric*** *on an open set U if their fundamental $(1,1)$-forms ω_A and ω_B satisfy $c\omega_A \leqslant \omega_B \leqslant C\omega_A$ on U for some positive constants c and C. For convenience, we also refer to the $(1,1)$-forms ω_A and ω_B as quasi-isometric. Metrics which are quasi-isometric have the same L_2-cohomology.*

Definition— *Let X be a singular subvariety of a compact complex manifold M and let ω be the fundamental $(1,1)$-form of a hermitian metric on M. Let $\pi : \tilde{M} \to M$ be a holomorphic map of a compact complex manifold*

$\tilde{M}$ to M whose exceptional set E is a divisor with normal crossings in $\tilde{M}$ and such that the restriction

$$\pi : \tilde{M} - E \to M - X_{\text{sing}}$$

is a biholomorphism. Let L_E be the line bundle on $\tilde{M}$ associated with E and let h be a hermitian metric on L_E. Let $s : \tilde{M} \to L_E$ be a global holomorphic section whose associated divisor (s) equals E (so s vanishes exactly on E) and let $\| s \|$ be the norm of s with respect to h.

A metric on $\tilde{M} - E$ which is quasi-isometric to a metric with fundamental $(1,1)$-form

$$l\pi^*\omega - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\log \| s \|^2)^2,$$

*for l a positive integer, will be called a **Saper-type** or **modified Saper metric**, distinguished with respect to the map π. The corresponding metric on $M - X_{\text{sing}} \cong \tilde{M} - E$ and its restriction to $X - X_{\text{sing}}$ are also called Saper-type or modified Saper. We will usually omit the phrase "distinguished with respect to π."*

We call the form

$$\nu = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\log \| s \|^2)^2$$

a Poincaré-type $(1,1)$-form. In [GM1], we discuss a more general class of modified Saper metrics, in which the Poincaré-type form ν is replaced by a finite sum of positive integer multiples of Poincaré-type forms. We will not need this more general description in this paper.

Local construction of Saper-type metrics

Before constructing Saper-type metrics, we will describe a Kähler metric for a local blow-up which is essentially the local model for our desingularizing metric.

Let U be an open set in $\mathbb{C}^n$ and let $\pi : \tilde{U} \to U$ be the blow-up of U along a coherent sheaf of ideals $\mathcal{I}$ such that $\tilde{U}$ is smooth. Let E be the exceptional divisor of π. Assume that U is small enough that $\mathcal{I}$ is generated by global sections on U and let

$$\iota_f : \tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$$

be the embedding associated with a collection of generators f. Let σ_1 and σ_2 be the projection maps

$$\begin{array}{ccc} U \times \mathbb{P}^{r-1} & \xrightarrow{\sigma_2} & \mathbb{P}^{r-1} \\ \sigma_1 \downarrow & & \\ U & & \end{array}$$

Suppose that ω is the Kähler form of a Kähler metric on U and let $\omega_{\text{Fub-St}}$ be the Kähler form of the Fubini-Study metric on $\mathbb{P}^{r-1}$.

Lemma 8.1. — *The embedding $\tilde{U} \hookrightarrow U \times \mathbb{P}^{r-1}$ induces a Kähler metric on $\tilde{U}$ whose Kähler form is*

$$\omega' = \pi^*\omega - c_1(L_E, h),$$

where $c_1(L_E, h)$ is a Chern form of the line bundle L_E (with respect to a metric h) of the type described in Lemma 7.1. If $f_1, ..., f_r$ are local holomorphic generators for $\mathcal{I}$ on U, then ω' is given on $\tilde{U} - E$ by

$$\omega' = \pi^*(\omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\sum_{j=1}^{r} \mid f_j \mid^2).$$

The corresponding Kähler metric on $U - V(\mathcal{I})$ has Kähler form

$$\tilde{\omega} = \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\sum_{j=1}^{r} \mid f_j \mid^2.$$

Proof. — The Kähler form on $\tilde{U}$ given by the restriction of the product metric on $U \times \mathbb{P}^{r-1}$ is

$$\begin{aligned} \omega' &= \iota_f^*(\sigma_1^*\omega + \sigma_2^*\omega_{\text{Fub-St}}) \\ &= \pi^*\omega + \iota_f^*\sigma_2^*\omega_{\text{Fub-St}} \\ &= \pi^*\omega - c_1(L_E, h) \end{aligned}$$

where $c_1(L_E, h)$ is given on the set $\tilde{U} - E$ by

$$c_1(L_E, h) = \pi^*(-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\sum_{j=1}^{r} \mid f_j \mid^2)$$

by Lemma 7.1. □

Remark 8.2. — The Kähler metric determined by $\tilde{\omega}$ is essentially the local model of our desingularizing metric.

The function $F = \sum_{j=1}^{r} | f_j |^2$ can also be used to construct a Saper-type metric on $U - V(\mathcal{I})$. We are particularly interested in the case of a coherent sheaf of ideals $\mathcal{I}$ which determines a resolution of singularities of a singular variety and which is supported on the singular locus of the variety. Theorems 8.4 and 8.6 describe local and global constructions, respectively, of Saper-type and desingularizing metrics for a singular variety. The main differences between the two theorems are that we must patch with a C^∞ partition of unity in the global case, and that our global desingularizing metrics may also require a multiple of the original metric.

The following lemma will be useful in constructing Saper-type metrics and describing their rates of growth (cf. [GM1, Theorem 9.2.1]).

LEMMA 8.3. — *Let F be a real C^∞ function on a complex manifold Y such that $0 \leqslant F < 1$ on Y and $F = 0$ exactly on a subvariety Z of Y. Let ω be the positive $(1,1)$-form of a hermitian metric on Y. Suppose that the $(1,1)$-form*

$$\tilde{\omega} = k\omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log F$$

is positive on Y for some positive integer k. Then for each point p in Z, there exists a neighborhood V of p in Y such that the $(1,1)$-form

$$\omega_S = \omega - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\log F)^2$$

is positive and quasi-isometric to

$$\omega_S' = \omega + \frac{\sqrt{-1}}{\pi}\frac{1}{(\log F)^2F^2}\partial F\wedge\overline{\partial}F + \frac{1}{|\log F|}\tilde{\omega}$$

on $V - Z$.

Proof. — Let $R = |\log F| = -\log F$ and expand ω_S as follows:

$$\begin{aligned}
\omega_S &= \omega + \frac{\sqrt{-1}}{\pi}\frac{1}{R^2}\partial R\wedge\overline{\partial}R - \frac{\sqrt{-1}}{\pi}\frac{1}{R}\partial\overline{\partial}R \\
&= \omega + \frac{\sqrt{-1}}{\pi}\frac{1}{R^2}\partial R\wedge\overline{\partial}R + \frac{\sqrt{-1}}{2\pi}\frac{2}{R}\partial\overline{\partial}\log F \\
&= \left(1 - \frac{2k}{R}\right)\omega + \frac{\sqrt{-1}}{\pi}\frac{1}{R^2}\partial R\wedge\overline{\partial}R + \frac{2}{R}\left(k\omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log F\right) \\
&= \left(1 - \frac{2k}{|\log F|}\right)\omega + \frac{\sqrt{-1}}{\pi}\frac{1}{(\log F)^2F^2}\partial F\wedge\overline{\partial}F + \frac{2}{|\log F|}\tilde{\omega}.
\end{aligned}$$

Since $\frac{1}{|\log F|} \to 0$ as we approach Z, we may choose a neighborhood V of p in Y such that the coefficient $1 - \frac{2k}{|\log F|}$ of ω is positive and bounded away from 0 on $V - Z$. Since ω and $\tilde{\omega}$ are positive and $\frac{\sqrt{-1}}{\pi}\partial F \wedge \overline{\partial} F$ is positive semi-definite, ω_S is positive and quasi-isometric to ω_S' on $V - Z$. □

THEOREM 8.4 LOCAL METRICS. — *Let X be a singular subvariety of a compact Kähler manifold M with singular locus X_{sing}. Let ω be the Kähler (1,1)-form of a Kähler metric on M. Let p be any point in X_{sing}. Then there exists a neighborhood U of p and a C^∞ function F on U, vanishing on $U \cap X_{\text{sing}}$, such that*

i. *the (1,1)-form*
$$\tilde{\omega} = \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log F$$
is the Kähler form of an incomplete metric on $U - U \cap X_{\text{sing}}$ which determines an embedded resolution of singularities locally over the neighborhood U, and

ii. *the (1,1)-form*
$$\omega_S = \omega - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\log F)^2$$
on $U - U \cap X_{\text{sing}}$ is the Kähler form of a Saper-type metric on $U - U \cap X_{\text{sing}}$.

Furthermore, the function F may be constructed to be of the form

$$F = \sum_{j=1}^{r} |\ f_j\ |^2,$$

where $f_1, ..., f_r$ are holomorphic functions on U which are local generators of a coherent ideal sheaf $\mathcal{I}$ on M, such that blowing up M along $\mathcal{I}$ gives embedded desingularization of X.

Proof. — Part (i) is a consequence of Corollary 6.3, Lemma 8.1, and Lemma 7.1.

Positivity of ω_S is a consequence of Lemma 8.3. Let $\pi : \tilde{U} \to U$ be the blow-up of U along $\mathcal{I}$. By the remark following Lemma 7.1, $\pi^* F = ||\ s\ ||^2$, where s is a holomorphic section over $\tilde{U}$ of the line bundle L_E associated to the exceptional divisor E of π, s vanishes exactly on $E \cap \tilde{U}$, and $||\ s\ ||^2$ is the norm-squared of s with respect to a metric h on the restriction of

L_E to $\tilde{U}$. Thus the metric determined by ω_S is Saper-type. Completeness of Saper-type metrics is proved in [GM1] (Theorem 9.2.1) and is essentially due to the term of order

$$\frac{\sqrt{-1}}{\pi}\frac{1}{(\log F)^2F^2}\partial F\wedge\overline{\partial}F$$

in the description of the quasi-isometry class of ω_S in Lemma 8.3. This term gives us a lower bound on the growth of the metric near X_{sing} in terms of the growth of the Poincaré metric on the punctured disc. □

Remark 8.5. — Positivity of ω_S in a neighborhood of $p\in X_{\text{sing}}$ could also be proved for F of this form using Lemma 9.4, which implies that the form $-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\log F)^2$ is positive semi-definite in a small enough neighborhood of p.

Global construction of Saper-type metrics

To construct global metrics we patch together our local metrics using C^∞ partitions of unity on M.

THEOREM 8.6 GLOBAL METRICS. — *Let X be a singular subvariety of a compact Kähler manifold M with singular locus X_{sing}. Let ω be the Kähler (1,1)-form of a Kähler metric on M. There exists a global C^∞ function F on M, vanishing exactly on X_{sing}, such that for k a large enough positive integer*

i. *the (1,1)-form*

$$\tilde{\omega}=k\omega+\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log F$$

is the Kähler form of an incomplete Kähler metric on $M-X_{\text{sing}}$ which is a desingularizing metric for X (i.e. the completion of $X-X_{\text{sing}}$ with respect to $\tilde{\omega}$ is nonsingular), and

ii. *the (1,1)-form*

$$\omega_S=\omega-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\log F)^2$$

on $M-X_{\text{sing}}$ is the Kähler form of a complete Kähler Saper-type metric.

Furthermore, the function F may be constructed to be of the form

$$F=\prod_\alpha F_\alpha^{\rho_\alpha},$$

where $\{\rho_\alpha\}$ *is a* C^∞ *partition of unity subordinate to an open cover* $\{U_\alpha\}$ *of* M*,* F_α *is a function on* U_α *of the form*

$$F_\alpha = \sum_{j=1}^{r} \mid f_j \mid^2,$$

and $f_1, ..., f_r$ *are holomorphic functions on* U_α*, vanishing exactly on* $X_{\mathrm{sing}} \cap U_\alpha$*. More specifically,* $f_1, ..., f_r$ *are local holomorphic generators of a coherent ideal sheaf* $\mathcal{I}$ *on* M *such that blowing up* M *along* $\mathcal{I}$ *gives embedded desingularization of* X*.*

Before proving Theorem 8.6, we prove a lemma which we will apply to the functions F_α.

Lemma 8.7. — *Let* $\mathcal{I}$ *be a nonzero coherent sheaf of ideals on a complex manifold* M*. Suppose that on an open neighborhood* U *of a point* p *in* M *there are collections of holomorphic functions* $\{f_1, ..., f_r\}$ *and* $\{g_1, ..., g_s\}$*, each of which generates* I *over* U*. Then there are positive constants* C_1 *and* C_2 *such that, on some neighborhood* V *of* p *in* M*,*

$$C_1 \sum_{i=1}^{s} \mid g_i \mid^2 \leqslant \sum_{j=1}^{r} \mid f_j \mid^2 \leqslant C_2 \sum_{i=1}^{s} \mid g_i \mid^2.$$

Proof. — Since the collection $\{f_j\}$ generates $\mathcal{I}$ over U, there exist holomorphic functions $\{a_{ij}\}$ on U such that, for each i,

$$g_i = \sum_{j=1}^{r} a_{ij} f_j \qquad \text{on } U. \tag{8.1}$$

Similarly, there exist holomorphic functions $\{b_{jk}\}$ on U such that, for each j,

$$f_j = \sum_{k=1}^{s} b_{jk} g_k \qquad \text{on } U. \tag{8.2}$$

From equation (8.2) we have

$$\mid f_j \mid^2 \leqslant \sum_{k=1}^{s} \mid b_{jk} \mid^2 \sum_{k=1}^{s} \mid g_k \mid^2$$

by the Schwartz inequality. Thus

$$\sum_{j=1}^{r} \mid f_j \mid^2 \leqslant \left(\sum_{j=1}^{r} \sum_{k=1}^{s} \mid b_{jk} \mid^2 \right) \left(\sum_{k=1}^{s} \mid g_k \mid^2 \right).$$

Let V be a neighborhood of the point p with compact closure $\overline{V}$ contained in U. Let

$$C_2 = \max\left\{\sum_{j=1}^{r}\sum_{k=1}^{s} \mid b_{jk}(x) \mid^2\right\} \quad \text{for } x \in \overline{V}.$$

Clearly the constant C_2 cannot be zero, because that would imply that all the functions b_{jk} are zero on V and hence all the functions f_j are zero on V, which is impossible because $\mathcal{I}$ is not the zero sheaf. On V we have

$$\sum_{j=1}^{r} \mid f_j \mid^2 \leqslant C_2 \sum_{k=1}^{s} \mid g_k \mid^2.$$

Similarly, for some positive constant C_3, we have

$$\sum_{i=1}^{s} \mid g_i \mid^2 \leqslant C_3 \sum_{j=1}^{r} \mid f_j \mid^2$$

on V. Letting $C_1 = \frac{1}{C_3}$, we obtain

$$C_1 \sum_{i=1}^{s} \mid g_i \mid^2 \leqslant \sum_{j=1}^{r} \mid f_j \mid^2 \leqslant C_2 \sum_{i=1}^{s} \mid g_i \mid^2$$

on V. □

Proof of Theorem 8.6. — To prove part (i), we will show that we can patch our local metrics using a C^∞ partition of unity from M and adding a high enough multiple of the original metric from M to obtain a positive (1,1)-form.

Let $\mathcal{I}$ be a coherent sheaf of ideals on M such that blowing up M along $\mathcal{I}$ gives embedded desingularization of X. Let $\{U_\alpha\}$ be an open cover of M such that $\mathcal{I}$ is generated by global holomorphic sections on U_α. For each U_α, pick a set of holomorphic generators $f_1, ..., f_r$ of $\mathcal{I}$ over U_α and let

$$F_\alpha = \sum_{j=1}^{r} \mid f_j \mid^2.$$

Let $\{\rho_\alpha\}$ be a C^∞ partition of unity subordinate to the open cover $\{U_\alpha\}$ of M and set

$$F = \prod_\alpha F_\alpha^{\rho_\alpha}.$$

As in Proposition 7.5, we may choose the neighborhoods U_α and functions f_j so that $F < 1$ on M.

To prove part (i) of the theorem, we will first note that the $(1,1)$-forms

$$\omega'_\alpha = \pi^*(\omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log F_\alpha)$$

are well-defined and positive on the open sets $\tilde{U}_\alpha = \pi^{-1}U_\alpha$ in $\tilde{M}$ by Lemma 8.1. We next show that the form

$$\begin{aligned}\omega' &= \pi^*(k\omega + \frac{\sqrt{-1}}{2\pi}\sum_\alpha \partial\overline{\partial}(\rho_\alpha \log F_\alpha)) \\ &= \pi^*(k\omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log F)\end{aligned}$$

is positive on $\tilde{M}$ for k a large enough positive integer. It follows that

$$\tilde{\omega} = k\omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log F$$

is the Kähler form of a desingularizing metric for X.

Let p be a point in $\tilde{M}$ and let $q = \pi(p)$ be its image in M. There is some β such that $\rho_\beta(q) > 0$ (and consequently $\pi^*\rho_\beta(p) > 0$). Let V be an open set in U_β containing q such that $\rho_\beta(q) > 0$ on V and such that the closure of V is compact.

The $(1,1)$-form $\partial\overline{\partial}(\rho_\alpha \log F_\alpha)$ is the sum of the following three terms:

$$\rho_\alpha \partial\overline{\partial}\log F_\alpha, \tag{8.3}$$

$$\partial\rho_\alpha \wedge \overline{\partial}(\log F_\alpha) + \partial(\log F_\alpha)\wedge\overline{\partial}\rho_\alpha, \qquad \text{and} \tag{8.4}$$

$$(\log F_\alpha)\partial\overline{\partial}\rho_\alpha. \tag{8.5}$$

By our choice of the set V, the $(1,1)$-form

$$\pi^*(\omega + \frac{\sqrt{-1}}{2\pi}\sum_\alpha \rho_\alpha\partial\overline{\partial}\log F_\alpha)$$

is positive on V.

Next note that

$$\sum_\alpha \partial\rho_\alpha = 0$$

since $\sum_\alpha \rho_\alpha = 1$. Then

$$\begin{aligned}\sum_\alpha \partial\rho_\alpha \wedge \overline{\partial}\log F_\alpha &= \sum_\alpha \partial\rho_\alpha \wedge \overline{\partial}\log F_\alpha - (\sum_\alpha \partial\rho_\alpha) \wedge \overline{\partial}\log F_\beta \\ &= \sum_\alpha \partial\rho_\alpha \wedge \overline{\partial}(\log F_\alpha - \log F_\beta) \\ &= \sum_\alpha \partial\rho_\alpha \wedge \overline{\partial}(\log \frac{F_\alpha}{F_\beta}).\end{aligned}$$

Each form $\partial\rho_\alpha$ is bounded on V since ρ_α is C^∞ and V is compact. The quotients $\frac{F_\alpha}{F_\beta}$ are positive, bounded, bounded away from 0, and C^∞ (where defined), so the forms $\overline{\partial}(\log \frac{F_\alpha}{F_\beta})$ are also bounded on V. Thus

$$k\omega + \frac{\sqrt{-1}}{2\pi}\sum_\alpha \partial\rho_\alpha \wedge \overline{\partial}\log F_\alpha$$

is positive on V for k a large enough positive integer. A similar argument applies to the terms of the form $\partial(\log F_\alpha) \wedge \overline{\partial}\rho_\alpha$.

Finally we apply this argument to the terms $(\log F_\alpha)\partial\overline{\partial}\rho_\alpha$, noting that

$$\sum_\alpha \partial\overline{\partial}\rho_\alpha = 0$$

so that

$$\begin{aligned}\sum_\alpha (\log F_\alpha)\partial\overline{\partial}\rho_\alpha &= \sum_\alpha (\log F_\alpha)\partial\overline{\partial}\rho_\alpha - \log F_\beta \sum_\alpha \partial\overline{\partial}\rho_\alpha \\ &= \sum_\alpha (\log F_\alpha - \log F_\beta)\partial\overline{\partial}\rho_\alpha \\ &= \sum_\alpha (\log \frac{F_\alpha}{F_\beta})\partial\overline{\partial}\rho_\alpha\end{aligned}$$

which is bounded on V. Thus

$$k\omega + \frac{\sqrt{-1}}{2\pi}\sum_\alpha (\log F_\alpha)\partial\overline{\partial}\rho_\alpha$$

is positive on V for k a large enough positive integer.

Pulling back all the terms to $\pi^{-1}(V)$ in $\tilde{M}$, we see that the form

$$\omega' = \pi^*(k\omega + \frac{\sqrt{-1}}{2\pi}\sum_\alpha \partial\overline{\partial}(\rho_\alpha \log F_\alpha))$$

$$= \quad \pi^*(k\omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log F)$$

is positive on V for k a large enough positive integer. Since $\tilde{M}$ is compact, we may choose a finite covering of $\tilde{M}$ by such open sets V, and choose a large enough positive integer k such that ω' is positive on all of $\tilde{M}$.

Positivity of ω_S is a consequence of Lemma 8.3. Let $\pi : \tilde{M} \to M$ be the blow-up of M along $\mathcal{I}$. By Proposition 7.5, $\pi^*F = \| s \|^2$, where s is a holomorphic section of the line bundle L_E associated to the exceptional divisor E of π, s vanishes exactly on E, and $\| s \|^2$ is the norm-squared of s with respect to a metric h on L_E. Thus the metric determined by ω_S is Saper-type. Completeness of Saper-type metrics is proved in [GM1] (Theorem 9.2.1) and is essentially due to the term of order

$$\frac{\sqrt{-1}}{\pi}\frac{1}{(\log F)^2 F^2}\partial F \wedge \overline{\partial} F$$

in the description of the quasi-isometry class of ω_S in Lemma 8.3, which gives us a lower bound on the growth of the metric near X_{sing} in terms of the growth of the Poincaré metric on the punctured disc. $\square$

Quasi-isometry of local and global Saper-type metrics

We show that our local and global Saper-type metrics are locally quasi-isometric. Both are locally quasi-isometric to a local Euclidean metric near points not in the singular locus of X, so we need only prove quasi-isometry near points of X_{sing}.

As above, let $\mathcal{I}$ be a coherent sheaf of ideals on M such that blowing up M along $\mathcal{I}$ gives embedded desingularization of X. Let $\{U_\alpha\}$ be an open cover of M such that $\mathcal{I}$ is generated by global holomorphic sections on each U_α and let $\{\rho_\alpha\}$ be a C^∞ partition of unity subordinate to $\{U_\alpha\}$.

For each set U_α, we pick a collection of holomorphic generators $f_1, ..., f_r$ of $\mathcal{I}$ and let

$$F_\alpha = \sum_{j=1}^{r} | f_j |^2.$$

We construct a global function F given by

$$F = \prod_\alpha F_\alpha^{\rho_\alpha}.$$

Before proving quasi-isometry of our local and global metrics, we compare the rates of growth of our local and global generating functions F_α and F (Lemma 8.8) and their logarithms (Corollary 8.9).

Lemma 8.8. — *For each point $p \in X_{\mathrm{sing}} \cap U_\alpha$ there exists a neighborhood V of p in U_α and positive constants c and C such that*

$$cF_\alpha \leqslant F \leqslant CF_\alpha.$$

Proof. — Let p be a point in $X_{\mathrm{sing}} \cap U_\alpha$ and suppose that p is also in U_β, for some β. By Lemma 8.7 there exists a neighborhood V of p in $U_\alpha \cap U_\beta$ and positive constants c_β and C_β such that

$$c_\beta F_\alpha \leqslant F_\beta \leqslant C_\beta F_\alpha.$$

Let $\Lambda = \{\beta : p \in U_\beta\}$. Note that $\rho_\gamma = 0$ if $p \notin U_\gamma$ so $\sum_{\beta\in\Lambda} \rho_\beta = 1$ and $F = \prod_{\beta\in\Lambda} F_\beta^{\rho_\beta}$. Set

$$c = \min_{\beta\in\Lambda} c_\beta \quad \text{and} \qquad C = \max_{\beta\in\Lambda} C_\beta.$$

Then

$$cF_\alpha \leqslant F_\beta \leqslant CF_\alpha \qquad \text{for } \beta \in \Lambda$$

and

$$cF_\alpha = \prod_{\beta\in\Lambda} (cF_\alpha)^{\rho_\beta} \leqslant \prod_{\beta\in\Lambda} F_\beta^{\rho_\beta} \leqslant \prod_{\beta\in\Lambda} (CF_\alpha)^{\rho_\beta} = CF_\alpha,$$

i.e.

$$cF_\alpha \leqslant F \leqslant CF_\alpha. \qquad \square$$

Corollary 8.9. — *For each point $p \in X_{\mathrm{sing}} \cap U_\alpha$ there exists a neighborhood V of p in U_α and positive constants k and K such that*

$$-k \log F_\alpha \leqslant -\log F \leqslant -K \log F_\alpha.$$

Proof. — By Lemma 8.8, we have

$$\log(cF_\alpha) \leqslant \log F \leqslant \log(CF_\alpha),$$

i.e.

$$\log c + \log F_\alpha \leqslant \log F \leqslant \log C + \log F_\alpha.$$

Note that $\log F_\alpha$ and $\log F$ are negative close enough to X_{sing}. Multiplying by -1 and dividing by $-\log F_\alpha$, we have

$$\frac{-\log C}{-\log F_\alpha} + 1 \leqslant \frac{-\log F}{-\log F_\alpha} \leqslant \frac{-\log c}{-\log F_\alpha} + 1.$$

Since $\frac{1}{-\log F_\alpha} \to 0$ as we approach X_{sing},

$$\frac{-\log F}{-\log F_\alpha} \to 1$$

so we can find a neighborhood V of p in U_α and positive constants k and K such that

$$-k \log F_\alpha \leqslant -\log F \leqslant -K \log F_\alpha. \qquad \square$$

Recall that our global Saper-type metric on $M - X_{\text{sing}}$ is given by

$$\omega_S = \omega - \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log(\log F)^2$$

where ω is the positive hermitian $(1,1)$-form of a metric on M.

If $(z_1, ..., z_n)$ are local holomorphic coordinates on an open set U_α and F_α is defined as above, we have a local Saper-type metric on $U_\alpha - X_{\text{sing}} \cap U_\alpha$ given by

$$\omega_{S,\alpha} = \omega_\alpha - \frac{\sqrt{-1}}{2\pi} \partial\overline{\partial} \log(\log F_\alpha)^2$$

where

$$\omega_\alpha = \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^{n} dz_i \wedge d\overline{z}_i.$$

Proposition 8.10. — *For each point $p \in X_{\text{sing}} \cap U_\alpha$ there exists a neighborhood V of p in U_α, such that the metrics determined by ω_S and $\omega_{S,\alpha}$ are quasi-isometric on $V - X_{\text{sing}} \cap V$.*

Proof. — We wish to show that there exist positive constants c and C and a neighborhood V of p in U_α such that for all tangent vectors ξ on $V - X_{\text{sing}} \cap V$ we have

$$c\omega_{S,\alpha}(\xi,\xi) \leqslant \omega_S(\xi,\xi) \leqslant C\omega_{S,\alpha}(\xi,\xi).$$

Letting $R_\alpha = -\log F_\alpha$, we expand $\omega_{S,\alpha}$ as

$$\omega_{S,\alpha} = \omega_\alpha + \frac{\sqrt{-1}}{\pi} \frac{1}{R_\alpha^2} \partial R_\alpha \wedge \overline{\partial} R_\alpha - \frac{\sqrt{-1}}{\pi} \frac{1}{R_\alpha} \partial\overline{\partial} R_\alpha.$$

By Lemma 8.8, on a neighborhood of $p \in X_{\mathrm{sing}} \cap U_\alpha$, we may write

$$F_\alpha = Fh$$

where h is a bounded positive C^∞ function which is bounded away from 0. Then $R_\alpha = R - \log h$, where $R = -\log F$, and

$$\begin{aligned}\omega_{S,\alpha} &= \omega_\alpha + \frac{\sqrt{-1}}{\pi}\frac{1}{R_\alpha^2}\left(\partial R - \frac{1}{h}\partial h\right) \wedge \left(\overline{\partial} R - \frac{1}{h}\overline{\partial} h\right) \\ &\qquad - \frac{\sqrt{-1}}{\pi}\frac{1}{R_\alpha}\left(\partial\overline{\partial} R + \frac{1}{h^2}\partial h \wedge \overline{\partial} h - \frac{1}{h}\partial\overline{\partial} h\right).\end{aligned}$$

We rewrite $\omega_{S,\alpha}$ as the sum of three terms

$$\omega_{S,\alpha} = \omega'_{S,\alpha} + \sigma + \phi$$

where

$$\omega'_{S,\alpha} = \omega_\alpha + \frac{\sqrt{-1}}{\pi}\frac{1}{R_\alpha^2}\partial R \wedge \overline{\partial} R - \frac{\sqrt{-1}}{\pi}\frac{1}{R_\alpha}\partial\overline{\partial} R,$$

$$\sigma = \frac{\sqrt{-1}}{\pi}\left[\left(\frac{1}{R_\alpha^2} - \frac{1}{R_\alpha}\right)\frac{1}{h^2}\partial h \wedge \overline{\partial} h + \frac{1}{R_\alpha h}\partial\overline{\partial} h\right],$$

and

$$\phi = -\frac{\sqrt{-1}}{\pi}\frac{1}{R_\alpha^2 h}\left(\partial h \wedge \overline{\partial} R + \partial R \wedge \overline{\partial} h\right).$$

Comparing $\omega'_{S,\alpha}$ to the expansion of ω_S in terms of $R = -\log F$,

$$\omega_S = \omega + \frac{\sqrt{-1}}{\pi}\frac{1}{R^2}\partial R \wedge \overline{\partial} R - \frac{\sqrt{-1}}{\pi}\frac{1}{R}\partial\overline{\partial} R,$$

and applying Corollary 8.9, we see that there exists a neighborhood V of p in U_α such that $\omega'_{S,\alpha}$ is positive and $\omega'_{S,\alpha}$ and ω_S are quasi-isometric on $V - X_{\mathrm{sing}} \cap V$. The term σ approaches 0 as we approach X_{sing} since $\frac{1}{R_\alpha} \to 0$ and h is bounded away from 0, so σ is dominated by ω_α near X_{sing}. To study the behavior of ϕ near X_{sing}, we further expand, using $R = -\log F$, to obtain

$$\phi = \frac{\sqrt{-1}}{\pi}\frac{1}{R_\alpha^2 h}\left(\partial h \wedge \frac{1}{F}\overline{\partial} F + \frac{1}{F}\partial F \wedge \overline{\partial} h\right).$$

Let ξ be a tangent vector in $V - X_{\mathrm{sing}}$, for V a small neighborhood of p in U_α, and set

$$a = \partial h(\xi) \qquad \text{and} \qquad b = \frac{1}{F}\partial F(\xi).$$

Then

$$\begin{aligned}
\mid \phi(\xi,\xi) \mid &= \frac{1}{\pi R_\alpha^2 h} \mid a\overline{b} + b\overline{a} \mid \\
&\leqslant \frac{2}{\pi R_\alpha^2 h} \mid a \mid\mid b \mid \\
&\leqslant \frac{1}{\pi R_\alpha^2 h}(\mid a \mid^2 + \mid b \mid^2) \\
&= \frac{1}{\pi R_\alpha^2 h}\left(\mid \partial h(\xi) \mid^2 + \frac{1}{F^2}\mid \partial F(\xi) \mid^2\right).
\end{aligned}$$

Furthermore,

$$\mid \partial h(\xi) \mid^2 \leqslant \mid \partial h \mid^2 \mid \xi \mid^2,$$

where $\mid \partial h \mid$ and $\mid \xi \mid$ denote the norms of ∂h and ξ with respect to the usual Euclidean metric. Since $\frac{1}{R_\alpha^2} \to 0$ as we approach X_{sing}, there exists a neighborhood V of p in U_α such that

$$\frac{1}{\pi R_\alpha^2 h}\mid \partial h(\xi) \mid^2 \leqslant \omega(\xi,\xi)$$

for all tangent vectors ξ on $V - X_{\mathrm{sing}} \cap V$.

We wish to show that the second term of $\mid \phi(\xi,\xi) \mid$ is bounded by a multiple of $\omega_S(\xi,\xi)$ near X_{sing}. Recall from Lemma 8.3 that we can choose a small enough neighborhood V of p in U_α that ω_S is quasi-isometric on $V - X_{\mathrm{sing}} \cap V$ to

$$\omega_S' = \omega + \frac{\sqrt{-1}}{\pi}\frac{1}{R^2F^2}\partial F \wedge \overline{\partial} F + \frac{1}{R}\tilde{\omega}$$

where

$$\tilde{\omega} = k\omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log F$$

is positive for some positive integer k, by Theorem 8.6. Hence

$$\omega_S'(\xi,\xi) = \omega(\xi,\xi) + \frac{1}{\pi R^2 F^2}\mid \partial F(\xi) \mid^2 + \frac{1}{R}\tilde{\omega}(\xi,\xi).$$

Since $\tilde{\omega}$ is positive and $R = -\log F$ is positive near X_{sing},

$$\omega_S'(\xi,\xi) \geqslant \frac{1}{\pi R^2 F^2}\mid \partial F(\xi) \mid^2$$

near $p \in X_{\mathrm{sing}}$.

Comparing with our bound on $\mid \phi(\xi,\xi) \mid$ and our estimates of the other terms of $\omega_{S,\alpha}$, we conclude that there is a neighborhood V of p in U_α and a constant $c > 0$ such that $\omega_S(\xi,\xi) \geqslant c\omega_{S,\alpha}(\xi,\xi)$ for all tangent vectors ξ on $V - X_{\text{sing}} \cap V$.

The proof that $C\omega_{S,\alpha} \geqslant \omega_S$ locally for some $C > 0$ is similar. □

9. Ohsawa's boundedness condition, examples

In this section we discuss a boundedness criterion of Ohsawa's: that the gradient of a generating function of the fundamental (1,1)-form of the metric is locally bounded with respect to the metric. We prove that the local model for our Saper-type metrics on M satisfies this condition, so that our Saper-type metrics are locally quasi-isometric to metrics with generating functions satisfying Ohsawa's condition (and thus have the same local L_2-cohomology). In view of results of Donnelly-Fefferman [DF], Ohsawa [O], and Gromov [Gro] on vanishing of certain L_2-cohomology groups, we hope that this property would allow us to apply Goresky-MacPherson's work on the axiomatic characterization of intersection cohomology for the purpose of identification of the latter (for the middle perversity) with the L_2-cohomology groups for our Saper-type metrics.

Gradient of a C^∞ function with respect to a metric. — Let U be a complex manifold of dimension n and let ω be the fundamental (1,1)-form of a hermitian metric on U. Let H be a C^∞ function on U. The **gradient of H with respect to ω** is the vector field $\text{grad}_\omega H$ defined by the property that, for any holomorphic tangent vector ξ on U,

$$\partial H(\xi) = \xi \cdot_\omega \text{grad}_\omega H,$$

where $\cdot_\omega$ denotes the inner product with respect to the metric determined by ω. The gradient vector field is dual to the 1-form ∂H with respect to the inner products determined by ω on the tangent and cotangent spaces.

Denoting by $| \cdot |^2_\omega$ the norm-squared with respect to ω, we have

$$| \text{grad}_\omega H |^2_\omega = \partial H(\text{grad}_\omega H) = | \partial H |^2_\omega .$$

If $\omega = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}G$, where G is a C^∞ function on U, we call ω the **complex Hessian** of G and we call G a **generating function** for ω.

Ohsawa's Condition. — Let U be a complex manifold of dimension n and let

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}G$$

be the fundamental (1,1)-form of a hermitian metric on U, where G is a C^∞ function on U. We will say that the generating function G **satisfies Ohsawa's condition on** U if $\mid \partial G \mid_\omega^2$ is bounded. Equivalently, the gradient of G with respect to ω has bounded sup norm with respect to ω.

THEOREM 9.1 (DONNELLY AND FEFFERMAN [DF], [O, THEOREM 1.1]). *Let ω be the fundamental (1,1)-form of a hermitian metric on an open set U. If ω has a generating function which satisfies Ohsawa's condition on U and if the metric on U determined by ω is complete, then the L_2-cohomology of U with respect to ω vanishes in all positive dimensions except possibly the middle dimension, i.e.,*

$$H^r_{(2)}(U) = 0 \qquad \text{if } r \neq 0, n.$$

We note that L_2-cohomology depends only on the quasi-isometry class of the metric.

INNER PRODUCTS WITH RESPECT TO THE METRIC DETERMINED BY ω. — If the fundamental $(1,1)$-form of a hermitian metric is given in local coordinates $(z_1, ..., z_n)$ by

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} h_{ij} dz_i \wedge d\overline{z}_j,$$

the associated metric is given by

$$ds^2 = \sum_{i,j=1}^{n} h_{ij} dz_i \otimes d\overline{z}_j,$$

i.e., the inner product of tangent vectors $\frac{\partial}{\partial z_i}$ and $\frac{\partial}{\partial z_j}$ is

$$\frac{\partial}{\partial z_i} \cdot_\omega \frac{\partial}{\partial z_j} = h_{ij}.$$

Thus, the inner product of any two tangent vectors ξ and η with respect to the metric determined by ω is

$$\xi \cdot_\omega \eta = -2\sqrt{-1}\, \omega(\xi \wedge \overline{\eta}).$$

EXAMPLE 9.2. — The Euclidean metric on $\mathbb{C}^n$ with Kähler (1,1)-form

$$\omega_{\mathrm{E}} = \frac{\sqrt{-1}}{2\pi} \sum_{i=1}^{n} dz_i \wedge d\overline{z}_i$$

has a generating function $G = \sum_{i=1}^{n} \mid z_i \mid^2$ which satisfies Ohsawa's condition on bounded open sets.

Proof. — Let H be any C^∞ function on U. We first calculate the gradient of H and its norm-squared with respect to ω_E. Suppose that

$$\mathrm{grad}_E H = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i},$$

and let ξ be a holomorphic vector, given by

$$\xi = \sum_{i=1}^{n} b_i \frac{\partial}{\partial z_i}.$$

Then

$$\partial H(\xi) = \sum_{i=1}^{n} \frac{\partial H}{\partial z_i} b_i$$

and

$$\xi \cdot_E \mathrm{grad}_E H = -2\sqrt{-1}\ \omega_E(\xi \wedge \overline{\mathrm{grad}_E H}) = \frac{1}{\pi} \sum_{i=1}^{n} b_i \overline{a}_i$$

so that $a_i = \pi \frac{\partial H}{\partial \overline{z}_i}$ and

$$\mathrm{grad}_E H = \pi \sum_{i=1}^{n} \frac{\partial H}{\partial \overline{z}_i} \frac{\partial}{\partial z_i}.$$

It follows that

$$\mid \partial H \mid_E^2 = \mid \mathrm{grad}_E H \mid_E^2 = \partial H(\mathrm{grad}_E H) = \pi \sum_{i=1}^{n} \mid \frac{\partial H}{\partial z_i} \mid^2.$$

Next we check Ohsawa's condition. A generating function for ω_E is

$$G = \sum_{i=1}^{n} \mid z_i \mid^2.$$

The gradient of G with respect to ω_E is

$$\mathrm{grad}_{\mathrm{E}} G = \pi \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}$$

and the norm-squared of ∂G with respect to ω_{E} is

$$\mid \partial G \mid_{\mathrm{E}}^2 = \mid \mathrm{grad}_{\mathrm{E}} G \mid_{\mathrm{E}}^2 = \pi \sum_{i=1}^{n} \mid z_i \mid^2$$

which is bounded on bounded open sets. □

Example 9.3. — The Poincaré metric on the punctured unit disc with Kähler $(1,1)$-form

$$\omega_P = \frac{\sqrt{-1}}{\pi} \frac{dz \wedge d\overline{z}}{(-\log \mid z \mid^2)^2 \mid z \mid^2}$$

has a generating function $-\log(-\log \mid z \mid^2)^2$ which satisfies Ohsawa's condition.

Proof. — We write the given generating function as

$$G = -2 \log(-\log \mid z \mid^2).$$

The associated 1-form is

$$\partial G = \frac{2}{(-\log \mid z \mid^2) z} dz$$

and we calculate, as in the previous example, that the gradient of G with respect to ω_P is

$$\mathrm{grad}_P G = \pi(-\log \mid z \mid^2) z \frac{\partial}{\partial z}.$$

The norm-squared of ∂G with respect to ω_P is

$$\mid \partial G \mid_P^2 = \mid \mathrm{grad}_P G \mid_P^2 = \partial G(\mathrm{grad}_P G) = 2\pi,$$

which is bounded. □

We will show (Example 9.6) that a generating function for the following metric satisfies Ohsawa's condition, using Proposition 9.5.

Lemma 9.4. — *The* $(1,1)$*-form*

$$\omega_Q = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(-\log(-\log\sum_{j=1}^{n} \mid z_i \mid^2)^2)$$

determines a Kähler metric on $V - (0,...,0)$*, for* V *a small enough neighborhood of* $(0,...,0)$ *in* $\mathbb{C}^n$*.*

Proof. — Our generating function is

$$G = -2\log(-\log\sum_{j=1}^{n} \mid z_i \mid^2).$$

Letting $R = -\log \mid z \mid^2$, where $\mid z \mid^2 = \sum_{j=1}^{n} \mid z_i \mid^2$, we calculate that

$$\partial\overline{\partial}G = \left(-\frac{1}{R}+\frac{1}{R^2}\right)\frac{2}{\mid z \mid^4}\left(\sum_{i=1}^{n}\overline{z}_i dz_i\right)\wedge\left(\sum_{i=1}^{n} z_i d\overline{z}_i\right)+\frac{2}{R\mid z \mid^2}\sum_{i=1}^{n} dz_i\wedge d\overline{z}_i.$$

Suppose that $\xi = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i}$ is a holomorphic tangent vector on $\mathbb{C}^n$. Then

$$\partial\overline{\partial}G(\xi\wedge\overline{\xi}) = \left(-\frac{1}{R}+\frac{1}{R^2}\right)\frac{2}{\mid z \mid^4}\left(\sum_{i=1}^{n}\overline{z}_i a_i\right)\left(\sum_{i=1}^{n} z_i \overline{a}_i\right)+\frac{2}{R\mid z \mid^2}\sum_{i=1}^{n} \mid a_i \mid^2.$$

Setting $Z = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}$, we can rewrite this equation as

$$\partial\overline{\partial}G(\xi\wedge\overline{\xi}) = \left(-\frac{1}{R}+\frac{1}{R^2}\right)\frac{2}{\mid z \mid^4}\mid (\xi\cdot Z) \mid^2 + \frac{2}{R\mid z \mid^2}\mid \xi \mid^2,$$

where $\cdot$ denotes the usual dot product with respect the Euclidean metric on $\mathbb{C}^n$ and $\mid \xi \mid^2$ is the usual norm-squared of ξ with respect to the Euclidean metric on $\mathbb{C}^n$.

Using the inequality $\mid (\xi\cdot Z) \mid^2 \leqslant \mid \xi \mid^2 \mid Z \mid^2$ and the identity $\mid Z \mid^2 = \mid z \mid^2 = \sum_{i=1}^{n} \mid z_i \mid^2$, we obtain, for $\mid z \mid^2$ near 0,

$$\partial\overline{\partial}G(\xi\wedge\overline{\xi}) \geqslant \frac{2\mid \xi \mid^2}{R^2\mid z \mid^2} = \frac{2\mid \xi \mid^2}{(-\log\mid z \mid^2)^2\mid z \mid^2}.$$

Hence

$$\mid \xi \mid_Q^2 = -2\sqrt{-1}\,\omega_Q(\xi\wedge\overline{\xi}) = \frac{1}{\pi}\partial\overline{\partial}G(\xi\wedge\overline{\xi}) \geqslant \frac{2\mid \xi \mid^2}{\pi R^2\mid z \mid^2}. \qquad (9.1)$$

In particular, ω_Q is positive on $V-(0,...,0)$ for V a small enough neighborhood of $(0,...,0)$ in $\mathbb{C}^n$, so ω_Q determines a Kähler metric. □

We will use following proposition to show that our generating function of the metric of Lemma 9.4 satisfies Ohsawa's condition. We will also apply it to local models for desingularizing metrics (Example 9.7) and modified Poincaré metrics (Example 9.8) to show that our generating functions for these local models do not satisfy Ohsawa's condition.

Proposition 9.5. — *Suppose that* $\omega = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}G$ *is the fundamental* $(1,1)$*-form of a hermitian metric on an open set* U *in* $\mathbb{C}^n$*, such that* G *is of the form* $G = g\circ F$*, where* $F : \mathbb{C}^n \to \mathbb{R}$ *is given by* $F(z_1,...,z_n) = \sum_{i=1}^n \mid z_i \mid^2$ *and* $g : \mathbb{R} \to \mathbb{R}$ *is* C^∞*. Then the gradient of* G *with respect to* ω *is*

$$grad_\omega G = \frac{\pi g'(F)}{Fg''(F)+g'(F)}\sum_{i=1}^n z_i \frac{\partial}{\partial z_i}$$

and the norm-squared of $grad_\omega G$ *with respect to* ω *is*

$$\mid grad_\omega G \mid_\omega^2 = \frac{\pi F g'(F)^2}{Fg''(F)+g'(F)}.$$

In particular, we obtain a criterion for Ohsawa's boundedness condition in terms of the real C^∞ *function* g*. Let* x *be a coordinate for* $\mathbb{R}$*. The generating function* G *of* ω *satisfies Ohsawa's boundedness condition on* U *if and only if the expression*

$$\frac{xg'(x)^2}{xg''(x)+g'(x)}$$

is bounded on the image of U *in* $\mathbb{R}$*.*

Proof. — Let

$$Z = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}$$

and

$$\eta = \left(\frac{\pi g'(F)}{Fg''(F)+g'(F)}\right) Z.$$

To show that η is the gradient of G with respect to ω, we will show that for any holomorphic tangent vector ξ on U, $\partial G(\xi) = \xi \cdot_\omega \eta$, i.e.,

$$\partial G(\xi) = \frac{1}{\pi}\partial\overline{\partial}G(\xi \wedge \overline{\eta}).$$

Expanding the left side, we obtain

$$\begin{aligned} \partial G(\xi) &= g'(F)\partial F(\xi) \\ &= g'(F)(\xi \cdot Z), \end{aligned} \tag{9.2}$$

where $\xi \cdot Z$ denotes the usual dot product in $\mathbb{C}^n$. Similarly, expanding the right side gives

$$\begin{aligned} \frac{1}{\pi}\partial\overline{\partial}G(\xi \wedge \overline{\eta}) &= \frac{1}{\pi}\left(g''(F)\partial F \wedge \overline{\partial}F + g'(F)\partial\overline{\partial}F\right)(\xi \wedge \overline{\eta}) \\ &= \frac{1}{\pi}\left(g''(F)(\xi \cdot Z)(Z \cdot \eta) + g'(F)(\xi \cdot \eta)\right) \\ &= \left(\frac{g'(F)}{Fg''(F) + g'(F)}\right)\left(g''(F)(\xi \cdot Z)(Z \cdot Z) + g'(F)(\xi \cdot Z)\right) \\ &= \left(\frac{g'(F)}{Fg''(F) + g'(F)}\right)\left(g''(F)(\xi \cdot Z)F + g'(F)(\xi \cdot Z)\right) \\ &= g'(F)(\xi \cdot Z). \end{aligned}$$

Hence $\eta = \operatorname{grad}_\omega G$.

Using equation (9.2) above, we calculate that the norm-squared of the gradient of G with respect to ω is

$$\begin{aligned} \mid \operatorname{grad}_\omega G \mid^2 &= \partial G(\operatorname{grad}_\omega G) \\ &= g'(F) \operatorname{grad}_\omega G \cdot Z \\ &= \left(\frac{\pi g'(F)^2}{Fg''(F) + g'(F)}\right) Z \cdot Z \\ &= \frac{\pi F g'(F)^2}{Fg''(F) + g'(F)}. \qquad \square \end{aligned}$$

EXAMPLE 9.6. — The $(1,1)$-form

$$\omega_Q = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(-\log(-\log\sum_{j=1}^{n} \mid z_i \mid^2)^2)$$

determines a Kähler metric whose generating function $-\log(-\log\sum_{j=1}^{n} \mid z_i \mid^2)^2$ satisfies Ohsawa's condition on $V - (0,...,0)$, for V a small enough neighborhood of $(0,...,0)$ in $\mathbb{C}^n$.

Proof. — The given generating function is of the form $G = g \circ F$, where $F = \sum_{i=1}^{n} \mid z_i \mid^2$ and

$$g(x) = -2\log(-\log(x)).$$

Thus, by Proposition 9.5, the norm-squared of the gradient of G with respect to ω_Q is

$$\mid \mathrm{grad}_Q G \mid_Q^2 = 2\pi,$$

and G satisfies Ohsawa's condition. □

The next example is a local model for our desingularizing metric. Let $\tilde{\omega}$ be the $(1,1)$-form on $\mathbb{C}^n - (0,...,0)$ given by

$$\tilde{\omega} = \frac{\sqrt{-1}}{2\pi}\left(\sum_{i=1}^{n} dz_i \wedge d\overline{z}_i + \partial\overline{\partial}\log(\sum_{i=1}^{n} \mid z_i \mid^2)\right).$$

Let $\pi : \tilde{\mathbb{C}}^n \to \mathbb{C}^n$ be the blow-up of $\mathbb{C}^n$ at the origin. The pullback $\pi^*\tilde{\omega}$ of $\tilde{\omega}$ extends smoothly to a form ω' defined on all of $\tilde{\mathbb{C}}^n$. The form ω' is the Kähler form of the restriction to $\tilde{\mathbb{C}}^n$ of the usual product metric on $\mathbb{C}^n \times \mathbb{P}^{n-1}$, determined by the Euclidean metric on $\mathbb{C}^n$ and the Fubini-Study metric on $\mathbb{P}^{n-1}$, under the embedding $\tilde{\mathbb{C}}^n \hookrightarrow \mathbb{C}^n \times \mathbb{P}^{n-1}$ (see Lemma 8.1). Thus $\tilde{\omega}$ is a positive $(1,1)$-form determining a Kähler metric on $\mathbb{C}^n - (0,0)$.

Example 9.7 (Local model for our desingularizing metric). — The generating function

$$G = \sum_{i=1}^{n} \mid z_i \mid^2 + \log(\sum_{i=1}^{n} \mid z_i \mid^2)$$

of the metric determined by $\tilde{\omega} = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}G$ on $\mathbb{C}^n - (0,0)$ does not satisfy Ohsawa's condition. The norm-squared of the gradient of G with respect to $\tilde{\omega}$ is

$$\mid \mathrm{grad}_{\tilde{\omega}} G \mid_{\tilde{\omega}}^2 = \pi\left(\sum_{i=1}^{n} \mid z_i \mid^2 + 2 + \frac{1}{\sum_{i=1}^{n} \mid z_i \mid^2}\right).$$

Proof. — Once again, we apply Proposition 9.5, with $G = g \circ F$ and

$$g(x) = x + \log(x). \qquad \square$$

The metric of the following example is a local model for a complete metric which is bounded below by a multiple of a desingularizing metric, rather than by a multiple of the original metric on the space M in which our singular variety X is embedded. It is a simple local model of Poincaré and modified Poincaré metrics (see [GM1]), which may be constructed by adding terms of Poincaré-type growth to a desingularizing metric. Our Saper-type metrics of this paper, in contrast, are constructed by adding a term of

Poincaré-type growth to the original metric on M, and are locally quasi-isometric, by Proposition 8.10, to the metric of Example 9.11.

If $\pi : \tilde{\mathbb{C}}^n \to \mathbb{C}^n$ is the blow-up of $\mathbb{C}^n$ at the origin and E is the exceptional divisor, the pullback of the following metric to $\tilde{\mathbb{C}}^n - E$ is locally quasi-isometric, near points of E, to the Poincaré metric of Example 9.3 on a punctured disc, times a Euclidean metric in $n-1$ variables (by [GM1], Proposition 5.4.1(ii) or Corollary 7.2.4).

Example 9.8 (Local example of a Poincaré metric) . — Let $\omega = \tilde{\omega} + \omega_Q$, where $\tilde{\omega}$ is the $(1,1)$-form of Example 9.7 and ω_Q is the $(1,1)$-form of Example 9.6. The generating function

$$G = \sum_{i=1}^{n} \mid z_i \mid^2 + \log(\sum_{i=1}^{n} \mid z_i \mid^2) - \log(-\log \sum_{j=1}^{n} \mid z_i \mid^2)^2$$

of the metric determined by ω on $\mathbb{C}^n - (0,0)$ does not satisfy Ohsawa's condition.

Proof. — Apply Proposition 9.5 with $G = g \circ F$ and

$$g(x) = x + \log x - 2\log(-\log(x)).$$

We calculate that

$$\frac{xg'(x)^2}{xg''(x) + g'(x)} = \frac{(1 + x + 2S)^2}{x + 2S^2}, \tag{9.3}$$

where $S = -\frac{1}{\log(x)}$. Since $S \to 0$ as $x \to 0$, the expression in line (9.3) is unbounded near 0, and thus $\mid \mathrm{grad}_\omega G \mid^2$ is unbounded near the origin. □

Lemma 9.9. — *Suppose that $f : X \to Y$ is a holomorphic map of complex manifolds and ω_X and ω_Y are the positive $(1,1)$-forms of hermitian metrics on X and Y respectively. Let ω be the positive $(1,1)$-form of a new metric on X determined by*

$$\omega = \omega_X + f^*\omega_Y.$$

Let H and G be C^∞ functions on X and Y respectively and let

$$K = H + f^*G.$$

Then

$$\mid gradK \mid \leqslant \mid grad_X H \mid_X + \mid grad_Y G \mid_Y$$

where $\mid gradK \mid$ *denotes the norm with respect to* ω *of the gradient of* K *with respect to* ω*, and similarly* $\mid grad_X H \mid_X$ *and* $\mid grad_Y G \mid_Y$ *denote the norms with respect to* ω_X *and* ω_Y*, respectively, of the gradients of* H *and* G *with respect to* ω_X *and* ω_Y *respectively.*

Proof. — For any tangent vector ξ on X,

$$\begin{aligned} \omega(\xi \wedge \overline{\xi}) &= \omega_X(\xi \wedge \overline{\xi}) + f^*\omega_Y(\xi \wedge \overline{\xi}) \\ &= \omega_X(\xi \wedge \overline{\xi}) + \omega_Y(f_*\xi \wedge \overline{f_*\xi}), \end{aligned}$$

i.e.,

$$\mid \xi \mid^2 = \mid \xi \mid_X^2 + \mid f_*\xi \mid_Y^2 . \tag{9.4}$$

If $K = H + f^*G$, then

$$\begin{aligned} \partial K(\xi) &= \partial H(\xi) + \partial(f^*G)(\xi) \\ &= \partial H(\xi) + f^*(\partial G)(\xi) \\ &= \partial H(\xi) + \partial G(f_*\xi). \end{aligned}$$

Hence, by the definition of the gradient,

$$\omega(\xi \wedge \overline{\mathrm{grad} K}) = \omega_X(\xi \wedge \overline{\mathrm{grad}_X H}) \; + \omega_Y(f_*\xi \wedge \overline{\mathrm{grad}_Y G}).$$

Let $\xi = \mathrm{grad} K$, $\xi_X = \mathrm{grad}_X H$, and $\xi_Y = \mathrm{grad}_Y G$. Then

$$\begin{aligned} \mid \xi \mid^2 &= \xi \cdot_X \xi_X + f_*\xi \cdot_Y \xi_Y \\ &= \mid \xi \cdot_X \xi_X + f_*\xi \cdot_Y \xi_Y \mid \\ &\leqslant \mid \xi \cdot_X \xi_X \mid + \mid f_*\xi \cdot_Y \xi_Y \mid \\ &\leqslant \mid \xi \mid_X \mid \xi_X \mid_X + \mid f_*\xi \mid_Y \mid \xi_Y \mid_Y . \end{aligned}$$

By (9.4) above, $\mid \xi \mid_X \leqslant \mid \xi \mid$ and $\mid f_*\xi \mid_Y \leqslant \mid \xi \mid$. Thus

$$\mid \xi \mid^2 \leqslant \mid \xi \mid \mid \xi_X \mid_X + \mid \xi \mid \mid \xi_Y \mid_Y$$

so

$$\mid \xi \mid \leqslant \mid \xi_X \mid_X + \mid \xi_Y \mid_Y,$$

i.e.

$$\mid \mathrm{grad} K \mid \leqslant \mid \mathrm{grad}_X H \mid_X + \mid \mathrm{grad}_Y G \mid_Y . \qquad \square$$

Proposition 9.10. — *Suppose that $f : X \to Y$ is a holomorphic map of complex manifolds and*

$$\omega_X = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}H \qquad \text{and} \qquad \omega_Y = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}G$$

are the Kähler forms of Kähler metrics on X and Y respectively. Let ω be the Kähler form of a Kähler metric on X determined by

$$\omega = \omega_X + f^*\omega_Y.$$

If the generating functions H and G of ω_X and ω_Y satisfy Ohsawa's condition, then so does the generating function $H + f^*G$ of ω.

Proof. — Apply Lemma 9.9 with $\omega = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}K$, where $K = H + f^*G$. □

Let X be a singular subvariety of a compact Kähler manifold M with singular locus X_{sing}. Let $\mathcal{I}$ be a coherent sheaf of ideals on M such that blowing up M along $\mathcal{I}$ gives embedded desingularization of X. Let p be a point in X_{sing} and let U be an open coordinate neighborhood of p with local holomorphic coordinates $(z_1, ..., z_n)$ and such that $\mathcal{I}$ is generated on U by holomorphic functions $f_1, ..., f_r$. Set

$$F = \sum_{j=1}^{r} \mid f_i \mid^2.$$

Let ω_E be the Kähler $(1,1)$-form

$$\omega_E = \frac{\sqrt{-1}}{2\pi}\sum_{i=1}^{n} dz_i \wedge d\overline{z}_i = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\sum_{i=1}^{n} \mid z_i \mid^2$$

of a Euclidean metric on U.

Proposition 9.11. — *The Saper-type metric on $U - X_{\text{sing}} \cap U$ with Kähler form given by*

$$\omega_S = \omega_E - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\log F)^2,$$

has a generating function which satisfies Ohsawa's condition on $U - X_{\text{sing}} \cap U$, for U a small enough neighborhood of $p \in X_{\text{sing}}$.

Proof. — Let f be the function

$$f = (f_1, ..., f_r) : U \to \mathbb{C}^r.$$

Let $(w_1, ..., w_r)$ be holomorphic coordinates on $\mathbb{C}^r$ and let V be a small enough neighborhood of $(0, ..., 0)$ in $\mathbb{C}^r$ such that the $(1,1)$-form

$$\omega_Y = \frac{\sqrt{-1}}{2\pi}(-\log(-\log \sum_{j=1}^{r} | w_i |^2)^2)$$

determines a Kähler metric on $Y = V - (0, ..., 0)$ satisfying Ohsawa's condition (by Lemma 9.4 and Example 9.6). Choose U as above and small enough that f maps U into V. Let $X = U$ and $\omega_X = \omega_E$. Now apply Proposition 9.10. □

10. Example

The cuspidal cubic (cf. [GM2] and [BM2]). — Let $M = \mathbb{P}^2$ and let X be the cuspidal cubic given in homogeneous coordinates by $\xi_0\xi_2^2 - \xi_1^3 = 0$. In local coordinates x, y in a neighborhood $U \cong \mathbb{C}^2$ of the singular point, X is given by

$$y^2 - x^3 = 0.$$

We may obtain embedded resolution of X by three blow-ups of points. We will show that these three blow-ups are equivalent to a single blow-up along the ideal sheaf given locally by

$$\mathcal{I} = (x, y)(x^2, y)(x^3, x^2y, y^2).$$

First blow-up π_1. — The center C_1 for the first blow-up is the point $x = y = 0$ and its ideal is $\mathcal{I}_{C_1} = (x, y)$. The blow-up $U_1 = \pi_1^{-1}(U)$ may be covered by two coordinate charts, which we will call the x- and y-coordinate charts, according to whether the chart is a complement in U_1 of the strict transform of $x = 0$ or $y = 0$. (The exceptional divisor is given by the vanishing of the x-coordinate in the x-chart and the y-coordinate in the y chart.) On the x-coordinate chart, π_1 is given by

$$\pi_1(x_1, y_1) = (x_1, x_1y_1) = (x, y)$$

and the exceptional divisor E_1 is given by $x_1 = 0$. The inverse image $\pi_1^{-1}(X)$ is given by $x_1^2y_1^2 - x_1^3 = 0$. The strict transform X_1 of X is obtained from the inverse image by removing all copies of E_1, i.e. by dividing by the highest possible power of x_1, which gives

$$y_1^2 - x_1 = 0.$$

Although X_1 is smooth, it does not have normal crossings with the divisor E_1 at the point $x_1 = y_1 = 0$, so we must blow up again at this point. Before doing so, we note that in the y-coordinate chart, the strict transform X_1 is smooth and has normal crossings with E_1, so there is no need to blow up further at any points in that chart.

Second blow-up π_2. — The center C_2 for the second blow-up is the point $x_1 = y_1 = 0$ in the x-coordinate chart of U_1, and its ideal is $\mathcal{I}_{C_2} = (x_1, y_1)$. In the x-coordinate chart of π_2 we have normal crossings, so there is no need to blow up further at any points in that chart. In local coordinates (x_2, y_2) for the y-coordinate chart of π_2, we have

$$\pi_2(x_2, y_2) = (x_2 y_2, y_2) = (x_1, y_1)$$

and $\mathcal{I}_{E_2} = (y_2)$. The strict transform X_2 of X_1 is given by

$$y_2 - x_2 = 0$$

and the strict transform $\tilde{E}_1$ of E_1 by $x_2 = 0$. The total exceptional divisor of the first two blow-ups, which is the union of E_2 and $\tilde{E}_1$, does not have normal crossings with X_2 so we blow up again.

Third blow-up π_3. — The center C_3 for the third blow-up is the point $x_2 = y_2 = 0$ with ideal $\mathcal{I}_{C_2} = (x_2, y_2)$. After this third blow-up, the strict transform of X and all three components of the total exceptional divisor have normal crossings.

Construction of $\mathcal{I}$. — We will construct $\mathcal{I}$ as a product $\mathcal{I} = \mathcal{I}_1\mathcal{I}_2\mathcal{I}_3$ of ideals corresponding to the centers of the blow-ups. We begin by choosing $\mathcal{I}_1 = \mathcal{I}_{C_1} = (x, y)$.

To obtain $\mathcal{I}_2$, we start with $\mathcal{I}_{C_2}$ and multiply by a high enough power of $\mathcal{I}_{E_1}$ such that taking the direct image under π_1 and then the inverse image does not change the ideal. We define $\mathcal{I}_2$ to be the direct image of the resulting product under the map π_1.

Locally, in the x-coordinate chart of π_1, $\mathcal{I}_{C_2}$ is given by (x_1, y_1) and $\mathcal{I}_{E_1}$ by (x_1), where $x_1 = x$ and $y_1 = \frac{y}{x}$. Thus $\mathcal{I}_{C_2}$ is not the inverse image of an ideal sheaf, but $\mathcal{I}_{C_2}\mathcal{I}_{E_1}$ is, since

$$\pi_1^{-1}(x^2, y) = \mathcal{I}_{C_2}\mathcal{I}_{E_1}.$$

The direct image $\pi_{1*}(\mathcal{I}_{C_2}\mathcal{I}_{E_1})$ is the largest ideal sheaf whose inverse image is contained in $\mathcal{I}_{C_2}\mathcal{I}_{E_1}$, so $\pi_{1*}(\mathcal{I}_{C_2}\mathcal{I}_{E_1})$ contains (x^2, y). It is easily checked that x^2 and y generate $\pi_{1*}(\mathcal{I}_{C_2}\mathcal{I}_{E_1})$, since they are the only monomials whose pullbacks are sections of $\mathcal{I}_{C_2}\mathcal{I}_{E_1}$. Thus

$$\mathcal{I}_2 = \pi_{1*}(\mathcal{I}_{C_2}\mathcal{I}_{E_1}) = (x^2, y).$$

Similarly, to obtain $\mathcal{I}_3$ we start with $\mathcal{I}_{C_3}$, given locally by (x_2, y_2), and recall that $x_2 = \frac{x_1}{y_1}$ and $y_2 = y_1$. Hence $\mathcal{I}_{C_3}\mathcal{I}_{E_2}$ is the inverse image of an ideal sheaf $\mathcal{J}$ given locally on U_1 by (x_1, y_1^2), and $\mathcal{J}\mathcal{I}_{E_1}^2$ is the inverse image of the ideal sheaf (x^3, y^2). Since $\pi_2^{-1}(\mathcal{I}_{E_1}) = \mathcal{I}_{\tilde{E}_1}\mathcal{I}_{E_2}$, it follows that

$$\pi_2^{-1}\pi_1^{-1}(x^3, y^2) = \mathcal{I}_{C_3}\mathcal{I}_{\tilde{E}_1}^2\mathcal{I}_{E_2}^3.$$

In local coordinates, $\pi_2^{-1}\pi_1^{-1}(x^3, y^2) = (x_2, y_2)(x_2^2)(y_2^3)$. We define $\mathcal{I}_3$ to be the direct image $\pi_{1*}\pi_{2*}(\mathcal{I}_{C_3}\mathcal{I}_{\tilde{E}_1}^2\mathcal{I}_{E_2}^3)$, and note that $\mathcal{I}_3$ contains (x^3, y^2), since $\mathcal{I}_3$ is the largest ideal sheaf whose inverse image is contained in $\mathcal{I}_{C_3}\mathcal{I}_{\tilde{E}_1}^2\mathcal{I}_{E_2}^3$. To find any remaining generators of $\mathcal{I}_3$, we test monomials not generated by x^3 or y^2 to see which pull back to sections of $\mathcal{I}_{C_3}\mathcal{I}_{\tilde{E}_1}^2\mathcal{I}_{E_2}^3$. It is easily checked that x, y, x^2, and xy are not in $\mathcal{I}_3$, but x^2y is in $\mathcal{I}_3$ since $x^2y = x_1^3y_1 = x_2^3y_2^4$. Thus

$$\mathcal{I}_3 = \pi_{1*}\pi_{2*}(\mathcal{I}_{C_3}\mathcal{I}_{\tilde{E}_1}^2\mathcal{I}_{E_2}^3) = (x^3, x^2y, y^2).$$

We define the ideal $\mathcal{I}$ to be the product of $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$

$$\mathcal{I} = (x, y)(x^2, y)(x^3, x^2y, y^2).$$

Blowing up along $\mathcal{I}$ is equivalent to blowing up sequentially along the centers C_1, C_2, and C_3.

The method used in this example has been generalized to any locally toric complex analytic variety (see[GM2] and [BM2]).

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