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Regularity of the solution of some transmission problems in domains with cuspidal points(*)

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Abstract. — Regularity results for transmission problems in domains with (outgoing) cuspidal points are considered. We prove in some special but generic situations that the solution is piecewise in H^2.

Résumé. — Nous considérons des résultats de régularité des solutions de problèmes de transmission dans des domaines à points cuspides. Nous démontrons que la solution est H^2 par morceaux dans des situations particulières mais génériques.

1. Introduction

In our days, regularity results for boundary value problems on nonsmooth domains with a Lipschitz boundary are well known. These regularity results are due to the singular points of the domain, i.e. corners, edges, etc... [8, 4, 2], but also to the discontinuities of the coefficients of the operator (so called transmission problems) [9, 11, 12]. Usually one obtains a decomposition of the (weak) solution into a regular part and a singular one, this last one being related to the geometrical singularities of the boundary and/or the discontinuities of the coefficients.

For domains with outgoing cusps (the boundary being not Lipschitz), regularity results for boundary value problems with smooth coefficients were obtained by different authors [6, 7, 10, 14, 5, 3, 1]. Roughly speaking since the angle at the cusp is zero, we can expect good regularity of the solution, which is mainly the results obtained by these authors for different domains and operators. Surprisingly (to our knowledge) no regularity results exist

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for transmission problems on domains with cusps. We therefore fill this gap and prove the piecewise H^2 regularity of the solution of the Laplace transmission problem in dimension 2 in some particular situations. We finally show that the 2D piecewise H^2 regularity directly yields the piecewise H^2 edge regularity in three-dimensional domains. The extension of such results to polyhedral domains with cusps requires more investigations and will be the object of forthcoming works.

As a motivation of our results, let us consider the following "standard" transmission problem [9, 11, 12]: Fix the finite cone $C = C_1 \cup C_2 \cup \Gamma$, where

$$\begin{aligned} C_1 &= \{(x,y) \in \mathbb{R}^2; 0 < r < 1, -\omega_1 < \theta < 0\}, \\ C_2 &= \{(x,y) \in \mathbb{R}^2; 0 < r < 1, 0 < \theta < \omega_2\}, \\ \Gamma &= \{(x,0); 0 < x < 1\}, \end{aligned}$$

where (r,θ) are the polar coordinates of (x,y) and $\omega_1 > 0$ and $\omega_2 > 0$ are the respective opening of the cone C_1 and C_2.

Let $u \in H_0^1(C)$ be the variational solution of the following Dirichlet interface problem for the Laplace operator:

$$\left\{ \begin{array}{ll} -\Delta u_i = f_i & \text{in } C_i,\ i = 1,2, \\ u_i = 0 & \text{on } \partial C_i \setminus \Gamma, \\ u_1 = u_2 & \text{on } \Gamma, \\ p_1 \frac{\partial u_1}{\partial y} = p_2 \frac{\partial u_2}{\partial y} & \text{on } \Gamma, \end{array} \right. \tag{1.1}$$

where $f \in L^2(C)$, u_i means the restriction of u to C_i, $i = 1,2$ and p_1 and p_2 are two positive real numbers, supposed to be different.

It is well known [9, 11, 12] that u behaves like r^{λ_0} near $(0,0)$, where $\lambda_0 > 0$ is the smallest positive root of

$$p_2 \cos(\lambda\omega_2) \sin(\lambda\omega_1) + p_1 \cos(\lambda\omega_1) \sin(\lambda\omega_2) = 0.$$

A careful analysis of this transcendental equation shows that λ_0 satisfies

$$\lambda_0 \geqslant \frac{\pi}{2\omega_1},$$

if we assume that $\omega_2 \leqslant \omega_1$. Consequently we get that

$$\lambda_0 \to \infty \text{ as } \omega_1 \text{ and } \omega_2 \to 0.$$

From the results from [9, 11, 12], we can expect good regularity properties of the solution of a problem similar to (1.1) on a domain with a cusp at $(0,0)$ (since it corresponds to the limit case $\omega_1 = \omega_2 = 0$).

The proofs of our two-dimensional regularity results consist in three main steps:

1. As in [14, 6, 5, 3] we perform appropriate changes of variables to transform the bounded domain into an infinite domain, similar to a strip.
2. We use a diadic covering to reduce the regularity problem to a bounded domain.
3. We use regularity results for transmission problems on bounded domain and prove uniform bounds.

The schedule of the paper is the following one: Section 2 recalls the transmission problem we have in mind and gives the piecewise H^2 regularity result for a straight interface. A similar result is obtained in section 3, when the interface is curved. Finally using a standard Fourier transform technique we show in section 4 the piecewise H^2 regularity for three-dimensional domains with a cuspidal edge.

Let us finish this introduction with some notation used in the whole paper: As usual, we denote by $L^2(.)$ the Lebesgue spaces and by $H^s(.)$, $s \geqslant 0$, the standard Sobolev spaces. The usual norm and seminorm of $H^s(D)$ are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$. The space $H_0^1(\Omega)$ is defined, as usual, by $H_0^1(\Omega) := \{v \in H^1(\Omega)/v = 0 \text{ on } \partial\Omega\}$.

2. Transmission problem in a domain with a cuspidal point: Straight interface

Let U be the following bounded domain of the plane, with boundary containing a turning point (or outgoing cusp):

$$U = \{(x, y) \in \mathbb{R}^2; 0 < x < a, \varphi_1(x) < y < \varphi_2(x)\},$$

where φ_1 and φ_2 are two functions satisfying the conditions :

$$\begin{cases} \varphi_1, \varphi_2 \in C^1([0, a]) \cap C^\infty(]0, a]), \\ \varphi_1 < 0 < \varphi_2 \text{ on }]0, a], \\ \varphi_1(0) = \varphi_2(0) = 0, \\ \varphi_1'(0) = \varphi_2'(0) = 0. \end{cases}$$

In addition we suppose that $\lim\limits_{x\to 0} \frac{\varphi_1(x)}{\varphi_2(x)}$ is finite (it even may vanish). The case $\lim\limits_{x\to 0} \frac{\varphi_1(x)}{\varphi_2(x)} = +\infty$ can be treated by exchanging the indices 1 and 2 below.

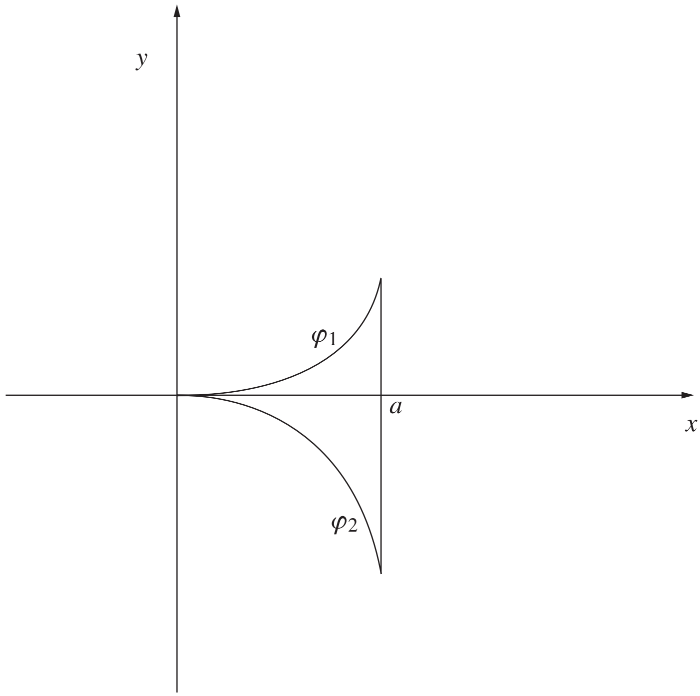

Figure 1. — The domain U

U is actually divided into two parts U_1 and U_2, separated by a straight interface Σ_0, namely (see Fig 1)

$$\begin{aligned} U_1 &= \{(x,y) \in \mathbb{R}^2; 0 < x < a, \varphi_1(x) < y < 0\}, \\ U_2 &= \{(x,y) \in \mathbb{R}^2; 0 < x < a, 0 < y < \varphi_2(x)\}, \\ \Sigma_0 &= \{(x,0) \in \mathbb{R}^2; 0 < x < a\}. \end{aligned}$$

In this section, we consider the variational solution $u \in H_0^1(U)$ of the following Dirichlet interface problem for the Laplace operator

$$\begin{cases} -\Delta u_i = f_i & \text{in } U_i,\ i = 1,2, \\ u_i = 0 & \text{on } \partial U_i \setminus \Sigma_0, \\ u_1 = u_2 & \text{on } \Sigma_0, \\ \sum\limits_{i=1}^{2} p_i \frac{\partial u_i}{\partial \nu_i} = 0 & \text{on } \Sigma_0, \end{cases} \tag{2.1}$$

where $f \in L^2(U)$, u_i means the restriction of u to U_i, $i = 1, 2$ and p_1 and p_2 are two positive real numbers, supposed to be different. ν_i denotes the unit normal vector to Σ_0 directed outside U_i. In other words, $u \in H_0^1(U)$ is the unique solution of

$$\int_U p\nabla u \cdot \nabla v \, dx = \int_U pfv \, dx, \forall v \in H_0^1(U),$$

where the function p is defined by

$$p = \begin{cases} p_1 \text{ in } U_1, \\ p_2 \text{ in } U_2. \end{cases}$$

In this section we will show that the variational solution u of (2.1) has the improved regularity $PH^2(U)$, where

$$PH^2(U) := \{u \in H^1(U) : u_i \in H^2(U_i), i = 1, 2\},$$

is the space of piecewise H^2 functions on U.

2.1. The change of variables

Following [14], we set

$$t = -\int_x^{+\infty} \frac{d\sigma}{\varphi_2(\sigma)}, \qquad \theta = \frac{y}{\varphi_2(x)},$$

where φ_2 is extended to $[a, \infty)$ so that φ_2 remains positive and $1/\varphi_2$ belongs to $L^1(a, \infty)$.

The image of U by this change of variables is the (semi-infinite) domain $\Omega = \Omega_1 \cup \Omega_2 \cup \Sigma$, where

$$\begin{aligned} \Omega_1 &= \{(t, \theta) \in \mathbb{R}^2; t < b, \varphi(t) < \theta < 0\}, \\ \Omega_2 &= \{(t, \theta) \in \mathbb{R}^2; t < b, 0 < \theta < 1\}, \\ \Sigma &= \{(t, 0) \in \mathbb{R}^2; t < b\}, \end{aligned}$$

and

$$b = -\int_a^{+\infty} \frac{d\sigma}{\varphi_2(\sigma)}, \qquad \varphi(t) = \frac{\varphi_1(x)}{\varphi_2(x)}.$$

Let us set

$$v(t, \theta) = u(x, y), \qquad g(t, \theta) = f(x, y),$$

or more precisely

$$u(x,y) = v\left(-\int\limits_{x}^{+\infty} \frac{d\sigma}{\varphi_2(\sigma)}, \frac{y}{\varphi_2(x)}\right), \quad f(x,y) = g\left(-\int\limits_{x}^{+\infty} \frac{d\sigma}{\varphi_2(\sigma)}, \frac{y}{\varphi_2(x)}\right).$$

Direct calculations yield

$$D_y u_i = \frac{1}{\varphi_2} D_\theta v_i, \qquad D_y^2 u_i = \frac{1}{\varphi_2^2} D_\theta^2 v_i,$$

and

$$\begin{aligned}
D_x u_i &= \frac{1}{\varphi_2} D_t v_i - y\frac{\varphi_2'}{\varphi_2^2} D_\theta v_i, \\
D_x^2 u_i &= D_x\left[\frac{1}{\varphi_2} D_t v_i - \frac{y\varphi_2'}{\varphi_2^2} D_\theta v_i\right] \\
&= \frac{\varphi_2'}{\varphi_2^2} D_t v_i + \frac{1}{\varphi_2} D_x D_t v_i - y\left[\frac{\varphi_2''\varphi_2^2 - 2\varphi_2{\varphi_2'}^2}{\varphi_2^4} D_\theta v_i + \frac{\varphi_2'}{\varphi_2^2} D_x D_\theta v_i\right] \\
&= \frac{1}{\varphi_2^2}\Big[D_t^2 v_i + \theta^2 {\varphi_2'}^2 D_\theta^2 v_i - 2\theta\varphi_2' D_{t\theta}^2 v_i - \varphi_2' D_t v_i \\
&+ \qquad \theta\left(2{\varphi_2'}^2 - \varphi_2\varphi_2''\right) D_\theta v_i\Big].
\end{aligned}$$

Consequently problem (2.1) becomes

$$\begin{cases}
(-\Delta + P)v_i = \varphi_2^2 g_i(t,\theta) & \text{in } \Omega_i, i = 1,2, \\
v_i = 0 & \text{on } \partial\Omega_i \setminus \Sigma, \\
v_1 = v_2 & \text{on } \Sigma, \\
p_1 \frac{\partial v_1}{\partial\theta} = p_2 \frac{\partial v_2}{\partial\theta} & \text{on } \Sigma,
\end{cases} \tag{2.2}$$

where we have set

$$Pv = -\theta^2 {\varphi_2'}^2 D_\theta^2 v + 2\theta\varphi_2' D_{t\theta}^2 v + \varphi_2' D_t v - \theta\left(2{\varphi_2'}^2 - \varphi_2\varphi_2''\right) D_\theta v.$$

Since we are interested in regularity results on Sobolev spaces, it will be convenient to study the effect of the same change of variables on these functional spaces, in particular on L^2 spaces. In that case, we obviously have the

LEMMA 2.1. — *For $i = 1,2$, $f_i \in L^2(U_i)$ if and only if $\varphi_2 g_i \in L^2(\Omega_i)$.*

In view of this Lemma, we define a new function $w = \varphi_2^{-1} v$ and a new right-hand side $h = \varphi_2 g \in L^2(U)$.

Now we look at the boundary value problem solved by w. Since we have

$$\begin{aligned} D_\theta v_i &= \varphi_2 D_\theta w_i, \qquad D_\theta^2 v_i = \varphi_2 D_\theta^2 w_i, \\ D_t v_i &= \varphi_2 \varphi_2' w_i + \varphi_2 D_t w_i, \\ D_t^2 v_i &= \left(\varphi_2 {\varphi_2'}^2 + \varphi_2^2 \varphi_2''\right) w_i + 2\varphi_2 \varphi_2' D_t w_i + \varphi_2 D_t^2 w_i, \\ D_{t\theta}^2 v &= D_t\left(\varphi_2 D_\theta w_i\right) = \varphi_2 \varphi_2' D_\theta w_i + \varphi_2 D_{t\theta}^2 w_i, \end{aligned}$$

problem (2.2) implies that

$$\begin{aligned} &-\Big\{\varphi_2 D_\theta^2 w_i + (\varphi_2 {\varphi_2'}^2 + \varphi_2^2 \varphi_2'') w_i + 2\varphi_2 \varphi_2' D_t w_i + \varphi_2 D_t^2 w_i \\ &+\theta^2 {\varphi_2'}^2 \varphi_2 D_\theta^2 w_i - 2\theta \varphi_2' (\varphi_2 \varphi_2' D_\theta w_i + \varphi_2 D_{t\theta}^2 w_i) - \varphi_2' (\varphi_2 \varphi_2' w_i + \varphi_2 D_t w_i) \\ &+\theta (2{\varphi_2'}^2 - \varphi_2 \varphi_2'') \varphi_2 D_\theta w_i \Big\} \\ &= \varphi_2^2 g_i(t,\theta). \end{aligned}$$

This equation is equivalent to

$$(-\Delta + L) w_i = h_i,$$

where L is the differential linear operator of second order with bounded coefficients defined by

$$Lw = -\varphi_2 \varphi_2'' w - \varphi_2' D_t w + \theta \varphi_2 \varphi_2'' D_\theta w - \theta^2 {\varphi_2'}^2 D_\theta^2 w + 2\theta \varphi_2' D_{t\theta}^2 w.$$

Summing up, we have established the following proposition

Proposition 2.2. — *There exists a differential linear operator of second order with bounded coefficients L such that problem (2.1) is equivalent to*

$$\begin{cases} (-\Delta + L) w_i = h_i(t,\theta) & \text{in } \Omega_i, \\ w_i = 0 & \text{on } \partial\Omega_i \setminus \Sigma, \\ w_1 = w_2 & \text{on } \Sigma, \\ p_1 \frac{\partial w_1}{\partial \theta} = p_2 \frac{\partial w_2}{\partial \theta} & \text{on } \Sigma, \end{cases} \tag{2.3}$$

where we have set $h = \varphi_2 f$ and $w = \varphi_2^{-1} u$.

2.2. The reference problem

In this subsection we shall prove the following result

THEOREM 2.3. — *For $f \in L^2(\Omega)$, there exists a unique solution $u \in H_0^1(\Omega) \cap PH^2(\Omega)$ of the problem*

$$\begin{cases} -\Delta u_i = f_i & \text{in } \Omega_i, \\ u_1 = u_2 & \text{on } \Sigma, \\ p_1 \frac{\partial u_1}{\partial \theta} = p_2 \frac{\partial u_2}{\partial \theta} & \text{on } \Sigma. \end{cases}$$

Proof. — This problem admits a unique variational solution $u \in H_0^1(\Omega)$ (since Poincaré's inequality remains valid in Ω because it is bounded in the direction of θ). Note further that there exists a positive constant C_0 such that

$$\sum_{i=1}^{2} p_i \|u_i\|_{1,\Omega_i}^2 \leqslant C_0 \sum_{i=1}^{2} p_i \|f_i\|_{0,\Omega_i}^2. \tag{2.4}$$

In order to study the $PH^2(\Omega)$ regularity of the variational solution, we used the technique of a dyadic covering.

Let $(\eta_j)_{j=-\infty}^{0}$ be a sequence of C^∞ functions on $\mathbb{R}$ such that

$$\eta_j(t) = \begin{cases} 1 & \text{if} \quad j-1+b \leqslant t \leqslant j+b \\ 0 & \text{if} \quad t \leqslant j-2+b \quad \text{or} \quad t \geqslant j+1+b \end{cases}$$

and

$$\sum_{j=-\infty}^{0} \eta_j(t) = 2 \text{ on }]-\infty, b].$$

Clearly we can take $\eta_j(t) = \eta(t-j-b)$, for an appropriate cut-off function η such that supp $\eta = [-2, 1]$ and $\eta \equiv 1$ on $[-1, 0]$.

Therefore the solution u can be written

$$u = \frac{1}{2} \sum_{j=-\infty}^{0} \eta_j u$$

and we have

$$-\Delta(\eta_j u_i) = \eta_j f_i - \eta_j'' u_i - 2\eta_j' D_t u_i \in L^2(\Omega_i).$$

Let us now set

$$\begin{array}{rcll} u_j & = & \eta_j u, & u_{ij} = u_j|_{\Omega_i}, \\ g_j & = & \eta_j f - \eta_j'' u - 2\eta_j' D_t u, & g_{ij} = g_j|_{\Omega_i}, \end{array}$$

$$\begin{array}{rcl} Q_j & = & \Omega \cap \{(t,\theta) \in \mathbb{R}^2; j-2+b<t<t_j\}, \\ \hat{Q}_j & = & \Omega \cap \{(t,\theta) \in \mathbb{R}^2; j-1+b<t<j+b\}, \\ Q_{ij} & = & Q_j \cap \Omega_i, \end{array}$$

$$\begin{array}{rcl} \Sigma_j & = & \{(t,0); j-2+b<t<t_j\}, \\ \Gamma_{j,\varphi} & = & \{(t,\varphi(t)); j-2+b<t<t_j\}, \end{array}$$

where $t_j = b + \min\{j+1, 0\}$ ($t_j = b+j+1$ if $j<0$ and $t_0 = b$). It is clear that u_j belongs to $H_0^1(Q_j)$ and is solution of the transmission problem

$$\begin{cases} -\Delta u_{ij} = g_{ij} & \text{in } Q_{ij}, i=1,2, \\ u_{1j} = u_{2j} & \text{on } \Sigma_j, \\ p_1 \frac{\partial u_{1j}}{\partial \theta} = p_2 \frac{\partial u_{2j}}{\partial \theta} & \text{on } \Sigma_j. \end{cases}$$

Moreover for all j, u_j belongs to $PH^2(Q_j)$ since Q_j has a piecewise C^2 boundary with convex angles at the exterior boundary and the angles of Q_{ij} at the interface Σ_j are equal to $\pi/2$ (see [9, 12]).

Moreover we take advantage of the following result which will be proved later on. □

PROPOSITION 2.4. — *There exists a positive constant C independent of j such that*

$$\sum_{i=1}^{2} p_i \|u_{ij}\|^2_{2,Q_{ij}} \leqslant C \sum_{i=1}^{2} p_i \|g_{ij}\|^2_{0,Q_{ij}}. \tag{2.5}$$

Thanks to this proposition, we are now able to estimate $\sum\limits_{i=1}^{2} p_i \|u_i\|^2_{2,\Omega_i}$. As $u = u_j$ on $\hat{Q}_j$ since $\eta_j = 1$, we may write

$$\begin{aligned} &\sum_{i=1}^{2} p_i \|u_i\|^2_{2,\Omega_i} \\ &= \sum_j \left\{ \sum_{i=1}^{2} p_i \sum_{|\alpha|=2} \int_{\hat{Q}_j \cap \Omega_i} |D^\alpha u_{ij}|^2 dt d\theta + \int_{\hat{Q}_j} p |\nabla u_j|^2 dt d\theta + \int_{\hat{Q}_j} p |u_j|^2 dt d\theta \right\} \\ &\leqslant \sum_j \left\{ \sum_{i=1}^{2} p_i \sum_{|\alpha|=2} \int_{Q_{ij}} |D^\alpha u_{ij}|^2 dt d\theta + \int_{Q_j} p |\nabla u_j|^2 dt d\theta + \int_{Q_j} p |u_j|^2 dt d\theta \right\} \\ &= \sum_j \sum_{i=1}^{2} p_i \|u_{ij}\|^2_{2,Q_{ij}}. \end{aligned}$$

Therefore, Proposition 2.4 leads to

$$\sum_{i=1}^{2} p_i \|u_i\|^2_{2,\Omega_i} \leqslant C \sum_j \sum_{i=1}^{2} p_i \|g_{ij}\|^2_{0,Q_{ij}}. \tag{2.6}$$

By the definition of g_{ij} we have

$$\|g_{ij}\|^2_{0,Q_{ij}} = \int\limits_{Q_{ij}} |\eta_j f_i - \eta_j'' u_i - 2\eta_j' D_t u_i|^2 dt d\theta$$

and since $\eta_j(t) = \eta(t-j)$ we get

$$\|g_{ij}\|^2_{0,Q_{ij}} \leqslant 4\|\eta\|_{C^2(\mathbb{R})} \left[\int\limits_{Q_{ij}} |f_i|^2 + \int\limits_{Q_{ij}} |u_i|^2 + \int\limits_{Q_{ij}} |D_t u_i|^2 \right].$$

Multiplying this identity by p_i and summing up on i and j, we obtain

$$\begin{aligned} \sum_j \sum_{i=1}^{2} p_i \int\limits_{Q_{ij}} |g_{ij}|^2 dt d\theta &= \sum_j \int\limits_{Q_j} p|g_j|^2 dt d\theta \\ &\leqslant 4\|\eta\|_{C^2(\mathbb{R})} \sum_j \left\{ \int\limits_{Q_j} p|f|^2 dt d\theta + \int\limits_{Q_j} p|u|^2 dt d\theta + \int\limits_{Q_j} p|\nabla u|^2 dt d\theta \right\}. \end{aligned}$$

Taking into account that $Q_j = \hat{Q}_{j-1} \cup \hat{Q}_j \cup \hat{Q}_{j+1}$ for $j < 0$ and $Q_0 = \hat{Q}_{-1} \cup \hat{Q}_0$ (implying a finite overlaping) and $\Omega = \cup_{j=-\infty}^{0} \hat{Q}_j$ we conclude

$$\sum_j \sum_{i=1}^{2} p_i \int\limits_{Q_{ij}} |g_{ij}|^2 dt d\theta \leqslant 12\|\eta\|_{C^2(\mathbb{R})} \sum_{i=1}^{2} p_i \left\{ \|f_i\|^2_{0,\Omega_i} + \|u_i\|^2_{1,\Omega_i} \right\}.$$

Finally, making use of (2.4) we arrive at

$$\sum_j \sum_{i=1}^{2} p_i \|g_{ij}\|^2_{0,Q_{ij}} \leqslant 12\|\eta\|_{C^2(\mathbb{R})}(1 + C_0) \sum_{i=1}^{2} p_i \|f_i\|^2_{0,\Omega_i}.$$

This estimate in (2.6) leads to

$$\sum_{i=1}^{2} p_i \|u_i\|^2_{2,\Omega_i} \leqslant K \sum_{i=1}^{2} p_i \|f_i\|^2_{0,\Omega_i} \tag{2.7}$$

where $K = 12\|\eta\|_{C^2(\mathbb{R})} C(1 + C_0)$.

Remark 2.5. — The cut-off function η and also the constants C, C_0 do not depend on b, and consequently K is independent of b.

2.3. Proof of Proposition 2.4

The proof of Proposition 2.4 is based on three main steps that are summarized in three Lemmas whose proofs are postponed to the end of this subsection.

The first Lemma gives a bound for the norm of u_{ij} in the space $H^1(Q_{ij})$ (and mainly follows from Poincaré's inequality).

LEMMA 2.6. — *There exists a constant C_1 (independent of j) such that*

$$\left(\sum_{i=1}^{2} p_i \|u_{ij}\|^2_{1,Q_{ij}}\right)^{\frac{1}{2}} \leqslant C_1 \left(\sum_{i=1}^{2} p_i \|g_{ij}\|^2_{0,Q_{ij}}\right)^{\frac{1}{2}}. \tag{2.8}$$

To obtain (2.5), it remains to bound the L^2-norm of the second derivatives of u_{ij}. We proceed as in section 3.3 of [4] where the Dirichlet problem for the Laplace equation in a domain with turning points (without interface) is considered.

For a fixed j, we set

$$v_i = D_t u_{ij}, \qquad w_i = D_\theta u_{ij} \quad i = 1, 2. \tag{2.9}$$

The functions v_i, w_i belong to $H^1(Q_{ij})$, and we approximate them by functions belonging to $H^2(Q_{ij})$ in order to apply Theorem 3.1.1.2 of [4]. We observe that

$$u_{ij} = 0 \text{ so } \frac{\partial u_{ij}}{\partial \tau_i} = 0 \text{ on } \partial Q_{ij} \setminus \Sigma_j.$$

In view of (2.9) this means

$$\tau_1 v_i + \tau_2 w_i = 0 \qquad \text{on} \quad \partial Q_{ij} \setminus \Sigma_j.$$

On the interface Σ_j we have

$$u_{1j} = u_{2j} \quad \text{and consequently} \quad \frac{\partial u_{1j}}{\partial t} = \frac{\partial u_{2j}}{\partial t} \text{ or } v_1 = v_2,$$

$$p_1 \frac{\partial u_{1j}}{\partial \theta} = p_2 \frac{\partial u_{2j}}{\partial \theta} \quad \text{or} \quad p_1 w_1 = p_2 w_2.$$

In summary (v_i, w_i) fulfils the following boundary and transmission conditions

$$\begin{cases} v_2 = 0 & \text{for } \theta = 1,\ j-2+b < t < t_j, \\ w_1 = 0 & \text{for } t = j-2+b,\ t = t_j,\ \varphi(t) < \theta < 0, \\ w_2 = 0 & \text{for } t = j-2+b, t = t_j,\ 0 < \theta < 1, \\ \tau_1 v_1 + \tau_2 w_1 = 0 & \text{on } \Gamma_{j,\varphi}, \\ v_1 = v_2, p_1 w_1 = p_2 w_2 & \text{on } \Sigma_j. \end{cases}$$

LEMMA 2.7. — *There exists a sequence of pairs of functions $(v_{k,i}, w_{k,i}) \in PH^2(Q_i)$, $k = 1, 2, \ldots$ such that*

$$v_{k,i} \to v_i, w_{k,i} \to w_i \text{ in } H^1(Q_{ij}) \text{ as } k \to \infty,$$

and satisfying

$$\begin{cases} v_{k,2} = 0 & \text{for } \theta = 1, j-2+b < t < t_j, \\ w_{k,1} = 0 & \text{for } t = j-2+b, t = t_j, \varphi(t) < \theta < 0, \\ w_{k,2} = 0 & \text{for } t = j-2+b, t = t_j, 0 < \theta < 1, \\ \tau_1 v_{k,1} + \tau_2 w_{k,1} = 0 & \text{on } \Gamma_{j,\varphi}, \\ v_{k,1} = v_{k,2}, p_1 w_{1,k} = p_2 w_{k,2} & \text{on } \Sigma_j. \end{cases} \tag{2.10}$$

Applying the identity (3.1.1.10) of [4] (valid in a domain Ω with a piecewise C^2 boundary) to the vector function $V_{k,i} = (v_{k,i}, w_{k,i})$ we obtain

$$\int_{Q_{1j}} |D_t v_{k,1} + D_\theta w_{k,1}|^2 dt d\theta - \int_{Q_{1j}} \left[|D_t v_{k,1}|^2 + |D_\theta w_{k,1}|^2 + 2 D_t w_{k,1} D_\theta v_{k,1} \right] dt d\theta \tag{2.11}$$
$$= \int_{\Sigma_j} \{ \operatorname{div}_T (V_{k,1})_\nu (V_{k,1})_T - 2 (V_{k,1})_T \nabla_T (V_{k,1})_\nu \} \, d\sigma - \int_{\Gamma_{j,\varphi}} (\operatorname{tr} B)(V_{k,1})_\nu^2 d\sigma,$$

$$\int_{Q_{2j}} |D_t v_{k,2} + D_\theta w_{k,2}|^2 dt d\theta - \int_{Q_{2j}} \left[|D_t v_{k,2}|^2 + |D_\theta w_{k,2}|^2 + 2 D_t w_{k,2} D_\theta v_{k,2} \right] dt d\theta \tag{2.12}$$
$$= \int_{\Sigma_j} \{ \operatorname{div}_T (V_{k,2})_\nu (V_{k,2})_T - 2 (V_{k,2})_T \nabla_T (V_{k,2})_\nu \} \, d\sigma,$$

where $(V_{k,i})_T$ means the tangential component of $V_{k,i}$, $(V_{k,i})_\nu$ the normal component of $V_{k,i}$ and B is the second fundamental quadratic form along the boundary of Q_{1j}. Moreover we recall that (2.10) implies that $(V_{k,i})_T = 0$ on $\partial Q_{ij} \setminus \Sigma_j$ and $(V_{k,1})_T = (V_{k,2})_T$ and $p_1 (V_{k,1})_\nu = p_2 (V_{k,2})_\nu$ on Σ_j.

Adding (2.11) multiplied by p_1 and (2.12) multiplied by p_2 we obtain

$$\sum_{i=1}^{2} p_i \left\{ \int_{Q_{ij}} |D_t v_{k,i} + D_\theta w_{k,i}|^2 dt d\theta - \int_{Q_{ij}} \left[|D_t v_{k,i}|^2 + |D_\theta w_{k,i}|^2 + 2 D_t w_{k,i} D_\theta v_{k,i} \right] dt d\theta \right\}$$

$$= -p_1 \int_{\Gamma_{j,\varphi}} (\operatorname{tr} B)(V_{k,1})_\nu^2 d\sigma.$$

Then, taking the limit in k we obtain

$$\sum_{i=1}^{2} p_i \left\{ \int_{Q_{ij}} |D_t v_i + D_\theta w_i|^2 dt d\theta - \int_{Q_{ij}} \left[|D_t v_i|^2 + |D_\theta w_i|^2 + 2 D_t w_i D_\theta v_i \right] dt d\theta \right\}$$

$$= -p_1 \int_{\Gamma_{j,\varphi}} (\operatorname{tr} B)(V_1)_\nu^2 d\sigma,$$

and consequently, using (2.9) we have

$$\sum_{i=1}^{2} p_i \left\{ \int_{Q_{ij}} |g_{ij}|^2 dt d\theta - \int_{Q_{ij}} \left[|D_t^2 u_{ij}|^2 + |D_\theta^2 u_{ij}|^2 + 2 |D_t D_\theta u_{ij}|^2 \right] dt d\theta \right\}$$

$$= -p_1 \int_{\Gamma_{j,\varphi}} (\operatorname{tr} B) |\gamma \frac{\partial u_{1j}}{\partial \nu}|^2 d\sigma.$$

Since $\operatorname{tr} B$ is bounded by $|\varphi''(t)|$ at the point $(t, \varphi(t))$, this identity implies that

$$\sum_{i=1}^{2} p_i \int_{Q_{ij}} \left[|D_t^2 u_{ij}|^2 + |D_\theta^2 u_{ij}|^2 + 2|D_{t\theta}^2 u_{ij}|^2\right] dtd\theta$$
$$\leqslant \sum_{i=1}^{2} p_i \int_{Q_{ij}} |g_{ij}|^2 dtd\theta + p_1 C \int_{\Gamma_{j,\varphi}} |\varphi''(t)||\frac{\partial u_{1j}}{\partial \nu}|^2 d\sigma. \tag{2.13}$$

The claim follows if we can estimate the integral in $\Gamma_{j,\varphi}$.

Lemma 2.8. — *There exists a positive constant C_2 independent of j such that*

$$\int_{\Gamma_{j,\varphi}} |\varphi''(t)||\frac{\partial u_{1j}}{\partial \nu}|^2 d\sigma$$
$$\leqslant C_2 \sum_{i=1}^{2} p_i \left\{ \epsilon^{\frac{1}{2}} \sum_{|\alpha|=2} \int_{Q_{ij}} |D^\alpha u_{ij}|^2 dtd\theta + \epsilon^{-\frac{1}{2}} \int_{Q_{ij}} |\nabla u_{ij}|^2 dtd\theta \right\},$$

for all $\epsilon \in]0,1[$.

This Lemma applied to (2.13) leads to

$$\sum_{i=1}^{2} p_i \sum_{|\alpha|=2} \int_{Q_{ij}} |D^\alpha u_{ij}|^2 dtd\theta$$
$$\leqslant \frac{1}{1 - C_2 \epsilon^{\frac{1}{2}}} \sum_{i=1}^{2} p_i \left\{ \|g_{ij}\|_{0,Q_{ij}}^2 + C_2 \epsilon^{-\frac{1}{2}} \|u_{ij}\|_{1,Q_{ij}}^2 \right\}, \tag{2.14}$$

where we chose ϵ small enough so that $1 - C_2 \epsilon^{\frac{1}{2}} > 0$. The combination of (2.8) and (2.14) implies (2.5).

2.3.1 Proof of Lemma 2.6

Integrating by parts $(\Delta u_{ij}) u_{ij}$ we obtain

$$- \int_{Q_{ij}} (\Delta u_{ij}) u_{ij} dtd\theta = \int_{Q_{ij}} |\nabla u_{ij}|^2 dtd\theta - \int_{\partial Q_{ij}} u_{ij} \frac{\partial u_{ij}}{\partial \nu_{ij}} d\sigma.$$

Taking into account the boundary and transmission conditions, we get

$$-\sum_{i=1}^{2} p_i \int_{Q_{ij}} (\Delta u_{ij}) u_{ij} dt d\theta = \sum_{i=1}^{2} p_i \int_{Q_{ij}} |\nabla u_{ij}|^2 dt d\theta.$$

Therefore, applying Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned}\sum_{i=1}^{2} p_i \int_{Q_{ij}} |\nabla u_{ij}|^2 dt d\theta &\leqslant \sum_{i=1}^{2} p_i \|g_{ij}\|_{0,Q_{ij}} \cdot \|u_{ij}\|_{0,Q_{ij}} \\ &\leqslant \left(\sum_{i=1}^{2} p_i \|g_{ij}\|^2_{0,Q_{ij}}\right)^{\frac{1}{2}} \left(\sum_{i=1}^{2} p_i \|u_{ij}\|^2_{0,Q_{ij}}\right)^{\frac{1}{2}}. \qquad (2.15)\end{aligned}$$

On the other hand

$$\left(\sum_{i=1}^{2} p_i \|u_{ij}\|^2_{0,Q_{ij}}\right)^{\frac{1}{2}} \leqslant C_1 \sum_{i=1}^{2} p_i \|\nabla u_{ij}\|^2_{0,Q_{ij}},$$

thanks to Poincaré's inequality. This estimate in (2.15) implies (2.8).

2.3.2 Proof of Lemma 2.7

We first recall a density result from [4]. Let Ω be a polygon of $\mathbb{R}^2$ with boundary $\Gamma = \cup_{j=1}^{n} \overline{\Gamma}_j$. We denote by $G^s(\Omega)$ the space of $(v,w) \in (H^s(\Omega))^2$ satisfying the following boundary conditions

$$\alpha_j v + \beta_j w = 0 \qquad \text{on } \Gamma_j, j = 1, \cdots, n,$$

where (α_j, β_j) are n couples of real numbers such that $\alpha_j^2 + \beta_j^2 \neq 0$. Then Lemma 4.3.1.2 of [4] can be formulated as follows:

LEMMA 2.9. — *$G^2(\Omega)$ is dense in $G^1(\Omega)$ for the norm induced by $H^1(\Omega) \times H^1(\Omega)$.*

Following [9] (Proof of Lemma II.2.2) we define two functions v, w on $Q_j = Q_{1,j} \cup \Sigma_j \cup Q_{2,j}$ as follows

$$v = \begin{cases} v_1 & \text{in} \quad Q_{1,j} \\ v_2 & \text{in} \quad Q_{2,j} \end{cases}, \qquad w = \begin{cases} p_1 w_1 & \text{in} \quad Q_{1,j} \\ p_2 w_2 & \text{in} \quad Q_{2,j}. \end{cases}$$

By definition (v,w) belongs to $(H^1(Q_j))^2$ and satisfies the boundary conditions

$$\begin{cases} v=0 & \text{for } \theta=1,\ j-2+b<t<t_j,\\ w=0 & \text{for } t=j-2+b, t=t_j,\ \varphi(t)<\theta<1,\\ \alpha v+\beta w=0 & \text{on } \Gamma_{j,\varphi},\end{cases}$$

where $(\alpha,\beta)=(\tau_1,\frac{\tau_2}{p_1})$. This shows that $(v,w)\in G^1(Q_j)$ (for appropriate pairs (α_j,β_j)). Applying Lemma 2.9 we deduce the existence of a sequence of vector functions $(v_k,w_k)\in G^2(Q_j)$ such that $v_k\to v$ and $w_k\to w$ in $H^1(Q_j)$ as $k\to\infty$. Moreover $(v_k,w_k)\in G^2(Q_j)$ means that

$$\begin{cases} v_k,w_k\in H^2(Q_j) & \\ v_k=0 & \text{for } \theta=1,\ j-2+b<t<t_j,\\ w_k=0 & \text{for } t=j-2+b, t=t_j,\ \varphi(t)<\theta<1,\\ \alpha v_k+\beta w_k=0 & \text{on } \Gamma_{j,\varphi},\end{cases}$$

By setting

$$v_{k,i}=v_k|_{Q_{ij}},\qquad\qquad w_{k,i}=\frac{w_k}{p_i}|_{Q_{ij}},$$

it is clear that $v_{k,i},w_{k,i}\in H^2(Q_{ij})$ and in addition $v_{k,i}\to v_i$ and $w_{k,i}\to w_i$ in $H^1(Q_{ij})$ as $k\to\infty$. It remains to show that $v_{k,i},w_{k,i}$ satisfies the conditions (2.10). Indeed

$$\begin{aligned} v_{k,2}|_{\theta=1} &= \left(v_k|_{Q_{2j}}\right)|_{\theta=1}=0,\\ w_{k,i}|_{t=j-2+b,t_j} &= \left(\frac{w_k}{p_i}|_{Q_{ij}}\right)|_{t=j-2+b,t_j}=0,\\ \tau_1v_{k,1}+\tau_2w_{k,1}|_{(t,\varphi(t))} &= \left(\tau_1v_{k,1}+\tau_2\frac{w_k}{p_1}|_{Q_{1j}}\right)_{(t,\varphi(t))}\\ &= \left(\alpha v_k+\beta w_k|_{Q_{1j}}\right)_{(t,\varphi(t))}=0.\end{aligned}$$

Finally, since v_k, w_k belong to $H^1(Q_j)$ we obtain

$$v_{k,1}|_{\Sigma_j}=(v_k|_{Q_{1,j}})|_{\Sigma_j}=(v_k|_{Q_{2,j}})|_{\Sigma_j}=v_{k,2}|_{\Sigma_j},$$

$$p_1w_{k,1}|_{\Sigma_j}=(w_k|_{Q_{1,j}})|_{\Sigma_j}=(w_k|_{Q_{2,j}})|_{\Sigma_j}=p_2w_{k,2}|_{\Sigma_j}.$$

2.3.3 Proof of Lemma 2.8

Firstly, we show that $\varphi''(t)$ is (uniformly) bounded. Indeed since $\varphi(t)=\frac{\varphi_1(x)}{\varphi_2(x)}$, we have

$$\varphi'(t) = \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial t}=\varphi_2(x)\frac{\partial\varphi}{\partial x}$$

$$\begin{aligned} &= \varphi_1'(x) - \frac{\varphi_1(x)}{\varphi_2(x)}\varphi_2'(x), \\ \varphi''(t) &= \varphi_2 \frac{\partial}{\partial x}(\varphi'(t)) \\ &= \varphi_2(x)\varphi_1''(x) - \varphi_1(x)\varphi_2''(x) - \varphi_1'(x)\varphi_2'(x) + \frac{\varphi_1(x)}{\varphi_2(x)}\left[\varphi_2'(x)\right]^2 . \end{aligned}$$

From this identity we deduce that

$$\sup_{t\in]-\infty,0[} |\varphi''(t)| < \infty. \tag{2.16}$$

It therefore remains to estimate $\int\limits_{\Gamma_{j,\varphi}} |\frac{\partial u_{1j}}{\partial \nu}|^2 d\sigma$. Since

$$|\frac{\partial u_{1j}}{\partial \nu}| \leqslant |\frac{\partial u_{1j}}{\partial t}| + |\frac{\partial u_{1j}}{\partial \theta}| = |v_1| + |w_1|,$$

we are reduced to estimate

$$\int\limits_{\Gamma_{j,\varphi}} |v_1|^2 d\sigma \text{ and } \int\limits_{\Gamma_{j,\varphi}} |w_1|^2 d\sigma.$$

Let $\nu^i = (\nu_1^i, \nu_2^i)$ be the unit outward normal vector at a point (t,θ) of the boundary of Q_{ij}. It is clear that

$$\nu_2^1(t,\varphi(t)) = \left(\frac{\varphi'(t)}{\sqrt{1+(\varphi'(t))^2}}, \frac{-1}{\sqrt{1+(\varphi'(t))^2}} \right).$$

We now define on $\overline{Q}_j$, the function μ as follows

$$\mu(t,\theta) = \psi(\theta)\nu_2^1(t,\varphi(t)) \qquad\qquad \forall (t,\theta) \in \overline{Q}_j,$$

where $\psi(\theta)$ is a C^∞ function in $\mathbb{R}$ such that

$$\psi(\theta) = \begin{cases} 1 & if \quad \theta \leqslant \frac{1}{4}, \\ 0 & if \quad \theta \geqslant \frac{3}{4}. \end{cases}$$

Fix for the moment an arbitrary function $v \in PH^2(Q_j)$. Leibniz's rule yields

$$\int\limits_{Q_{ij}} \frac{\partial}{\partial \theta}|v_i|^2 \mu(t,\theta) dt d\theta = 2 \int\limits_{Q_{ij}} v_i \frac{\partial v_i}{\partial \theta} \mu(t,\theta) dt d\theta.$$

On the other hand, applying Green's formula we obtain

$$\int_{Q_{ij}} \frac{\partial}{\partial\theta}|v_i|^2\mu(t,\theta)dtd\theta = \int_{\partial Q_{ij}} |v_i|^2\mu(t,\theta)\nu_2^i(t,\theta)d\sigma - \int_{Q_{ij}} |v_i|^2\frac{\partial\mu}{\partial\theta}(t,\theta)dtd\theta.$$

These two identities give

$$\begin{aligned}\int_{\partial Q_{ij}} |v_i|^2\mu(t,\theta)\nu_2^i(t,\theta)d\sigma &= 2\int_{Q_{ij}} v_i D_\theta v_i \psi(\theta)\nu_2^1(t,\varphi(t))dtd\theta \\ &+ \int_{Q_{ij}} |v_i|^2\frac{\partial}{\partial\theta}\left(\psi(\theta)\nu_2^1(t,\varphi(t))\right)dtd\theta. \qquad (2.17)\end{aligned}$$

Since on $\partial Q_{2j}\setminus\Sigma_j$,

$$\begin{cases}\psi(\theta)=0 & \text{for } \theta=1,\\ \nu_2^2(t,\theta)=0 & \text{for } t=j-2+b, t=t_j,\end{cases}$$

we have

$$\int_{\partial Q_{2j}} |v_2|^2\mu(t,\theta)\nu_2^2(t,\theta)d\sigma = \int_{\Sigma_j} |v_2|^2\nu_2^1(t,\varphi(t))\nu_2^2(t,0)d\sigma. \qquad (2.18)$$

On the other hand, on $\partial Q_{1j}\setminus\Sigma_j$

$$\nu_2^1(t,\theta)=0 \qquad \text{for } t=j-2+b, t=t_j,$$

and consequently

$$\int_{\partial Q_{1j}} |v_1|^2\mu(t,\theta)\nu_2^1(t,\theta)d\sigma =$$

$$\int_{\Sigma_j} |v_1|^2\nu_2^1(t,\varphi(t))\nu_2^1(t,0)d\sigma + \int_{\Gamma_{j,\varphi}} |v_1|^2(\nu_2^1(t,\varphi(t)))^2d\sigma. \qquad (2.19)$$

We now distinguish the two following cases

First case. — If $v_1 = v_2$ on Σ_j (which is the case for $v_1 = D_t u_{1j}$, $v_2 = D_t u_{2j}$) we get by adding (2.18) and (2.19) and using (2.17)

$$\int_{\Gamma_{j,\varphi}} |v_1|^2 \left(\nu_2^1(t,\varphi(t))\right)^2 d\sigma$$

$$= \sum_{i=1}^{2} \left\{ 2 \int_{Q_{ij}} v_i D_\theta v_i \psi(\theta) \nu_2^1(t,\varphi(t)) dt d\theta + \int_{Q_{ij}} |v_i|^2 \frac{\partial}{\partial \theta} \left(\psi(\theta)\nu_2^1(t,\varphi(t))\right) dt d\theta \right\}.$$

As

$$|\nu_2^1(t,\varphi(t))| = \frac{1}{\sqrt{1+(\varphi'(t))^2}} \leqslant 1$$

and

$$\frac{\partial}{\partial \theta}\left(\psi(\theta)\nu_2^1(t,\varphi(t))\right) = \psi'(\theta)\nu_2^1(t,\varphi(t)),$$

because $\nu_2^1(t,\varphi(t))$ does not depend on θ, it follows that

$$\int_{\Gamma_{j,\varphi}} |v_1|^2 \left(\nu_2^1(t,\varphi(t))\right)^2 d\sigma$$

$$\leqslant \sum_{i=1}^{2} \left\{ 2 \max \psi(\theta) \int_{Q_{ij}} |v_i||D_\theta v_i| dt d\theta + \max \psi'(\theta) \int_{Q_{ij}} |v_i|^2 dt d\theta \right\}.$$

Then applying Cauchy-Schwarz's inequality, we obtain

$$\int_{\Gamma_{j,\varphi}} |v_1|^2 \left(\nu_2^1(t,\varphi(t))\right)^2 d\sigma$$

$$\leqslant \|\psi\|_{C^1(\mathbb{R})} \sum_{i=1}^{2} \left\{ 2 \left(\int_{Q_{ij}} |v_i|^2 dt d\theta \right)^{\frac{1}{2}} \left(\int_{Q_{ij}} |D_\theta v_i|^2 dt d\theta \right)^{\frac{1}{2}} + \int_{Q_{ij}} |v_i|^2 dt d\theta \right\}$$

and then, by Young's inequality

$$\int_{\Gamma_{j,\varphi}} |v_1|^2 \left(\nu_2^1(t,\varphi(t))\right)^2 d\sigma$$

$$\leqslant \|\psi\|_{C_1(\mathbb{R})} \sum_{i=1}^{2} \left\{ \epsilon^{\frac{1}{2}} \int_{Q_{ij}} |D_\theta v_i|^2 dt d\theta + (1+\epsilon^{-\frac{1}{2}}) \int_{Q_{ij}} |v_i|^2 dt d\theta \right\}.$$

As $\epsilon \in]0,1[$, this inequality clearly implies that

$$\int_{\Gamma_{j,\varphi}} |v_1|^2 \left(\nu_2^1(t,\varphi(t))\right)^2 d\sigma$$

$$\leqslant C\|\psi\|_{C_1(\mathbb{R})} \sum_{i=1}^{2} \left\{ \epsilon^{\frac{1}{2}} \int_{Q_{ij}} |D_\theta v_i|^2 dt d\theta + \epsilon^{-\frac{1}{2}} \int_{Q_{ij}} |v_i|^2 dt d\theta \right\}, \quad (2.20)$$

with $C = 2$.

Second case. — If $p_1 v_1 = p_2 v_2$ on Σ_j (which is the case for $v_1 = D_\theta u_{1j}$, $v_2 = D_\theta u_{2j}$), proceeding as in the first case but replacing v_1 by $p_1 v_1$ and v_2 by $p_2 v_2$, we get

$$p_1^2 \int_{\Gamma_{j,\varphi}} |v_1|^2 \left(\nu_2^1(t,\varphi(t))\right)^2 d\sigma$$

$$= \sum_{i=1}^{2} p_i^2 \left\{ 2 \int_{Q_{ij}} v_i D_\theta(p_i v_i) \psi(\theta) \nu_2^1(t,\varphi(t)) dt d\theta \right.$$

$$\left. + \int_{Q_{ij}} |v_i|^2 \frac{\partial}{\partial\theta} \left(\psi(\theta)\nu_2^1(t,\varphi(t))\right) dt d\theta \right\}$$

$$\leqslant 2\|\psi\|_{C_1(\mathbb{R})} \sum_{i=1}^{2} p_i^2 \left\{ \epsilon^{\frac{1}{2}} \int_{Q_{ij}} |D_\theta v_i|^2 dt d\theta + \epsilon^{-\frac{1}{2}} \int_{Q_{ij}} |v_i|^2 dt d\theta \right\}.$$

This still proves (2.20) with $C > 0$ depending on the ratio p_2/p_1.

Since $\varphi'(t) = \varphi_1'(x) - \frac{\varphi_1(x)}{\varphi_2(x)} \varphi_2'(x)$, we get

$$\alpha = \sup_{t \in]-\infty, 0[} |\varphi'(t)| < \infty$$

and then $\left[\nu_2^1(t,\varphi(t))\right]^2 = \frac{1}{1+(\varphi'(t))^2} \geqslant \frac{1}{1+\alpha^2}$. As a consequence (2.20) becomes

$$\int_{\Gamma_{j,\varphi}} |v_1|^2 d\sigma \leqslant C(1+\alpha^2)\|\psi\|_{C^1(\mathbb{R})} \sum_{i=1}^{2} \left\{ \epsilon^{\frac{1}{2}} \int_{Q_{ij}} |\nabla v_i|^2 dtd\theta + \epsilon^{-\frac{1}{2}} \int_{Q_{ij}} |v_i|^2 dtd\theta \right\}.$$

Setting $K_0 = C(1+\alpha^2)\|\psi\|_{C^1(\mathbb{R})}$ and applying this estimation to $v_i = D_t u_{ij}$ and $v_i = D_\theta u_{ij}$ respectively, we get

$$\begin{aligned}
&\int_{\Gamma_{j,\varphi}} |D_t u_{1j}|^2 d\sigma \\
&\quad\leqslant \quad K_0 \sum_{i=1}^{2} \left\{ \epsilon^{\frac{1}{2}} \int_{Q_{ij}} \left[|D_t^2 u_{ij}|^2 + |D_{t\theta}^2 u_{ij}\right]| dtd\theta + \epsilon^{-\frac{1}{2}} \int_{Q_{ij}} |D_t u_{ij}|^2 dtd\theta \right\}, \\
&\int_{\Gamma_{j,\varphi}} |D_\theta u_{1j}|^2 d\sigma \\
&\quad\leqslant \quad K_0 \sum_{i=1}^{2} \left\{ \epsilon^{\frac{1}{2}} \int_{Q_{ij}} \left[|D_{t\theta}^2 u_{ij}|^2 + |D_\theta^2 u_{ij}\right]| dtd\theta + \epsilon^{-\frac{1}{2}} \int_{Q_{ij}} |D_\theta u_{ij}|^2 dtd\theta \right\}.
\end{aligned}$$

These two estimates lead to

$$\begin{aligned}
&\int_{\Gamma_{j,\varphi}} \left|\frac{\partial u_{1j}}{\partial \nu}\right|^2 d\sigma \\
&\qquad\leqslant 2K_0 \sum_{i=1}^{2} \left\{ \epsilon^{\frac{1}{2}} \int_{Q_{ij}} \left[|D_t^2 u_{ij}|^2 + |D_\theta u_{ij}|^2 + 2|D_{t\theta}^2 u_{ij}|^2\right] dtd\theta \right. \\
&\qquad\qquad \left. + \epsilon^{-\frac{1}{2}} \int_{Q_{ij}} |\nabla u_{ij}|^2 dtd\theta \right\}.
\end{aligned}$$

This last inequality with (2.16) leads to the estimation in Lemma 2.8.

2.4. Resolution of the transformed problem (2.3)

Thanks to Theorem 2.3 we deduce the following result

Theorem 2.10. — *For a small enough, the operator*

$$B: \left\{u \in H_0^1(\Omega) \cap PH^2(\Omega); u_1 = u_2, p_1 \frac{\partial u_1}{\partial \theta} = p_2 \frac{\partial u_2}{\partial \theta}\right\}$$

$$\to L^2(\Omega) : u \mapsto \{(-\Delta + L)u_i\}_{i=1,2}$$

is an isomorphism.

Proof . — By Theorem 2.3, we know that the operator

$$A: \left\{u \in H_0^1(\Omega) \cap PH^2(\Omega); u_1 = u_2, p_1 \frac{\partial u_1}{\partial \theta} = p_2 \frac{\partial u_2}{\partial \theta}\right\}$$

$$\to L^2(\Omega) : u \mapsto \{-\Delta u_i\}_{i=1,2}$$

is an isomorphism. In addition, the estimation (2.7) guarantees that the norm of A^{-1} does not depend on a, since the constant K is independent of a. On the other hand, it is clear that the norm of the operator L in $\mathcal{L}\left(PH^2(\Omega), L^2(\Omega)\right)$ goes to 0 as a goes to 0. Consequently for a small enough, this norm is less than the inverse of the norm of A. In other words, $A^{-1}L$ is a strict contraction and it follows that B is an isomorphism. □

It remains to derive the derivability properties of u from the corresponding ones of w.

2.5. The effect of the inverse change of variables

By definition we have $w_i = \varphi_2^{-1} v_i(t, \theta) = \varphi_2^{-1} u_i(x, y)$. In view of the previous calculations (see subsection 2.1) we get

$$\begin{aligned} D_\theta w_i &= \varphi_2^{-1} D_\theta v_i = D_y u_i, \\ D_\theta^2 w_i &= \varphi_2^{-1} D_\theta^2 v_i = \varphi_2 D_y^2 u_i. \end{aligned}$$

Since w_i, $D_\theta w_i$, $D_\theta^2 w_i$ belongs to $L^2(\Omega_i)$, Lemma 2.1 implies that $\varphi_2^{-2} u_i$, $\varphi_2^{-1} D_y u_i$ and $D_y^2 u_i$ belong to $L^2(U_i)$. As $D_x^2 u_i = -f_i + D_y^2 u_i$ it follows immediately that $D_x^2 u_i \in L^2(U_i)$. It remains to check that $D_x u_i$ and $D_{xy}^2 u_i$ belong to $L^2(U_i)$. Indeed

$$\begin{aligned} D_t w_i &= \varphi_2^{-1}(D_t v_i - \varphi_2 \varphi_2' w_i) \\ &= D_x u_i + \theta \varphi_2' \varphi_2^{-1} D_\theta v_i - \varphi_2' w_i, \end{aligned}$$

then

$$D_x u_i = D_t w_i - \theta \varphi_2' D_\theta w_i + \varphi_2' w_i.$$

As we know that w_i, $D_t w_i$, $D_\theta w_i \in L^2(\Omega_i)$, by Lemma 2.1 we conclude that $\varphi_2^{-1} D_x u_i \in L^2(U_i)$. Finally we have

$$\begin{aligned} D^2_{t\theta} &= D^2_{t\theta}\left(\varphi_2^{-1} u_i(x,y)\right) \\ &= D_t\left[\varphi_2^{-1} D_\theta u_i(x,y)\right] = D_t D_y u_i \\ &= \varphi_2^{-1} D^2_{xy} u_i + \theta\varphi_2\varphi_2' D^2_y u_i \\ &= \varphi_2 D^2_{xy} u_i + \theta\varphi_2' D^2_\theta w_i, \end{aligned}$$

so that

$$\varphi_2 D^2_{xy} u_i = D^2_{t\theta} w_i - \theta\varphi_2' D^2_\theta w_i \in L^2(\Omega_i).$$

Applying Lemma 2.1, we deduce that $D^2_{xy} u_i \in L^2(U_i)$.

Summing up, we have established the following results.

Proposition 2.11. — *The regularity property $w \in PH^2(\Omega)$ implies that $\varphi_2^{-2} u_i$, $\varphi_2^{-1} D_x u_i$, $\varphi_2^{-1} D_y u_i$, $D^2_x u_i$, $D^2_y u_i$, $D^2_{xy} u_i$ belong to $L^2(\Omega_i)$.*

Theorem 2.12. — *The operator $\left\{u \in H^1_0(U) \cap PH^2(U); u_1 = u_2, p_1 \frac{\partial u_1}{\partial \nu_1} + p_2 \frac{\partial u_2}{\partial \nu_2} = 0\right\} \to L^2(U) : u \mapsto \{-\Delta u_i\}_{i=1,2}$ is an isomorphism.*

Proof. — Direct consequence of Proposition 2.11, Theorem 2.10 and on the regularity results about standard transmission problem (far from the cuspidal point) [9, 12]. □

3. Transmission problem in a domain with a cuspidal point: Curved interface

We consider the same problem (2.1) as in section 1, but here the interface is not straight. We define $U = U_1 \cup U_2 \cup \Sigma_0$, where U_1, U_2 and Σ_0 are given by

$$\begin{aligned} U_1 &= \{(x,y) \in \mathbb{R}^2; 0 < x < a, \varphi_1(x) < y < \varphi_0(x)\}, \\ U_2 &= \{(x,y) \in \mathbb{R}^2; 0 < x < a, \varphi_0(x) < y < \varphi_2(x)\}, \\ \Sigma_0 &= \{(x, \varphi_0(x)); 0 < x < a\}, \end{aligned}$$

where the functions φ_i, $i = 0, 1, 2$ satisfy the conditions

$$\begin{cases} \varphi_0,\ \varphi_1,\ \varphi_2 \in C^1([0,a]) \cap C^\infty(]0,a]), \\ \varphi_1 < \varphi_0 < \varphi_2 \text{ on }]0,a], \\ \varphi_0(0) = \varphi_1(0) = \varphi_2(0) = 0, \\ \varphi_0'(0) = \varphi_1'(0) = \varphi_2'(0) = 0. \end{cases}$$

Moreover, we suppose that $\lim_{x\to 0} \frac{\varphi_1(x)-\varphi_0(x)}{\varphi_2(x)-\varphi_0(x)}$ is finite and does not vanish. Here contrary to the previous section we do not allow that this limit may vanish. The main reason is the use of a lifting trace result in a strip (see Theorem 3.2).

We shall study the regularity of the variational solution of (2.1) in that case. For this purpose, we firstly make a change of variables in order to come back to the case of straight interface. Unfortunately, we cannot directly take advantage of the results from section 2, since this change of variables leads also to a Dirichlet transmission problem but with nonhomogenous interface conditions (corresponding to tangential derivatives, see below). Therefore, we follow step by step the techniques from section 2 but with the necessary adaptations.

We skip the proof of some results due to their similarity with some proofs from section 2.

3.1. First change of variables

Let us set $X = x$, $Y = y - \varphi_0(x)$. The image of U by this change of variables is the open set $G = G_1 \cup G_2 \cup \Gamma$, where

$$\begin{aligned} G_1 &= \{(X,Y) \in \mathbb{R}^2; 0 < X < a, \varphi_1(x) - \varphi_0(x) < Y < 0\}, \\ G_2 &= \{(X,Y) \in \mathbb{R}^2; 0 < X < a, 0 < Y < \varphi_2(x) - \varphi_0(x)\}, \\ \Gamma &= \{(X,0) \in \mathbb{R}^2; 0 < X < a\}. \end{aligned}$$

We set

$$\begin{aligned} u_0(X,Y) &= u(x,y), \\ f_0(X,Y) &= f(x,y). \end{aligned}$$

Then we have $u(x,y) = u_0(x, y - \varphi_0(x))$ and therefore

$$\begin{aligned} D_x u_i &= D_X u_{0,i} - \varphi_0'(x) D_Y u_{0,i}, \\ D_x^2 u_i &= D_X^2 u_{0,i} - \varphi_0'(x) D_{XY}^2 u_{0,i} - \varphi_0''(x) D_Y u_{0,i} - \varphi_0'(x) \\ &\quad \left[D_X Y^2 u_{0,i} - \varphi_0'(x) D_Y^2 u_{0,i}\right], \\ &= D_X^2 u_{0,i} - 2\varphi_0' D_{XY}^2 u_{0,i} - \varphi_0'' D_Y u_{0,i} + {\varphi_0'}^2 D_Y^2 u_{0,i}, \\ D_y u_i &= D_Y u_{0,i}, \\ D_y^2 u_i &= D_Y^2 u_{0,i}. \end{aligned}$$

On the interface $y = \varphi_0(x)$ we have

$$\frac{\partial u}{\partial \nu_2} = \frac{\varphi_0'(x)}{\sqrt{1+(\varphi_0'(x))^2}} D_x u - \frac{1}{\sqrt{1+(\varphi_0'(x))^2}} D_y u.$$

Consequently the interface condition $\sum_{i=1}^{2} p_i \frac{\partial u_i}{\partial \nu_i} = 0$ is equivalent to

$$\sum_{i=1}^{2} (-1)^i p_i \left[\varphi_0'(x) D_x u_i - D_y u_i\right] = 0 \text{ on } \Sigma_0.$$

In view of the previous expressions of $D_x u_i$ and $D_y u_i$, this is equivalent to

$$\sum_{i=1}^{2} (-1)^i p_i \left\{ D_Y u_{0,i} - \frac{\varphi_0'}{1+{\varphi_0'}^2} D_X u_{0,i} \right\} = 0 \text{ on } \Gamma.$$

Summing up, the problem (2.1) becomes

$$\begin{cases} (-\Delta + P_0) u_{0,i} = f_{0,i} & \text{in } G_i, i = 1,2, \\ u_{0,i} = 0 & \text{on } \partial G_i \setminus \Gamma, \\ u_{0,1} = u_{0,2} & \text{on } \Gamma, \\ \sum_{i=1}^{2} (-1)^i p_i \left(D_Y u_{0,i} - \frac{\varphi_0'}{1+{\varphi_0'}^2} D_X u_{0,i} \right) = 0 & \text{on } \Gamma, \end{cases} \tag{3.1}$$

where

$$P_0 u = 2\varphi_0' D_{XY}^2 u + \varphi_0'' D_Y u - {\varphi_0'}^2 D_Y^2 u.$$

3.2. Second change of variables

Let us set $\varphi(x) = \varphi_2(x) - \varphi_0(x)$,

$$t = -\int_x^{+\infty} \frac{d\sigma}{\varphi(\sigma)}, \qquad \theta = \frac{Y}{\varphi(X)},$$

and let

$$w(t,\theta) = \varphi^{-1} u_0(X,Y),$$

$$g(t,\theta) = f_0(X,Y).$$

Using the calculations from section 2, we obtain the identities

$$\begin{aligned}
D_X u_{0,i} &= \varphi' w_i + D_t w_i - \theta\varphi' D_\theta w_i,\\
D_X^2 u_{0,i} &= \frac{1}{\varphi}\{\varphi\varphi'' w_i + \varphi' D_t w_i - \theta\varphi\varphi'' D_\theta w_i + D_t^2 w_i\\
&\quad +\theta^2 {\varphi'}^2 D_\theta^2 w_i - 2\theta\varphi' D_{t\theta}^2 w_i\},\\
D_Y u_{0,i} &= D_\theta w_i\\
D_Y^2 u_{0,i} &= \frac{1}{\varphi} D_\theta^2 w_i,\\
D_{XY}^2 u_{0,i} &= D_X(D_\theta w_i) = \frac{1}{\varphi}\{D_{t\theta}^2 w_i - \theta\varphi' D_\theta^2 w_i\}.
\end{aligned}$$

In the variables (t,θ) the interface condition

$$\sum_{i=1}^{2}(-1)^i\left[p_i D_Y u_{0,i} - p_i \frac{\varphi_0'}{1+{\varphi_0'}^2} D_X u_{0,i}\right] = 0 \quad \text{on } \Gamma$$

is equivalent to

$$\sum_{i=1}^{2}(-1)^i p_i D_\theta w_i = \frac{\varphi_0'}{1+{\varphi_0'}^2}\sum_{i=1}^{2}(-1)^i p_i(\varphi' w_i + D_t w_i) \quad \text{on } \Sigma.$$

Therefore, with the second change of variables, problem (3.1) becomes

$$\left\{\begin{array}{ll}
(-\Delta + L)w_i = \varphi g_i(t,\theta) & \text{in } \Omega_i,\\
w_i = 0 & \text{on } \partial\Omega_i \setminus \Sigma,\\
w_1 = w_2 & \text{on } \Sigma,\\
\sum_{i=1}^{2}(-1)^i p_i D_\theta w_i = h_i & \text{on } \Sigma,
\end{array}\right. \tag{3.2}$$

where

$$\begin{aligned}
Lw_i = &-\varphi\varphi'' w_i - \varphi' D_t w_i + \varphi(\theta\varphi'' + \varphi_0'') D_\theta w_i\\
&- \left[{\varphi_0'}^2 + \theta\varphi'(\theta\varphi' + 2\varphi_0')\right] D_\theta^2 w_i + 2(\theta\varphi' + \varphi_0') D_{t\theta}^2 w_i,
\end{aligned}$$

$$h_i = \frac{\varphi_0'}{1+{\varphi_0'}^2}\sum_{i=1}^{2}(-1)^i p_i(\varphi' w_i + D_t w_i).$$

Then we get a transmission problem similar to the transformed problem in section 2 with the difference that the second transmission condition is not homogeneous.

In order to obtain a similar regularity result, it therefore suffices to show in that case that the operator

$$T_a : u \mapsto \left\{(-\Delta u_i)_{i=1,2}, \sum_{i=1}^{2} p_i \frac{\partial u_i}{\partial \nu_i}|_\Sigma\right\}$$

is an isomorphism from $H_0^1(\Omega) \cap PH^2(\Omega)$ onto $L^2(\Omega) \times \tilde{H}^{\frac{1}{2}}(]-\infty, b[)$ and that the norm of T_a^{-1} is independent of a.

First we need to establish a lifting result. Let us set $l = -\lim_{x\to 0} \frac{\varphi_1(x)-\varphi_0(x)}{\varphi_2(x)-\varphi_0(x)}$ (notice that $l > 0$), $d < \min\{l, 1\}$ and

$$B = \left\{(t, \theta) \in R^2; t < b, -\frac{2d}{3} < \theta < \frac{2d}{3}\right\}.$$

LEMMA 3.1. — *For $h \in \tilde{H}^{\frac{1}{2}}(]-\infty, b[)$, there exists $v \in H_0^1(B) \cap PH^2(B)$ satisfying*

$$v_1 = v_2 \ \text{on}\ \Sigma,$$
$$\sum_{i=1}^{2} p_i \frac{\partial v_i}{\partial \nu_i} = h \ \text{on}\ \Sigma.$$

Moreover there exists a constant $C > 0$ such that

$$\|v\|_{PH^2(\Omega)} \leqslant C \|h\|_{\tilde{H}^{\frac{1}{2}}(]-\infty,b[)}. \tag{3.3}$$

Proof. — We use the sequence of cut-off functions $(\eta_j)_{j=-\infty}^{0}$ defined in the proof of Theorem 2.3, and write

$$h = \frac{1}{2} \sum_{j=-\infty}^{0} h_j,$$

where $h_j = \eta_j h$. As $h \in \tilde{H}^{\frac{1}{2}}(]-\infty, b[)$, we conclude that $h_j \in \tilde{H}^{\frac{1}{2}}(]j-2+b, t_j[) = V_0^{\frac{1}{2}}(]j-2+b, t_j[)$ (see Theorem 1.35 of [11]). Applying Theorem 3.14 of [11], we deduce the existence of a function $v_j \in PH^2(B_j)$ (where $B_j = \{(t, \theta) \in B : j-2+b < t < t_j\}$) satisfying

$$\begin{cases} v_{j,i} = \frac{\partial v_{j,i}}{\partial \nu_i} = 0 & \text{on } \partial B_j, \\ v_{j,1} = v_{j,2} & \text{on } \Sigma_j, \\ \sum_{i=1}^{2} p_i \frac{\partial v_{j,i}}{\partial \nu_i} = h_j & \text{on } \Sigma_j, \end{cases} \tag{3.4}$$

and the existence of $C > 0$ (independent of j, since B_j is isomorphic to B_0) such that

$$\|v_j\|_{PH^2(B_j)} \leqslant C\|h_j\|_{\tilde{H}^{\frac{1}{2}}(]j-2+b,t_j[)}. \tag{3.5}$$

The claim follows by setting $v = \sum_j v_j$ (v_j being extended by zero outside B_j). Indeed one has

$$\begin{aligned} \sum_{i=1}^{2} p_i \frac{\partial v_i}{\partial \nu_i} &= \sum_{i=1}^{2} p_i \frac{\partial(\sum_j v_{j,i})}{\partial \nu_i} \\ &= \sum_j \sum_{i=1}^{2} p_i \frac{\partial v_{j,i}}{\partial \nu_i} \\ &= \sum_j h_j = h. \end{aligned}$$

The estimation (3.3) is obtained using (3.5) and the finite covering property of the B_j. □

We are now in position to state the main result of this section.

THEOREM 3.2. — *The operator T_a is an isomorphism from $H_0^1(\Omega) \cap PH^2(\Omega)$ onto $L^2(\Omega) \times H^{\frac{1}{2}}(]-\infty, b[)$ and the norm of T_a^{-1} is independent of a.*

Proof. — Let $u \in H_0^1(\Omega)$ be the variational solution of the problem

$$\begin{cases} -\Delta u_i = f_i & \text{in } \Omega_i, \\ u_1 = u_2 & \text{on } \Sigma, \\ \sum_{i=1}^{2} p_i \frac{\partial u_i}{\partial \nu_i} = h_i & \text{on } \Sigma. \end{cases}$$

We define a cut-off function ψ as follows

$$\psi(\theta) = \begin{cases} 1 \text{ if } & -d/3 \leqslant \theta \leqslant d/3, \\ 0 \text{ if } & \theta \geqslant 2d/3 \quad \text{or} \quad \theta \leqslant -2d/3. \end{cases}$$

Therefore we may write

$$u = \psi u + (1-\psi)u$$

It is easy to check that $(1-\psi)u$ is solution of a Dirichlet problem for the Laplace equation in the domain

$$B_0 = \left\{ (t,\theta) \in R^2; t < b, \theta \in \left] \frac{\varphi_1 - \varphi_0}{\varphi_2 - \varphi_0}, -\frac{d}{3} \right[\cup \left] \frac{d}{3}, 1 \right[\right\}.$$

Theorem 2.3 shows (case without interface) that $(1-\psi)u \in H^2(B_0)$ and that

$$\|(1-\psi)u\|_{H^2(B_0)} \leqslant C\|f\|_{L^2(\Omega)}. \tag{3.6}$$

On the other hand the function ψu is solution of problem

$$\begin{cases} -\Delta(\psi u_i) = F_i & \text{in } B, \\ \psi u_i = 0 & \text{on } \partial B, \\ \psi u_1 = \psi u_2 & \text{on } \Sigma, \\ \sum_{i=1}^{2} p_i \frac{\partial(\psi u_i)}{\partial \nu_i} = h_i & \text{on } \Sigma, \end{cases}$$

where

$$F_i = \psi f_i - 2\psi' D_\theta u_i - \psi'' u_i \in L^2(B).$$

According to Lemma 3.1, there exists a function $v \in H_0^1(B) \cap PH^2(B)$ such that

$$\begin{cases} v_1 = v_2 & \text{on } \Sigma, \\ \sum_{i=1}^{2} p_i \frac{\partial v_i}{\partial \nu_i} = h_i & \text{on } \Sigma. \end{cases}$$

Let us set $u_0 = \psi u - v$. It is clear that $u_0 \in H_0^1(\Omega)$ and is solution of the transmission problem

$$\begin{cases} -\Delta u_0 = (F_i + \Delta v_i) \in L^2(B), & \\ u_{0,1} = u_{0,2} & \text{on } \Sigma, \\ \sum_{i=1}^{2} p_i \frac{\partial u_{0,i}}{\partial \nu_i} = 0 & \text{on } \Sigma. \end{cases}$$

Then we come back to a problem with homogeneous transmission conditions. We can then follow the same technique as in the proof of Theorem 2.3 (in a simpler case since the domain B is a straight half-strip) and show that $u_0 \in PH^2(B)$, with the estimation

$$\|u_0\|_{PH^2(B)} \leqslant C\|\Delta u_0\|_{L^2(B)}, \tag{3.7}$$

where the constant C does not depend on a.

As $\psi u = u_0 + v$, we conclude that $\psi u \in PH^2(B)$, and by (3.3) and (3.7) that it fulfils

$$\begin{aligned} \|\psi u\|_{PH^2(B)} &\leqslant C\left\{\|F\|_{L^2(\Omega)} + \|\Delta v\|_{L^2(B)} + \|h\|_{\tilde{H}^{\frac{1}{2}}(]-\infty,b[)}\right\} \\ &\leqslant C\left\{\|f\|_{L^2(\Omega)} + \|h\|_{\tilde{H}^{\frac{1}{2}}(]-\infty,b[)}\right\}. \end{aligned}$$

This estimate with (3.6) implies that

$$\|u\|_{PH^2(B)} \leqslant C\left\{\|f\|_{L^2(\Omega)} + \|h\|_{\tilde{H}^{\frac{1}{2}}(]-\infty,b[)}\right\}.$$

Arguing as in subsection 2.4. we deduce that for a sufficiently small, the operator

$$u \mapsto \left\{ [(-\Delta + L)u_i]_{i=1,2}, \sum_{i=1}^{2} p_i \frac{\partial u_i}{\nu_i} + Mu \right\}$$

where

$$Mu = -\frac{\varphi_0'}{1 + {\varphi_0'}^2} \sum_{i=1}^{2} (\varphi' u_i + D_t u_i)$$

is an isomorphism between the same spaces. That proves the analogue of Theorem 2.10. In addition, the same arguments as in subsection 2.5 lead to the same increase of regularity for u_0 as in Proposition 2.11, and consequently the same for the solution u of (2.1), because for the first change of variables we have $u \in L^2(U)$ if and only if $u_0 \in L^2(G)$. $\square$

In summary, we have showed that Theorem 2.12 holds in the case of a curved interface.

4. Edge behavior in 3D

Since we want to describe the regularity along a cuspidal edge, it suffices to consider the infinite three dimensional domain $Q = U \times \mathbb{R}$ with basis U defined either as in section 2 or as in section 3. The coordinates will be denoted by $x = (x_1, x_2, x_3)$ with $x' = (x_1, x_2) \in U$ and $x_3 \in \mathbb{R}$.

We shall consider the following interface problem in Q

$$\begin{cases} -\Delta u_i = g_i & \text{in } Q_i,\ i = 1, 2, \\ u = 0 & \text{on } \partial Q, \\ u_1 = u_2 & \text{on } \Sigma_0 \times \mathbb{R}, \\ \sum_{i=1}^{2} p_i \frac{\partial u_i}{\partial \nu_i} = 0 & \text{on } \Sigma_0 \times \mathbb{R}, \end{cases} \tag{4.1}$$

where $f \in L^2(Q)$, $p(x) = p_i$ if $x \in Q_i = U_i \times \mathbb{R}$, ν_i denotes the unit normal vector to $\Sigma_0 \times \mathbb{R}$ directed outside Q_i.

We easily check that this problem admits a unique variational solution $u \in H_0^1(Q)$ which satisfies

$$\int_Q p \nabla u \cdot \nabla \bar{v}\, dx = \int_Q p g \bar{v}\, dx, \forall v \in H_0^1(Q).$$

Let $\hat{u}$, $\hat{g}$ be the partial Fourier transform with respect to x_3 of u, g respectively. Then $\hat{u}$ is the variational solution of

$$\int_U p\nabla'\hat{u}\cdot\nabla'\bar{v}\,dx' + \xi^2\int_U p\hat{u}\cdot\bar{v}\,dx' = \int_U p\hat{g}\bar{v}\,dx', \forall v \in H_0^1(U),$$

where ∇' denotes the (partial) gradient in x'. In the above identity taking $v = \hat{u}$ we directly get

$$|\xi|^2\|\hat{u}\|_{0,U} \leqslant C\|\hat{g}\|_{0,U}, \tag{4.2}$$

and

$$\|\nabla'\hat{u}\|_{0,U}^2 \leqslant C\|\hat{g}\|_{0,U}\|\hat{u}\|_{0,U},$$

for some positive constant C. The two last estimates lead to

$$|\xi||\hat{u}|_{1,U} \leqslant \|\hat{g}\|_{0,U}. \tag{4.3}$$

On the other hand, $\hat{u}$ satisfies (2.1) with $f = \hat{g} - \xi^2\hat{u} \in L^2(U)$. Applying Theorem 2.12, we get $\hat{u} \in PH^2(U)$ and

$$\|\hat{u}\|_{PH^2(U)} \leqslant C\|\hat{g} - \xi^2\hat{u}\|_{0,U}.$$

With (4.2) and (4.3) this yields

$$\|\hat{u}\|_{PH^2(U)} + |\xi||\hat{u}|_{H^1(U)} + |\xi|^2\|\hat{u}\|_{L^2(U)} \leqslant C\|\hat{g}\|_{L^2(U)}.$$

By inverse Fourier transform, this estimate shows that $u \in PH^2(Q)$.

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